

# TOWARDS A TOPOLOGICAL PROOF OF THE FOUR COLOR THEOREM IV

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ABSTRACT. We formulate conditions how to simplify  $S(x)$  so that it becomes Eulerian. One major issue is the “end game”. What happens if we have very few odd degree vertices left so that its no more possible to subdivide? Fortunately, there are constraints, which prevent a lock up.

Notation from [1, 2] and previous notes. The standing aim is to show:

**Proposition 1.** *Given  $S \in \mathcal{S}_2$  and a subgraph  $U \subset S$  which is a disk for which every vertex has even degree, as well as a triangle  $T$  in  $S \setminus U$ . Without cutting edges in  $U \cup T$  we can render all vertices in  $S \setminus U$  be Eulerian.*

This is no problem if  $U$  is small. But at the very end, if  $U$  is large we have to worry about situations like that  $U \cup T$  covers everything. This would be a disaster as we could no more cut. We need a lemma:

**Lemma 2.** *The support of the set of odd degree vertices of a graph  $G \in \mathcal{S}_2$  can not be part of a triangle.*

*Proof.* We only need to show this for pairs  $d = (a, b)$  as the number of odd degree vertices is even. A graph in  $\mathcal{S}_2$  defines a generalized geodesic flow for which odd degree vertices are fixed points. Rays can get out into any directions from such a point. If we look at an orbit starting at an odd degree vertex, it ends at such vertex. We have a permutation of the edge set. Two neighboring odd degree vertices produce one fixed point. Through every edge there is a geodesic path all closed or getting into  $W$ . This implies we can find a consistent flow on the edge set. Assume we have only two odd degree vertices and they are neighboring. We can now continue the geodesic flow through this set. Can we show that all paths starting and ending have even length? [We can also remove the connection between the two points and get an

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Eulerian disc for which all curvatures are a multiple of  $1/3$ . ] Lets look at a function which is constant on the odd degree vertex  $d$ . The level curves have even degree from now on. If there would be no other odd degree vertex, we ended up with an odd number of edges.

An other proof: If there are exactly two odd degree vertices, there is an Eulerian path starting at one and ending at the other. If the two vertices are adjacent, then there is an Eulerian cycle. Contradiction.  $\square$

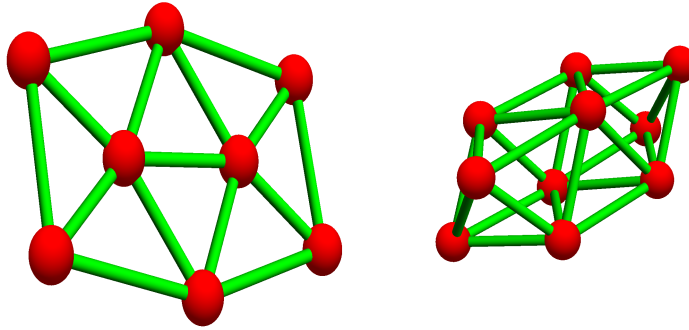


FIGURE 1. The left figure shows the Birkhoff diamond, a configuration of two degree 5 vertices which are neighboring. A double Birkhoff diamond is obtained by gluing two such diamonds together at the boundary.

## REFERENCES

- [1] O. Knill. Coloring graphs using topology. <http://arxiv.org/abs/1410.3173>, 2014.
- [2] O. Knill. Graphs with eulerian unit spheres. <http://arxiv.org/abs/1501.03116>, 2015.

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