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§3 Integrals via Asymptotics; the Störmer Problem

(a) Integrals of the Störmer Problem

If the solutions of a Hamiltonian system all escape to infinity it is usually very easy to establish the existence of integrals in involution. We illustrate this observation with the example of a charged particle in a dipole field, which is described by the differential equations

(3.1) $$\frac{d^2}{dt^2} q = \frac{dq}{dt} \wedge B(q) \ , \quad q \in R^3 \ ,$$

where

(3.2) $$B(q) = \nabla (q_3 r^{-3}) = - \frac{\partial}{\partial q_3} \left(\frac{q}{r^3}\right) = \frac{1}{r^5} (-3q\, q_3 + e_3 r^2)$$

with $r = |q|$.

This system has the Hamiltonian

(3.3) $$H = \frac{1}{2}\left(\left(p_1 - \frac{q_2}{r^3}\right)^2 + \left(p_2 + \frac{q_1}{r^3}\right)^2 + p_3^2 \right)$$

as one integral (section 2 in Chapter I) and as a second integral the angular momentum:

$$G = q_1 p_2 - q_2 p_1 = q_1 \dot{q}_2 - q_2 \dot{q}_1 - \frac{q_1^2 + q_2^2}{r^3} \ ,$$

where the first term on the right hand side is the angular momentum ordinarily encountered, while the second term is the contribution due to the dipole. Is it possible to find a third integral, or more precisely, are there 3 integrals in involution which would make this system an integrable one?

We will show that in the domain given by

(3.4) $$H > 0 \quad , G > 0$$

there exist indeed 3 integrals in involution. On the other hand in some regions where $G < 0$ one has evidence that no such three integrals exist. This state of affairs which seems paradox at first has a very simple explanation. We will show that in the domain (3.4) all solutions escape and have the asymptotic behavior

$$q(t) \sim at + b + O(t^{-1}) \quad \text{as } t \to \infty \quad .$$

The vectors $a, b \in R^3$ can be viewed as functions of the initial values $q(0)$, $\dot{q}(0)$, and, moreover, a_1 , a_2 , a_3 will turn out to be 3 integrals in involution.

On the other hand there are other open regions in the phase space in which particles are trapped and in which these integrals are not defined.

One can view the situation also in another way: Locally, in a sufficiently small domain Ω of a 2n-dimensional space one can always find n integrals in involution, say $G_j(z)$. Using then the fact that

$$G_j(\phi^t(z)) = G_j(z) \quad ,$$

one can extend their domain of definition to the region of accessibility $\bigcup_t \phi^t(\Omega)$. But this may give rise to multiple valuedness of the extended functions if the orbits through Ω return to Ω. If, however, they escape to ∞ without recurrence then such an extension is indeed possible. Thus the difficulties of finding integrals in the large are closely tied to the recurrence of the orbits. The point of this section is to show that this difficulty disappears in case the orbits escape. This example is presented for its instructional value; otherwise it has little importance. The differential equations (3.1), however, played an important role in the study of the motion of charged particles in the magnetic field of the earth. Early numerical studies were made by Störmer and therefore it is often referred to as the Störmer problem.

(b) Escape of the Solutions

Next we show that any solution with $G > 0$, $H > 0$ satisfies

$$(3.5) \qquad q(t) = at + b + O(t^{-1}) \qquad \text{for } t \to \infty .$$

To prove this remark we compute

$$(3.6) \qquad \frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 = \langle q, \ddot{q}\rangle + |\dot{q}|^2 = \langle q, \dot{q}\wedge B\rangle + |\dot{q}|^2 .$$

By (3.2) we have

$$\langle q, \dot{q}\wedge B\rangle = -\langle \dot{q}, q \wedge B\rangle = \frac{-1}{r^3}\langle \dot{q}, q\wedge e_3\rangle = \frac{q_1\dot{q}_2 - q_2\dot{q}_1}{r^3}$$

$$= \frac{1}{r^3}\left(G + \frac{q_1^2 + q_2^2}{r^3}\right) > 0$$

since

$$\dot{q}_1 = p_1 - \frac{q_2}{r^3} , \quad \dot{q}_2 = p_2 + \frac{q_1}{r^3} , \quad \dot{q}_3 = p_3$$

hence

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 \quad > \quad |\dot{q}|^2 = 2H$$

and $H = \frac{1}{2}|\dot{q}|^2$. Therefore the velocity is a constant and without loss of generality we may take $|\dot{q}| = 1$ or $H = \frac{1}{2}$. Then

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 \geq 1$$

and by integration

$$|q|^2 \geq t^2 - c_1|t| - c_2 \quad \text{for all } t.$$

With this information we go into the differential equation (3.1). Using $B(q) = 0(|q|^{-3}) = 0(t^{-3})$ we obtain $\ddot{q} = 0(t^{-3})$ which yields (3.5) by integration. We also obtain

$$(3.7) \quad \dot{q}(t) = a + 0(t^{-2}) \; ; \quad p(t) = a + 0(t^{-2}).$$

We remark that the solution does not pass through $q = 0$, since

$$|q| \geq \frac{G}{\sqrt{2H}} > 0$$

in the region $H > 0$ and $G > 0$. This estimate is a simple consequence from the formula (3.10) below and the following discuss under (c).

We reformulate this result: We combine (q,p) to a vector $z \in R^6$ and denote the flow by $z \to \phi^t(z)$. Similarly, let

$$\phi_0^t(z) = \begin{pmatrix} q + tp \\ p \end{pmatrix}$$

denote the "free flow" corresponding to $B = 0$. Then (3.5) and (3.7) can be expressed by saying that the limit

$$(3.8) \qquad \phi_0^{-t} \circ \phi^t(z) \to \psi(z) \qquad \text{for } t \to +\infty$$

exists and maps $z = (q,p)$ into the vector (b,a).

We will use - but not prove here - that in (3.8) also the derivatives of $\phi_0^{-t} \circ \phi^t$ converge to the corresponding derivatives of ψ uniformly for z in a compact domain. Since ϕ_0^{-t} and ϕ^t both are canonical maps, it therefore follows that also ψ is a canonical map. This map ψ assigns the "scattering data at $t = +\infty$" $(b,a) = \psi(q,p)$ to the initial values (q,p) at $t = 0$ of a solution. This way the components a_j, b_j of a,b become differentiable functions of q,p, moreover

$$\{a_j, a_k\} = 0, \quad \{a_j, b_k\} = -\delta_{jk}, \quad \{b_j, b_k\} = 0,$$

since ψ is canonical. Finally we show that

$$(3.9) \qquad \psi \circ \phi^s = \phi_0^s \circ \psi,$$

i.e. ψ maps the given flows ϕ^t into the free flow ϕ_0^t. To prove this we replace t by $t + s$ in (3.8) so that

$$\begin{aligned}
\psi(z) &= \lim_{t \to +\infty} \phi_0^{-(t+s)} \circ \phi^{t+s}(z) \\
&= \lim_{t \to +\infty} \phi_0^{-s} \circ (\phi_0^{-t} \circ \phi^t)(\phi^s(z)) \\
&= \phi_0^{-s} \circ \psi \circ \phi^s(z)
\end{aligned}$$

proving our claim.

Since the free flow is described by the Hamiltonian

$$H_0 = \frac{1}{2} |a|^2$$

we conclude from (3.9) that

$$H_0 \circ \psi = H$$

on the domain $H > 0$ and $G > 0$. Furthermore, since a_j are integrals in involution of $\dot{X}_{H_0}$ we have in

$$F_j(q,p) = a_j \circ \psi\ (q,p) \quad , \quad j = 1, 2, 3 \ ,$$

three integrals in involution of the given system. It is also clear that dF_j are linearly independent, and our claim is proven, namely that (3.1) is integrable in $G > 0$, $H > 0$.

(c) Allowed region for the Störmer problem

We give a rough description of the cases when $G < 0$. For this purpose it is useful to introduce cylinder coordinates

$$q_1 = \rho \cos \theta \quad , \quad q_2 = \rho \sin \theta \quad , \ q_3 = z$$

and extend it to a canonical transformation with the generating function

$$W = \rho(p_1 \cos \theta + p_2 \sin \theta) + p_3 z \quad .$$

The equation

$$p_\theta = \frac{\partial W}{\partial \theta} \quad , \quad p_\rho = \frac{\partial W}{\partial \rho} \quad , \ p_z = \frac{\partial W}{\partial z}$$

can be solved:

$$\begin{cases} p_1 = -\dfrac{p_\theta}{\rho} \sin\theta + p_\rho \cos\theta \\ p_2 = \dfrac{p_\theta}{\rho} \cos\theta + p_\rho \sin\theta \\ p_3 = p_z \end{cases},$$

and one finds

$$\begin{cases} H = \frac{1}{2} \left(p_\rho^2 + p_z^2 + \left(\dfrac{p_\theta}{\rho} + \dfrac{\rho}{r^3} \right)^2 \right) \\ G = p_\theta \end{cases}.$$

Hence θ does not occur explicitly in H, i.e. θ is an ignorable variable, which simply expresses that $p_\theta = G$ is an integral. If we assign G a constant value

$$G = 2\gamma$$

the Hamiltonian system can be written as

$$\ddot{\rho} = -\frac{1}{2}\frac{\partial V}{\partial \rho}, \qquad \ddot{z} = -\frac{1}{2}\frac{\partial V}{\partial z}, \qquad V = \left(\frac{2\gamma}{\rho} + \frac{\rho}{r^3} \right)^2$$

with Hamiltonian

(3.10) $$H = \frac{1}{2} \left(\dot{\rho}^2 + \dot{z}^2 + V(\rho, z) \right).$$

As above we restrict ourselves to the domain

$$D(\gamma) = \{ q, p \mid G = 2\gamma ,\ H = \tfrac{1}{2} \}.$$

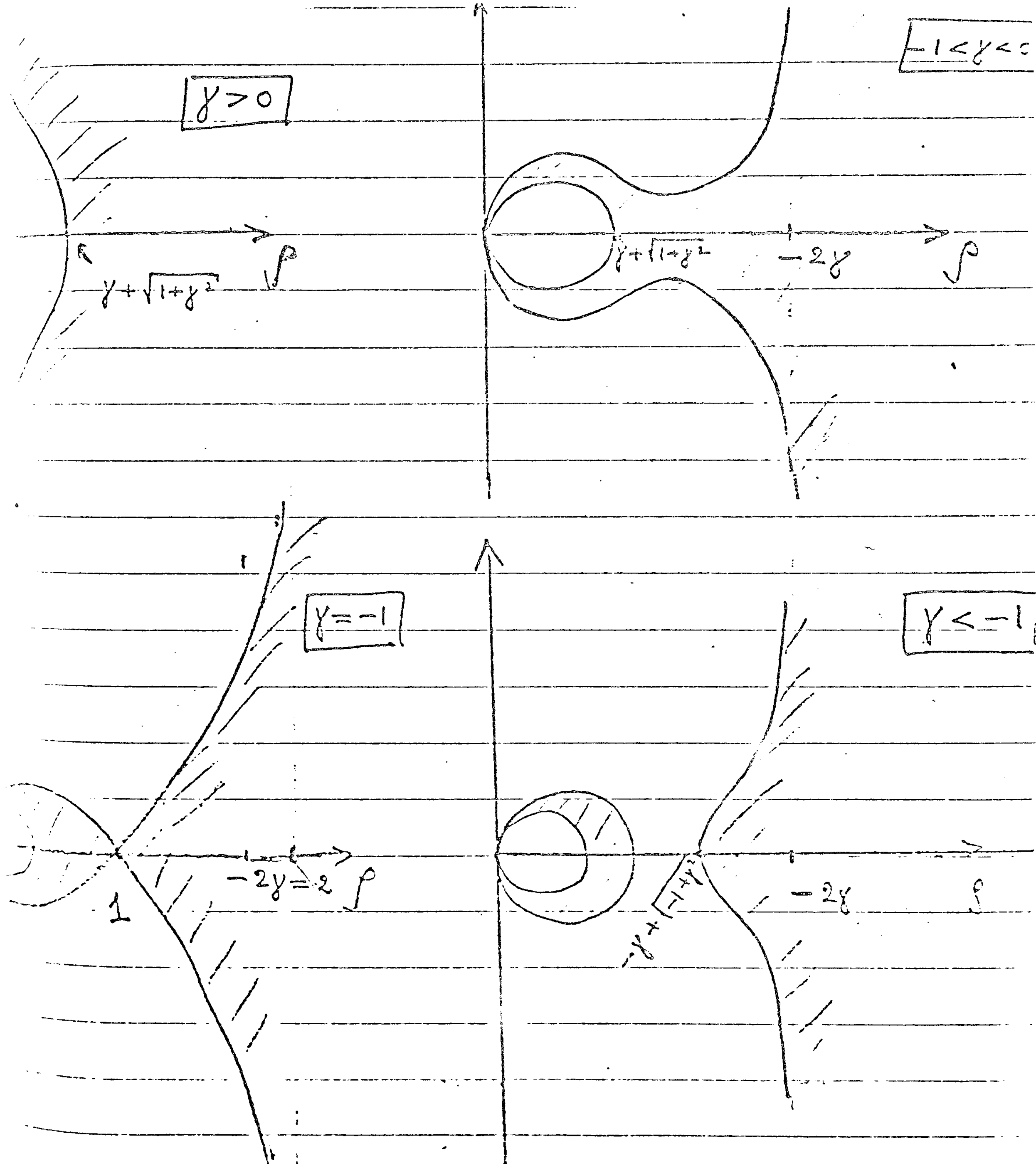
We project this domain into the configuration space and obtain by (3.10)

$$\Omega(\gamma) = \{ \rho, z \mid \rho \geq 0 ,\ V \leq 1 \} ;$$

on the boundary one has $\dot{\rho}^2 + \dot{z}^2 = 0$. For this reason the boundary curves of $\Omega(\gamma)$ are called zero-velocity curves. They are given by

$$\frac{2\gamma}{\rho} + \frac{\rho}{r^3} = \pm 1 .$$

Clearly for $\gamma \geq 0$ only the plus sign can occur. But for $\gamma < 0$ both signs occur and $\Omega(\gamma)$ has two components if $\gamma < -1$. Below the allowable regions $\Omega(\gamma)$ are sketched in four cases $\gamma > 0$, $-1 < \gamma < 0$, $\gamma = -1$, $\gamma < -1$.



Incidentally, for $\gamma = -1$ the point of intersection $(\rho, z) = (1,0)$ corresponds to a circular periodic orbit for the original problem (3.1).

(d) The case $\gamma < -1$

The phase space is restricted by prescribing the values of the integrals H, G. In particular, for $\gamma < -1$ the domain $D(\gamma)$ has two components:

$$D(\gamma) = D_1(\gamma) \cup D_2(\gamma) \quad ,$$

where $D_1(\gamma)$ is the bounded region and $D_2(\gamma)$ the unbounded region. If we abandon the normalization of $H = \frac{1}{2}$ the condition $\gamma < -1$ has to be replaced by $\gamma < -\sqrt{2H}$ or $G < -\sqrt{H/2}$. We conclude that the domain

$$\{ q, p \mid G < -\sqrt{H/2} \} = I \cup II$$

has two components I, II: I bounded, II unbounded.

In conclusion we show that our system is integrable in the unbounded component II while one has evidence for nonexistence of such integrals in I - due to the presence of homoclinic orbits.

In the domain II all solutions escape again and the previous argument is applicable. Indeed, going back to the normalization $H = \frac{1}{2}$ we have in II

$$p = \sqrt{q_1^2 + q_2^2} \geq -\gamma + \sqrt{-1 + \gamma^2} \geq 1 + \delta \ , \quad \delta > 0 \tag{3.11}$$

and

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 = \langle q, \dot{q} \wedge B \rangle + |\dot{q}|^2$$

$$= \frac{1}{r^3}\left(2\gamma + \frac{\rho^2}{r^3}\right) + 1$$

$$= \frac{\rho}{r^3}\left(\frac{2\gamma}{\rho} + \frac{\rho}{r^3}\right) + 1$$

$$\geq -\frac{\rho}{r^3} + 1 \geq -\frac{1}{\rho^2} + 1 ,$$

since

$$\frac{2\gamma}{\rho} + \frac{\rho}{r^3} \geq -1$$

in II. Hence, by (3.11)

$$\frac{1}{2}\left(\frac{d}{dt}\right)^2 |q|^2 \geq 1 - \frac{1}{(1+\delta)^2} > 0 ,$$

which shows again that $|q| \to \infty$ and $|q|^{-1} = 0(t^{-1})$. The rest of the argument is the same.