A TREATISE

ON THE

THEORY OF DETERMINANTS

WITH GRADUATED SETS OF EXERCISES

FOR USE IN COLLEGES AND SCHOOLS

BY

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London

MACMILLAN AND CO.

1882

CHAPTER IV.

HISTORICAL AND BIBLIOGRAPHICAL SUMMARY.

§ 201. In dealing with the solution of a set of simultaneous linear equations with literal coefficients any inquiring student might experience a desire to find the law of formation of the functions which appear as numerator and denominator in the resulting values. The first mathematician who when thus occupied had genius enough to obtain a glimpse of the possibility of a Theory of Determinants was LEIBNITZ. His idea, published in 1693, does not however appear to have been developed by him, and it soon sank into oblivion. In 1750 CRAMER, independently of Leibnitz, came somewhat nearer the full conception. He gave a rule (§ 19), long afterwards known by his name, for the formation of the said functions; but like his predecessor he failed to make the most of the discovery. A simpler rule (§ 15) than Cramer's was given by BEZOUT in 1764, when studying the allied subject of Elimination: in other respects no advance was then made. In 1771, however, new ground was broken by VANDERMONDE, who made the first step towards a *notation*. Taking

$$\frac{a}{a}, \frac{a}{b}, \frac{a}{c}, \cdots$$

for the coefficients of the first equation

$$\frac{\beta}{a}, \frac{\beta}{b}, \frac{\beta}{c}, \cdots$$

for those of the second and so on, he then denoted the functions by

$$\frac{\alpha \mid \beta}{a \mid b}, \quad \frac{\alpha \mid \beta \mid \gamma}{a \mid b \mid c}, \quad \ldots\ldots$$

and wrote the law of development in the form

$$\frac{\alpha \mid \beta}{a \mid b} = \frac{\alpha}{a} \cdot \frac{\beta}{b} - \frac{\alpha}{b} \cdot \frac{\beta}{a},$$

$$\frac{\alpha \mid \beta \mid \gamma}{a \mid b \mid c} = \frac{\alpha}{a} \cdot \frac{\beta \mid \gamma}{b \mid c} + \frac{\alpha}{b} \cdot \frac{\beta \mid \gamma}{c \mid a} + \frac{\alpha}{c} \cdot \frac{\beta \mid \gamma}{a \mid b},$$

&c. Vandermonde also gave the theorem (§ 36)

$$\frac{\alpha \mid \beta \mid \gamma}{a \mid b \mid c} = -\frac{\alpha \mid \beta \mid \gamma}{b \mid a \mid c} = \ldots\ldots;$$

noted the result of making 'two letters of the same alphabet equal' (§ 27); and gave the development equation

$$\frac{\alpha \mid \beta \mid \gamma \mid \delta}{a \mid b \mid c \mid d} = \frac{\alpha \mid \beta}{a \mid b} \cdot \frac{\gamma \mid \delta}{c \mid d} - \ldots\ldots + \frac{\alpha \mid \beta}{c \mid d} \cdot \frac{\gamma \mid \delta}{a \mid b},$$

and other instances of the theorem stated in modern form in § 77. Nearly simultaneously with Vandermonde, LAPLACE made the very same important advances. He gave the same theorems, and indeed it is his name which one of them (§ 77) still bears: he introduced a make-shift notation, writing

$$(abc) \text{ for } ab'c'' + a''bc' + a'b''c - a''b'c - a'bc'' - ab''c',$$

and he made a beginning of a nomenclature, calling such functions as the said $(a\,b\,c)$ *resultants*. In 1773 some service to the subject was rendered by LAGRANGE, who incidentally gave in the ordinary non-determinant notation certain identities which are now easily recognizable as the special case of § 69 and of § 96 where $n = 3$. In the same indirect way the subject occurs in 1779 in a second work by Bezout, where there are given many simple special instances of theorems, such as that of § 85. Passing over Rothe and other writers of the Hindenburg school we next come to Gauss (1801), whose connection with the subject was also quite similar to that of Lagrange. The terms '*Determinant*' and '*adjugate*' have their origin in Gauss's work. He defined

$$b^2 - ac,$$
$$ab^2 + a'b'^2 + a''b''^2 - aa'a'' - 2bb'b'',$$

as the *determinants* of

$$ax^2 + 2bxy + cy^2,$$
$$ax^2 + a'x'^2 + a''x''^2 + 2bx'x'' + 2b'xx'' + 2b''xx',$$

respectively, and the name thus given to a special form of the functions was afterwards adopted for the functions in general. Under the name of the "fonctions *Schin*" (ש) the subject was apparently familiar to Wronski in 1811: he did not however treat of it directly till 1815, and by that time he had been forestalled by the first great master of it, Cauchy.

§ 202. Cauchy's standpoint was entirely different from that of any of his predecessors. His memoir (1812) deals professedly with *Alternating Functions*, and at the end of the First Part, the subject of which is alternating functions in general, he intimates that he will now examine an important special class of these functions, instances of which have appeared in the solution of simultaneous linear equations, in the theory of elimination, and in the investigation of the properties of binary forms. He refers to Laplace, Vandermonde, Bezout and Gauss, and says he will adopt from the latter the name 'determinant' for functions of the special kind referred to. The Second Part upon which he then enters is nothing short of a methodically arranged treatise on determinants, extending to about 60 quarto pages. His definition is of course that which is founded on the theorem of § 118: his notation is

$$S(\pm a_{1\cdot1}\, a_{2\cdot2}\, a_{3\cdot3} \ldots a_{n\cdot n}) \quad \text{and} \quad (a_{1\cdot n}).$$

He arranges $a_{1\cdot1}$, $a_{1\cdot2}$, ... in a square; speaks of 'lignes horizontales' and 'colonnes verticales'; calls $a_{2\cdot3}$ and $a_{3\cdot2}$ 'conjugués'; applies the word 'principal' to $a_{1\cdot1}$, $a_{2\cdot2}$, ..., $a_{n\cdot n}$ and to their product; denotes the co-factor of $a_{\mu,\nu}$ by $b_{\nu,\mu}$, and forming the determinant (b_{1n}) calls it 'le système adjoint au système (a_{1n})'. The theorems of §§ 24, 36, 46, 52, 62 are established by him in order, and the example of pp. 73, 74 is given as an instance of the application of the theory. Further on we find the theorem of § 95, followed by a special case of the theorem of § 96, and towards the end a more important result still, the multiplication theorem (§ 67). He even enters on the subject of the compounds *(systèmes dérivés)* of (a_{1n}) and obtains the identity

$$\Delta_m \Delta_{n-m} = \Delta^{C_{n,m}}$$

which appears on page 211.

In the light of all this and bearing in mind the isolated character of the results obtained before his time, it is not to be wondered that Cauchy has been claimed as the real founder of the theory of determinants. His predecessors had left scattered here and there stones of varied mass and usefulness; Cauchy brought them together, laid the foundation, and made progress with the superstructure.

§ 203. Simultaneously with Cauchy, BINET obtained some of the results just mentioned, the most notable being the multiplication theorem. He calls the functions 'resultants', following Laplace instead of Gauss. In 1815 Wronski discussed the Schin functions at considerable length, as has been already stated. In 1821 Cauchy, returning to the subject (which indeed he did more than once, even as late as 1847) gave a short exposition of it in his Course of Analysis for the Polytechnic School; and from about this time forward determinants were never long lost sight of. Advantage was taken of them in analytical and geometrical investigations by JACOBI, REISS (1829, 1838), LEBESGUE (1837) and CATALAN (1839). For twenty years Jacobi's writings alone would have sufficed to keep the subject before the world. In memoirs on various matters (1827, 1833, 1835 &c.) we find him repeatedly using determinants, and at length in 1841 he made them the subject of a masterly monograph. After Cauchy's his is the next great name.

§ 204. Interest in determinants would now, doubtless, never have declined: but a sudden powerful impulse was given to the study of them by the researches which the English mathematicians began about 1840 into the theory of the linear transformation of quantics. In this theory the great instrument is determinants; and men who, like Cayley and Sylvester, worked with the instrument from day to day were sure to have new properties of it and new special forms of it brought before their notice. With Cayley, there came into use what we may call 'determinant brackets', viz. the now familiar pair of upright lines, and the determinant of a system of quantities was denoted by the said system itself. This facilitated the study of special forms: and, speaking generally, the work of the forty years from then till now has been work of this kind.

§ 205. The originator of the theory and application of Continuants was Sylvester. A number of the identities in Simple Continuants were however first given in non-determinant form by Euler.

§ 206. The first start in the theory of Alternants, as may be inferred from § 202 was made by Cauchy. Since his time Jacobi, Trudi, Nägelsbach and Garbieri have made the most important advances. The name 'Alternant' is adopted from Sylvester.

§ 207. Perhaps the first theorem regarding Axi-symmetric determinants occurs in a paper of Jacobi's: almost all that is known of them is due to Lebesgue, Sylvester and Hesse.

Persymmetric determinants were so called by Sylvester. They were afterwards studied by Hankel who gave the expression in non-determinant notation for a large number of special instances.

The first theorem regarding Circulants was given by Catalan: the fundamental theorem (§ 149) however is due to Spottiswoode. Additions to the theory have recently been made by Glaisher and Scott.

For the conception of Centro-symmetric determinants we are indebted to Zehfuss, who also gave the fundamental theorem regarding them.

§ 208. The author of the important theory of Skew Determinants and Pfaffians is Cayley. The latter functions first occur with their law of development in a paper by Jacobi dealing with the subject of Pfaff's problem; hence the name which Cayley gave to them.

§ 209. Compound Determinants occur in Cauchy, as has been already noted; the name however and the origination of the theory are due to Sylvester. Of subsequent investigators the most eminent are Reiss and Picquet: the admirable but neglected work of the former probably contains all that is as yet known on the subject.

§ 210. Jacobians were so called by Sylvester in well-deserved honour of Jacobi, who was the first to introduce them and who investigated their properties so thoroughly that little has been added to the theory since his time. The second definition of a Jacobian (§ 182) is due to Bertrand: the notation following upon this to Donkin.

Wronskians were first used by Wronski and appear in his well-known expansion-theorem. After being almost unheard of for about sixty years they have recently come into marked prominence through the researches of Christoffel and Frobenius, to whom in the main we are indebted for the discovery of their properties. Wronski's early connection with determinants has been so long unrecognised, and it is so convenient to have a short name for functions of which we repeatedly require to speak, that I think mathematicians will be glad to adopt the designation which I have ventured to propose.

§ 211. The first separately published treatise on Determinants was written by Spottiswoode, and appeared in 1851. After this came Brioschi's in 1854, Baltzer's (still a standard work) in 1857, Salmon's in 1859, Trudi's in 1862, Garbieri's in 1874, Günther's in 1875, Dostor's in 1877, Baraniecki's (the most extensive of all) in 1879, and Scott's in 1880. During the same time a large number of smaller works suited for schools have also appeared: good specimens are Mansion's in French, Bartl's in German, Mollame's in Italian, and Thomson's in English.

§ 212. The student who desires detailed information regarding the History of the subject should consult the memoirs of Mellberg and Studnička and the above-mentioned text-book of Günther.

§ 213. Perhaps the fullest possible materials for both a complete Theory and a complete History are to be found in the chronologically arranged "List of Writings on Determinants" (1693-1880) published by me in the Quarterly Journal of Mathematics for October, 1881.