# THE FOUR COLOR THEOREM

By

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Final Edition, May 1977

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To understand this proof, no knowledge except that of English and a few fundamental facts from plane geometry and algebra are needed. The necessary mathematics could be had intuitively without formal schooling; in fact, today's education may hinder, rather than help.

Figure 1 shows a loop. It separates the points of the plane into three sets: those in the loop, those inside it and those outside. The picture is map $M_2$ with two faces, each bounded by the loop. Choose two unequal points in the loop as vertices. Join them by an arc of inside points. The operation bonds the outside face to itself across the inside. The result, Figure 2, is a map $M_3$ with three faces, called lunes, each bounded by a loop; two vertices, each at three faces and at two sides of which it is the meet; and three sides, each at two faces which it joins and at two vertices which it joins. The arc can instead consist of outside points, Figure 3; it then bonds the inside face to itself across the outside. The two $M_3$ are to be regarded as equal. In an $M_3$, choose a face $F$. Bonding equal or unequal faces across $F$ gives two $M_4$. The faces of the first, Figure 4, are two lunes and two quadrilaterals; it is called the Quonset hut because it can be deformed into the figure so named in three dimensions. The faces of the second, Figure 5, are four triangles; it is called the tetrahedron. Clearly, the two $M_4$ are to be regarded as unequal. Repetition, Figure 6, defines by induction map $M_n$ with $n$ faces. Passage backwards from $M_{n+1}$ to $M_n$ is erasure.

Except when they obviously have their usual meaning, combinations of Arabic digits will be given special chores. 1 is reserved for the unit in the Galois field of order three, $GF(3)$. Here a word of caution: don't try to pick daisies with the cherrypicker designed to construct one hundred story buildings. Think of $GF(3)$ as the set of three numbers 0, 1, $-1$ which are added and subtracted by manipulating

$$x + 0 = x; \qquad x + x + x = 0;$$

and multiplied and divided by manipulating

$$x \cdot 0 = 0; \qquad x \neq 0, \qquad xx = 1; \qquad xxx = x.$$

Writing the last two in modern notation $x^2 = 1, x^3 = x$ assigns special roles to 2, 3; but 45678 . . . are only assumed to be distinguishable. Call 4567 colors. A map is colored if one of them is assigned to each face so that the colors on the two faces at each side are unequal, Figure 7. The four color theorem is that every map can be colored with at most four colors.

The only step not obvious in the proof was taken by P. J. Heawood. At each vertex he put one of the abbreviations $+$, $-$ for the non-zero numbers in $GF(3)$. Omit from $M$ with at least three faces the two faces at any side. Heawood proved the theorem is equivalent to: the sum of the numbers in the loop of each of the remaining faces is 0. This fundamental result is momentarily assumed. A proof is given in the appendix. After pioneer Heawood blazed the start of the trail, the obvious way of proceeding is to place unknowns at the vertices and solve the algebraic system which expresses that the sums are 0. Call the linear system $Y$. Since the unknowns are to be non-zero, the system to be solved is $S = Y + Q$, where $Q$ is the quadratic system expressing that the square of each unknown is to be 1. Thus $Y$ is to be solved not in $GF(3)$, but in $GF(3) - 0$, the field with 0 omitted. The unknowns are $8, 9, 10, \cdots$. Because of $Q$, besides being distinguishable they now have square equal to 1.

System $Y$ for the cube shown in Figure 8 is:

$$
\begin{aligned}
8 + 9 + 12 + 13 &= 0, \\
9 + 10 + 11 + 12 &= 0, \\
11 + 12 + 13 + 14 &= 0, \\
8 + 13 + 14 + 15 &= 0.
\end{aligned} \tag{1}
$$

The simple-minded way of solving linear $Y$ involves only transposition and substitution; it could have occurred to Ahmes. Transposition in the first two equations, followed by transposition and substitution in the other two, gives

$$
\begin{aligned}
-8 - 9 - 12 &= 13, \\
-9 - 11 - 12 &= 10, \quad -8 + 9 - 12 + 14 = 10, \\
8 + 9 - 14 &= 11, \\
9 + 12 - 14 &= 15.
\end{aligned} \tag{2}
$$

The second equation is rewritten by substitution from the third. The right members are the principal unknowns. Formulas (2) solve linear $Y$ in $GF(3)$. The solution is to be put in $Q$. Equation $13^2 = 1$ can be written

$$(8 + 9)^2 - (8 + 9)12 + 12^2 = 1,$$

whence

$$(8 + 9)^2 = (8 + 9)12, \quad 8 + 9 = (8 + 9)^2 12.$$

Hence $Q$ becomes

$$(3)\qquad \begin{aligned} 8 + 9 &= (8 + 9)^2 12, \\ -8 + 9 - 12 &= (-8 + 9 - 12)^2 14, \\ -(8 + 9) &= (8 + 9)^2 14, \\ -(9 + 12) &= (9 + 12)^2 14. \end{aligned}$$

These equations condition 12, 14, called mediate unknowns. Obviously, making $8 = 9$ gives $-8 = 12$; and making $-8 = 9$ leaves 12 arbitrary, so that $p = 12$, where $p$ is a non-zero parameter. Similarly, each of the other equations in (3) individually gives a non-zero 14 if the unknowns on the left are given arbitrary non-zero values. The three values of 14 must be equal. The three equations on 14 have the form

$$A = A^2 14, \qquad B = B^2 14, \qquad C = C^2 14.$$

Operations used from the beginning of algebra are addition and multiplication of two simultaneous equations member by member. The second of these eliminates 14 and gives

$$(4)\qquad BC = B^2C^2, \qquad CA = C^2A^2, \qquad AB = A^2B^2$$

as necessary conditions. They are also sufficient: if one of $ABC$ is 0, the three equations reduce to one; if none of $ABC$ is 0, they become $A = B = C$. If one of (4) is not an identity, it is written

$$D + Ex = 0$$

and, since $D^2$ must equal $E^2$, conditions another mediate $x$ thus:

$$-DE = D^2E^2x.$$

The operation is repeated and ultimately ends because the number of unknowns and equations is limited. The final form of $S$ is called passive. It gives the solution, which as the set of all roots is necessarily unique, although its appearance varies. The system is inconsistent or consistent according as the solution is empty or not; that is, according as the passive form contains $0 = 1$ or not.

The foregoing is ideal as an illustrative example because solving the system for the general map requires no new operation; only repetition is needed.

Returning to (4), we find they are identically satisfied. Thus a passive form of (1) is

$$(5)\qquad \begin{aligned} -8 - 9 - 12 &= 13, & 8 + 9 &= (8 + 9)^2 12, \\ -8 + 9 - 12 + 14 &= 10, & -8 + 9 - 12 &= (-8 + 9 - 12)^2 14, \\ 8 + 9 - 14 &= 11, & -(8 + 9) &= (8 + 9)^2 14, \\ 9 + 12 - 14 &= 15, & -(9 + 12) &= (9 + 12)^2 14, \end{aligned}$$

and (1) is consistent. The principal unknowns are 10, 11, 13, 15; the mediate are 12, 14; and the parametric are 8, 9. To get a root, the parametric are assigned arbitrary non-zero values; next the mediates are found; and finally the principal. The solution gives all information about coloring the map.

If any equation $H$ is omitted from $Y$ or is replaced by $H^2 = 1$, the same procedure yields a passive form for $S - H$ or for $S - H + (H^2 - 1)$ with all the above properties. In solving linear $Y$, it is pointless to include $H$ if in $GF(3) - 0$ every root of $Y - H$ is a root of $H$; it is understood that such an $H$ is omitted in reducing $S$ to passive form. Under this restriction, reducing $X$ to passive form is the same as reducing $S$, although, as for the tetrahedron, the rank of $X$ may be $n - 1$.

Since the map depends on positive integer $n$ which is defined by induction, the proof must be by induction; there is no alternative.

For fixed $n$, assume $S$ consistent for all maps with at most $n$ faces. Let $M'$ be had by bonding $F$ to $H$ across $G$ in $M$ with $n$ faces by side $s$ joining vertices marked $x$, $y$, as in Figure 6. Solving $S$ for a few maps practically forces the conclusion that the theorem can be re-phrased:

(6) *For a map with n faces system S is consistent and its solution has exactly* $n - 2$ *principal unknowns.*

If face $F$ = face $H$, a passive form of $S$ is converted into a passive form of $S'$ by adjoining $-x = y$ with the additional principal unkonwn $y$.

Suppose face $F \neq$ face $H$. Evaluation of sums $F$, $H$ for a root of $S'$ gives $F \neq 0$, $H \neq 0$. Hence a necessary condition is that

$$S^* = S - H + (F^2 - 1) + (H^2 - 1)$$

be consistent. The equation $F^2 - 1$ can be omitted: if a root of $S - H + (H^2 - 1)$ made $F = 0$, that root would make zero all face sums except $G$, $H$ which join, whence the contradiction $H = 0$. Operating on $(S - H) + H$ gives a passive form of $S$. Hence $S - H$ is consistent because $S$ is. If every root of $S - H$ made $H = 0$ or if $S - H$ had fewer than $n - 3$ principal unknowns, $S$ would have fewer than the assumed $n - 2$. Hence $S^*$ is consistent and has at least $n - 3$ principal unknowns. Since $S - H$ has at most $n - 3$ and $H^2 - 1$ gives a mediate, $S^*$ has exactly $n - 3$ principal unknowns. Adjoining $x = -F$, $y = -H$ to the solution of $S^*$ gives the solution of consistent $S^{**}$ which is equivalent to $S'$ and has principal unknowns $x, y$ in addition to the $n - 3$ of $S^*$, that is, $(n - 3) + 2 = (n + 1) - 2$ in all. The proof is complete. The equation $H^2 = 1$ gives an unknown mediate for $S'$ and $F^2 = 1$ another dependent on the rest.

## Appendix

Another passive form of (1) with 8, 9, 10, 12 principal, 14 mediate and 11, 13, 15 parametric is:

$$-13 - 14 - 15 = 8, \qquad 13 + 15 = (13 + 15)^2 14,$$

$$11 + 13 - 14 + 15 = 9, \qquad -(11 + 13 + 15) = (11 + 13 + 15)^2 14,$$

$$-11 - 14 - 15 = 10, \qquad 11 + 15 = (11 + 15)^2 14,$$

$$-11 - 13 - 14 = 12, \qquad 11 + 13 = (11 + 13)^2 14.$$

Two faces are a two-ring if they join at more than one side; the joins are the bars of the ring. The two quadrilaterals of the hut are a two-ring. A side is erasable if and only if it is bar in no two-ring ($3 < n$). Each side of a lune in the hut is erasable; the other two sides are bars in a two-ring; deletion of neither gives a map because in the result the other bar is a side at only one face. Each side of the tetrahedron is erasable. Except in $M_3$, each bond at a vertex of a lune is bar in a two-ring and only in $M_3$ is erasable.

Let $FG$ be a two-ring in $M$ ($3 < n$). Starting from vertex $a$ at both $F$ and $G$ follow arc $f$ in loop $G$ but not in loop $F$ to vertex $b$ at both $F$ and $G$, see Figure 9. Then follow $g$ in loop $F$ but not in loop $G$, completing a loop $L$. Identify the outside of $L$ as containing $F$ and $G$. From $M$ delete all sides inside $L$ and all sides in arc $f$ to have map $M'$ in which $FG$ become $F'G'$. Then from the original $M$ delete all sides outside $L$ except those at $F$ to have map $M''$ in which $FG$ become $F''G''$. Coloring $M$ is thus reduced to coloring $M'M''$ so that the colors 45 on $F'G'$ equal those on $F''G''$, respectively. The colors on $FG$, which are the only faces of $M$ not in just one of $M'M''$, are made 45. Erasure of a bar from a three-ring, next to be defined, gives a two-ring. In proving existence, knowledge of erasable side $s$ can be assumed. If a map is proposed for coloring without giving this information, the proof can be made by using the above classical reduction, which in any event is assumed in the proof of (7) below.

If faces $FGH$ are not at a vertex and if each pair $GH$, $HF$, $FG$ has a unique join, the faces are a three-ring, of which the joins are bars. The three bonds at the vertices of a triangle are bars in a three-ring, except in the tetrahedron, where they are at a vertex. Just as a loop separates the points of the plane into three sets, so a three-ring separates the faces of a map into those in the ring, those inside it and those outside. A four-ring is had by introducing an additional bar.

Let $FGH$ be a three-ring in $M$. Deletion of the three bars gives loops $L'L''$ in Figure 10. By deletion shrink $L''$ to point $v'$ which is a vertex in map $M'$ where $FGH$ are replaced by $F'G'H'$. Similarly, shrink $L'$ to a vertex $v''$ in map $M''$. Coloring $M$ is thus reduced to coloring $M'M''$ so that the colors 456 on $F'G'H'$ equal those on $F''G''H''$.

The faces joining a face $F$ are its fringe. First get a necessary condition for coloring by treating $F$ and its fringe alone. Describe the loop of $F$ clockwise. Ignore all joins except those of $F$ with its fringe and those at consecutive faces of the fringe. At vertex $v$ put 4 on $F$ and 56 on the other two faces. As a vertex is passed at least one of 567 is available for the next face. Choose one of the available arbitrarily. Mark $v$ with 1, abbreviated to $+$, and make the marks from $+$ and $-$ at a side equal or unequal according as the colors on the two faces bonded by the side are unequal or equal. Generalize "bond" so that the arc of loop $F$ clockwise from $G$ to $H$ bonds face $G$ to face $H$. Then (i) the colors on two faces $GH$ in the fringe are equal or not according as the sum of the marks at the vertices in arc $GH$ and at its ends is 0 or not; (ii) the sum around the whole loop is 0; (iii) a color is determined for each face in the fringe by the colors on the other faces in the fringe and the marks. The proof is by induction. If into a fringe of $k$ faces a face is introduced between the two at $w$, the insertion is compensated by replacing $x$ at $w$ by $-x$, $-x$ with sum $x$ at the two vertices replacing $w$. This completes the induction if the colors on three consecutive faces of the given $k + 1$ are 567 (no two equal); if the colors are alternately 56, the signs also alternate and the result is obvious. A necessary condition for coloring the whole map is therefore that the sum of the marks about each face be 0. A linear, homogeneous system $X$ is thus had. Its consistency in $GF(3)$ is sufficient. A weaker condition is also, namely, the sum about each face, except about two $FG$ which join, is 0. This is Heawood's theorem:

(7) *If faces* FG *join, system* X − F − G *is equivalent to* X *in* GF(3) − 0.

In the proof, the context will decide whether $F$ means a face or the sum about its loop. Color $M - F - G$ by a root of $Y = X - F - G$, using (i)-(iii). If $n = 2$, the theorem is meaningless. If $n = 3$, the three sums are identically equal; if one is 0, all are. If $M$ is the tetrahedron $FGHJ$, the three marks at $J$ are equal to each other and to those at $H$, so that all four are equal; and $F = G = 0$. If fringe $F$ contains a face $H$ which is neither $G$ nor in fringe $G$, (iii) applied to fringe $H$ gives $F = 0$; and then $G$ is also 0. Hence assume that fringe $F$ less $G$ equals fringe $G$ less $F$. In the cases not covered, there is a ring and induction completes the proof.

The delay in obtaining a proof of the four color theorem should emphasize the neglect of the theory of equations, which should be to mathematics what the lever is to mechanics.

## References

P. J. Heawood, *Map-Colour Theorem*, Quarterly Journal of Mathematics, 24(1890), 332–338; *On the Four-Colour Map Theorem*, ibid., 29(1898), 270–285.

J. M. Thomas, *The Four Color Theorem*, Philadelphia, Pa., 1969; *A Primer on Roots*, Richmond, Va. 1974; *A Brief Proof*, Durham, N. C., 1976.

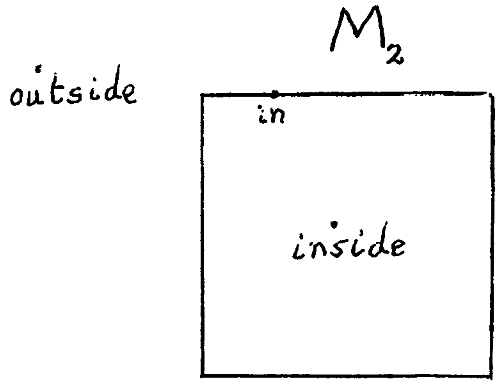


Figure 1

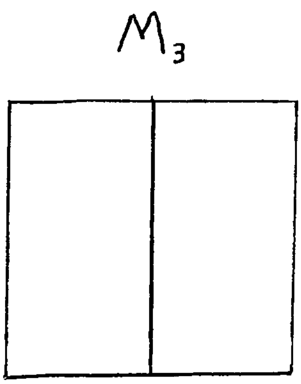


Figure 2

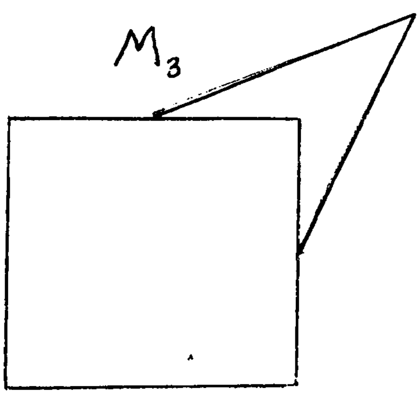


Figure 3

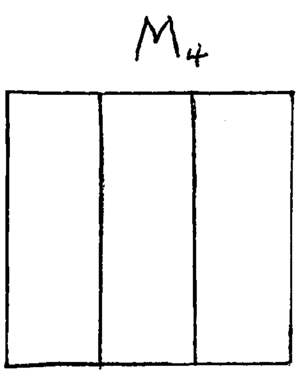


Figure 4

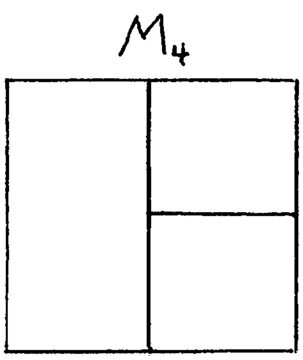


Figure 5

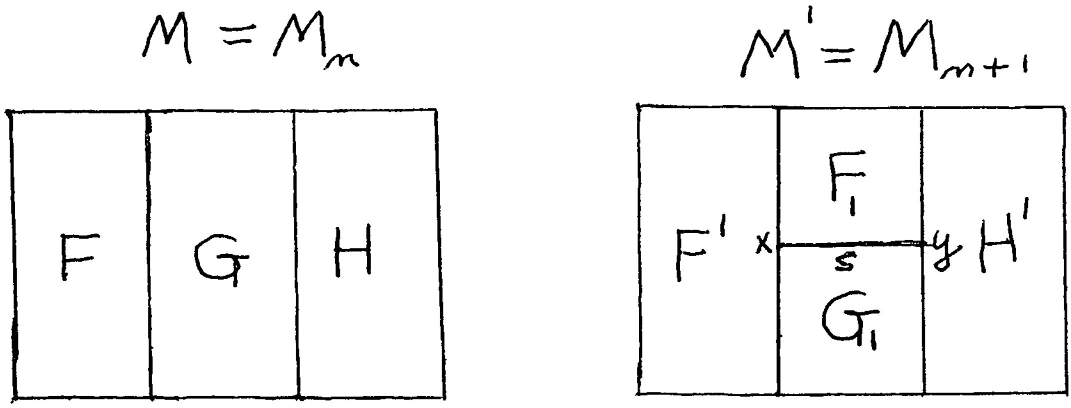


Figure 6

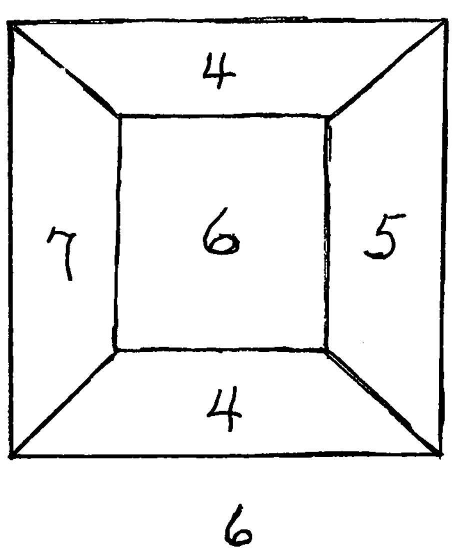


Figure 7

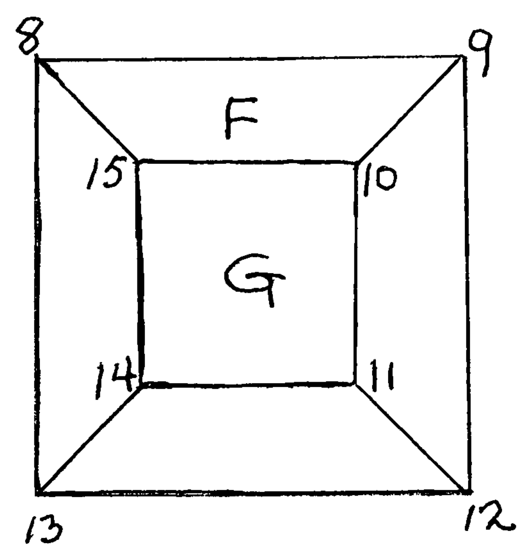


Figure 8

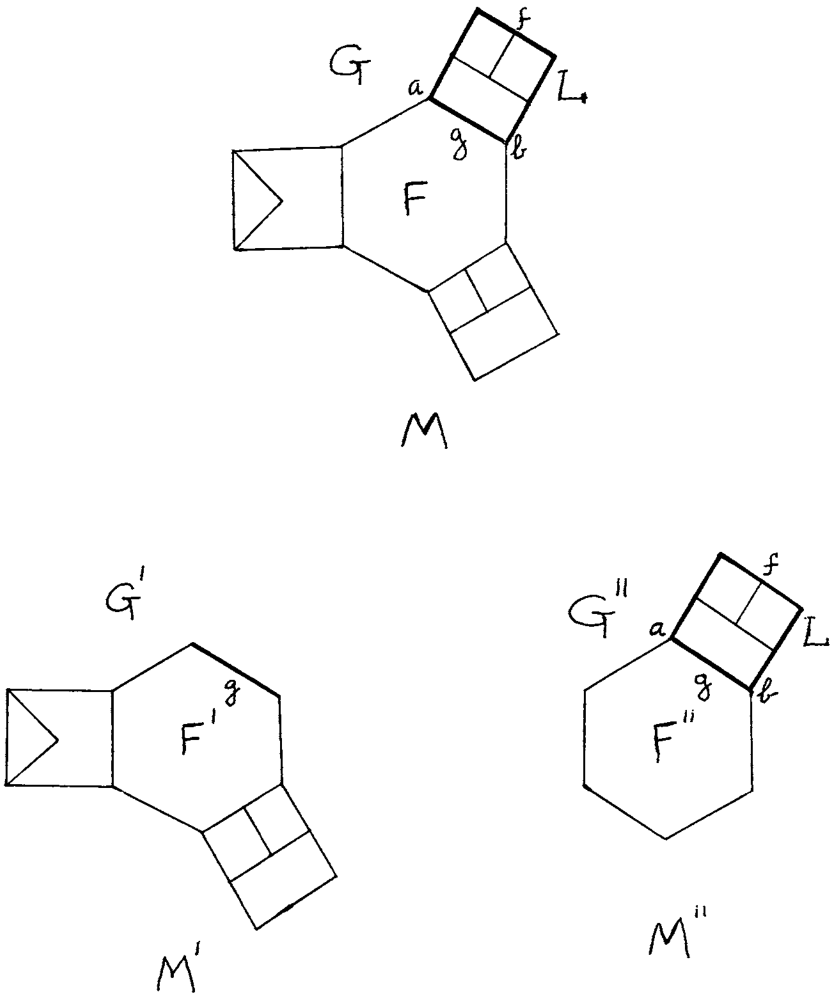


Figure 9

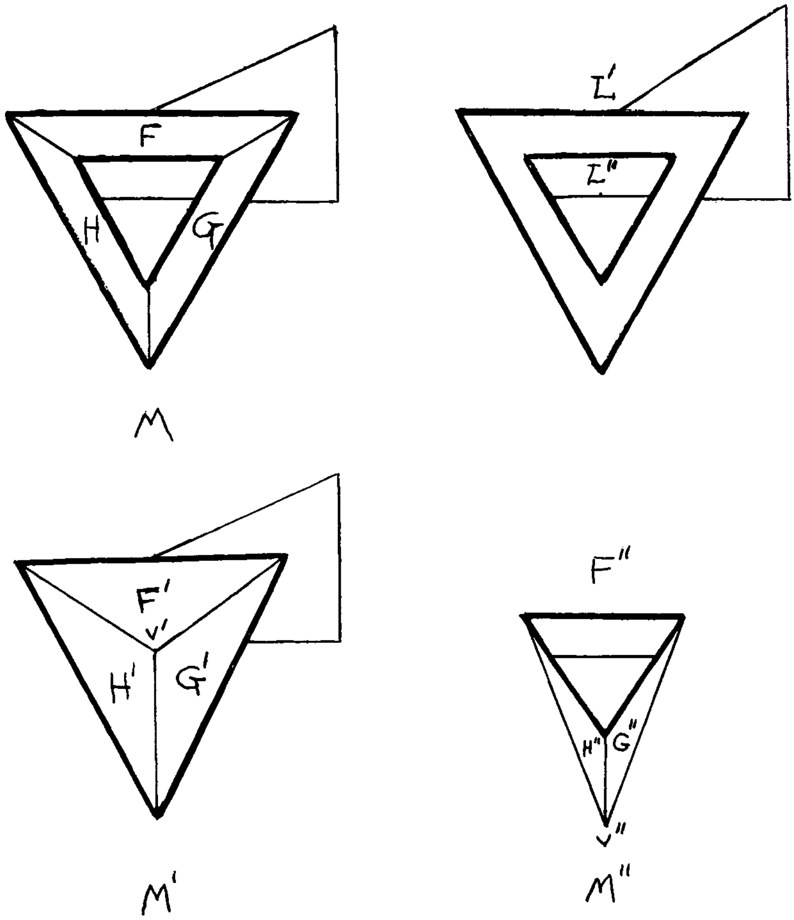


Figure 10