Structural Stability and the Störmer Problem

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Introduction. The motion of a charged particle under the influence of the earth's magnetic field may be calculated to first approximation by considering the field to be equivalent to that of a magnetic dipole situated at the center of the earth. The question of existence and uniqueness (for fixed values of energy and angular momentum) of an orbit entering the dipole has long been of interest to mathematicians and physicists. In the early 1900's, Störmer [4] found a formal series expansion, which, if it converged, would represent a trajectory running into the singularity. From this expansion, and many numerical calculations, he was led to conjecture the existence and *uniqueness* of a "Störmer orbit" entering the dipole. In 1944 the Swedish mathematician Malmquist [3] succeeded in proving the existence of at least one such solution whose asymptotic expansion was given by the formal series of Störmer. In this paper we will first present an alternate proof for the existence of a Störmer orbit, and then establish its uniqueness. We will then be able to "read off" the behavior of all trajectories in the vicinity of the dipole. Moreover, our method will be applicable to more general Hamiltonian systems of two degrees of freedom.

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1. A fundamental theorem.

a. In this section we prove a theorem concerning the behavior of solutions of a differential equation in the neighborhood of an equilibrium point, and in Section 2 we will apply this theorem to the dipole field. We state our result as follows:

Theorem. Let q, y_1, y_2 *be three real coordinates and set* $y = y_1 + iy_2$, $r =$

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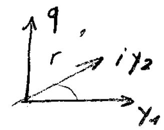

$(q^2 + |y|^2)^{1/2}$. *Consider the system of equations*

$$\frac{dq}{d\tau} = r^a + f(q, y, \bar{y}), \quad a > 1, \tag{1.1}$$

$$\frac{dy}{d\tau} = -ih(q, y, \bar{y})y + g(q, y, \bar{y}),$$

where h and f are real valued, g complex valued C^2 functions. Set

$$||\phi||_R = \sup_{r \le R} \sum_{k=0}^{2} r^k \, |\partial^k \phi|,$$

and assume

$$(*) \qquad ||f||_R = O(R^{a+\epsilon}); \qquad ||g||_R = O(R^N); \qquad ||h - h(o, o, o)||_R = O(R^c),$$

with $c, \epsilon > 0$.

Then for N sufficiently large, specifically

$$N > \max\,(a + 1, \quad 2a - c, \quad 3a - 2c - 1),$$

the orbits of (1.1) *are topologically equivalent to the orbits of* (1.1) *with* $f = g \equiv 0$. *In particular there exists a unique trajectory running into the origin* $r = 0$, *and a unique trajectory running out.* *End of Theorem.*

The structure of all solutions of (1.1) for $f = g \equiv 0$ (henceforth we shall denote this system by (1.1)′) is quite simple. All trajectories lie on the cylinders $y_1^2 + y_2^2 =$ constant and the coordinate q increases monotonically (see Figure 1) with time τ. The negative q axis is the unique trajectory entering the origin, and the positive q axis is the unique trajectory leaving the origin. The content of our theorem is simply that if the perturbation g is small enough, *i.e.* if all orbits, to a sufficiently high order, remain on the cylinders $|y|^2 =$ constant, then the orbit structure is essentially unchanged. Alternatively, we may say that the system (1.1)′ is *structurally stable* if we admit only perturbations f, g satisfying the conditions (*).

The first step in our proof is to establish the existence and uniqueness of an orbit running into the origin, and one running out of the origin. Observe that for $f = g \equiv 0$, the solid closed cusped like region C_+ (see Figure 2) flows into the positive q axis as $\tau \to \infty$, and C_- flows into the negative q axis as $\tau \to -\infty$. By this we mean the following: Given $\hat{q} > 0$, $\epsilon > 0$, there exists $T_0(\hat{q}, \epsilon)$ such that if $(q(\tau), y(\tau))$ is the image at time τ of a point (q_0, y_0) in C_+, and $q(\tau) < \hat{q}$, then $|y(\tau)| < \epsilon$. The main idea of our proof will be to show that if C_+ is small enough, then it flows under (1.1) into a unique solution leaving the origin, and C_- flows, as $\tau \to -\infty$, into a unique trajectory entering the origin. To this end we consider the mapping M induced by the differential equation by following all solutions from their initial values to their value at time $\tau = T$. This mapping assumes the form

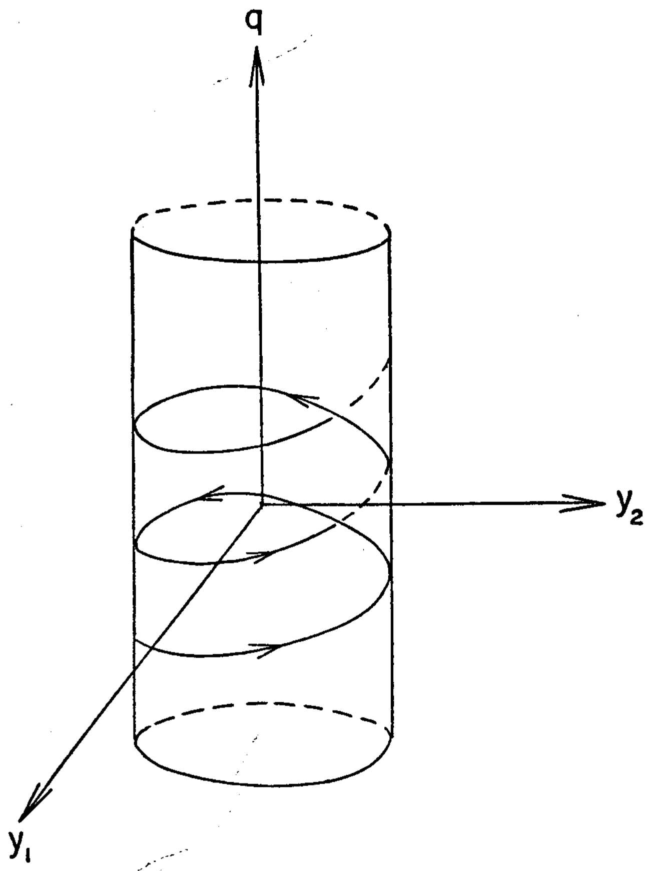

FIGURE 1. Orbits of (1.1) with $f = g \equiv 0$.

$$(1.2) \qquad \begin{aligned} q_1 &= q + Tr^a + \tilde{f}(q, y, \bar{y}) = F(q, y, \bar{y}), \\ M : y_1 &= e^{-i\omega}y + \tilde{g}(q, y, \bar{y}) = G(q, y, \bar{y}), \\ \omega &= \int_0^T h(q(\tau), y(\tau), \bar{y}(\tau))\, d\tau, \end{aligned}$$

where $\tilde{f}$ and $\tilde{g}$ satisfy the same assumptions as f and g in (*). Any trajectory of (1.1) entering the origin is an *invariant curve* of the mapping M. Conversely, any invariant curve of M entering the origin gives rise to a solution of (1.1) approaching the origin if we take the orbit issuing forth from any point on this curve. Similarly for orbits leaving the origin. Our method of proof will consist of first establishing the existence of invariant curves of the mapping M and then showing that these (2) invariant curves are indeed the desired solutions of (1.1) entering and leaving the origin.

b. Existence of Invariant Curves of M. In the following we will take $T > 0$ and prove the existence of an invariant curve $y = \phi(q)$ of the mapping M where

ϕ is a complex valued function defined for $q \geqq 0$ and $\phi(0) = 0$. In an exactly analogous manner one can show the existence of an invariant curve of M entering the origin. Our existence proof is based on an iteration procedure: We start with a trial curve $y = \phi_0(q)$ and look at its image curve $y = \phi_1(q)$ under the mapping M. From (1.2) ϕ_1 satisfies the functional equation

$$\phi_1(F(q, \phi_0, \bar{\phi}_0)) = G(q, \phi_0, \bar{\phi}_0). \tag{1.3}$$

We then define the curve $y = \phi_k(q)$ as the kth iterate under M of $y = \phi_0(q)$, and show that $\phi_k(q) \to \phi(q)$ uniformly in $0 \leqq q \leqq$ constant. The curve $y = \phi(q)$ will then be the desired invariant curve of M leaving the origin. This will be the justification of our intuitive idea that the region C_+ flows into a unique solution of (1.1) leaving the origin, since $y = \phi_k(q)$ is the image, at time $\tau = kT$ of the initial curve $y = \phi_0(q)$.

Uniform boundedness of iterates: We will now determine $\alpha > 0$ so that if $|\phi_k(q)| \leqq q^{1+\alpha}$, then also $|\phi_{k+1}(q)| \leqq q^{1+\alpha}$ or equivalently

$$|\phi_{k+1}(q_1)|^2 \leqq q_1^{2+2\alpha}. \tag{1.4}$$

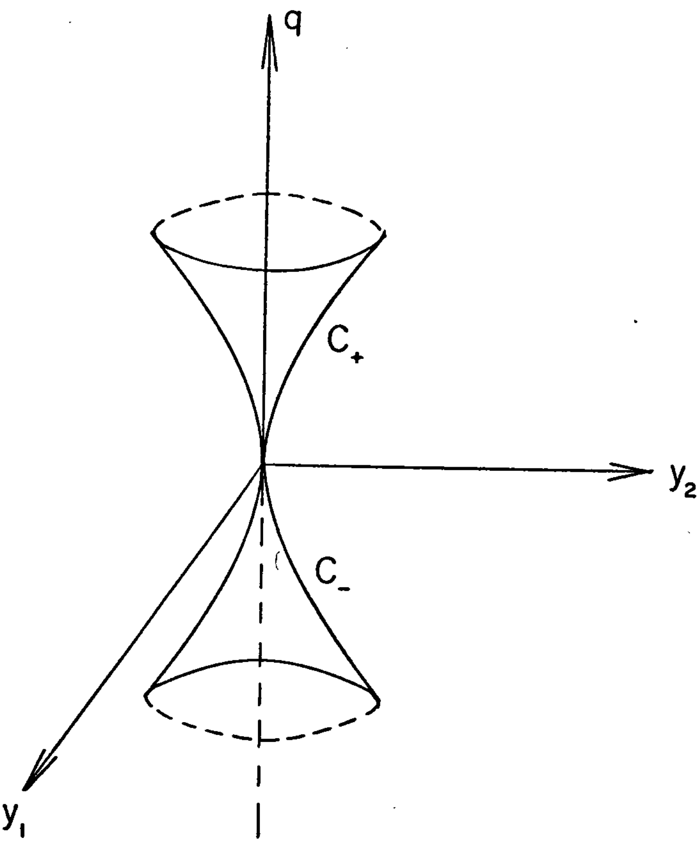

FIGURE 2

Let $x(q) = q^c + q^{2\alpha}$. Since

$$q_1 \geqq q + Tq^a - Ax(q)q^a,$$
$$|\phi_{k+1}(q_1)|^2 = |G(q, \phi_k, \bar{\phi}_k)|^2 \leqq q^{2+2\alpha} + Aq^{N+1+\alpha},$$

for q sufficiently small, $\alpha > 0$, and $A > 0$, equation (1.4) is implied by

$$(1.4)' \qquad q^{2+2\alpha} + Aq^{N+1+\alpha} \leqq q^{2+2\alpha}(1 + Tq^{a-1} - Ax(q)q^{a-1})^{1+\alpha}$$

which will be satisfied for sufficiently small $q \geqq 0$ provided we can choose $\alpha > 0$ so that

(i) $0 < \alpha < N - a$.

Thus, we require that N be greater than a.

Uniform boundedness of first derivatives of iterates. We now seek to determine $\beta > 0$ so that if $|\phi_k'| \leqq q^\beta$, then also $|\phi_{k+1}'| \leqq q^\beta$, where $\phi_k' = d\phi_k/dq$. By differentiating (1.3) and using the estimates for $\hat{f}$, $\hat{g}$ and h in (*) we see that $\beta > 0$ must be chosen so that the inequality

$$q^\beta + Aq^{\alpha+c} + Aq^{N-1} \leqq (q + Tq^a - Ax(q)q^a)^\beta(1 + aTq^{a-1} - Ax(q)q^{a-1}),$$

is satisfied for $q \geqq 0$ sufficiently small. Thus we have the additional relations

(ii) $0 < \beta < N - a$,

(iii) $\beta < 1 + \alpha + c - a$.

Uniform boundedness of second derivatives of iterates. In proving that the orbits of (1.1) are topologically equivalent to the orbits of (1.1)′ it will be crucial that the invariant curve $y = \phi(q)$ is differentiable at $q = 0$. To guarantee this we now show that

$$|\phi_k''| \leqq \gamma \Rightarrow |\phi_{k+1}''| \leqq \gamma,$$

for any fixed $\gamma > 0$ and q sufficiently small. By differentiating (1.3) twice with respect to q we are led to the inequality

$$(1.5) \qquad |\phi_{k+1}''(q_1)| \leqq \frac{\gamma + \hat{A}(q^{c-1+\beta} + q^{N-2} + q^{a-2+\beta})}{(1 + aTq^{a-1} - \hat{A}x(q)q^{a-1+\beta})^2},$$

for some constant $\hat{A} > 0$ (here we have assumed $\alpha \geqq \beta$). Inequality (1.5) will certainly be satisfied for $q \geqq 0$ sufficiently small if

(iv) $a - 1 < a - 2 + \beta$; *i.e.* $\beta > 1$,

(v) $a - 1 < N - 2$; *i.e.* $N > a + 1$,

(vi) $a - 1 < c - 1 + \beta$; *i.e.* $\beta > a - c$.

Note that (ii) and (iv) imply (v) for $a > 1$, and that (i)–(vi) can be satisfied if

$$N > \max(a + 1, 2a - c, 3a - 2c - 1).$$

From the preceding we can conclude that a subsequence ϕ_{n_k} of ϕ_k converges uniformly to a differentiable function $\phi(q)$, $0 \leqq q \leqq$ const., with $\phi'(0) = 0$. However, we cannot conclude as yet that $y = \phi(q)$ is an invariant curve for the mapping M, since the entire sequence $\phi_k(q)$ may not converge. To show that this cannot happen, we resort to a device first utilized by Hadamard [2]. Namely we will prove that if $\phi_n(q)$ and $\tilde{\phi}_n(q)$ are the n-th iterates under M of $\phi_0(q)$ and $\tilde{\phi}_0(q)$ respectively, then

$$|\phi_n(q) - \tilde{\phi}_n(q)| \to 0, \quad \text{uniformly for} \quad 0 \leqq q \leqq \text{const.}$$

provided

$$|\phi_0(q)| \leqq q^{1+\alpha}; \; |\phi_0'(q)| \leqq q^{\beta},$$

$$|\tilde{\phi}_0(q)| \leqq q^{1+\alpha}; \; |\tilde{\phi}_0'(q)| \leqq q^{\beta}.$$

Thus the entire sequence $\phi_k(q)$ converges to $\phi(q)$, $0 \leqq q \leqq$ const.

Proof. We make the substitution

$$v = \frac{y}{q^{\delta}},$$

where $0 < \delta < 1 + \alpha$. In terms of the coordinates q, v the mapping M assumes the form

$$(1.6) \qquad \begin{aligned} q_1 &= q + Tq^{a} + q\tilde{f}(q, q^{\delta}v, q^{\delta}\bar{v}) = \tilde{F}(q, q^{\delta}v, q^{\delta}\bar{v}), \\ v_1 &= \frac{e^{-i\omega}v + \tilde{g}(q, q^{\delta}v, q^{\delta}\bar{v})}{(1 + Tq^{a-1} + \tilde{f})^{\delta}} = \tilde{G}(q, v, \bar{v}), \end{aligned}$$

where

$$\tilde{g} = \frac{\hat{g}}{q^{\delta}} = O(q^{N-\delta}),$$

$$\tilde{f} \leqq Ax(q)q^{a-1} = O(q^{a-1+\epsilon} + q^{a-1+2\alpha}),$$

in the region $|v| \leqq q^{1+\alpha-\delta}$. The sequence of curves $y = \phi_k(q)$ goes over into the sequence of curves

$$v = \psi_k(q) = \frac{\phi_k(q)}{q^{\delta}}.$$

Hence,

$$|\psi_k(q)| \leqq q^{1+\alpha-\delta},$$

$$|\psi_k'(q)| \leqq Aq^{\beta-\delta},$$

where A will always denote a fixed large positive constant. The same estimates hold for $\tilde{\psi}_k(q)$. To show that $|\psi_k(q) - \tilde{\psi}_k(q)| \to 0$ we proceed as follows:

$$(1.7) \qquad \begin{aligned} |\psi_{k+1}(q_1) - \tilde{\psi}_{k+1}(q_1)| \leqq\; & |\psi_{k+1}(q_1) - \tilde{\psi}_{k+1}(\tilde{q}_1)| \\ & + |\tilde{\psi}_{k+1}(\tilde{q}_1) - \tilde{\psi}_{k+1}(q_1) \end{aligned}$$

where

$$q_1 = \tilde{F}(q, q^\delta \psi_k, q^\delta \bar{\psi}_k),$$
$$\bar{q}_1 = \tilde{F}(q, q^\delta \bar{\psi}_k, q^\delta \bar{\bar{\psi}}_k).$$

Since

$$|\bar{\psi}'_{k+1}(q_1)| \leqq A q_1^{\beta-\delta} \leqq A q^{\beta-\delta}, \qquad \text{(for a different } A\text{)}$$

and

$$|q_1 - \bar{q}_1| \leqq A x(q) q^{\alpha-1+\delta} |\psi_k(q) - \bar{\psi}_k(q)|,$$

the second part of (1.7) may be estimated by

$$(1.8) \qquad |\bar{\psi}_{k+1}(\bar{q}_1) - \bar{\psi}_{k+1}(q_1)| \leqq A x(q) q^{\beta-\delta} q^{\alpha-1+\delta} |\psi_k(q) - \bar{\psi}_k(q)|.$$

Applying the Mean Value Theorem to $\tilde{G}$ in (1.6) we obtain

$$(1.9) \qquad |\psi_{k+1}(q_1) - \bar{\psi}_{k+1}(\bar{q}_1)| \leqq \left(1 - \frac{3\delta T}{4} q^{\alpha-1}\right) |\psi_k(q) - \bar{\psi}_k(q)|.$$

Inequality (1.9) follows easily from (ii)–(iii), the estimates for v and $\tilde{g}$, and the fact that

$$\frac{1}{(1 + Tq^{\alpha-1} + \tilde{f})^\delta} = 1 - \delta T q^{\alpha-1} + O(x(q) q^{\alpha-1}).$$

Hence (1.8) and (1.9) yield

$$(1.10) \qquad |\psi_{k+1}(q_1) - \bar{\psi}_{k+1}(q_1)| \leqq \left(1 - \frac{\delta T}{2} q^{\alpha-1}\right) |\psi_k(q) - \bar{\psi}_k(q)|,$$

for $q \geqq 0$ sufficiently small. Note that the difference at the point q_1 is estimated by the difference at the point q. Iterating (1.10) k times we see that

$$|\psi_{k+1}(q) - \psi_{k+1}(q)| \leqq \mu_0 \prod_{\nu=1}^{k} \left(1 - \frac{\delta T}{2} q_{-\nu}^{\alpha-1}\right),$$

where μ_0 is an upper bound for $|\psi_0(q) - \bar{\psi}_0(q)|$. (One can take $\mu_0 = 2$.) To complete our proof we must show that

$$\prod_{\nu=0}^{k} \left(1 - \frac{\delta T}{2} q_{-\nu}^{\alpha-1}\right) \to 0 \quad \text{as} \quad k \to \infty.$$

This follows immediately from the inequality

$$(1.11) \qquad q_{-k}^{\alpha-1} \geqq \frac{q^{\alpha-1}}{1 + \alpha k T q^{\alpha-1}},$$

which we now prove. Let $u = q^{\alpha-1}$. From the relation

$$q_1 = q + Tq^\alpha + f^*(q); \qquad f^*(q) = O(q^{\alpha+\epsilon} + q^{\alpha+2\alpha}),$$

we see that

$$u_1 = u\left(1 + Tu + \frac{f^*}{q}\right)^{a-1},$$

or, more generally,

$$u_1 = u[1 + (a-1)Tu + O(u)].$$

Hence

$$\frac{1}{u_1} = \frac{1}{u}[1 - (a-1)Tu + O(u)] \geqq \frac{1}{u} - aT,$$

or equivalently,

$$\frac{1}{u} \leqq \frac{1}{u_1} + aT,$$

for u sufficiently small. Iterating this inequality k times we find that

$$\frac{1}{u_{-k}} \leqq \frac{1}{u} + akT.$$

Therefore

$$u_{-k} \geqq \frac{u}{1 + akuT},$$

which proves (1.11).

Remark. Actually, we can prove something stronger; namely that q_{-k}^{a-1} has exact order $1/k$. This will be shown in (*c*).

Thus for N sufficiently large, we have shown the existence of a differentiable curve $y = \phi(q)$, $\phi(0) = 0$ $(0 \leqq q \leqq \text{const.})$ which is an invariant curve for the mapping M. In an exactly analogous manner we prove the existence of an invariant curve of M which runs into the origin. Below we will show that these invariant curves are indeed the unique solutions entering and leaving the origin.

c. Uniqueness of invariant curves. From the discussion at the end of (a) it follows immediately that the orbits entering and leaving the origin are unique if we can establish the uniqueness of the corresponding invariant curves of the mapping M. Actually the uniqueness of these curves among the curves satisfying $|\phi| \leqq |q|^{1+\alpha}$, $|\phi'| \leqq |q|^{\beta}$, $|\phi''| \leqq \gamma$ is already established by the above argument. Now we prove uniqueness in a stronger sense. Any point $(q, y, \bar{y})$ for which the iterates under M approach the origin must lie on the above curve. Our proof proceeds as follows: From our previous work we know the existence of a curve $y = \phi(q)$ defined in a neighborhood of $q = 0$, which is invariant under M and which satisfies

$$|\phi(q)| \leqq |q|^{1+\alpha},$$
$$|\phi'(q)| \leqq |q|^{\beta}.$$

Setting $\xi = y - \phi(q)$, our mapping in the coordinates q, ξ assumes the form

(1.12) $$q_1 = q + T[q^2 + |\xi + \phi|^2]^{a/2} + \mathfrak{f}(q, \xi + \phi, \bar{\xi} + \bar{\phi}),$$

(1.13) $$\begin{aligned} \xi_1 &= y_1 - \phi(q_1) \\ &= e^{-i\omega}(\xi + \phi) + \mathfrak{g}(q, \xi + \phi, \bar{\xi} + \bar{\phi}) - \phi(q_1) \\ &= e^{-i\omega}\xi + e^{-i\omega}\phi(q) + \mathfrak{g}(q, \xi + \phi, \bar{\xi} + \bar{\phi}) - \phi(q_1). \end{aligned}$$

Note that now ω is to be considered as a function of q, ξ, $\bar{\xi}$, and in these variables satisfies the same estimates as $h(q, y, \bar{y})$ in (*). Since

$$\xi = y - \phi(q) = 0,$$

is an invariant curve of M, *i.e.* $\xi = 0 \Rightarrow \xi_1 = 0$, we may estimate the last three terms of (1.13) by the Mean Value Theorem: If $s^2 = q^2 + |\xi|^2$ and ∂ denotes a partial derivative with respect to ξ or $\bar{\xi}$, then

$$|\partial e^{-i\omega}\phi| \leqq \frac{As^{1+\alpha}}{s^{1-c}} = As^{\alpha+c},$$

$$|\partial\mathfrak{g}(q, \xi + \phi, \bar{\xi} + \bar{\phi})| \leqq As^{N-1},$$

$$|\partial\phi(q_1)| \leqq As^{a-1+\beta}.$$

Thus it is easily seen that

(1.14) $$\xi_1 = e^{-i\omega}\xi + g^*(q, \xi, \bar{\xi}),$$

where

(1.14)′ $$|g^*(q, \xi, \bar{\xi})| \leqq A|\xi|s^{\lambda},$$

for

$$\lambda = \min(\alpha + c, N - 1, a - 1 + \beta) = a - 1 + \beta,$$

by (ii) and (iii). (Strictly speaking, in using the Mean Value Theorem, the derivatives must be evaluated at an intermediate point ζ, with $0 < |\zeta| < |\xi|$. In our case, though, since $\lambda > 0$ it follows trivially that (1.14)′ is valid.) We will now show that the invariant curve approaching the origin is unique; *i.e.* $q_k, \xi_k \to 0 \Leftrightarrow \xi_0 = 0$. The uniqueness of the invariant curve leaving the origin will follow in an exactly analogous manner.

Setting $\eta_k = |\xi_k|^2$ we see from (1.14) that

$$\eta_{k+1} = \eta_k(1 + G^*(q_k, \xi_k, \bar{\xi}_k)) = \eta_k(1 + G_k^*),$$

where $|G_k^*| \leqq As_k^{\lambda}$. Hence

$$\eta_{k+1} = \eta_0 \prod_{\nu=0}^{k} (1 + G_\nu^*).$$

Now, if we can show that

$$\sum_{k=0}^{\infty} s_k^{\lambda} < \infty, \quad (\text{for } s_k \to 0), \tag{1.15}$$

then it follows immediately that $\eta_0 = 0$ since $\prod_{\nu=0}^{\infty} (1 + G_\nu^*)$ converges to a non-zero number. Hence $\eta_k \to 0$ implies $\eta_0 = 0$ as we wanted to show.

The proof of (1.15) will be divided into two parts. Firstly, we will show that

$$|\xi_k| \leqq \sigma |q_k|, \qquad \sigma > 0; \quad \text{for } s_k \to 0 \tag{1.16}$$

and secondly we will establish the inequality

$$|q_k|^{a-1} \leqq \frac{2\,|q|^{a-1}}{2 + \delta k(a-1)\,|q|^{a-1}}; \quad \delta = T(1 + \sigma^2)^{a/2}. \tag{1.17}$$

The proof of (1.15) will then follow immediately for

$$\sum s_k^{\lambda} \leqq \text{const} \sum |q_k|^{\lambda} \leqq \text{const} \sum \left(\frac{1}{k}\right)^{(\lambda/a-1)} < \infty,$$

since $\lambda > a - 1$.

Remark. From (1.17) and (1.11) we may deduce that q_{-k}, $(q > 0)$ and q_k $(q < 0)$ each have exact order $1/k$. Namely, we observe that $|q_1|$ and q_{-1} both satisfy similar equations, *i.e.*

$$|q_1| = |q| - \delta\,|q|^a + O(|q|^{a+\epsilon} + |q|^{a+2\alpha}), \quad q < 0,$$

and

$$q_{-1} = q - Tq^a + O(q^{a+\epsilon} + q^{a+2\alpha}), \quad q > 0.$$

Hence we can estimate both q_k and q_{-k} from above and below by const/k.

Proof of (1.16). Since q_k approaches zero we see from (1.12) that

$$-q_k = \sum_{\nu=k}^{\infty} (Ts_\nu^a + z_\nu),$$

where

$$\begin{aligned} z_\nu &= z_\nu(q_\nu, \xi_\nu, \bar{\xi}_\nu) \\ &= \tilde{f}(q_\nu, \xi_\nu + \phi(q_\nu), \bar{\xi}_\nu + \bar{\phi}(q_\nu)) + T[q_\nu^2 + |\xi + \phi(q_\nu)|^2]^{a/2} - Ts_\nu^a, \end{aligned}$$

and

$$\frac{z_\nu}{s_\nu^a} \to 0 \quad \text{for} \quad s_\nu \to 0.$$

(From this we may infer that $\sum_{\nu=k}^{\infty} s_\nu^a < \infty$.) From (1.14) we may write

$$|\xi_{k+1}| \geqq |\xi_k| - As_k^{\lambda+1},$$

and consequently

$$|\xi_k| \leqq A \sum_{\nu=k}^{\infty} s_\nu^{\lambda+1} \leqq A \sum_{\nu=k}^{\infty} s_\nu^{a+\beta},$$

as $\lambda + 1 > a + \beta$. Thus for k sufficiently large we have

$$|q_k| \geqq \frac{T}{2} \sum_{\nu=k}^{\infty} s_\nu^a ,$$

$$|\xi_k| \leqq \frac{A}{2} \sum_{\nu=k}^{\infty} s_\nu^a .$$

Hence

$$|\xi_k| \leqq \frac{A}{T} |q_k| = \sigma |q_k|,$$

which proves (1.16).

Proof of (1.17). By (1.16) we may restrict ourselves to the region $|\xi| \leqq \sigma|q|$. Thus, we may write (1.12) in the form

(1.18) $\quad q_1 = q + \delta|q|^a + O(x(q)|q|^a); \qquad x(q) = |q|^\epsilon + |q|^{2\alpha}.$

Setting

$$u = (-q)^{a-1},$$

we see from (1.18) that

$$\begin{aligned} u_1 &= (-q)^{a-1}[1 - \delta(-q)^{a-1} + o(x(q)q^{a-1})]^{a-1}, \\ &= u[1 - \delta(a-1)u + O(u)], \end{aligned}$$

which implies that

$$\frac{1}{u_1} \geqq \frac{1}{u} + \frac{\delta(a-1)}{2},$$

for u sufficiently small. Iterating this inequality k times we see that

$$\frac{1}{u_k} \geqq \frac{1}{u} + \frac{k\,\delta(a-1)}{2} = \frac{2 + k\,\delta(a-1)u}{2u}.$$

Hence

$$|q_k|^{a-1} \leqq \frac{2\,|q|^{a-1}}{2 + k\,\delta(a-1)\,|q|^{a-1}},$$

as we wanted to show. Thus, the invariant curves of M entering and leaving the origin are unique.

Remark 1. From the uniqueness of the invariant curves of M entering and leaving the origin, it now follows that these 2 curves are the unique solutions of

(1.1) entering and leaving the origin. Namely, we make the following observations:

(1) There exists an open ball S_2 containing the origin such that if all the forward iterates (q_k, y_k) of a point (q, y) lie in S_2, and $(q_k, y_k) \to (0, 0)$, then $y = \phi(q)$.

(2) There exists a smaller ball $S_1 \subset S_2$ such that for time $\tau \leqq T$ the orbit (under (1.1)) of any point starting in S_1 remains in S_2.

(3) About each point P of S_1 there exists a ball of radius $\delta(P)$ such that for time $\tau \leqq T$ the orbit beginning at P remains in this ball. Furthermore, $\delta(P) \to 0$ as $P \to$ the origin. Now, let Γ denote the curve $y = \phi(q)$, const $\leqq q \leqq 0$. Choose P_0 in $\Gamma \cap S_1$, and let L be the orbit through P_0. It is clear from (2) and (3) that L remains in S_2 for all $\tau \geqq 0$, and approaches the origin as $\tau \to \infty$. Hence L is an invariant curve of the mapping M, and from (1), L must coincide with Γ. In exactly the same manner, we show that the invariant curve of M leaving the origin is also an orbit of (1.1).

Remark 2. It is now possible to prove our earlier conjecture that the cusped like region C_+ flows into the singular solution as $\tau \to +\infty$. Namely, let T_k be an increasing sequence of times with $T_k \to \infty$ and $|T_k - T_{k-1}|$ bounded from above and below. The mapping M_k which takes solutions starting at time $\tau = T_k$ with initial values (q, y) into their values (q_1, y_1) at time $\tau = T_{k+1}$ can be written in the form

$$M_k: \quad \begin{aligned} q_1 &= q + (T_{k+1} - T_k)r^a + f^*, \\ y_1 &= e^{-i\omega}y + g^*, \\ \omega &= \int_{T_k}^{T_{k+1}} h(q(\tau), y(\tau), \bar{y}(\tau))\, d\tau, \end{aligned}$$

where f^* and g^* satisfy the same estimates as $\hat{f}$ and $\hat{g}$ in (1.2). Let $M^k \equiv M_k \circ M_{k-1} \circ \cdots \circ M_1$, and let $y = \phi_k(q)$ be the image under M^k of an initial curve $y = \phi_0(q)$. In exactly the same manner as before we can show that $\phi_k(q)$ converges uniformly for $0 \leqq q \leqq$ const. to a *unique* curve $y = \psi(q)$. Since $y = \phi(q)$ is a solution of the differential equation (1.1), it is an invariant curve for all the M_k, and hence all the M^k. Consequently, by uniqueness, $\psi(q) \equiv \phi(q)$.

Remark 3. To prove the existence and uniqueness of the solutions entering and leaving the origin, we need only require that (i)–(iii) be satisfied; *i.e.* the singular solutions need not be differentiable at the origin. However, we need differentiability of the singular solution in order to complete the proof of our theorem. To interpret the conditions (i)–(vi) assume that the function $h(q, y, \bar{y})$ in (1.1) is identically constant. In this case the constant c in (*) may be taken as large as desired and the inequalities (i)–(vi) reduce to the single inequality $N > a + 1$. (The conditions (i)–(iii) reduce to $N > a$.) This is rather surprising since one would intuitively expect that the theorem is true for $N > a$. The requirement that N be greater than a is certainly necessary: As a counterexample consider the system of equations

$$\frac{dq}{d\tau} = q^2 + s^2, \quad s^2 = y_1^2 + y_2^2,$$

$$\frac{dy_1}{d\tau} = y_2 + 2qy_1,$$

$$\frac{dy_2}{d\tau} = -y_1 + 2qy_2.$$

For this system (with $a = 2$, $N = 2$),

$$\frac{ds}{d\tau} = \frac{y_1 \frac{dy_1}{d\tau} + y_2 \frac{dy_2}{d\tau}}{[y_1^2 + y_2^2]^{1/2}} = 2qs.$$

Hence for

$$\xi = q + s,$$

$$\eta = q - s,$$

we have the equations

$$\frac{d\xi}{d\tau} = \xi^2,$$

$$\frac{d\eta}{d\tau} = \eta^2,$$

with the solution

$$\xi = \frac{\xi_0}{1 - \xi_0 \tau}, \quad \eta = \frac{\eta_0}{1 - \eta_0 \tau}.$$

Therefore any orbit with $\xi_0 \leqq 0$, $\eta_0 \leqq 0$ must run into the origin. The locus of points $\xi \leqq 0$, $\eta \leqq 0$ is the closed solid right circular cone C illustrated in Figure 3. Any orbit starting inside or on C must run into the origin as $\tau \to +\infty$. This phenomena is known as *funneling*. Similarly, all orbits with $\xi_0 \geqq 0$, $\eta_0 \geqq 0$ run backwards into the origin.

d. The structure of solutions in the neighborhood of $r = 0$. In this section we complete the proof of our theorem by describing completely all solutions in the neighborhood of $r = 0$. This will be accomplished by constructing a local homeomorphism Ψ which maps the trajectories of (1.1) onto the trajectories of (1.1)′ with a change of parametrization. To this end we set $\xi = y - \phi(q)$, $r^2 = q^2 + |\xi|^2$ and write the equation (1.1) in the form

$$\frac{dq}{d\tau} = r^a + f^*(q, \xi, \bar{\xi}), \ |f^*| \leqq Ar^{a+\epsilon}; \epsilon > 0,$$

(1.20) $$\frac{d\xi}{d\tau} = -ih\xi + g^*(q, \xi, \bar{\xi}), \ |g^*| \leqq A\ |\xi| r^\lambda; \lambda > a - 1,$$

$$h = h(q, \xi + \phi, \bar{\xi} + \bar{\phi}),$$

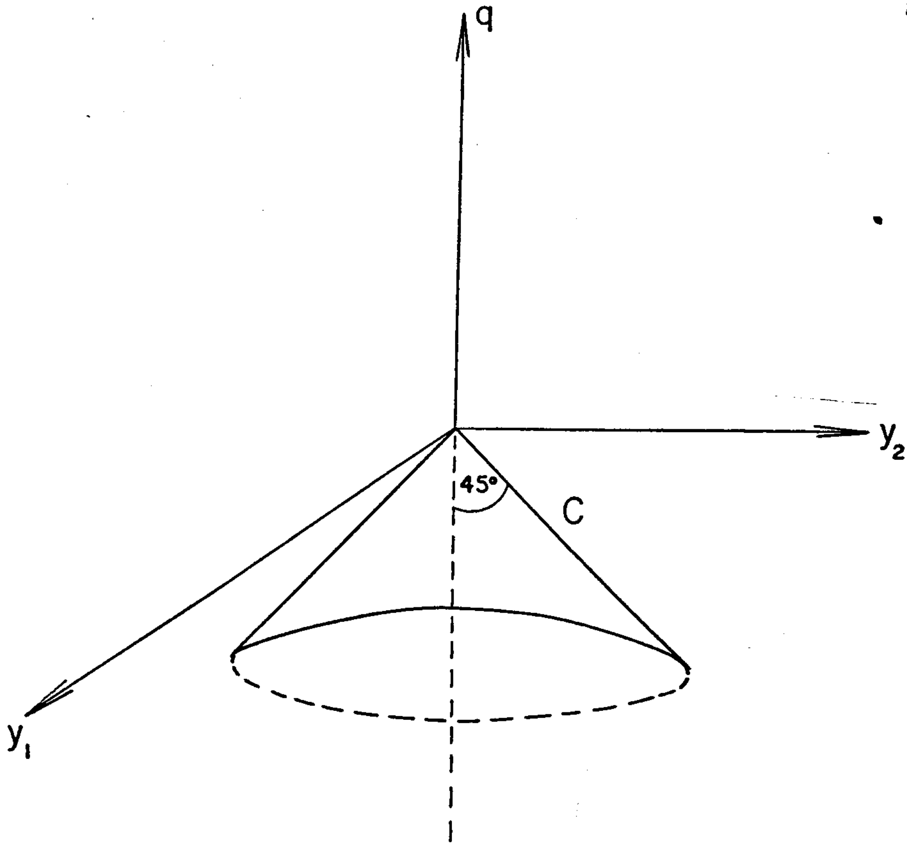

FIGURE 3

and denote by (1.20)′ the system (1.20) with $f^* = g^* \equiv 0$. The main tool in our proof will be the following lemma:

Lemma. *There exists* $\gamma > 0$, $\delta > 0$ *such that the orbit* $(q(\tau), \xi(\tau))$ *of* (1.20) *with initial values* (q_0, ξ_0), $|q_0| < \delta$, $|\xi_0| < \gamma$ *satisfies*

$$\tfrac{2}{3}|\xi_0| \leqq |\xi(\tau)| \leqq 2|\xi_0|, \tag{1.21}$$

provided $|q(\tau)| \leqq \delta$.

Proof of Lemma. There exists $\gamma_1 > 0$ and $\delta > 0$ such that $dq/d\tau \geqq r^a/2$ for $|q| \leqq \delta$, $|\xi| \leqq \gamma_1$. Take $\gamma = \gamma_1/2$. Now, the solution $\xi(\tau)$ of (1.20) is given by

$$\xi(\tau) = \exp\left(-i\int_0^\tau h\right)\xi_0 + \exp\left(-i\int_0^\tau h\right)\int_0^\tau \exp\left(i\int_0^\tau h\right)g^*\,dt.$$

Hence

$$\max_{0\leq t\leq\tau} |\xi| \leqq |\xi_0| + A \max_{0\leq t\leq\tau} |\xi(t)| \int_0^\tau r^\lambda\,dt',$$

$$\max_{0 \leq t \leq \tau} |\xi| \geq |\xi_0| - A \max_{0 \leq t \leq \tau} |\xi(t)| \int_0^\tau r^\lambda \, dt'.$$

To estimate $\int_0^\tau r^\lambda \, dt'$ observe that

$$\int_0^\tau r^\lambda \, dt' = \int_{q_0}^{q} r^\lambda \frac{dt'}{dq} \, dq \leq 2 \int_{-\delta}^{\delta} r^{\lambda - a} \, dq,$$

for $|q(\tau)| \leq \delta$, $|\xi| \leq \gamma_1$. For $\lambda \geq a$ the integral on the right is trivially estimated by 2δ. For $a - 1 < \lambda < a$ we have

$$\int_{-\delta}^{\delta} r^{\lambda - a} \, dq \leq 2 \int_0^\delta \frac{1}{q^{a-\lambda}} \, dq = \frac{2\delta^{\lambda + 1 - a}}{\lambda + 1 - a}.$$

Thus it is clear that we can choose $\delta > 0$ so small so that

$$\tfrac{2}{3}|\xi_0| \leq |\xi(\tau)| \leq 2|\xi_0|,$$

provided $|q(\tau)| \leq \delta$. (If $|\xi_0| < \gamma_1/2$ then $|\xi(\tau)|$ will remain less than γ_1 for $|q| \leq \delta$.) This completes the proof of the Lemma.

To construct the homeomorphism Ψ, let N_1 be the interior of the cylinder $|q| \leq \delta$, $|\xi| = \gamma$. Let C_2 be the surface generated for $|q| \leq \delta$ by the flow under (1.20) of the circle $|\xi| = \gamma$ in the $q = 0$ plane. Define N_2 to be the open region bounded by C_2 and the planes $q = \pm\delta$. It is clear from the construction of N_2, together with the existence and uniqueness of the singular solution $\xi \equiv 0$ that any orbit in N_2 with $\xi_0 \neq 0$ must leave N_2 at $q = \delta$ for some $\tau_1 > 0$ and at $q = -\delta$ for some $\tau_2 < 0$. (see Figure 4). We now define a transformation Ψ taking the trajectories of (1.20)′ in N_1 onto the trajectories of (1.20) in N_2 by identifying points with the same q-value; *i.e.* the point (q, ξ) in N_1, $\xi \neq 0$, which lies on the trajectory of (1.20) with initial value $(0, \xi_0)$ is mapped by Ψ onto the point $(q, \psi\xi)$, where $\psi\xi$ is the ξ value of the trajectory of (1.20) with initial value $(0, \xi_0)$ when it reaches q. On the q-axis, Ψ is defined to be the identity. It is clear that the transformation Ψ so defined maps N_1 onto N_2 in a $1 - 1$ manner, and that Ψ and Ψ^{-1} are continuous for $\xi \neq 0$. To prove that Ψ is indeed a homeomorphism, we must verify continuity at $\xi = 0$; *i.e.* we must show that

$$\xi \to 0 \Leftrightarrow \psi\xi \to 0.$$

But this follows immediately from inequality (1.21). The proof of our theorem is now complete.

Remark. Note that our theorem is true for $N > a$ if the function g in (1.1) can be estimated by

$$|g| \leq A|y|r^{N-1},$$

i.e. if $y \equiv 0$ remains a solution of the perturbed equation. An open question is whether the theorem is true for $|g| \leq Ar^N$, $a < N \leq a + 1$.

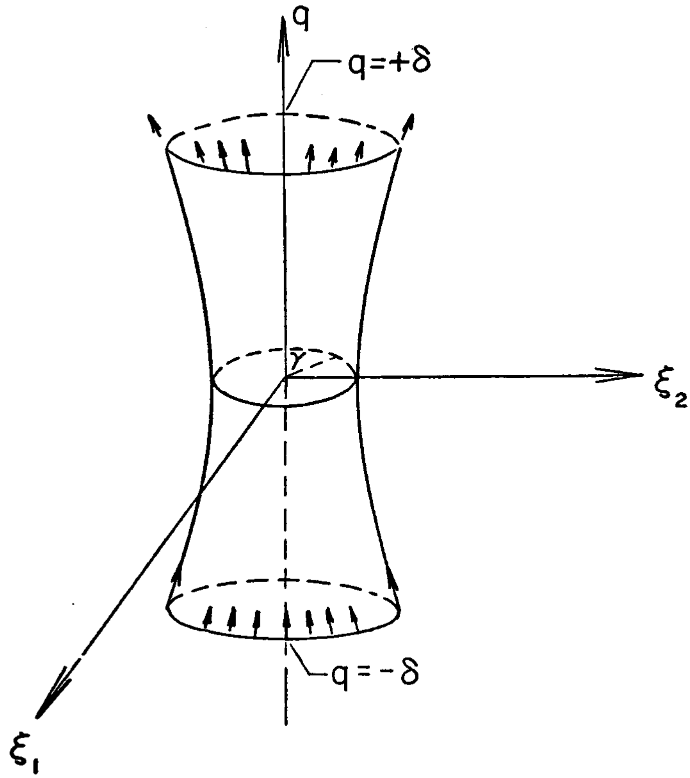

FIGURE 4. The region N_1

2. Application to the dipole field.

a. The earth's magnetic field is assumed here to be equivalent to the field produced by a magnetic dipole situated at the center of the earth. Such a field can be described in cylindrical coordinates ρ, z, ϕ by the equations

$$\mathbf{B} = \text{curl } \mathbf{A},$$

$$\mathbf{A} = \frac{M\rho}{r^3}\hat{\phi} \qquad (r^2 = \rho^2 + z^2),$$

where M is the moment of the magnetic dipole, which points in the negative z direction, and $\hat{\phi}$ is a unit vector in the ϕ direction (see Figure 5). The magnetic lines of force are given by

$$r = a\cos^2\theta,$$

$$\phi = \text{constant}.$$

To write the differential equations of motion for the Störmer problem it is most convenient to employ a canonical formulation described by the Hamiltonian

$$H = \frac{1}{2m}\left[p_\rho^2 + p_z^2 + \left(\frac{p_\phi}{\rho} - \frac{qM\rho}{r^3}\right)^2\right],$$

where

$$p_\rho = m\dot{\rho},$$

$$p_z = m\dot{z},$$

$$p_\phi = m\rho^2\dot{\phi} + q\rho A,$$

and m and q denote the mass and charge of the particle. Since H is independent of time, the energy

$$H = \tfrac{1}{2}mv^2 = E,$$

is a constant of the motion. A second integral is obtained by noting that H is independent of the angle ϕ. Hence, the canonical angular momentum

$$p_\phi = qM\Gamma,$$

where Γ is defined by this equation is a constant of the motion, and

$$\phi(t) = \phi(0) + \frac{1}{m}\int_0^t \left(\frac{qM\Gamma}{\rho^2} - \frac{qM}{r^3}\right) dt'.$$

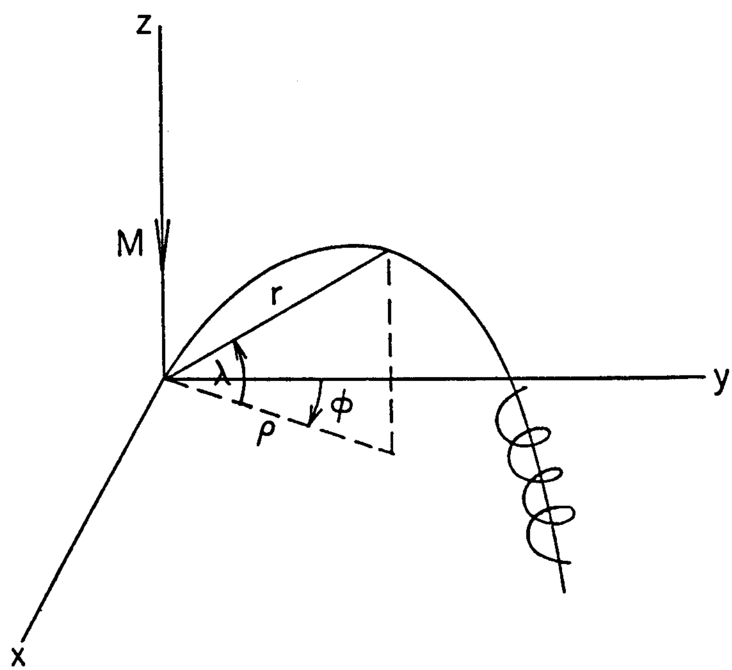

FIGURE 5

We now restrict ourselves to the case $\Gamma > 0$, and introduce the dimensionless variables

$$z' = \Gamma z,$$

$$\rho' = \Gamma \rho,$$

$$t' = \frac{\Gamma^3 qM}{m} t.$$

It is easily seen that the equations of motion for these dimensionless variables are derived from the new Hamiltonian

$$H = \tfrac{1}{2}(p_\rho^2 + p_z^2) + \frac{1}{2}\left(\frac{1}{\rho} - \frac{\rho}{r^3}\right)^2, \tag{2.1}$$

where we have omitted the primes for convenience. The potential

$$V(\rho, z) = \frac{1}{2}\left(\frac{1}{\rho} - \frac{\rho}{r^3}\right)^2,$$

vanishes along the curve $r = \cos^2 \theta$ and is positive elsewhere. Since the Hamiltonian is a constant of the motion, the particle is restricted to lie in the region $0 \leqq V \leqq H$. This region assumes three different forms depending on whether H is less than, equal to, or greater than 1/32. These regions are illustrated in Figure 6.

In the following we shall establish the existence and uniqueness, for each value of $H > 0$, of the Störmer orbit which enters the singularity from above the equator ($\theta > 0$), and describe completely all solutions in the vicinity of the dipole. Note that $\theta \to \pi/2$ for an orbit which enters the origin. The situation is completely analogous for those orbits below the equator, *i.e.* $\theta < 0$.

b. In polar coordinates r, θ, p_r, p_θ, the Hamiltonian (2.1) assumes the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{1}{2r^2 \cos^2 \theta}\left(\frac{r - \cos^2 \theta}{r}\right)^2. \tag{2.1$'$}$$

We now introduce new orthogonal coordinates $\frac{\partial a}{\partial r}\frac{\partial b}{\partial r} + \frac{1}{r^2}\frac{\partial a}{\partial \theta}\frac{\partial b}{\partial \theta} = 0$

$$a = a(r, \theta) = \frac{r - \cos^2 \theta}{r},$$

$$b = b(r, \theta) = \frac{r}{(\sin \theta)^{1/2}},$$

via the generating function

$$F(r, \theta, p_a, p_b) = \left(1 - \frac{\cos^2 \theta}{r}\right)p_a + \frac{r}{(\sin \theta)^{1/2}} p_b,$$

where

$$p_r = \frac{\partial F}{\partial r} = \frac{\cos^2 \theta}{r^2} p_a + \frac{p_b}{(\sin \theta)^{1/2}},$$

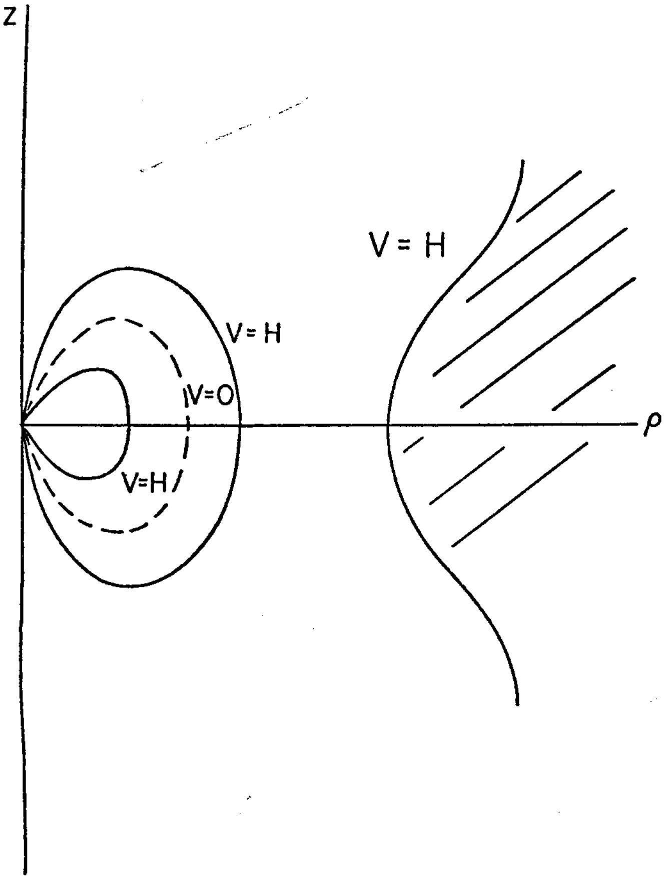

FIGURE 6a. Allowed region $V(\rho, z) < \frac{1}{32}$.

$$p_\theta = \frac{\partial F}{\partial \theta} = \frac{2 \sin \theta \cos \theta}{r} p_a - \frac{1}{2} \frac{r \cos \theta}{(\sin \theta)^{3/2}} p_b .$$

(a and b are not to be confused with the "dipolar" coordinates a and b of Dragt [1].)

In terms of the new canonical variables the Hamiltonian H assumes the form

$$(2.2) \qquad H = \left[\frac{4(1-a)}{r^3} - \frac{3(1-a)^2}{r^2}\right] \frac{p_a^2}{2} + \left[\frac{1}{[1 - r(1-a)]^{1/2}} + \frac{1}{4} \frac{r(1-a)}{[1 - r(1-a)]^{3/2}}\right] \frac{p_b^2}{2} + \frac{a^2}{2(1-a)r^3},$$

where r is assumed to be expressed in terms of a and b.

c. In order to apply our theorem to the dipole field, we change the singular point into an equilibrium point. This is accomplished by the change of time scale

$$\tau = \int \frac{dt}{r^3}.$$

Our trajectories are now the zero energy solutions of the Hamiltonian

$$(2.3)\qquad K = r^3\left(H - \frac{E}{2}\right) = [4(1-a) - 3r(1-a)^2]\frac{p_a^2}{2} + \frac{a^2}{2(1-a)}$$
$$+ \frac{r^3}{2}\left\{\left[\frac{1}{(1-r(1-a))^{1/2}} + \frac{r(1-a)}{4(1-r(1-a))^{3/2}}\right]p_b^2 - E\right\},$$

where $E/2$ is the constant value (energy) of (2.2). To solve for $r = r(a, b)$ observe that

$$b = \frac{r}{(\sin\theta)^{1/2}} = \frac{r}{[1-r(1-a)]^{1/4}}.$$

Setting $y = b(1-a)$, $x = r(1-a)$ we see that

$$y = \frac{x}{(1-x)^{1/4}}.$$

Since $dy/dx = 1$ at $x = 0$, we may solve for x as an analytic function of y, in a neighborhood of $y = 0$. Thus, we have

$$r = b + \frac{r_1(a, b)}{1-a},$$

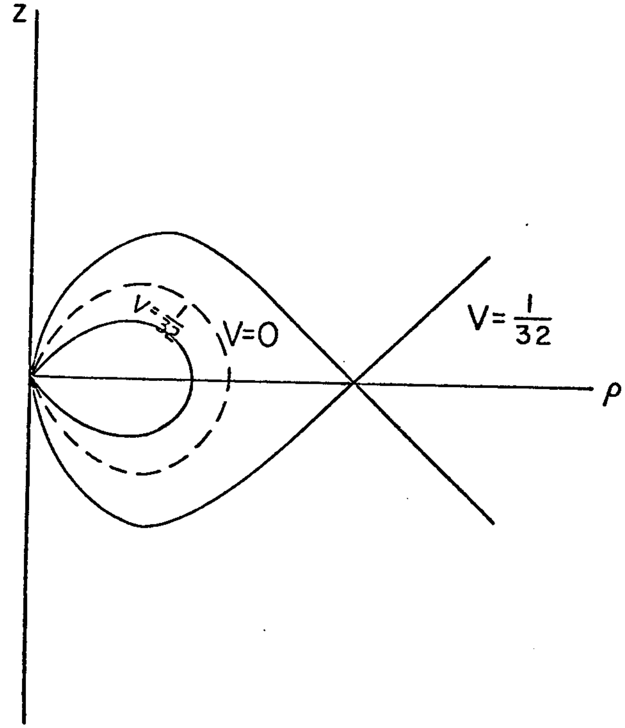

FIGURE 6b. Allowed region $V(\rho, z) \leq \frac{1}{32}$.

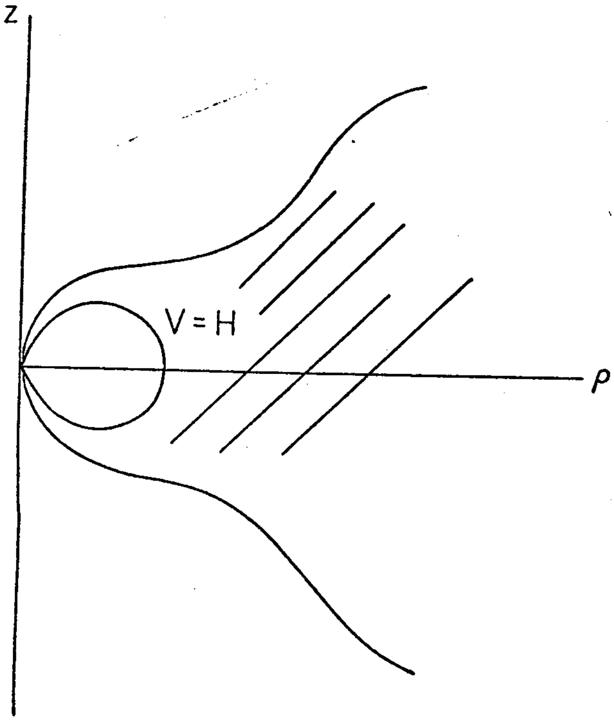

FIGURE 6c. Allowed region $V(\rho, z) > \frac{1}{32}$.

where r_1 is a power series in $b(1 - a)$ beginning with second order terms. Hence, we may write

$$K = \frac{a^2 + 4p_a^2}{2} + b^3 \frac{(p_b^2 - E)}{2} + K_1(a, b, p_a, p_b, E),$$

where

$$K_1 = \sum_{j,k,l} c_{jkl}(p_b, E) a^j b^k p_a^l,$$

with

$$j + \frac{2k}{3} + l \geqq \frac{8}{3}.$$

For our later convenience we will make the transformation

$$a \to 2^{1/2} a$$

$$p_a \to \frac{p_a}{2^{1/2}}$$

so that

$$a^2 + 4p_a^2 \to 2(a^2 + p_a^2).$$

d. To lowest order our trajectories are now the zero energy solutions of the Hamiltonian

$$K_0 = a^2 + p_a^2 + \tfrac{1}{2}b^3(p_b^2 - E).$$

These solutions are quite simple to describe since all orbits lie on the cylinders

$$a^2 + p_a^2 = c^2.$$

Moreover, in the b, p_b plane, these orbits are the curves (see Figure 7),

$$b^3(p_b^2 - E) = -2c^2.$$

Notice that in this approximation there is a unique orbit ($p_b = -(E^{1/2})$, $a = p_a = 0$) entering the origin, and a unique orbit ($p_b = (E^{1/2})$, $a = p_a = 0$) leaving the origin. To describe the motion more fully, set

$$q = b^{3/2}p_b\,, \qquad y = a + ip_a\,.$$

It is easily verified then that

$$\frac{dq}{d\tau} = \frac{3}{2E^{1/6}}(q^2 + 2\,|y|^2)^{7/6},$$
$$\frac{dy}{d\tau} = -2iy,$$

which is essentially the system (1.20)′ with the exponent $a = 7/3$.

e. Our goal now is to show that the perturbation K_1 does not destroy this "structure." To this end we first show that on the energy surface $K = 0$ we may solve for b as a function of the variables q, a, p_a.

Proof. The equation $K = 0$ yields

$$b = \frac{[q^2 + 2(a^2 + p_a^2) + 2K_1(a, b, p_a, p_b)]^{1/3}}{E^{1/3}}. \tag{2.4}$$

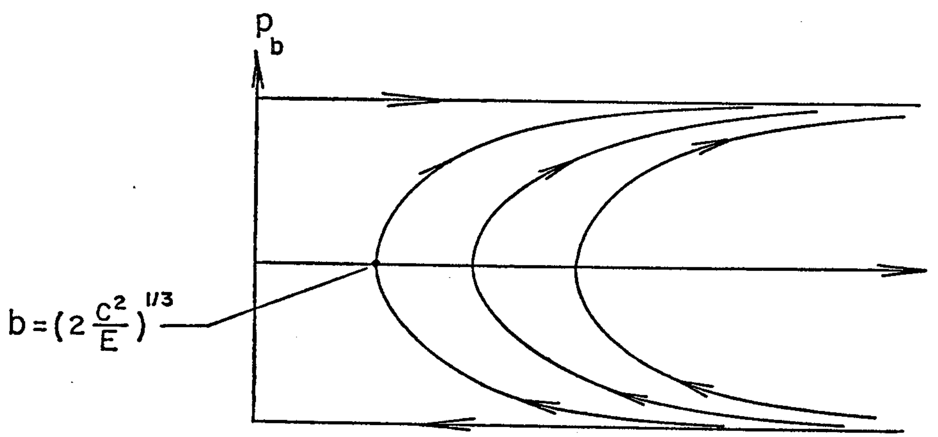

FIGURE 7. The integral curves $b^3(p_b^2 - E) = -2c^2$.

We now replace p_b in (2.4) by

$$p_b = \frac{q}{b^{3/2}}.$$

Since the only terms of K_1 involving p_b^2 are of the form

$$c_{jkl}(E)a^j b^k p_a^l p_b^2; \qquad k \geqq 4,$$

we see that K_1 is an analytic function of a, b, p_a, q. (Later we will solve (2.4) with a K_1 depending on p_b, $n \geqq 2$. However, this Hamiltonian will have the property that after replacing p_b by $q/b^{3/2}$, it is an analytic function of a, b, p_a, q, with order 8/3.) To solve for b we write (2.4) in the form

(2.4)′ $$b = Tb,$$ Suchen Fixpunkt mit Kontraktion

and proceed by iteration, *i.e.* set

$$b_{n+1} = Tb_n .$$

In the region

(2.5) $$b_n \leqq \frac{2}{E^{1/3}}[q^2 + 2(a^2 + p_a^2)]^{1/3},$$

we may estimate $K_1(a, b, p_a, q)$ by

$$|K_1| \leqq A[q^2 + 2(a^2 + p_a^2)]^{4/3}.$$

Hence for b_n satisfying (2.5)

$$0 \leqq b_{n+1} \leqq \frac{[q^2 + 2(a^2 + p_a^2]^{1/3}[1 + 2A(q^2 + 2(a^2 + p_a^2))^{1/3}]^{1/3}}{E^{1/3}}$$

$$\leqq \frac{2[q^2 + 2(a^2 + p_a^2)]^{1/3}}{E^{1/3}},$$

for $[q^2 + 2(a^2 + p_a^2)]^{1/3} \leqq 7/A$. It is now a simple matter to show that

$$|Tb_{n+1} - Tb_n| \leqq \hat{A}(q^2 + a^2 + p_a^2)^{1/3}|b_{n+1} - b_n|,$$

if b_0 satisfies (2.5). Thus, T is a contraction for $r^2 = q^2 + a^2 + p_a^2$ small enough, and the iterates converge to a non-negative function $b = b(a, p_a, q)$ with

$$b \sim [q^2 + 2(a^2 + p_a^2)]^{1/3}. \quad \sim r^{2/3}$$

Also, one easily verifies now that

$$\partial b \sim \frac{1}{r^{1/3}}, \quad \partial^2 b \sim \frac{1}{r^{4/3}}, \quad r^2 = q^2 + a^2 + p_a^2 .$$

In terms of the variables q, $y = a + ip_a$, the differential equations of motion assume the form

$$\frac{dq}{d\tau} = \frac{3}{2E^{1/6}}(q^2 + 2|y|^2)^{7/6} + f(q, y, \bar{y}),$$

$$\frac{dy}{d\tau} = -2iy + g(q, y, \bar{y}),$$

where

$$|\partial^k f| \leq Ar^{3-k}, \qquad k = 0, 1, 2.$$

Unfortunately, the function g is proportional to $r^{5/3}$, and we cannot use our theorem since already $N = 5/3$ is less than $\alpha = 7/3$. To rectify this situation, we will now devise an averaging method for the Hamiltonian (2.3) so that K will be a function of $a^2 + p_a^2$ alone through any desired order in r.

f. Averaging method. To each term

$$c_{jkl}(p_b, E)a^j b^k p_a^l,$$

in the power series expansion of K, we assign the *weight*

$$s = j + \frac{2k}{3} + l.$$

In this manner, the Hamiltonian (2.3) may be written in the form

$$K = K^{(2)} + K^{(8/3)} + K^{(9/3)} + \cdots$$

where

$$K^{(2)} = a^2 + p_a^2 + \tfrac{1}{2}b^3(p_b^2 - E),$$

and $K^{(s)}$ includes all terms of K of order s. We seek new canonical variables $\xi_1, \eta_1, \xi_2, \eta_2$ (where ξ_1 and η_1 will have weight 1, and ξ_2 will have weight 2/3), so that the transformed Hamiltonian $\Gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ will be a function of $\xi_1^2 + \eta_1^2$ alone, through a given order. To this end we construct our transformation with the aid of a generating function

$$\begin{aligned} W(a, b, \eta_1, \eta_2) &= a\eta_1 + b\eta_2 + \sum W_{jkl}(\eta_2)a^j b^k \eta_1^l, \quad j + \frac{2k}{3} + l \geq \tfrac{8}{3} \\ &= W^{(2)} + W^{(8/3)} + W^{(9/3)} + \cdots, \end{aligned}$$

with the canonical relations

$$\begin{aligned} p_a &= \frac{\partial W}{\partial a} = \eta_1 + \frac{\partial W^{(8/3)}}{\partial a} + \cdots, \\ p_b &= \frac{\partial W}{\partial b} = \eta_2 + \frac{\partial W^{(8/3)}}{\partial b} + \cdots, \\ \xi_1 &= \frac{\partial W}{\partial \eta_1} = a + \frac{\partial W^{(8/3)}}{\partial \eta_1} + \cdots, \\ \xi_2 &= \frac{\partial W}{\partial \eta_2} = b + \frac{\partial W^{(8/3)}}{\partial \eta_2} + \cdots. \end{aligned} \tag{2.6}$$

Taking a, b, η_1, and η_2 as independent variables, we may write

$$K\left(a, b, \frac{\partial W}{\partial a}, \frac{\partial W}{\partial b}\right) = \Gamma\left(\frac{\partial W}{\partial \eta_1}, \frac{\partial W}{\partial \eta_2}, \eta_1, \eta_2\right). \tag{2.7}$$

Proceeding by induction, we assume that the $W^{(\sigma)}$, $\sigma < s$ have already been determined so that $\Gamma^{(\sigma)}$, $\sigma < s$, has the desired "normal form." Equating terms of order s in (2.7) we obtain the equation

$$DW^{(s)} = \Gamma^{(s)} + P^{(s)}(a, b, \eta_1, \eta_2), \tag{2.8}$$

for $W^{(s)}$, where

$$D = 2\eta_1 \frac{\partial}{\partial a} - 2a \frac{\partial}{\partial \eta_1},$$

and $P^{(s)}$ is a polynomial of weight s which is known to us in terms of K and $W^{(\sigma)}$, $\sigma < s$, which have already been determined. (What's important for the success of this averaging method is that $b^3(p_b^2 - E)$ gives rise, in the variables a, b, η_1, η_2, to terms of order s involving only $W^{(\sigma)}$, with $\sigma \leqq s - 4/3$.) To discuss the equation (2.8) we express all polynomials in terms of

$$\zeta = a + i\eta_1, \qquad \bar{\zeta} = a - i\eta_1,$$

and note that

$$D(\zeta^j \bar{\zeta}^l) = 2i(l - j)\zeta^j \bar{\zeta}^l.$$

We decompose any polynomial P into two parts

$$P = P_N + P_R,$$

where P_N contains all terms of P which are admitted in the normal form of Γ, (*i.e.* P_N only contains terms of the form $c(b, \eta_2)\zeta^k \bar{\zeta}^l$, $k = l$), while P_R contains the remaining terms. This decomposition is clearly unique and we have $DP_N = 0$. Observe that DW, for any polynomial W, belongs to the second part of the decomposition. Therefore, the equation (2.8) reduces to

$$DW^{(s)} = P_R^{(s)},$$
$$\Gamma^{(s)} = -P_N^{(s)}.$$

The first equation is solvable for $W^{(s)}$ since $P_R^{(s)}$ is in the range of D. The second equation says that $\Gamma^{(s)}$ is in normal form. For Γ to be in normal form through order s, we simply truncate W after $W^{(s)}$. Notice that this averaging method yields the *asymptotic expansion* of the orbit entering the origin. Namely, by setting $\xi_1 = \eta_1 = 0$ in (2.6) and using the energy relation $K = 0$, we can calculate the asymptotic expansion of the singular orbit to any desired order. In addition, by truncating the normalized Hamiltonian Γ at order s, we see that *all* trajectories lie on the invariant cylinders

$$\xi_1^2 + \eta_1^2 = \text{constant},$$

to order s. Thus, this averaging method provides a formal description of all orbits to any desired order.

We are now ready to apply our theorem to the dipole field. Firstly, we average out the Hamiltonian K through order 17/3, *i.e.* we find new canonical variables $\xi_1, \xi_2, \eta_1, \eta_2$ such that the new Hamiltonian

$$\Gamma = (\xi_1^2 + \eta_1^2) + \frac{\xi_2^3}{2}(\eta_2^2 - E) + \Gamma^{(8/3)} + \cdots,$$

is a function of $\xi_1^2 + \eta_1^2, \xi_2, \eta_2$ through order 17/3. We then solve for ξ_2 as a function of ξ_1, η_1, and $q = \xi_2^{3/2}\eta_2$, as previously described. (Since K_1 only depends on p_b^2 through the combination $b^n p_b^2$, $n \geqq 4$, one easily verifies that $b \geqq 0 \Leftrightarrow \xi_2 \geqq 0$, and that Γ is analytic in the variables ξ_1, η_1, ξ_2, q after the substitution $\eta_2 = q/\xi_2^{3/2}$.) With

$$q = \left(\frac{3}{2E^{1/6}}\right)^{3/10} \xi_2^{3/2}\eta_2,$$

$$y = \frac{1}{2^{1/2}}\left(\frac{3}{2E^{1/6}}\right)^{3/10}(\xi_1 + i\eta_1),$$

our differential equations of motion become

$$\text{(2.9)} \qquad \begin{aligned} \frac{dq}{d\tau} &= (q^2 + |y|^2)^{7/6} + f(q, y, \bar{y}), \\ \frac{dy}{d\tau} &= -iyh(q, |y|^2) + g(q, y, \bar{y}), \ (h \text{ real}) \end{aligned}$$

where the functions f, g, h satisfy the inequalities (*) of (1.1) with $\epsilon = 2/3$, $c = 2/3$, $N > 14/3$. Since

$$\Gamma^{(8/3)} = \frac{3E\xi_2^4}{4} - \frac{\xi_2^4\eta_2^2}{4} - \frac{3\xi_2}{4}(\xi_1^2 + \eta_1^2),$$

we find

$$\dot{q} = r^{7/3} + \text{terms of order} \geqq 3,$$

$$\dot{y} = -ihy + \text{terms of order} > 14/3,$$

where

$$h = (2 - \tfrac{3}{2}\xi_2 + \cdots),$$

and $\xi_2 = A(\xi_1^2 + \eta_1^2 + \frac{1}{2}q^2)^{1/3} + \cdots$ is of order 2/3. Thus, we cannot take c greater than 2/3. One easily verifies that for $a = 7/3$, $c = 2/3$, the relations (i)–(vi) of the previous section can be satisfied if $N > 14/3$. Hence our theorem applies to equation (2.9) and we have a complete description of all solutions in the neighborhood of the dipole.

g. Uniqueness of asymptotic orbit. What we have proven concerning the uniqueness of the Störmer orbit entering the dipole may be stated precisely as follows: For each value of energy and angular momentum there exists a $\delta > 0$ and a differentiable curve C such that any orbit starting in the region

$$R : a^2 + b^2 + p_a^2 < \delta^2,$$

and not on C must leave R. Since

$$p_a = \frac{r^2 p_r \cos\theta + 2rp_\theta \sin\theta}{\cos^3\theta + 2\sin\theta\sin 2\theta},$$

we see that a, b, and p_a will be sufficiently small if we are close enough to the dipole, *i.e.* if r is close to 0, and θ is close to $\pi/2$. *A priori*, it is entirely possible that a trajectory can enter and leave R in such a manner that for a sequence of times $t_k \to \infty$, $r(t_k) \to 0$. However, one can easily show that such a trajectory (if it exists) must cross the equator infinitely often. Namely, we first observe (Störmer [4], pp. 240–241) that the quantity z/r^3 cannot achieve a minimum (maximum) for $z > 0$ ($z < 0$). Any such orbit must enter and leave the region

$$\frac{z}{r^3} \geqq \delta_0 ,$$

for some fixed $\delta_0 > 0$ infinitely often. Consequently, along this trajectory, the function z/r^3 achieves infinitely many maxima and minima, which can only be possible if the particle crosses the equatorial plane infinitely often. We summarize this result as follows:

Theorem. *Let* $H = E$ *and the angular momentum* $P_\phi = \gamma > 0$ *be given. If an orbit enters into the domain*

$$D_+ = \{(\rho, z) \mid V(\rho, z) \leqq E, z > 0\},$$

then it leaves this domain D_+ *in a finite time, or otherwise agrees with the unique asymptotic orbit approaching the dipole from above.*

h. Generalizations. Consider now Hamiltonian systems of two degrees of freedom which can be written in the form

$$H = \frac{a^2 + p_a^2}{b^\sigma} + p_b^2 + H_1 , \tag{2.10}$$

where $\sigma > 2/3$, and $b^\sigma H_1$ in an analytic function of a, b, p_a , p_b which is small compared to $a^2 + p_a^2 + b^\sigma$. We regularize (2.10) by the change of time scale

$$\tau = \int \frac{dt}{b^\sigma} ;$$

our solutions are then the zero energy solutions of the Hamiltonian

$$K = a^2 + p_a^2 + b^\sigma(p_b^2 - E) + K_1 , \quad K_1 = b^\sigma H_1 . \tag{2.11}$$

Setting $q = b^{\sigma/2} p_b$, $y = a + ip_a$, and neglecting the perturbation term K_1 in (2.11) the differential equations of motion are

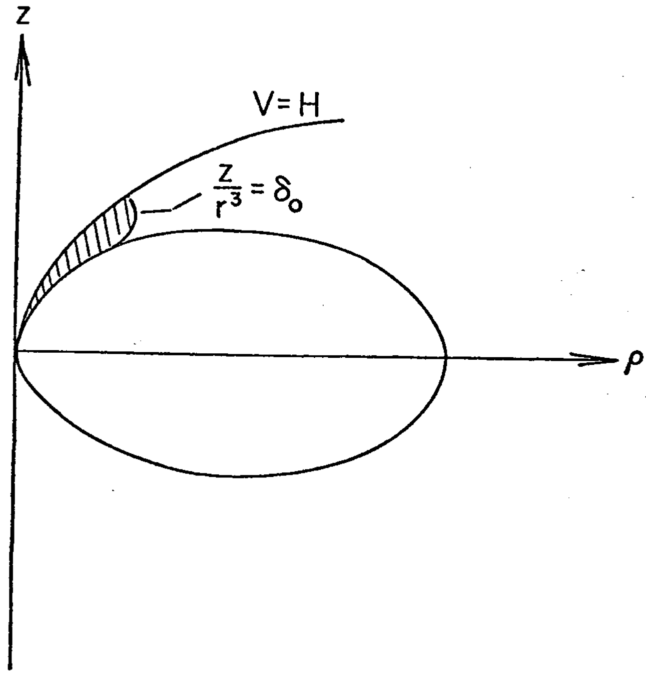

FIGURE 8. The region (shaded) $z/r^3 \geq \delta_0$.

$$\frac{dq}{d\tau} = \sigma E\left(\frac{a^2 + p_a^2 + q^2}{E}\right)^{3\sigma-2/2\sigma},$$

$$\frac{dy}{d\tau} = -2iy.$$

Here the exponent a of (1.1) is $(3\sigma - 2)/\sigma$. In order to apply our theorem to the full Hamiltonian K, we assign to the variables a, p_a, b the weights 1, 1, $2/\sigma$ respectively, and by our averaging method find new variables ξ_1, η_1, ξ_2, η_2, so that K only depends on $\xi_1^2 + \eta_1^2$ through a sufficiently high order. (It will certainly suffice to average out through order $3a = 9 - 6/\sigma$.) If we now solve for

$$\xi_2 = \xi_2(\xi_1, \eta_1, q = \xi_2^{\sigma/2}\eta_2),$$

on the energy surface $K = 0$, then the differential equations of motion for the variable q, $y = \xi_1 + i\eta_1$ will satisfy the hypotheses of our theorem.

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