MOD p POINTS ON SHIMURA VARIETIES OF ABELIAN TYPE

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Abstract. We show that the mod p points on a Shimura variety of abelian type with hyperspecial level, have the form predicted by the conjectures of Kottwitz and Langlands-Rapoport. Along the way we show that the isogeny class of a mod p point contains the reduction of a special point.

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Introduction

In his paper [La 1], which describes some of the ideas flowing from Kronecker's *Jugendtraum*, Langlands made a conjecture about the structure of the mod p points on a Shimura variety. This conjecture, which followed work of Ihara [Ih 1], [Ih 2] in the case of inner forms of $\mathrm{GL}_{2/\mathbb{Q}}$, is an essential part of Langlands' program [La 2], [La 3], [La 4] to express the zeta function of a Shimura variety as a product of automorphic L-functions.

The conjecture was made more precise by Kottwitz [Ko 3] and further refined by Langlands-Rapoport [LR]. To explain the statement, suppose that (G, X) is a Shimura datum, p a prime, and $\mathsf{K}^p \subset G(\mathbb{A}_f^p)$ and $\mathsf{K}_p \subset G(\mathbb{Q}_p)$ compact open subgroups. We will be concerned with the case when K_p is *hyperspecial*[1] , which means that G extends to a reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, and $\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. In this case the Shimura variety $\mathrm{Sh}_{\mathsf{K}_p}(G, X) = \varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}(G, X)$ is conjectured to have a canonical, smooth integral model $\mathscr{S}_{\mathsf{K}_p}(G, X)$ over any prime $\lambda|p$ of its reflex field. Fix such a prime λ, and an algebraic closure $\bar{\mathbb{F}}_p$ of its residue field $\kappa(\lambda)$. The conjecture asserts[2] there is a bijection

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{\phi} S(\phi) \tag{0.1}$$

where

$$S(\phi) = \varprojlim_{\mathsf{K}^p} I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/\mathsf{K}^p. \tag{0.2}$$

This requires some explanation. Each of the sets $S(\phi)$ is intended to correspond to the points in a fixed isogeny class in the left side of (0.1), with $X^p(\phi)$ and $X_p(\phi)$ corresponding to the isogenies of prime to p-order, and p-power order respectively. The set $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$ torsor, while $X_p(\phi)$ can be identified with a subset of $G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$ which has an explicit group theoretic description (see §3.3 below). The term I_ϕ is a certain algebraic group over $\mathbb{Q}$. When $\mathscr{S}_{\mathsf{K}_p}(G, X)$ is a moduli space for abelian varieties, I_ϕ corresponds to the automorphism group (in the isogeny category) of an abelian variety in the isogeny class corresponding to the parameter ϕ. Each of the sets $S(\phi)$ is naturally equipped with an action of $G(\mathbb{A}_f^p)$ and an action of the Frobenius in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\kappa(\lambda))$, and (0.1) is required to respect these actions.

In Kottwitz's formulation[3] the terms in the indexing set are (equivalence classes of) triples $(\gamma_0, (\gamma_l)_{l\neq p}, \delta)$ where $\gamma_0 \in G(\mathbb{Q})$, $\gamma_l \in G(\mathbb{Q}_l)$ for $l \neq p$, and $\delta \in G(\mathbb{Q}_p^{\mathrm{ur}})$. These triples are required to satisfy certain compatibilities, and each triple appears with a certain finite multiplicity in the indexing set. When $\mathscr{S}_{\mathsf{K}_p}(G, X)$ is a moduli space for abelian varieties the elements γ_l and δ corresponds to the relative and absolute Frobenius acting on the l-adic and crystalline cohomology of an abelian variety.

The formulation of Langlands-Rapoport is more abstract, and inspired by the formalism of motives. The indexing set consists of morphisms $\phi : \mathfrak{Q} \to G$, satisfying

[1]Langlands-Rapoport formulate their conjecture under more general conditions on K_p.

[2]Both Kottwitz and Langlands-Rapoport formulate the conjecture assuming that G^{der} is simply connected. However, it turns out that it is not difficult to remove this condition - see the discussion at the end of this introduction and §3.3 below.

[3]Kottwitz's formulation assumes that the maximal $\mathbb{R}$-split torus of Z_G is $\mathbb{Q}$-split.

certain conditions, where $\mathfrak{Q}$ is an explicitly constructed groupoid[4] which, morally, ought to be the fundamental groupoid of the category of motives over $\bar{\mathbb{F}}_p$. When Kottwitz's condition on Z_G is satisfied, it was shown by Langlands-Rapoport [LR, Satz 5.25] that one can pass between the two points of view. The two formulations of the conjecture are then almost equivalent, the only difference being that Langlands-Rapoport give a precise description of the action of $I_\phi(\mathbb{Q})$ on $X_p(\phi)\times X^p(\phi)$, whereas Kottwitz's formulation allows one to describe this action only up to conjugation by an element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$. Fortunately, this ambiguity should be inessential for the application to the zeta function.

Following advances by a number of authors [Mo], [De 3], [Mi 1], [Zi 2], [ReZ], Kottwitz proved the conjecture (in his formulation) for PEL Shimura varieties of types A and C [Ko 4]. The main aim of this paper is to prove the conjecture for Shimura varieties of abelian type. We use the formulation of Langlands-Rapoport, but we prove a statement only up to conjugation by an element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ as in Kottwitz's formulation. More precisely we show (see (4.6.7) below)

Theorem (0.3). *Suppose that $p > 2$ and that (G, X) is of abelian type. Then there is a bijection*

$$\mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{\phi} \varprojlim_{\mathsf{K}^p} I_\phi(\mathbb{Q})\backslash X_p(\phi)\times X^p(\phi)/\mathsf{K}^p$$

respecting the action of Frobenius and $G(\mathbb{A}_f^p)$, where the action of $I_\phi(\mathbb{Q})$ on $X_p(\phi)\times X^p(\phi)$ is obtained from the natural action by conjugating by an element $\tau(\phi)\in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$.

Recall that the class of Shimura varieties of abelian type are exactly those considered by Deligne in [De 2], and include almost all Shimura varieties where G is a classical group.

We now give some details about the techniques and organization of the paper. The theorem is proved by first studying the case where (G, X) is of *Hodge type*, so that $\mathrm{Sh}_{\mathsf{K}_p}(G,X)$ is a moduli space for abelian varieties equipped with certain Hodge cycles. Showing that an isogeny class on $\mathscr{S}_{\mathsf{K}_p}(G,X)$ has the form predicted by (0.2), involves solving several problems which do not arise for PEL Shimura varieties.

To explain the first of these, recall [Ki 2] that the integral model $\mathscr{S}_{\mathsf{K}_p}(G,X)$ is obtained by taking an embedding $\mathrm{Sh}_{\mathsf{K}_p}(G,X)\hookrightarrow \mathrm{Sh}_{\mathsf{K}'_p}(\mathrm{GSp},S^{\pm})$ into a Siegel modular variety, and taking the normalization of the closure of $\mathrm{Sh}_{\mathsf{K}_p}(G,X)$ in a suitable model $\mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp},S^{\pm})$ of $\mathrm{Sh}_{\mathsf{K}'_p}(\mathrm{GSp},S^{\pm})$. Given a point $x\in\mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$ one obtains a set $X_p(x)$, as in (0.2). If $\mathcal{A}_x$ denotes the polarized abelian variety corresponding to x, then by considering suitable p-power isogenies of $\mathcal{A}_x$, one obtains a map $\iota_x : X_p(x)\to\mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp},S^{\pm})$. However since these p-power isogenies do not, in general, lift to characteristic 0, it is not clear that this map factors through $\mathscr{S}_{\mathsf{K}_p}(G,X)$; unlike the PEL case one has no explicit moduli theoretic description of $\mathscr{S}_{\mathsf{K}_p}(G,X)$.

We solve this problem in §1. It turns out that $X_p(x)$ has a kind of geometric structure. Using a deformation theoretic argument we show that if ι_x factors

[4] More precisely, Langlands-Rapoport use the language of Galois gerbs which will also be convenient for us below.

through $\mathscr{S}_{\mathsf{K}_p}(G,X)$ at some point y, then it factors through $\mathscr{S}_{\mathsf{K}_p}(G,X)$ in a formal neighborhood of y, and hence on the "connected component" of $X_p(x)$ containing y. We then show that one can move between connected components of $X_p(x)$ using isogenies which lift to characteristic 0. This uses in a crucial way the main result of [CKV] which gives an explicit description of these connected components.

Attached to the abelian variety $\mathcal{A}_x$ are conjugacy classes $\gamma_l \in G(\mathbb{Q}_l)$, $l \neq p$, and a Frobenius conjugacy class $\delta \in G(\mathbb{Q}_p^{\mathrm{ur}})$, corresponding to the action of the p^r-Frobenius and absolute Frobenius on l-adic and crystalline cohomology respectively. Here r is an integer such that x is defined over $\mathbb{F}_{p^r}$. The conjecture implies that there ought to exist an element $\gamma_0 \in G(\mathbb{Q})$ such that γ_0 is conjugate to γ_l in $G(\bar{\mathbb{Q}}_l)$ and to $\gamma_p := \delta\sigma(\delta)\dots\sigma^{r-1}(\delta)$ in $G(\bar{\mathbb{Q}}_p)$. Here σ denotes the Frobenius on $\mathbb{Q}_p^{\mathrm{ur}}$. In the PEL cases considered by Kottwitz this turns out to be a consequence of the fact that the characteristic polynomials of γ_l and γ_p lie in $\mathbb{Z}[X]$ and are equal, but this does not suffice in general.

We also have the group I of automorphisms of $\mathcal{A}_x$ in the isogeny category, preserving the specializations of the Hodge cycles used to define $\mathrm{Sh}_{\mathsf{K}_p}(G,X)$. This is an algebraic group over $\mathbb{Q}$, and the existence of an isomorphism as in (0.1) implies that I should be an inner twisting of the centralizer of $\gamma_0 \in G(\mathbb{Q})$. When these Hodge cycles are endomorphisms and polarizations, one can deduce this from Tate's theorem [Ta 2] on endomorphisms of abelian varieties over finite fields, but this does not seem to be possible with arbitrary Hodge cycles.

In §2 we give a different proof of Tate's theorem which works for automorphisms preserving a collection of Hodge cycles. This allows us to show, in particular, that I has the same rank as G. Using this we deduce the following result of independent interest.

Theorem (0.4). *If (G,X) is of Hodge type, and $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$ then the isogeny class of x contains a point which lifts to a special point on $\mathrm{Sh}_{\mathsf{K}_p}(G,X)$.*

Finally, it follows from the above theorem and the reciprocity law, that the element $\gamma_0 \in G(\mathbb{Q})$ above exists, and is stably conjugate to γ_l and γ_p. We remark that for PEL Shimura varieties a weaker version of (0.4), where the isogeny is not required to respect polarisations, was proven by Zink [Zi 2]. The same weaker result is implicitly recovered in [Ko 4]. Thus, (0.4) is new even for Shimura varieties of PEL type.[5]

At this point it remains to match the isogeny classes on the left side of (0.1) with the indexing set on the right side, and to deduce the case of abelian type Shimura data from the case of Hodge type. These two steps are in fact carried out together. To explain this, recall that (G,X) is of abelian type if there is a Shimura datum of Hodge type (G',X') equipped with a morphism $G'^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism $(G'^{\mathrm{ad}},X'^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}},X^{\mathrm{ad}})$. The integral model $\mathscr{S}_{\mathsf{K}_p}(G,X)$ is constructed in [Ki 2] from the integral model $\mathscr{S}_{\mathsf{K}'_p}(G',X')$ (for a suitable choice of hyperspecial K'_p), following Deligne's formalism of connected Shimura varieties [De 2]. This uses, in particular, the action of $G'^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ = G'^{\mathrm{ad}}(\mathbb{Z}_{(p)}) \cap G'(\mathbb{R})^+$ on $\mathscr{S}_{\mathsf{K}'_p}(G',X')$.

In §3 we show that the right side of (0.1) satisfies an analogous formalism: it admits an action $G'^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, and there is a subset $(\coprod_\phi S(\phi))^+$, corresponding to the connected Shimura variety, which can be used to reconstruct $\coprod_\phi S(\phi)$. We use

[5]See §2.1 for further historical remarks.

this in §4 to show that for Shimura data (G', X') of Hodge type, there is a refined form of the bijection (0.1), which apart from respecting the action of $G'(\mathbb{A}_f^p)$, and Frobenius, also respects the action of $G'(\mathbb{Z}_{(p)})^+$, and is compatible with the passage to connected components on both sides. The main theorem follows.

The idea of emulating Deligne's formalism for $\coprod_\phi S(\phi)$ was introduced by Pfau [Pf], who carried out the above constructions subject to a certain cohomological condition on $Z_{G'}$. This was used by Milne [Mi 2, 4.19] to reduce the Langlands-Rapoport conjecture to the case of simply connected derived group. Unfortunately, although Pfau's condition greatly simplifies the cohomological calculations, it is almost never satisfied for Shimura varieties of Hodge type; in particular the Shimura varieties with simply connected G^{der} to which Milne reduces the conjecture are almost never of Hodge type. Thus we cannot use the results of Milne and Pfau directly.

As noted above, Kottwitz and Langlands-Rapoport formulate their conjectures when G^{der} is simply connected. It turns out that this condition is easy to remove: one of the conditions imposed in [LR] on the morphism $\phi : \mathfrak{Q} \to G$, is that the induced morphism $\phi : \mathfrak{Q} \to G^{\mathrm{ab}}$ is equal to a fixed morphism determined by the conjugacy class of cocharacters μ_h attached to (G, X). We replace this with a condition on the induced morphism $\phi : \mathfrak{Q} \to G/\tilde{G}$ to the stack $G/\tilde{G}$, where $\tilde{G}$ is the simply connected cover of G^{der}. This is very natural since the conjugacy class of $[\mu_h]$ determines not just an element of $X_*(G^{\mathrm{ab}})$, but an element of $\pi_1(G)$, which can be identified with $X_*(G/\tilde{G})$.

In [Mi 2, 4.4] Milne formulated an extension of the conjecture to non-simply connected derived groups, which requires that the morphism ϕ is *special*. This means that ϕ factors through a suitable torus in G, and is the analogue of the statement that every isogeny class contains a point with a special lifting. Our results show *a posteriori* that our formulation is equivalent to that of Milne, at least when (G, X) is of Hodge type. However, Milne's condition seems too inexplicit for applications to computing the zeta function of the Shimura variety.

Our main results are in terms of the formulation of the conjecture by Langlands-Rapoport, however we also use the point of view of Kottwitz and its relation to that of [LR], in an essential way. To construct an analogue of Deligne's formalism for $\coprod_\phi S(\phi)$ it seems essential to use the point of view of [LR] - this has been emphasized by Milne [Mi 2]. However, to relate these structures to $\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$ we need to make use of the more concrete formulation of Kottwitz.

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§1 Isogenies and Shimura varieties mod p.

(1.1) Classification of p-divisible groups In this section we assume that $p > 2$.[6] The aim of this subsection is to recall various results about p-divisible groups over perfect fields, and their deformations, especially when the Dieudonné

[6]Thanks to recent improvements to some of the arguments in [Ki 2], by G. Pappas and W. Kim, this is likely to soon be unnecessary.

modules of these p-divisible groups are equipped with a collection of φ-invariant tensors.

(1.1.1) Let k be a perfect field of characteristic p, $W = W(k)$ its ring of Witt vectors and $K_0 = W(k)[1/p]$. Let K be a finite totally ramified extension of K_0, and $\mathcal{O}_K$ its ring of integers. Fix an algebraic closure $\bar{K}$ of K, and set $G_K = \mathrm{Gal}(\bar{K}/K)$.

We denote by $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ the category of crystalline G_K-representations, and by $\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ}$ the category of G_K-stable $\mathbb{Z}_p$-lattices spanning a representation in $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$. For V a crystalline representation, recall Fontaine's functors

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \textit{and} \quad D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Fix a uniformiser $\pi \in K$, and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for π. We set $\mathfrak{S} = W[\![u]\!]$ equipped with a Frobenius φ which acts as the usual Frobenius σ on W and sends u to u^p.

Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

$$1 \otimes \varphi : \varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)].$$

We denote by $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ the full sub-category of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those modules such that $1 \otimes \varphi$ maps $\varphi^*(\mathfrak{M})$ into $\mathfrak{M}$, and the image of this map contains $E(u)\mathfrak{M}$.

For $i \in \mathbb{Z}$ we set

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}) = (1 \otimes \varphi)^{-1}(E(u)^i \mathfrak{M}) \cap \varphi^*(\mathfrak{M}).$$

Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}_{(p)}$, and denote by $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ the category of finite free $\mathcal{O}_{\mathcal{E}}$-modules M equipped with an isomorphism $\varphi^*(M) \xrightarrow{\sim} M$. We have a functor

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}; \quad \mathfrak{M} \mapsto \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}.$$

By [Ki 2, 1.2.1], we have the following

Theorem (1.1.2). *There exists a fully faithful tensor functor*

$$\mathfrak{M}(\cdot) : \mathrm{Rep}_{G_K}^{\mathrm{cris}\circ} \to \mathrm{Mod}^{\varphi}_{/\mathfrak{S}},$$

which is compatible with formation of symmetric and exterior powers. If L is in $\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ}$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, *and* $\mathfrak{M} = \mathfrak{M}(L)$, *then*

(1) *There are canonical isomorphisms*

$$D_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathfrak{M}/u\mathfrak{M}[1/p] \quad \textit{and} \quad D_{\mathrm{dR}}(V) \xrightarrow{\sim} \varphi^*(\mathfrak{M}) \otimes_{\mathfrak{S}} K,$$

where the map $\mathfrak{S} \to K$ is given by $u \mapsto \pi$. The first isomorphism is compatible with Frobenius and the second maps $\mathrm{Fil}^i \varphi^*(\mathfrak{M}) \otimes_W K_0$ *onto* $\mathrm{Fil}^i D_{\mathrm{dR}}(V)$ *for $i \in \mathbb{Z}$.*

(2) *There is a canonical isomorphism*

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M},$$

where $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is a certain faithfully flat, and formally étale $\mathcal{O}_{\mathcal{E}}$-algebra.

(1.1.3) For a ring R and a finite free R-module M, we denote by $M^{\otimes}$ the direct sum of the R-modules obtained from M by taking duals, tensor products, symmetric and exterior powers.[7] The stabilizer of a collection of tensors $(s_\alpha) \subset M^{\otimes}$ is then a closed subgroup scheme of $\mathrm{GL}(M)$.

Suppose L is in $\mathrm{Rep}^{\mathrm{criso}}_{G_K}$ and $(s_\alpha) \subset L^{\otimes}$ is a collection of G_K-invariant tensors defining a subgroup $G_{\mathbb{Z}_p} \subset \mathrm{GL}(L)$. We may view the s_α as morphisms $\mathbf{1} \to L^{\otimes}$ and apply the functor of (1.1.2) to obtain tensors $\tilde{s}_\alpha \in \mathfrak{M}(L)^{\otimes}$. By functoriality of the isomorphisms in (1.1.2)(1),

$$s_{\alpha,0} := \tilde{s}_\alpha|_{u=0} \in (\mathfrak{M}/u\mathfrak{M})^{\otimes} \subset D_{\mathrm{cris}}(L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\otimes}$$

is the image of s_α under the p-adic comparison isomorphism.

We have [Ki 2, 1.3.4, 1.3.6]

Theorem (1.1.4). *Suppose that $G_{\mathbb{Z}_p}$ is reductive, then*

(1) *The tensors $(\tilde{s}_\alpha)$ define a reductive subgroup in* $\mathrm{GL}(\mathfrak{M})$.
(2) *If k is separably closed or G is connected and k is finite, then there exists an isomorphism*

$$L \otimes_{\mathbb{Z}_p} \mathfrak{S} \xrightarrow{\sim} \mathfrak{M}$$

taking s_α to $\tilde{s}_\alpha$.

(1.1.5) For any p-divisible group $\mathscr{G}$ over a scheme T on which p is locally nilpotent, we denote by $\mathbb{D}(\mathscr{G})$ the corresponding (contravariant) F-crystal, and write φ for the action of Frobenius on $\mathbb{D}(\mathscr{G})$. We denote by $\mathscr{G}^*$ the Cartier dual of $\mathscr{G}$, and by $T_p\mathscr{G}$ the Tate module of $\mathscr{G}$.

Let S be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the kernel of $W(k)[u] \xrightarrow{u \mapsto \pi} \mathcal{O}_K$. We equip S with a Frobenius given by the usual Frobenius on $W(k)$ and sending u to u^p. We view S as a $\mathfrak{S}$-algebra by sending u to u, and we view W as an S-algebra by sending u to 0.

If $\mathfrak{M}$ is in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ then we set $\mathcal{M}(\mathfrak{M}) = S \otimes_{\mathfrak{S}} \varphi^*(\mathfrak{M})$, and equip $\mathcal{M}(\mathfrak{M})$ with the induced Frobenius φ, and with an S-submodule

$$\mathrm{Fil}^1\mathcal{M}(\mathfrak{M}) = \{x \in S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} : 1 \otimes \varphi(x) \in \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

One checks easily that

$$\mathrm{Fil}^1\mathcal{M}(\mathfrak{M}) = S \cdot \mathrm{Fil}^1(\varphi^*(\mathfrak{M})) + \mathrm{Fil}^1 S \cdot \mathcal{M}.$$

We denote by $(p\text{-div}/\mathcal{O}_K)$ the category of p-divisible groups over $\mathcal{O}_K$. Then by [Ki 2, 1.4.2] we have the following[8]

[7]When R is a mixed characteristic discrete valuation ring, and $G \subset \mathrm{GL}(M)$ is a closed, R-flat subgroup with reductive generic fibre, it is shown in [Ki 2, 1.3.2] that G is the pointwise stabilizer of a collection of elements of $M^{\otimes}$. The proof in *loc. cit* uses symmetric and exterior powers, so we include them in the definition here, however Deligne has explained to us an argument using only duals and tensor products

[8]In fact $T_p\mathscr{G}^*$ should be replaced by $T_p\mathscr{G}^*(-1)$, the linear dual of $T_p\mathscr{G}$, in (1.4.2), (1.4.3) and (1.5.11) of *loc. cit.* We have made this correction here.

Theorem (1.1.6). *The functor $\mathfrak{M}(\cdot)$ of (1.1.2) induces an exact equivalence*

$$\mathfrak{M}(\cdot) : (p\text{-div}/\mathcal{O}_K) \to \mathrm{BT}^{\varphi}_{/\mathfrak{S}}; \quad \mathscr{G} \mapsto \mathfrak{M}(\mathscr{G}) := \mathfrak{M}(T_p\mathscr{G}^*(-1))$$

If $\mathscr{G}$ is a p-divisible group over $\mathcal{O}_K$ then there is a canonical isomorphism

$$\mathbb{D}(\mathscr{G})(S) \xrightarrow{\sim} \mathcal{M}(\mathfrak{M}(\mathscr{G})).$$

compatible with φ and filtrations. In particular, there is a canonical φ-compatible isomorphism

$$\mathbb{D}(\mathscr{G})(W) \xrightarrow{\sim} \varphi^*(\mathfrak{M}(\mathscr{G})/u\mathfrak{M}(\mathscr{G})).$$

Corollary (1.1.7). *Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$, and $(s_\alpha) \subset T_p\mathscr{G}^{\otimes}$ a collection of G_K-invariant tensors defining a reductive subgroup $G \subset \mathrm{GL}(T_p\mathscr{G})$. Suppose that either k is separably closed or that $G_{\mathbb{Z}_p}$ is connected and k is finite. Let $s_{\alpha,0} \in \mathbb{D}(\mathscr{G})(W)^{\otimes} \otimes_W K_0$ denote the image of s_α under the p-adic comparison isomorphism. Then*

1. $(s_{\alpha,0}) \subset \mathbb{D}(\mathscr{G})(W)^{\otimes}$
2. *There is an isomorphism $T_p\mathscr{G}^*(-1) \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G})$ taking s_α to $s_{\alpha,0}$.*
3. *The filtration on $\mathbb{D}(\mathscr{G})(k)$ is induced by a G_W-valued cocharacter.*

Proof. This is [Ki 2, 1.4.3], the first two parts following from (1.1.4) and (1.1.6) □

(1.1.8) Suppose $\mathscr{G}$ is a p-divisible group over k, equipped with a collection of φ-invariant tensors $(s_{\alpha,0}) \subset \mathbb{D}(\mathscr{G})(W)^{\otimes}$ which defines a reductive subgroup $G_W = G(s_{\alpha,0}) \subset \mathrm{GL}(\mathbb{D}(\mathscr{G}))$. We also assume that the filtration on $\mathbb{D}(\mathscr{G})(k)$ is induced by a G_W-valued cocharacter. We denote by μ_0 the *inverse* of this cocharacter.

Let $\tilde{\mathscr{G}}$ be a deformation of $\mathscr{G}$ to $\mathcal{O}_K$. We say that $\tilde{\mathscr{G}}$ is G_W*-adapted* if there exist φ-invariant tensors $\tilde{s}_\alpha \in \mathbb{D}(\tilde{\mathscr{G}})(S)^{\otimes}$ lifting $s_{\alpha,0}$, such that $(\tilde{s}_\alpha)$ defines a reductive subgroup $G_S \subset \mathrm{GL}(\mathbb{D}(\tilde{\mathscr{G}})(S))$ and the image of $\tilde{s}_\alpha$ in $\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K)^{\otimes}$ is in $\mathrm{Fil}^0(\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K)^{\otimes})$.

We remark that there is a unique φ-invariant isomorphism [Ki 1, 1.2.6]

$$\mathbb{D}(\mathscr{G})(W) \otimes_W S[1/p] \xrightarrow{\sim} \mathbb{D}(\tilde{\mathscr{G}})(S) \otimes_S S[1/p]$$

lifting the identity on $\mathbb{D}(\mathscr{G})(W) \otimes_W K_0$ and this isomorphism must send $s_{\alpha,0}$ to $\tilde{s}_\alpha$. In particular, the subgroup of $\mathrm{GL}(\mathbb{D}(\tilde{\mathscr{G}})(S) \otimes_S S[1/p])$ defined by the $\tilde{s}_\alpha$ may be identified with $G_W \otimes_W S[1/p]$.

Lemma (1.1.9). *Let $\tilde{\mathscr{G}}$ be a G_W-adapted deformation of $\mathscr{G}$. Then the filtration on $\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$ is induced by a $G_W \otimes_W K$-valued cocharacter, and any such cocharacter is conjugate to μ_0^{-1}.*

Proof. Note that we are identifying the subgroup of $\mathrm{GL}(\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K)$ defined by the images of $\tilde{s}_\alpha$ with $G_W \otimes_W K$.

By [Ki 2, 1.4.5] the filtration of the lemma is defined by a $G_W \otimes_W K$-valued cocharacter, whose inverse we denote by μ. If μ' is another such cocharacter, then μ^{-1} and μ'^{-1} define the same parabolic P in $G_W \otimes_W K$, and induce the same P/U valued cocharacter, where U denotes the unipotent radical of P (see [Ki 2, 1.1.5] and its proof). Let L and L' respectively denote the centralizers of μ and μ'

in $G_W \otimes_W K$. Then L, L' are Levi subgroups of P, and hence are conjugate by an element of U [DG, XXVI 1.8]. Hence μ' is conjugate to a cocharacter $\mu'' : \mathbb{G}_m \to L$, which induces the same P/U-valued cocharacter as μ. As $L \xrightarrow{\sim} P/U$, $\mu'' = \mu$.

This shows that all μ as above are conjugate. By [Ki 2, 1.1.4] we may choose μ to be a $G_W \otimes_W \mathcal{O}_K$-valued cocharacter such that μ^{-1} defines the filtration on $\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K)$. Since the filtration on $\mathbb{D}(\tilde{\mathscr{G}})(\mathcal{O}_K)$ reduces to that on $\mathbb{D}(\mathscr{G})(k)$, the same argument as above shows that μ and μ_0 induce conjugate $G_W \otimes_W k$ valued cocharacters. It follows from [DG, IX 3.3] that μ and μ_0 are conjugate. □

Proposition (1.1.10). *Let* Spf R *be the versal deformation space of* $\mathscr{G}$. *Then there is a quotient* R_G *of* R, *which is formally smooth over* W, *and such that for any* K, *as above, a map of* W*-algebras* $\varpi : R \to \mathcal{O}_K$ *factors through* R_G *if and only if the* p*-divisible group* $\mathscr{G}_\varpi$ *induced by* ϖ *is* G_W*-adapted.*

Proof. This is [Ki 2, Prop. 1.5.8]. □

(1.1.11) Now suppose that $\mathscr{G}$ and $(s_{\alpha,0}) \subset \mathbb{D}(\mathscr{G})(W)^\otimes$ is as above, but drop the assumption that the filtration on $\mathbb{D}(\mathscr{G})(k)$ is given by a G_W-valued cocharacter.

Suppose that we are given a finite free $\mathbb{Z}_p$-module U, together with an isomorphism $U \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G})(W)$ such that under this isomorphism $s_{\alpha,0} \in U^\otimes$. These tensors then define a reductive $\mathbb{Z}_p$-subgroup $G = G_{\mathbb{Z}_p} \subset \mathrm{GL}(U)$ which is equipped with an isomorphism $G_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} G_W$.

If $R \to R'$ is a map of commutative rings and X is an R-module, or a scheme over R we shall write $X_R = X \otimes_R R'$. In particular, for R a $\mathbb{Z}_p$-algebra $G_R = G \otimes_{\mathbb{Z}_p} R$.

Since $s_{\alpha,0} \in U^\otimes$ are φ-invariant, φ on $\mathbb{D}(\mathscr{G})(W) \otimes_W W[1/p]$ has the form $b\sigma$ for some $b \in G(W[1/p])$. If μ is a cocharacter we will sometimes write p^μ for $\mu(p)$.

Lemma (1.1.12). *The filtration on* $\mathbb{D}(\mathscr{G})(k)$ *is given by a* G_W*-valued cocharacter* μ_0^{-1}, *and* $b \in G_W(W)p^{\upsilon_0}G_W(W)$ *where* $\upsilon_0 = \sigma(\mu_0^{-1})$.

Proof. By the Cartan decomposition there is a G_W-valued cocharacter υ_0', and $h_1, h_2 \in G_W(W)$ such that $b = h_1 p^{\upsilon_0'} h_2 = h_1 h_2 p^{h_2^{-1}\upsilon_0' h_2}$. Hence the filtration on $\mathbb{D}(\mathscr{G})(k)$ is given by $\sigma^{-1}(\upsilon_0)$, where $\upsilon_0 = h_2^{-1}\upsilon_0' h_2$. □

Proposition (1.1.13). *With the above assumptions, a deformation* $\tilde{\mathscr{G}}$ *of* $\mathscr{G}$ *to* $\mathcal{O}_K$ *is* G_W*-adapted if and only if, via the* p*-adic comparison isomorphism*

$$\mathbb{D}(\tilde{\mathscr{G}})(W) \otimes_W B_{\mathrm{cris}} \xrightarrow{\sim} T_p\tilde{\mathscr{G}}^*(-1) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}},$$

the $s_{\alpha,0}$ *correspond to tensors* $s_\alpha \in T_p\tilde{\mathscr{G}}^\otimes$ *which defined a reductive subgroup of* $\mathrm{GL}(T_p\tilde{\mathscr{G}})$. *If this condition holds, then there exists a* W*-linear isomorphism*

$$\mathbb{D}(\mathscr{G})(W) \xrightarrow{\sim} T_p\tilde{\mathscr{G}}^*(-1) \otimes_{\mathbb{Z}_p} W \tag{1.1.14}$$

taking $s_{\alpha,0}$ *to* s_α.

Proof. If $s_\alpha \in T_p\tilde{\mathscr{G}}^\otimes$, and these tensors define a reductive subgroup, then $\tilde{\mathscr{G}}$ is G_W-adapted by [Ki 2, Prop. 1.5.11]. Moreover, in this case, there is an isomorphism as in (1.1.14) by [Ki 2, 1.4.3(3)]. Thus it remains to show the converse.

First note that if $\tilde{\mathscr{G}}$ is a G_W-adapted deformation of $\mathscr{G}$, then the $\tilde{s}_\alpha$ give rise to φ-invariant tensors which lie in $\mathrm{Fil}^0\mathbb{D}(\mathscr{G})(\mathcal{O}_K)^\otimes$. Hence, they correspond to G_K-invariant tensors $s_\alpha \in T_p\tilde{\mathscr{G}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $\tilde{\mathscr{G}}_{R_G}$ denote the universal p-divisible group on Spec R_G. By (1.1.17) below, the p-adic étale local system $T_p\tilde{\mathscr{G}}^{\otimes}_{R_G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is equipped with tensors s_{α,R_G} which specialize to s_α for every G_W-adapted lifting $\tilde{\mathscr{G}}$. Thus, it suffices to show that $s_{\alpha,R_G} \in T_p\tilde{\mathscr{G}}^{\otimes}_{R_G}$, and that these tensors define a reductive subgroup of $\mathrm{GL}(T_p\tilde{\mathscr{G}}_{R_G})$. Since $T_p\tilde{\mathscr{G}}_{R_G}$ is a local system (and Spec $R_G[1/p]$ is connected) this holds if and only if $s_\alpha \in T_p\tilde{\mathscr{G}}^{\otimes}$ and these tensors define a reductive subgroup for some G_W-adapted lifting $\tilde{\mathscr{G}}$.

We now construct the required G_W-adapted lifting. Let $\mathfrak{M} = \sigma^{-1*}(\mathbb{D}(\mathscr{G})(W))$, so that $\varphi^*(\mathfrak{M}) \xrightarrow{\sim} \mathbb{D}(\mathscr{G})(W) \otimes_W \mathfrak{S}$. Since the $s_{\alpha,0}$ are in $U^{\otimes}$, we may think of these tensors as in $\mathfrak{M}^{\otimes}$ and $\varphi^*(\mathfrak{M})^{\otimes}$. Now lift the cocharacter μ_0 to a cocharacter μ, valued in $G_{\mathfrak{S}} \subset \mathrm{GL}(\varphi^*(\mathfrak{M}))$, and consider the map

$$\varphi^*(\mathfrak{M}) \stackrel{c\cdot\mu^{-1}(E(u))}{\to} \varphi^*(\mathfrak{M}) \xrightarrow{\sim} \mathfrak{M},$$

where $c = \sigma^{-1}(b)\mu_0(E(0))$ and the final map is induced by the identity on $U \subset \varphi^*(\mathfrak{M})$. By what we saw in (1.1.12),

$$\sigma(c) = bv_0(E(0))^{-1} = (bv_0(p)^{-1})(v_0(p/E(0))) \in \mathrm{GL}(\mathbb{D}(\mathscr{G})(W)).$$

Hence $c \in \mathrm{GL}(\mathbb{D}(\mathscr{G})(W))$ and the map above gives $\mathfrak{M}$ the structure of an object of $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, which corresponds to a deformation $\tilde{\mathscr{G}}$ of $\mathscr{G}$, by (1.1.6). The filtration on $\mathcal{M}(\mathfrak{M}) = S \otimes_{\mathfrak{S}} \varphi^*(\mathfrak{M})$ is induced by the cocharacter μ^{-1}. Hence thinking of the $s_{\alpha,0}$ in $\mathcal{M}(\mathfrak{M}) = \mathbb{D}(\tilde{\mathscr{G}})(S)$, shows that $\tilde{\mathscr{G}}$ is a G_W-adapted lifting. Since $s_{\alpha,0} \in \mathfrak{M}^{\otimes}$, we have $s_\alpha \in T_p\tilde{\mathscr{G}}^{\otimes}$ by the full faithfulness in (1.1.2), and the s_α define a reductive subgroup by (1.1.2)(2). $\square$

(1.1.15) Before proving the next lemma we recall some of the constructions of [Fa 2, §2]. Let R be a Noetherian, normal, integral, p-adically complete, p-torsion free W-algebra. Fix an algebraic closure κ of $\mathrm{Fr}\,R$, and let $\bar{R}$ be the union of the finite, normal, R-subalgebras $R' \subset \kappa$ such that $R'[1/p]$ is finite étale over R. We write $\hat{\bar{R}}$ for the p-adic completion of $\bar{R}$. Let $\mathcal{R}(R) = \varprojlim \bar{R}/p$ where the transition maps are given by Frobenius. Then there is a canonical surjection $W(\mathcal{R}(R)) \to \hat{\bar{R}}$ and we denote by $A_{\mathrm{cris}}(R)$ the p-adic completion of the divided power envelope of $W(\mathcal{R}(R))$ with respect to the kernel of this map.

If $\mathscr{G}$ is a p-divisible group over R, then an element of $T_p(\mathscr{G}) := \mathrm{Hom}_{\bar{R}}(\mathbb{Q}_p/\mathbb{Z}_p, \mathscr{G})$ gives rise to a map

$$\mathbb{D}(\mathscr{G})(A_{\mathrm{cris}}(R)) \to \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{cris}}(R)) = A_{\mathrm{cris}}(R).$$

Hence we get a map

$$T_p(\mathscr{G}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(R) \to \mathrm{Hom}_{A_{\mathrm{cris}}(R)}(\mathbb{D}(\mathscr{G})(A_{\mathrm{cris}}(R)), A_{\mathrm{cris}}(R)). \tag{1.1.16}$$

By considering the analogous map for the Cartier dual $\mathscr{G}^*$ one can show that this map becomes an isomorphism after inverting the element $t \in A_{\mathrm{cris}} = A_{\mathrm{cris}}(W)$ [Fa 2, p114].

Lemma (1.1.17). *The local system $T_p\tilde{\mathscr{G}}^{\otimes}_{R_G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is equipped with sections $\tilde{s}_{\alpha,R_G}$ which specialize to $\tilde{s}_\alpha$.*

Proof. Let $M_{R_G} := \mathbb{D}(\mathscr{G}_{R_G})(R_G)$. By construction [Fa, p136], [Ki 2, 1.5.4], $\mathrm{Fil}^0 M_{R_G}^{\otimes}$ is equipped with parallel, φ-invariant tensors s_{α,R_G} such that for any K as above, and $\varpi : R_G \to \mathcal{O}_K$, each $\varpi(s_{\alpha,R_G})$ is equal to the image of $\tilde{s}_\alpha$ in $\mathbb{D}(\mathscr{G}_\varpi)(\mathcal{O}_K)^{\otimes}$.

Now fix a lifting of $R_G \to \widehat{\bar{R}}_G$ to a map $R_G \to A_{\mathrm{cris}}(R_G)$. Since the s_α are parallel, they induce φ-invariant sections of

$$\mathbb{D}(\mathscr{G}_{R_G})(A_{\mathrm{cris}}(R_G))^{\otimes} \xrightarrow{\sim} M_{R_G}^{\otimes} \otimes_{R_G} A_{\mathrm{cris}}(R_G)$$

(see e.g. [Ki 2, 1.5.4]).

Hence, using the isomorphism (1.1.16), we obtain sections

$$s_{\alpha,R_G} \in T_p(\mathscr{G})^{\otimes} \otimes_{\mathbb{Z}_p} \mathrm{Fil}^0 A_{\mathrm{cris}}(R_G)[1/t]^{\varphi=1} = T_p(\mathscr{G})^{\otimes} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where the final equality is proved in [Fa 2, p119]. Since these tensors are necessarily $\mathrm{Gal}(\bar{R}_G/R_G)$-invariant, the lemma follows. □

(1.1.18) For any $\mathbb{Z}$-linear category $\mathcal{C}$ we denote by $\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Q}$ the category obtained from $\mathcal{C}$ by tensoring the Hom-groups by $\otimes_{\mathbb{Z}}\mathbb{Q}$ and we call this the isogeny category of $\mathcal{C}$. An isomorphism in this category is called a *quasi-isogeny.* We will apply this, in particular to the category of category of p-divisible groups (over some base), and to Dieudonné modules of p-divisible groups.

In order to simplify notation we will write $\mathbb{D}(\mathscr{G})$ for $\mathbb{D}(\mathscr{G})(W)$ below.

Proposition (1.1.19). *Let $\mu : \mathbb{G}_m \to G$ be a cocharacter, defined over K, and conjugate to μ_0. Suppose that μ^{-1} induces an admissible filtration on $\mathbb{D}(\mathscr{G})_K := \mathbb{D}(\mathscr{G})(W) \otimes_W K$.*

Then there exists a finite extension K'/K with residue field k', a p-divisible group $\tilde{\mathscr{G}}'$ over $\mathcal{O}_{K'}$, and a quasi-isogeny $\theta : \mathscr{G} \to \mathscr{G}'$ where $\mathscr{G}' = \tilde{\mathscr{G}}' \otimes_{\mathcal{O}_{K'}} k'$, such that

1. *θ identifies the filtration on $\mathbb{D}(\mathscr{G}')_K$ corresponding to $\tilde{\mathscr{G}}'$ and the filtration on $\mathbb{D}(\mathscr{G})_K$ induced by μ^{-1}.*
2. *θ identifies $\mathbb{D}(\mathscr{G}')$ with $g \cdot \mathbb{D}(\mathscr{G})$ for some $g \in G(L)$ with*
$$g^{-1} b\sigma(g) \in G(\mathcal{O}_L) p^{v_0} G(\mathcal{O}_L).$$
3. *Viewing $s_{\alpha,0} \in \mathbb{D}(\mathscr{G}')^{\otimes}$ via θ, the deformation $\tilde{\mathscr{G}}'$ of $\mathscr{G}'$ is $gG_{W(k')}g^{-1}$-adapted, where $gG_{W(k')}g^{-1}$ denotes the stabilizer of $s_{\alpha,0} \in \mathbb{D}(\mathscr{G}')^{\otimes}$.*

Proof. By [Ki 1, 2.2.6] (cf. [Br, Thm. 1.4]) there exists a p-divisible group $\tilde{\mathscr{G}}'$ over $\mathcal{O}_K$ and an isomorphism of weakly admissible modules $\mathbb{D}(\mathscr{G}')_{K_0} \xrightarrow{\sim} \mathbb{D}(\mathscr{G})_{K_0}$, where the filtration on the left ($\otimes K$) corresponds to $\tilde{\mathscr{G}}'$, and the filtration on the right is induced by μ^{-1}. This isomorphism induces a quasi-isogeny $\theta : \mathscr{G} \to \mathscr{G}'$, which satisfies (1) by construction.

Let $\tilde{\mathscr{G}}$ be a G_W-adapted lifting of $\mathscr{G}$ over $\mathcal{O}_K$. and let $s_\alpha \in T_p\tilde{\mathscr{G}}^{\otimes}$ and $s'_\alpha \in (T_p\tilde{\mathscr{G}}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\otimes}$ the tensors corresponding to $s_{\alpha,0}$ via the p-adic comparison isomorphism. The scheme of isomorphisms $T_p\tilde{\mathscr{G}}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} T_p\tilde{\mathscr{G}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ taking s'_α to s_α is a $G_{\mathbb{Q}_p}$-torsor. By (1.1.9), the filtrations on $\mathbb{D}(\mathscr{G})_K$ corresponding to $\tilde{\mathscr{G}}$ and $\tilde{\mathscr{G}}'$, are given by conjugate G-valued cocharacters, so this torsor is trivial by [Wi, Cor. 4.5.3] (and the theorem of Colmez-Fontaine that weakly admissible implies admissible), and there exists a $\mathbb{Q}_p$-linear isomorphism

$$T_p\tilde{\mathscr{G}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} T_p\tilde{\mathscr{G}}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{1.1.20}$$

taking s'_α to s_α.

After replacing K be a finite extension, we may assume that the image of $T_p\tilde{\mathscr{G}}$ under (1.1.20) is stable by G_K. Then replacing $\mathscr{G}'$ by an isogenous p-divisible group, we may assume that (1.1.20) induces an isomorphism $T_p\tilde{\mathscr{G}} \xrightarrow{\sim} T_p\tilde{\mathscr{G}}'$. In particular $s'_\alpha \in T_p\tilde{\mathscr{G}}'^{\otimes}$ and these tensors define a connected reductive subgroup in $\mathrm{GL}(T_p\tilde{\mathscr{G}}')$. Hence by (1.1.7), there exist W-linear isomorphisms

$$\mathbb{D}(\mathscr{G}) \xrightarrow{\sim} T_p\tilde{\mathscr{G}}^*(-1)\otimes_{\mathbb{Z}_p} W \underset{(1.1.20)}{\xrightarrow{\sim}} T_p\tilde{\mathscr{G}}'^*(-1)\otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G}') \tag{1.1.21}$$

with the composite taking $s_{\alpha,0}$ to $\theta(s_{\alpha,0})$. We use (1.1.21) to identify $\mathbb{D}(\mathscr{G})$ and $\mathbb{D}(\mathscr{G}')$. Then $\mathbb{D}(\theta)$ fixes $s_{\alpha,0,}$ and hence is induced by an element $g \in G(K_0)$. In particular,

$$\mathbb{D}(\mathscr{G}') = g\cdot \mathbb{D}(\mathscr{G}) \subset \mathbb{D}(\mathscr{G})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Write $G_W^g = gG_Wg^{-1}$. By (1.1.13), $\tilde{\mathscr{G}}'$ is a G_W^g-adapted deformation of $\mathscr{G}'$. Let μ'_0 be a G_W^g-valued cocharacter such that $\mu_0'^{-1}$ induces the filtration on $\mathbb{D}(\mathscr{G}')(k)$. By (1.1.9), μ'_0 is conjugate to μ in $G_W^g(K)$, and hence to $g\mu_0g^{-1}$. Since μ'_0 and $g\mu_0g^{-1}$ are G_W^g-valued cocharacters, they are conjugate by an element of $G_W^g(W)$.

Now applying (1.1.12) to $\mathscr{G}'$ and the isomorphism

$$U\otimes W \xrightarrow{\sim} \mathbb{D}(\mathscr{G}) \underset{\cdot g}{\xrightarrow{\sim}} \mathbb{D}(\mathscr{G}'),$$

we find

$$g^{-1}b\sigma(g) \in G_W(W)p^{\sigma(g^{-1}\mu'_0g)}G_W(W) = G_W(W)p^{v_0}G_W(W).$$

□

(1.2) Affine Deligne-Lusztig varieties:

(1.2.1) Now fix an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let $k\subset\bar{\mathbb{F}}_p$ be a finite field, and set $L = W(\bar{\mathbb{F}}_p)[1/p]$ and $\mathcal{O}_L = W(\bar{\mathbb{F}}_p)$. Denote by $\sigma\in\Gamma := \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ the absolute Frobenius.

Let G be a connected reductive group over $\mathbb{Z}_p$. Then G is quasi-split. We denote by $G_{\mathbb{Q}_p}$ the generic fibre of G. Let $T\subset G$ be the centralizer of a maximal split torus, $B\supset T$ a Borel subgroup of G, and $\Omega=\Omega_G$ the Weyl group of T in G. The Cartan decomposition gives an isomorphism

$$X_*(T)/\Omega \xrightarrow{\sim} G(\mathcal{O}_L)\backslash G(L)/G(\mathcal{O}_L) \quad \mu\mapsto\mu(p) \tag{1.2.2}$$

and each element of $X_*(T)/\Omega$ has a unique dominant representative. For $b\in G(L)$ denote by $\mu_{G,b}\in X_*(T)$ (or simply μ_b if the group G is clear,) the dominant cocharacter which maps to the image of b under (1.2.2).

Let $\mu,\mu'\in X_*(T)$ and $\tilde\mu,\tilde\mu'\in X_*(T)$ the dominant cocharacters in the Ω-orbit of μ and μ' respectively. We will write $\mu'\preccurlyeq\mu$ if $\tilde\mu-\tilde\mu'$ is a nonnegative integral linear combination of positive coroots.

Let $\pi_1(G)$ denote the quotient of $X_*(T)$ by the space of coroots. The composite

$$\tilde\kappa_G : G(L) \overset{b\mapsto\mu_{G,b}}{\rightarrow} X_*(T)/\Omega \rightarrow \pi_1(G)$$

is a homomorphism. We denote by κ_G the composite of $\tilde\kappa_G$ and the projection $\pi_1(G)\rightarrow\pi_1(G)_\Gamma$.

Lemma (1.2.3). *Let $T \subset G_{\mathbb{Z}_p}$ be the centralizer of a maximal split torus. Then the map $X_*(T)^\Gamma \to \pi_1(G)^\Gamma$ is surjective. In particular the map $\tilde{\kappa}_G : G(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$ is surjective.*

Proof. The first claim is contained in [CKV, Cor. 2.5.12]. Since $T(\mathbb{Q}_p)$ surjects onto $X_*(T)^\Gamma$ ($\mu \mapsto p^\mu$ is a section) the second claim follows. $\square$

Lemma (1.2.4). *If $g^{\mathrm{ad}} \in G^{\mathrm{ad}}(\mathbb{Q}_p)$ and $\tilde{\kappa}_{G^{\mathrm{ad}}}(g^{\mathrm{ad}})$ lifts to an element of $\pi_1(G)^\Gamma$, then g^{ad} is in the image of*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p).$$

Proof. By (1.2.3) there is a $g' \in G(\mathbb{Q}_p)$ which maps to $\tilde{\kappa}_{G^{\mathrm{ad}}}(g^{\mathrm{ad}})$. Let g'^{ad} be the image of g' in $G^{\mathrm{ad}}(\mathbb{Q}_p)$. We may replace g^{ad} by $g^{\mathrm{ad}} g'^{\mathrm{ad}-1}$ and assume that $\tilde{\kappa}_{G^{\mathrm{ad}}}(g^{\mathrm{ad}}) = 0$.

By the Cartan decomposition, $g^{\mathrm{ad}} = p^\varepsilon h$ with $\varepsilon \in X_*(T)$ for some maximal torus T of G^{ad}, and $h \in G^{\mathrm{ad}}(\mathbb{Z}_p)$ (cf. the proof of (1.1.12) above). If $\tilde{T}$ is as in (1.2.3), then $\varepsilon \in X_*(\tilde{T})$ as $\tilde{\kappa}_{G^{\mathrm{ad}}}(g^{\mathrm{ad}}) = 0$, so p^ε may be regarded as an element of G which maps to g^{ad} in $G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p)$. $\square$

(1.2.5) Let R be a $\bar{\mathbb{F}}_p$-algebra. A *frame* for R is a p-torsion free, p-adically complete and separated $\mathcal{O}_L$-algebra $\mathscr{R}$ equipped with an isomorphism $\mathscr{R}/p\mathscr{R} \xrightarrow{\sim} R$, and a lift (again denoted σ) of the Frobenius σ to $\mathscr{R}$.[9] If $\theta : R \to R'$ is a map of $\bar{\mathbb{F}}_p$-algebras, then a frame for θ is a morphism of $\mathcal{O}_L$-algebras $\tilde{\theta} : \mathscr{R} \to \mathscr{R}'$ from a frame of R to a frame of R', which lifts θ, and is compatible with σ.

Note that there is a unique σ-equivariant map $\mathscr{R} \to W(R)$ to the ring of Witt vectors of R. Hence, for any morphism of k-algebras, $s : R \to R'$ we obtain a map $\mathscr{R} \to W(R')$, which we will often again denote by s.

We write $\mathscr{R}_L = \mathscr{R} \otimes_{\mathcal{O}_L} L$. Fix a frame $\mathscr{R}$ for R, and let $g \in G(\mathscr{R}_L)$. For a dominant $\mu \in X_*(T)$ let

$$S_\mu(g) = \{s \in \mathrm{Spec}\ R : s(g) \in G(W(\bar{\kappa}(s)))p^\mu G(W(\bar{\kappa}(s)))\}$$

where $\bar{\kappa}(s)$ denotes an algebraic closure of $\kappa(s)$. Set

$$S_{\preccurlyeq\mu}(g) = \cup_{\mu' \preccurlyeq \mu} S_\mu(g),$$

where μ' runs over dominant characters $\preccurlyeq \mu$. As above, if $s \in S_\mu(g)$ with μ dominant we write $\mu_{s(g)} = \mu$. We have the following [CKV, Lem. 2.1.6]

Lemma (1.2.6). *Let R be a Noetherian, formally smooth $\bar{\mathbb{F}}_p$-algebra, $\mathscr{R}$ a frame for R, and $g \in G(\mathscr{R}_L)$. Then*

1. *The set $S_{\preccurlyeq\mu}(g) \subset S = \mathrm{Spec}\ R$ is Zariski closed.*
2. *The set $S_\mu(g)$ is locally closed and is closed if μ is minuscule.*
3. *The function*

$$\mathrm{Spec}\ R \to \pi_1(G); \quad s \mapsto [\mu_{G,s(g)}]$$

is locally constant.

(1.2.7) By an *étale covering*, we mean a faithfully flat, étale morphism $R \to R'$. Let $\mathscr{R}$ be a frame for R. Then any étale morphism $R \to R'$ admits a canonical frame $\mathscr{R} \to \mathscr{R}'$ [CKV, Lem. 2.1.4]. We have [CKV, Prop. 2.1.14]

[9] This is a special case of Zink's definition [Zi 1, Defn. 1].

Lemma (1.2.8). *Let R be a Noetherian, formally smooth $\bar{\mathbb{F}}_p$-algebra, $\mathscr{R}$ a frame for R, and $g \in G(\mathscr{R}_L)$. Suppose that μ is minuscule and that $S_\mu(g)$ contains the generic points of* Spec R*. Then there exists an étale covering $R \to R'$ with canonical frame $\mathscr{R} \to \mathscr{R}'$ such that $g \in G(\mathscr{R}')p^\mu G(\mathscr{R}')$.*

(1.2.9) Let $b \in G(L)$. If R is a $\bar{\mathbb{F}}_p$-algebra, $S = \operatorname{Spec} R$ and $\mathscr{R}$ a frame for R. Then we set

$$X_{\preccurlyeq\mu}(b)(\mathscr{R}) = \{g \in G(\mathscr{R}_L)/G(\mathscr{R}) : S_{\preccurlyeq\mu}(g^{-1}b\sigma(g)) = S\},$$

and we defined $X_\mu(b)(\mathscr{R})$ in an analogous way, replacing $S_{\preccurlyeq\mu}$ by S_μ. We will sometimes write simply $X_{\preccurlyeq\mu}(b)$ for $X_{\preccurlyeq\mu}(b)(W(\bar{\mathbb{F}}_p))$. When we want to make the group G explicit we will write $X^G_{\preccurlyeq\mu}(b)$ for $X_{\preccurlyeq\mu}(b)$. Note that the sets $X_\mu(b)$ depend only on the σ-conjugacy class of b.

Let $g_0, g_1 \in X_{\preccurlyeq\mu}(b)(W(\bar{\mathbb{F}}_p))$, and R a smooth $\bar{\mathbb{F}}_p$-algebra with connected spectrum, and equipped with a frame $\mathscr{R}$. We say that g_0 is connected to g_1 via R, if there exists a $g \in X_{\preccurlyeq\mu}(b)(\mathscr{R})$ and $s_0, s_1 \in (\operatorname{Spec} R)(\bar{\mathbb{F}}_p)$ such that $s_0(g) = g_0$ and $s_1(g) = g_1$. We denote by $\sim$ the smallest equivalence relation on $X_{\preccurlyeq\mu}(W(\bar{\mathbb{F}}_p))$ such that $g_0 \sim g_1$ if g_0 is connected to g_1, via some R as above, and we write $\pi_0(X_{\preccurlyeq\mu}(b))$ for the set of equivalence classes under $\sim$. We call the elements of $\pi_0(X_{\preccurlyeq\mu}(b))$ the *connected components* of $X_{\preccurlyeq\mu}(b)$.

(1.2.10) Let $\mathbb{D}$ denote the pro-torus with character group $\mathbb{Q}$. Recall the Newton map

$$\nu_b : \mathbb{D} \to G$$

defined by Kottwitz [Ko 2, 4.2]. If $G = GL(V)$ for a $\mathbb{Q}_p$-vector space V, then ν_b is the cocharacter which induces the slope decomposition of $b\sigma$ acting on $V \otimes_{\mathbb{Q}_p} L$. In general ν_b is determined by requiring that it be functorial in the group G.

Let $E \subset L$ be a finite Galois extension of $\mathbb{Q}_p$ such that μ is defined over E, and set

$$\bar{\mu} = [E : \mathbb{Q}_p]^{-1} \sum_{\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)} \tau(\mu).$$

We denote by $\mu^\natural = \mu^{\natural,G}$ the image of μ in $\pi_1(G)_\Gamma$.

Let $\bar{\nu}_b \in X_*(T)$ be the dominant cocharacter conjugate to ν_b. If $\nu_1, \nu_2 \in X_*(T)$ we write $\nu_1 \leqslant \nu_2$ if $\nu_2 - \nu_1$ is a non-negative $\mathbb{Q}$-linear combination of positive coroots. We denote by $B(G, \mu)$ the set of $b \in G(L)$ such that

$$\kappa_G(b) = \mu^\natural \in \pi_1(G)_\Gamma, \text{ and } \bar{\nu}_b \leqslant \bar{\mu}. \tag{1.2.11}$$

By a result of Wintenberger [Wi 2], $X_\mu(b)$ is non-empty if and only if $b \in B(G, \mu)$.

(1.2.12) Let $b \in G(L)$, and let $M_b \subset G$ denote the centralizer of ν_b. For any $\bar{b} \in G^{\mathrm{ad}}(L)$, we define a $\mathbb{Q}_p$-group $J_{\bar{b}}$ by setting

$$J_{\bar{b}}(A) = J^G_{\bar{b}}(A) := \{g \in G(A \otimes_{\mathbb{Q}_p} L) : \sigma(g) = \bar{b}^{-1}g\bar{b}\}.$$

for A a $\mathbb{Q}_p$-algebra. There is an inclusion $J_{\bar{b}} \subset G$, defined over L, which is given on A-points (A an L-algebra) by the natural map $G(A \otimes_{\mathbb{Q}_p} L) \to G(A)$. If $b \in G(L)$ we write $J_b = J_{\bar{b}}$ where $\bar{b}$ denotes the image of b in $G^{\mathrm{ad}}(L)$. Then the inclusion $J_b \to G$ identifies J_b with M_b over L, and J_b is an inner form of M_b [Ko 5, 3.3], [RZ, 1.12].

(1.2.13) By [CKV, Lem. 2.5.2] there exists a b' in the σ conjugacy class of b such that $\nu_{b'} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\nu_{b'}$ is dominant and (hence) σ-invariant and $b' \in M_{b'}$.

If the set $X_{\preccurlyeq\mu}(b)$ is non-empty then $b \in B(G,\mu)$, and in particular, $\kappa_G(b) = \mu^\natural$. A *standard* Levi subgroup $M \subset G$ is one defined by a dominant cocharacter in $X_*(T)^\Gamma$. By [CKV, Prop. 2.5.4] if $M \supset M_{b'}$ is a standard Levi subgroup and $\kappa_M(b') = \mu^{\natural,M}$ then the natural map $X^M_\mu(b') \to X^G_\mu(b') = X_\mu(b)$ is a bijection, inducing a bijection on connected components. The pair (μ, b) is called *HN-indecomposable* if $\kappa_M(b') \neq \mu^{\natural,M}$ for any standard Levi $M \supset M_{b'}$. We have the following [CKV, Thm. 1.1]

Theorem (1.2.14). *Suppose that μ is minuscule, and that G is (adjoint and) $\mathbb{Q}_p$-simple. Let $b \in B(G,\mu)$ with (μ, b) HN-indecomposable, and let $c_{b,\mu} \in \pi_1(G)$ satisfy $\sigma(c_{b,\mu}) - c_{b,\mu} = [\mu] - \tilde\kappa_G(b)$. Then either $\tilde\kappa_G$ induces a bijection*

$$\pi_0(X_{\preccurlyeq\mu}(b)) \xrightarrow{\sim} c_{b,\mu}\pi_1(G)^\Gamma$$

or μ is trivial, b is σ-conjugate to the identity, and

$$X_\mu(b) \xrightarrow{\sim} X_{\preccurlyeq\mu}(b) \xrightarrow{\sim} \pi_0(X_\mu(b)) \xrightarrow{\sim} G(\mathbb{Q}_p)/G(\mathbb{Z}_p).$$

(1.2.15) Now let $\mathscr{G}$ be a p-divisible group over $\bar{\mathbb{F}}_p$, and write $\mathbb{D}(\mathscr{G})$ for $\mathbb{D}(\mathscr{G})(\mathcal{O}_L)$. Let $(s_{\alpha,0}) \subset \mathbb{D}(\mathscr{G})^\otimes$ be a collection of φ-invariant tensors, which define a reductive subgroup $G_{\mathcal{O}_L} \subset \mathbb{D}(\mathscr{G})^\otimes$.

Suppose there exists a finite free $\mathbb{Z}_p$-module U, equipped with an isomorphism $U \otimes_{\mathbb{Z}_p} \mathcal{O}_L \xrightarrow{\sim} \mathbb{D}(\mathscr{G})$ and such that $s_{\alpha,0} \in U^\otimes$. Then the $s_{\alpha,0}$ define a reductive subgroup $G = G_{\mathbb{Z}_p} \subset \mathrm{GL}(U)$. If U' is another such $\mathbb{Z}_p$-module, the scheme of isomorphisms $U \to U'$ respecting the $s_{\alpha,0}$ is a $G_{\mathbb{Z}_p}$-torsor, which is necessarily trivial.

The action of φ on $\mathbb{D}(\mathscr{G})$ is given by $b\sigma$ and the discussion above shows that $b \in G(L)$ is independent of the choice of U up to σ-conjugation by an element of $G_{\mathcal{O}_L}(\mathcal{O}_L)$. By (1.1.12) the filtration on $\mathbb{D}(\mathscr{G})(k)$ is induced by a $G_{\mathcal{O}_L}$-valued cocharacter μ_0^{-1} and $b \in G(\mathcal{O}_L)p^{\upsilon_0}G(\mathcal{O}_L)$ where $\upsilon_0 = \sigma(\mu_0^{-1})$. Note that with these conventions we have $1 \in X_{\upsilon_0}(b)$, so $c_{b,\mu} = 1$.

(1.2.16) Let K/L be a finite extension and fix a Galois closure $\bar K$ of K, with residue field $\bar{\mathbb{F}}_p$. Let $\tilde{\mathscr{G}}$ be a $G_{\mathcal{O}_L}$-adapted lifting of $\mathscr{G}$ to a p-divisible group over $\mathcal{O}_K$. By (1.1.13), we may take U to be $T_p\tilde{\mathscr{G}}^*(-1)$ equipped with the tensors s_α in the construction and we fix an isomorphism

$$T_p\tilde{\mathscr{G}}^*(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_L \xrightarrow{\sim} \mathbb{D}(\tilde{\mathscr{G}}) \tag{1.2.17}$$

Now let $g \in G(\mathbb{Q}_p)$. There is a finite extension K'/K in $\bar K$ such that $g^{-1} \cdot T_p\tilde{\mathscr{G}}$ is $G_{K'}$-stable, and hence corresponds to a p-divisible group $\tilde{\mathscr{G}}'$ over K'. Let $\mathscr{G}' = \tilde{\mathscr{G}}' \otimes \bar{\mathbb{F}}_p$. The quasi-isogeny $\theta : \tilde{\mathscr{G}} \to \tilde{\mathscr{G}}'$ identifies $\mathbb{D}(\mathscr{G}')$ with $g_0 \cdot \mathbb{D}(\mathscr{G})$ for some $g_0 \in \mathrm{GL}(\mathbb{D}(\mathscr{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

Lemma (1.2.18). *The association $g \mapsto g_0$ induces a well defined map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G(L)/G(\mathcal{O}_L).$$

For two choices of (1.2.17) the corresponding maps $g \mapsto g_0$ differ by an automorphism of $G(L)/G(\mathcal{O}_L)$ given by left multiplication by an element of $G(\mathcal{O}_L)$.

Moreover, for any g we have $g_0 \in X_{\upsilon_0}(b)$ and $\tilde{\kappa}_G(g_0) = \tilde{\kappa}_G(g) \in \pi_1(G)^\Gamma$.

Proof. Since $g \in G(\mathbb{Q}_p)$ we have $s_{\alpha,et} \in T_p\tilde{\mathscr{G}}'^{\otimes}$, and the $\mathbb{Z}_p$-linear bijection $T_p\tilde{\mathscr{G}} \xrightarrow{\sim} T_p\tilde{\mathscr{G}}'$ induced by g^{-1} respects the s_α. Hence (s_α) defines a reductive subgroup of $\mathrm{GL}(T_p\tilde{\mathscr{G}}')$ and, as in the proof of (1.1.19), there is an isomorphism $\mathbb{D}(\mathscr{G}) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}')$ taking $s_{\alpha,0}$ to $\theta(s_{\alpha,0})$. If we use this isomorphism to identify $\mathbb{D}(\mathscr{G}')$ and $\mathbb{D}(\mathscr{G})$ then $\mathbb{D}(\theta)$ fixes the $s_{\alpha,0}$ and is induced by $g_0 \in G(\mathcal{O}_L)$.

The construction depends only on g as an element of $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, and g_0 is well defined as an element of $G(L)/G(\mathcal{O}_L)$. The claim regarding the dependence of the map on the choice in (1.2.17) is easily checked. The proof that $g_0 \in X_{\upsilon_0}(b)$ is the same as the argument given at the end of (1.1.19).

To prove that $\tilde{\kappa}_G(g_0) = \tilde{\kappa}_G(g)$, we use (1.1.4) and the bijection $T_p\tilde{\mathscr{G}} \xrightarrow{\sim} T_p\tilde{\mathscr{G}}'$ to obtain a $\mathfrak{S}$-linear bijection $\mathfrak{M}(\tilde{\mathscr{G}}) \xrightarrow{\sim} \mathfrak{M}(\tilde{\mathscr{G}}')$ which takes $\tilde{s}_\alpha \in \mathfrak{M}(T_p\tilde{\mathscr{G}}^*(-1))$ to $\theta(\tilde{s}_\alpha)$. We use this isomorphism to identify $\mathfrak{M}(\tilde{\mathscr{G}}')$ and $\mathfrak{M}(\tilde{\mathscr{G}})$. Fix an isomorphism $\mathfrak{M}(\tilde{\mathscr{G}}) \xrightarrow{\sim} T_p\tilde{\mathscr{G}}^*(-1) \otimes_W \mathfrak{S}$, which takes $\tilde{s}_\alpha$ to $\tilde{s}_\alpha$, so that the subgroup of $\mathrm{GL}(\mathfrak{M}(\tilde{\mathscr{G}}))$ defined by $(\tilde{s}_\alpha)$ is identified with $G_{\mathfrak{S}}$. Then $\mathfrak{M}(\tilde{\mathscr{G}}') = \tilde{g} \cdot \mathfrak{M}(\tilde{\mathscr{G}})$ for some $\tilde{g} \in G(\mathfrak{S})$.

By (1.1.2)(2) $\tilde{g} = g$ in $G(\widehat{\mathcal{E}^{\mathrm{ur}}})/G(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$. Since $\tilde{g}|_{u=0} = \sigma^{-1}(g_0)$ by (1.1.6), we see that g and $\sigma^{-1}(g_0)$ have the same image in $\pi_1(G)$ by (1.2.6)(3). Since the image of g is Γ-invariant, this coincides with the image of g_0. □

Lemma (1.2.19). *Let $f : G_{\mathbb{Z}_p} \to H_{\mathbb{Z}_p}$ be a surjection of reductive groups over $\mathbb{Z}_p$, and suppose that the map $G_K \to H_{\mathbb{Z}_p}(\mathbb{Z}_p)$ obtained by composing the action of G_K on $T_p\tilde{\mathscr{G}}$ with f factors through $Z_H(\mathbb{Q}_p)$.*

Then (1.2.17) can be chosen so that $f(g) = f(g_0) \in H(L)/H(\mathcal{O}_L)$.

Proof. Let $G' \subset G_{\mathbb{Z}_p}$ denote connected component of the identity of $f^{-1}(Z_H)$. After increasing K, we may assume that $G_K \to G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ factors through $G'(\mathbb{Z}_p)$. Extend (s_α) to a collection of tensors $(t_\beta) \subset (T_p\tilde{\mathscr{G}})^{\otimes}$ whose stabilizer is G'. Then, as above, t_β corresponds to a φ-invariant $t_{\beta,0} \in \mathbb{D}(\mathscr{G})^{\otimes}$, and we may choose (1.2.17) so that t_β maps to $t_{\beta,0}$.

Now let $T \subset G_{\mathbb{Z}_p}$ be the centralizer of a maximal split torus. By the Cartan decomposition we may write $g = h_1 p^{\varepsilon} h_2 = p^{h_1 \varepsilon h_1^{-1}} h_1 h_2$ with $\varepsilon \in X_*(T)$ a cocharacter defined over $\mathbb{Q}_p$, and $h_1, h_2 \in G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Hence we may assume $g \in T'(\mathbb{Q}_p)$ with $T' = h_1 T h_1^{-1} \subset G_{\mathbb{Z}_p}$ a maximal torus.

Let G'' be the connected component of the identity in $f^{-1}f(T')$, so that $g \in G''(\mathbb{Q}_p)$. Note that if $(t_\gamma) \subset (T_p\tilde{\mathscr{G}})^{\otimes}$ is a collection of tensors whose stabilizer is G'', then the stabilizer of $(t_\beta, t_\gamma)_{\beta,\gamma}$ is G', so (1.2.17) takes t_γ to $t_{\gamma,0} \in \mathbb{D}(\tilde{\mathscr{G}})^{\otimes}$, because the scheme of isomorphisms having this property is again a G'-torsor. Thus we may apply (1.2.18) to G'', and we find $g = g_0$ in $\pi_1(G'')$ and hence that

$$f(g) = f(g_0) \in \pi_1(f(T')) = f(T')(L)/f(T')(\mathcal{O}_L) \subset H(L)/H(\mathcal{O}_L).$$

□

(1.2.20) Let $\mu \in X_*(T)$ be the dominant cocharacter conjugate to μ_0, and set $\upsilon = \sigma(\mu^{-1})$. Then υ and υ_0 are conjugate over $G(\mathcal{O}_L)$, and $X_\upsilon(b) = X_{\upsilon_0}(b)$. Note that we also have a natural bijection

$$X_\upsilon(b) \xrightarrow{\sim} X_{\mu^{-1}}(b); \quad g \mapsto \sigma^{-1}(b^{-1}g).$$

We will denote by a superscript "ad" the image of an element (resp. a cocharacter) of G in G^{ad} (resp. the set of cocharacters of G^{ad}). We remind the reader that with our present conventions $c_{b,\mu} = 1$.

Proposition (1.2.21). *Suppose that the pair (υ, b) is HN-indecomposable. Then the map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to X_\upsilon(b) \to \pi_0(X_\upsilon(b)); \quad g \mapsto g_0$$

is surjective.

Proof. Let $M \supset T$ be a standard Levi subgroup. Consider the maps

$$\pi_1(M/Z_G)^\Gamma \to \pi_1(G^{\mathrm{ad}})^\Gamma \overset{c \mapsto \sigma(\tilde{c}) - \tilde{c}}{\to} \ker(X_*(Z_G)_\Gamma \to \pi_1(G)_\Gamma),$$

where in the second map $\tilde{c} \in \pi_1(G)$ is an element lifting c. The first map is a surjection by (1.2.3), and the second map is a surjection by the snake lemma. The kernel of $\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma \oplus \pi_1(M/Z_G)_\Gamma$ is contained in the image of $\ker(X_*(Z_G)_\Gamma \to \pi_1(G)_\Gamma)$, and hence in the image of $\pi_1(M/Z_G)^\Gamma$, which is trivial. Hence $(\upsilon^{\mathrm{ad}}, b^{\mathrm{ad}})$ is HN-indecomposable.

Write $G^{\mathrm{ad}} = G_1 \times G_2$, where υ^{ad} induces the trivial cocharacter of G_1^{ad} and induces a non-trivial cocharacter in every $\mathbb{Q}_p$-simple factor of G_2^{ad}. Since υ induces the trivial cocharacter in G_1, the representation $G_K \to G_1(\mathbb{Q}_p)$ is unramified, and hence trivial as K has algebraically closed residue field. We choose the isomorphism (1.2.17) so that the conclusion of (1.2.19) holds with f the projection $G \to G_1$; of course the truth of the statement of the lemma does not depend on this choice.

Let $h \in X_\upsilon(b)$. By (1.2.14) we have

$$\pi_0(X_{\upsilon^{\mathrm{ad}}}(b^{\mathrm{ad}})) \xrightarrow{\sim} G_1(\mathbb{Q}_p)/G_1(\mathbb{Z}_p) \times \pi_1(G_2)^\Gamma \tag{1.2.22}$$

and by (1.2.3) there exists $g^{\mathrm{ad}} \in G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p)$ which maps to the image of h^{ad} in right hand side of (1.2.22). Since $\tilde{\kappa}_G(h) \in \pi_1(G)^\Gamma$ lifts $\tilde{\kappa}_{G^{\mathrm{ad}}}(g^{\mathrm{ad}})$, g^{ad} lifts to $g \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ by (1.2.4). It follows by (1.2.18) and (1.2.19) that g_0 and h^{ad} have the same image in $\pi_0(X_{\upsilon^{\mathrm{ad}}}(b^{\mathrm{ad}}))$.

The non-empty fibres of the map $\pi_0(X_\upsilon(b)) \to \pi_0(X_{\upsilon^{\mathrm{ad}}}(b^{\mathrm{ad}}))$ are $X_*(Z_G)^\Gamma$-torsors [CKV, Cor. 2.4.3]. Hence, $h = g_0 z$ for some $z \in Z_G(\mathbb{Q}_p)$. Using (1.2.18) and the functoriality of the construction in (1.2.16), we find $h = g_0 z = g_0 z_0 = (gz)_0$. $\square$

Proposition (1.2.23). *There exists a $G_{\mathcal{O}_L}$-adapted lifting $\tilde{\mathscr{G}}$ of $\mathscr{G}$ such that the map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \pi_0(X_\upsilon(b))$$

arising from $\tilde{\mathscr{G}}$ is surjective.

Proof. Let b' be as in (1.2.13), and $M \supset M_{b'}$ a standard Levi subgroup such that $\kappa_M(b') = \upsilon$ in $\pi_1(M)$, and (υ, b') is HN-indecomposable in M. Since the map $X_\upsilon^M(b') \to X_\upsilon^G(b)$ is a bijection, there exists $h \in G(\mathcal{O}_L)$ such that $h^{-1} b \sigma(h) \in M(\mathcal{O}_L) p^\upsilon M(\mathcal{O}_L)$. We may replace b by its σ conjugate by h, and assume that $b \in M(\mathcal{O}_L) p^\upsilon M(\mathcal{O}_L)$.

Now extend $(s_{\alpha,0})$ to a collection of tensors $(t_{\beta,0}) \in U^\otimes$ which define the subgroup $M \subset \mathrm{GL}(U)$. Then the $t_{\beta,0} \in \mathbb{D}(\mathscr{G})^\otimes$ are Frobenius invariant. By (1.1.12), the filtration on $\mathbb{D}(\mathscr{G})(k)$ is given by an M_W-valued cocharacter, so by (1.1.10), $\mathscr{G}$ admits a $M_{\mathcal{O}_L}$-adapted lifting $\tilde{\mathscr{G}}$. Note that any such lifting is $G_{\mathcal{O}_L}$-adapted.

By (1.1.13), the $t_{\beta,0}$ correspond to $t_\beta \in T_p\tilde{\mathscr{G}}^\otimes$, and we may choose (1.2.17) to take t_β to $t_{\beta,0}$. By (1.2.21), composite map

$$M(\mathbb{Q}_p)/M(\mathbb{Z}_p) \to X_v^M(b) \to \pi_0(X_v^M(b))$$

arising from $\tilde{\mathscr{G}}$ is surjective. Since $X_v^M(b) \to X_v^G(b)$ is a bijection, the proposition follows. □

(1.3) Shimura varieties: The aim of this subsection is to recall the construction of integral models of Shimura varieties of Hodge type, and to give a deformation theoretic description of a formal neighborhood of a point on such a model.

For the rest of the paper it will be convenient to fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, an algebraic closure $\bar{\mathbb{Q}}_v$ of $\mathbb{Q}_v$ for each place v of $\mathbb{Q}$, and embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$. As usual, we write $\bar{\mathbb{Q}}_\infty = \mathbb{C}$. For $L \subset \bar{\mathbb{Q}}$ a Galois extension of $\mathbb{Q}$, and every place v, we let L_v denote the closure of L in $\bar{\mathbb{Q}}_v$.

We denote by $\bar{\mathbb{F}}_p$ the residue field of $\bar{\mathbb{Q}}_p$. As in §1.2, we write $L = \operatorname{Fr} W(\bar{\mathbb{F}}_p)$, and $\mathcal{O}_L$ for the ring of integers of L. Let $\mathbb{Q}_p^{\mathrm{ur}} \subset L$ be the subfield of elements which are algebraic over $\mathbb{Q}_p$. We fix an algebraic closure $\bar{L}$ of L, and an embedding of $\mathbb{Q}_p^{\mathrm{ur}}$-algebras $\bar{\mathbb{Q}}_p \hookrightarrow \bar{L}$.

(1.3.1) Let G be a connected reductive group over $\mathbb{Q}$ and X a conjugacy class of maps of algebraic groups over $\mathbb{R}$

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}},$$

such that (G, X) is a Shimura datum [De 2, §2.1].

For any $\mathbb{C}$-algebra R, we have $R \otimes_{\mathbb{R}} \mathbb{C} = R \times c^*(R)$ where c denotes complex conjugation, and we denote by μ_h the cocharacter given on R-points by

$$R^\times \to (R \times c^*(R))^\times = (R \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{S}(R) \xrightarrow{h} G_{\mathbb{C}}(R).$$

We set $w_h = \mu_h^{-1}\mu_h^{c-1}$.

Let $\mathbb{A}_f$ denote the finite adeles over $\mathbb{Q}$, and $\mathbb{A}_f^p \subset \mathbb{A}_f$ the subgroup of adeles with trivial component at a prime p. Let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_f)$ where $\mathsf{K}_p \subset G(\mathbb{Q}_p)$, and $\mathsf{K}^p \subset G(\mathbb{A}_f^p)$ are compact open subgroups.

A theorem of Baily-Borel asserts that if K^p is sufficiently small then

$$\mathrm{Sh}_{\mathsf{K}}(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathsf{K}$$

has a natural structure of an algebraic variety over $\mathbb{C}$. We will always assume in the following that K^p is sufficiently small. Results of Shimura, Deligne, Milne and others imply that $\mathrm{Sh}_{\mathsf{K}}(G, X)_{\mathbb{C}}$ has a model $\mathrm{Sh}_{\mathsf{K}}(G, X)$ over a number field $E = E(G, X)$ [Mi 3, II §4,5], which is the minimal field of definition of the conjugacy class of μ_h. We view E as a subfield of $\bar{\mathbb{Q}}$, via the chosen embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

We will sometimes consider the pro-variety

$$\mathrm{Sh}_{\mathsf{K}_p}(G, X) = \varprojlim \mathrm{Sh}_{\mathsf{K}}(G, X),$$

where K runs through compact open subgroups as above with a fixed factor K_p at p.

(1.3.2) Fix a $\mathbb{Q}$-vector space V with a perfect alternating pairing ψ. For any $\mathbb{Q}$-algebra R, we write $V_R = V \otimes_{\mathbb{Q}} R$. Let $\mathrm{GSp} = \mathrm{GSp}(V, \psi)$ the corresponding group of symplectic similitudes, and let $S^{\pm}$ be the Siegel double space, defined as the set of maps $h : \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$ such that

(1) The $\mathbb{C}^{\times}$ action on $V_{\mathbb{R}}$ gives rise to a Hodge structure of type $(-1,0),(0,-1)$:

$$V_{\mathbb{C}} \xrightarrow{\sim} V^{-1,0} \oplus V^{0,-1}.$$

(2) $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

For the rest of this section we will assume that there is an embedding of Shimura data $\iota : (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$.

(1.3.3) We assume from now on that K_p is hyperspecial, so that $\mathsf{K}_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ for some reductive group $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ with generic fibre G. Then ι is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ for some $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ [Ki 2, 2.3.1, 2.3.2].

By Zarhin's trick, after replacing $V_{\mathbb{Z}_{(p)}}$ by $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(V_{\mathbb{Z}_{(p)}}, V_{\mathbb{Z}_{(p)}})^4$, we may assume that ψ induces a perfect pairing on $V_{\mathbb{Z}_{(p)}}$, which will again be denote by ψ. For R a $\mathbb{Z}_{(p)}$-algebra, we write $V_R = V_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} R$.

We write $\mathsf{K}'_p = \mathrm{GSp}(V_{\mathbb{Z}_{(p)}})(\mathbb{Z}_p) \subset \mathrm{GSp}(\mathbb{Q}_p)$. By [Ki 2, 2.1.2], for each K^p, as above, there exists $\mathsf{K}'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ compact open containing K^p and such that ι induces an embedding

$$\mathrm{Sh}_{\mathsf{K}}(G, X) \hookrightarrow \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})$$

of E-schemes, where $\mathsf{K}' = \mathsf{K}'_p \mathsf{K}'^p$.

(1.3.4) For a $\mathbb{Z}_{(p)}$-scheme T and an abelian scheme $\mathcal{B}$ over T we set

$$\widehat{V}^p(\mathcal{B}) = \varprojlim_{p \nmid n} \mathcal{B}[n]$$

viewed as an étale local system on T. Write $\widehat{V}^p(\mathcal{B})_{\mathbb{Q}} = \widehat{V}^p(\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Now consider the category obtained from the category of abelian schemes over T, by tensoring the Hom groups by $\otimes \mathbb{Z}_{(p)}$. An object in this category will be called an *abelian scheme up to prime to p isogeny.* An isomorphism in this category will be called a p'-quasi-isogeny. Note that $\widehat{V}^p(\mathcal{B})_{\mathbb{Q}}$ is functorial for p'-quasi-isogenies.

Let $\mathcal{A}$ be an abelian scheme up to prime to p-isogeny, and write $\mathcal{A}^*$ for the dual abelian scheme. By a *weak polarization* we mean an equivalence class of p'-quasi-isogenies $\lambda : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$ such that some multiple of λ is a polarization, and where two such λ's are equivalent if they differ by an element of $\mathbb{Z}_{(p)}^{\times}$.

For such a pair $(\mathcal{A}, \lambda)$, denote by $\underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \widehat{V}^p(\mathcal{A})_{\mathbb{Q}})$ the étale sheaf on T consisting of isomorphisms $V_{\mathbb{A}_f^p} \xrightarrow{\sim} \widehat{V}^p(\mathcal{A})_{\mathbb{Q}}$ which are compatible with the pairings induced by ψ and λ up to a $\mathbb{A}_f^{p\times}$-scalar.

A K'^p-level structure on $(\mathcal{A}, \lambda)$ is a section

$$\varepsilon^p_{\mathsf{K}'} \in \Gamma(T, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}})/\mathsf{K}'^p).$$

For K'^p sufficiently small the functor which assigns to T the set of isomorphism classes of triples $(\mathcal{A}, \lambda, \varepsilon^p_{\mathsf{K}'})$, as above is representable by a smooth $\mathbb{Z}_{(p)}$-scheme,

$\mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})$, whose fibre over $\mathbb{Q}$ is naturally identified with $\mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})$ (cf. [Ko 4, §5]). As before, we will always assume that K'^p is sufficiently small in what follows.

Let $\mathcal{O}$ be the ring of integers of E, $\mathcal{O}_{(p)}$ the localisation of $\mathcal{O}$ at the prime corresponding to the embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, and E_p the completion of E at that prime. We denote by $\mathscr{S}_{\mathsf{K}}(G, X)^-$ the closure of $\mathrm{Sh}_{\mathsf{K}}(G, X)$ in $\mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(p)}$. We write $\mathscr{S}_{\mathsf{K}}(G, X)$ for the normalization of $\mathscr{S}_{\mathsf{K}}(G, X)^-$.

Then we have [Ki 2, 2.3.5, 2.3.8]

Theorem (1.3.5). *Let $x \in \mathscr{S}_{\mathsf{K}}(G, X)^-$ be a closed point of characteristic p, and $\widehat{U}_x$ the formal completion of $\mathscr{S}_{\mathsf{K}}(G, X)^-$ at x. Then*

1. *The irreducible components of $\widehat{U}_x$ are formally smooth.*
2. *$\mathscr{S}_{\mathsf{K}}(G, X)$ is a smooth $\mathcal{O}_{(p)}$-scheme, and the $G(\mathbb{A}_f^p)$-action on* $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ *extends to*
$$\mathscr{S}_{\mathsf{K}_p}(G, X) := \varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}^p\mathsf{K}_p}(G, X).$$

(1.3.6) The subgroup $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is the scheme theoretic stabilizer of a collection of tensors $(s_\alpha) \subset V^{\otimes}_{\mathbb{Z}_{(p)}}$ [Ki 2, 1.3.2].

Let $h : \mathcal{A} \to \mathscr{S}_{\mathsf{K}}(G, X)$ denote the pullback of the universal polarized abelian scheme. Let $V_B = R^1 h^{\mathrm{an}}_* \mathbb{Z}_{(p)}$ where h^{an}_* denotes the map of complex analytic spaces associated to h. Since the tensors s_α are G-invariant, they correspond to sections $s_{\alpha,B} \in V_B^{\otimes}$.

Write $\mathcal{V} = R^1 h_* \Omega^\bullet$ for the first relative de Rham cohomology. Using the de Rham isomorphism, we may view the s_α as parallel sections of the complex analytic vector bundle associated to $\mathcal{V}^{\otimes}$, which lie in the Fil^0 part of the Hodge filtration. The sections in fact arise from sections $s_{\alpha,\mathrm{dR}} \in \mathcal{V}^{\otimes}$ defined over $\mathcal{O}_{(p)}$ [Ki 2, Prop. 2.2.2, 2.3.9].

Similarly, for a finite prime l, let $\mathcal{V}_l = R^1 h_{\text{ét}*} \mathbb{Q}_l$, if $l \neq p$, and $\mathcal{V}_p = R^1 h_{\eta\text{ét}*} \mathbb{Z}_p$, where h_η denotes the generic fibre of h. Then s_α may be viewed as a section $s_{\alpha,l} \in \mathcal{V}_l^{\otimes}$, a priori defined over $\mathbb{C}$, but which descends to E by [Ki 2, Lem. 2.2.1], and to $\mathcal{O}_{(p)}$ if $l \neq p$.

If T is an $\mathcal{O}_{(p)}$ scheme (resp. an E-scheme, resp. a $\mathbb{C}$-scheme), $x \in \mathscr{S}(G, X)(T)$, and $* = l$ or dR (resp. $* = p$, resp. $* = B$), we denote by $s_{\alpha,*,x}$ and the pullback of $s_{\alpha,*}$ to T. Similarly, we denote by $\mathcal{A}_x$ the pullback of $\mathcal{A}$ to x.

As in [Ki 2, 3.4.2], if $x \in \mathscr{S}_{\mathsf{K}}(G, X)(T)$, corresponds to a triple $(\mathcal{A}_x, \lambda, \varepsilon^p_{\mathsf{K}'})$, then $\varepsilon^p_{\mathsf{K}'}$ may be promoted to a section

$$\varepsilon^p_{\mathsf{K}} \in \Gamma(T, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}})/\mathsf{K}^p).$$

Moreover $\varepsilon^p_{\mathsf{K}}$ takes s_α to $(s_{\alpha,l})_{l \neq p}$.

Now let $k \subset \bar{\mathbb{F}}_p$ be a subfield containing the residue field k_E of E_p. Set $W = W(k)$, and let $K \subset \bar{L}$ be a finite, totally ramified extension of $W[1/p]$. Let $x \in \mathscr{S}_{\mathsf{K}}(G, X)(k)$ and $\tilde{x} \in \mathrm{Sh}_{\mathsf{K}}(G, X)(K)$ a point specializing to x.

Let $\bar{K}$ be the algebraic closure of K in $\bar{L}$. and denote by $\mathcal{A}_{\tilde{x}}$ (resp. $\mathcal{A}_{\tilde{x},\bar{K}}$) the fibre (resp. geometric fibre) over $\tilde{x}$. The tensors $s_{\alpha,p,\tilde{x}} \in H^1_{\text{ét}}(\mathcal{A}_{\tilde{x},\bar{K}}, \mathbb{Z}_p)^{\otimes}$ are $\mathrm{Gal}(\bar{K}/K)$-invariant.

Let $\mathscr{G}_x$ denote the p-divisible group attached to $\mathcal{A}_x$. The following result shows that $\mathscr{G}_x$ satisfies the conditions imposed in (1.1.8) and (1.2.15), so that we may apply the results of §1.1, 1.2 to this p-divisible group.

Proposition (1.3.7).

(1) *Under the p-adic comparison isomorphism*

$$H^1_{\text{ét}}(\mathcal{A}_{\tilde{x},\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \xrightarrow{\sim} H^1_{\text{cris}}(\mathcal{A}_x/W) \otimes_W B_{\text{cris}}$$

the $s_{\alpha,p,\tilde{x}}$ map to $\varphi-$invariant tensors $s_{\alpha,0,\tilde{x}} \in \text{Fil}^0(H^1_{\text{cris}}(\mathcal{A}_x/W)^{\otimes})$.

(2) *There is a W-linear isomorphism*

$$H^1_{\text{ét}}(\mathcal{A}_{\tilde{x},\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} H^1_{\text{cris}}(\mathcal{A}_x/W)$$

taking $s_{\alpha,p,\tilde{x}}$ to $s_{\alpha,0,\tilde{x}}$. In particular, the $s_{\alpha,0,\tilde{x}}$ define a reductive group scheme $G_W \subset \text{GL}(H^1_{\text{cris}}(\mathcal{A}_x/W))$ which is isomorphic to $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W$.

(3) *The filtration on $H^1_{\text{cris}}(\mathcal{A}_x/W) \otimes_W k$ is given by μ_0^{-1} where μ_0 is a G_W-valued cocharacter conjugate to μ_h for $h \in X$.*

Proof. Apart from the statement that μ_0 is conjugate to μ_h the proposition follows from (1.1.7). To see this last statement, let $\mathscr{G}_{\tilde{x}}$ be the p-divisible group over $\mathcal{O}_K$ attached to $\tilde{x}$. Then $\mathscr{G}_{\tilde{x}}$ is a G_W-adapted deformation of $\mathscr{G}_x$ by (1.1.13). By (1.1.9) the filtration on

$$\mathbb{D}(\mathscr{G}_{\tilde{x}})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} H^1_{\text{dR}}(\mathcal{A}_{\tilde{x}})$$

is given by a G-valued cocharacter, which is conjugate to μ_0^{-1}. By definition of the Hodge filtration on $H^1_{\text{dR}}(\mathcal{A}_{\tilde{x}})$ this cocharacter is conjugate to μ_h^{-1} for $h \in X$. □

(1.3.8) Fix a uniformizer $\pi \in K$. We denote by $E(u) \in W[u]$ the Eisenstein polynomial of π and by S the ring introduced in (1.1.5), equipped with its Frobenius φ.

Consider the p-divisible group $\mathscr{G}_x$ equipped with the tensors $(s_{\alpha,0,\tilde{x}})$. Recall that by (1.1.10), there is a subspace Spf R_G of the versal deformation space Spf R of $\mathscr{G}_x$ corresponding to G_W-adapted deformations of $\mathscr{G}_x$.

Proposition (1.3.9). *Suppose that the image of x has residue field k. Then, using the notation of (1.3.5) we have*

(1) *$\widehat{U}_x$ may be identified with a closed subspace of* Spf R *containing* Spf R_G.

(2) *Let $\tilde{x}' \in \mathscr{S}_{\mathsf{K}}(G,X)(K)$ be a point whose specialization $x' \in \mathscr{S}_{\mathsf{K}}(G,X)(k)$ maps to the image of x in $\mathscr{S}_{\mathsf{K}}(G,X)^-(k)$. Then (the images of) $\tilde{x}, \tilde{x}'$ lie on the same irreducible component of $\widehat{U}_x$ if and only if the tensors $s_{\alpha,0,\tilde{x}}, s_{\alpha,0,\tilde{x}'}$ constructed above are equal.*

(3) *A deformation $\mathscr{G}$ of $\mathscr{G}_x$ to $\mathcal{O}_K$ corresponds to a point of $\widehat{U}_x$ on the same irreducible component as $\tilde{x}$, if and only if $\mathscr{G}$ is G_W-adapted.*

Proof. The first two parts follow from [Ki 2, Prop. 2.3.5] and its proof and the third part from [Ki 2, Prop. 1.5.8]. Since we will make essential use of this proposition, we sketch the argument.

Let $\tilde{x} \in \mathscr{S}_{\mathsf{K}}(G,X)(K)$ be a point specializing to x. The p-divisible group over $\widehat{U}_x$ attached to the universal abelian variety is induced by a map $\widehat{U}_x \hookrightarrow$ Spf R. Suppose that $s_{\alpha,0,\tilde{x}} = s_{\alpha,0,\tilde{x}'}$ in (2). Then (1.1.13) implies that $\tilde{x}'$ factors through Spf R_G, and hence $\mathscr{G}_{\tilde{x}'}$, the p-divisible group over $\mathcal{O}_K$ corresponding to $\tilde{x}'$, is G_W-adapted.

A result of Blasius and Wintenberger [Bl] asserts that the p-adic comparison isomorphism takes $s_{\alpha,p,\tilde{x}}$ to $s_{\alpha,\text{dR},\tilde{x}}$. Hence under the isomorphism

$$H^1_{\text{dR}}(\mathcal{A}_{\tilde{x}}) \xrightarrow{\sim} H^1_{\text{cris}}(\mathcal{A}_x/W) \otimes_W K$$

$s_{\alpha,\mathrm{dR},\tilde{x}}$ is taken to $s_{\alpha,0,\tilde{x}}$.

Now let $Z \subset \widehat{U}_x$ be the irreducible component containing the image of $\tilde{x}$, and let $\tilde{x}' \in Z(K)$. The sections $s_{\alpha,\mathrm{dR}}$ are parallel, and the isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}}) \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_x/W) \otimes_W K \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}'})$$

is given by parallel transport. As the generic fibre of Z is connected, we see that $s_{\alpha,0,\tilde{x}} = s_{\alpha,0,\tilde{x}'}$.

This shows one implication in (2), and that $\tilde{x}' \in \mathrm{Spf}\, R_G$ by what we saw above. Since this holds for any finite extension K/E_p in $\bar{\mathbb{Q}}_p$, we see that $Z \subset \mathrm{Spf}\, R_G$ and this must be an isomorphism as both spaces have the same dimension. Hence (3) follows, as does the other implication in (2). $\square$

(1.3.10) The above proposition shows that $s_{\alpha,0,\tilde{x}}$ depends only on x, and not on $\tilde{x}$. Hence, we will denote this tensor by $s_{\alpha,0,x}$ below. From (1.3.9)(2) we then get

Corollary (1.3.11). *Let $x, x' \in \mathscr{S}_{\mathsf{K}}(G,X)(k)$ be two points having the same image in $\mathscr{S}_{\mathsf{K}}(G,X)^-(k)$. Then $x = x'$ if and only if $s_{\alpha,0,x} = s_{\alpha,0,x'}$.*

(1.4) Points in a mod p isogeny class: The aim of this section is to show that an isogeny class in a mod p Shimura variety of Hodge type admits maps from sets of the form $X_v(b)$ where v depends on the Shimura datum and b depends on the isogeny class.

(1.4.1) We continue to use the notation of the previous section. Write $r = r_E = [k_E : \mathbb{F}_p]$.

Let $x \in \mathscr{S}_{\mathsf{K}}(G,X)(k)$. As above, we write $\mathcal{A}_x$ for the fibre of $\mathcal{A}$ at x, and $\mathscr{G}_x$ for the p-divisible group of $\mathcal{A}_x$. We will write simply $\mathbb{D}(\mathscr{G}_x)$ for $\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L)$.

Let $\tilde{x} \in \mathscr{S}_{\mathsf{K}}(G,X)(W)$ be a lifting of x. Then there is an isomorphism $T_p\mathscr{G}_{\tilde{x}} \xrightarrow{\sim} V_{\mathbb{Z}_p}$ which takes $s_{\alpha,p,\tilde{x}}$ to s_α. Choosing such an isomorphism, allows us to identify $G_{\mathbb{Z}_p} = G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ with the subgroup of $\mathrm{GL}(T_p\mathscr{G}_{\tilde{x}})$ defined by $(s_{\alpha,p,\tilde{x}})$. This identification is canonical up to inner conjugation. We often write G for $G_{\mathbb{Z}_p}$ when this is unlikely to cause confusion.

Note that there is a canonical isomorphism $H^1_{\mathrm{cris}}(\mathcal{A}_x/W) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x)(W)$. By (1.3.7)(2) and the remarks above, there is an isomorphism

$$V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x)(W)$$

taking s_α to the φ-invariant tensors $s_{\alpha,0,x}$, which define a subgroup

$$G_W \xrightarrow{\sim} G_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W \subset \mathrm{GL}(\mathbb{D}(\mathscr{G}_x)(W)).$$

Here $V^*_{\mathbb{Z}_p}$ denotes the dual of $V_{\mathbb{Z}_p}$. Fixing such an isomorphism, the action of φ has the form $b\sigma$ where $b \in G_W(K_0)$, and σ denotes the absolute Frobenius on $W(\bar{\mathbb{F}}_p)$. The element b is independent of choices up to Frobenius conjugation by elements in $G_W(W)$.

As in (1.2.1), fix a Borel $B \subset G_{\mathbb{Z}_p}$, and a maximal torus $T \subset B$. We denote by $\mu \in X_*(T)$ the dominant cocharacter conjugate to the cocharacters μ_h introduced in (1.3.1). Since the conjugacy class of μ_h is defined over E_p and $G_{\mathbb{Z}_p}$ is quasi-split, μ is defined over E_p. Hence μ is fixed by σ^r.

Note that, with the notation of (1.3.7), μ and μ_0 are conjugate, and μ is the dominant representative of the conjugacy class of μ_0. In particular, since μ and μ_0 are G_W valued they are conjugate by an element of $G(W)$. Hence, if we set $\upsilon_0 = \sigma(\mu_0^{-1})$, and $\upsilon = \sigma(\mu^{-1})$, then using (1.1.12), we find

$$b \in G(W)p^{\upsilon_0}G(W) = G(W)p^{\upsilon}G(W).$$

(1.4.2) Since υ is minuscule we have

$$X_{\preccurlyeq\upsilon}(b) = X_\upsilon(b) = \{g \in G(L)/G(\mathcal{O}_L) : g^{-1}b\sigma(g) \in G(\mathcal{O}_L)p^{\upsilon}G(\mathcal{O}_L)\}.$$

If $g \in X_\upsilon(b)$ define $\Phi(g) = \Phi_r(g)$ by

$$\Phi(g) = (b\sigma)^r(g) = b\sigma(b)\dots\sigma^{r-1}(b)\sigma^r(g).$$

Then

$$\Phi(g)^{-1}b\sigma(\Phi(g)) = \sigma^r(g^{-1}b\sigma(g)) \in G(\mathcal{O}_L)p^{\upsilon}G(\mathcal{O}_L)$$

since υ if fixed by σ^r, and Φ defines a map

$$\Phi : X_\upsilon(b) \to X_\upsilon(b); \quad g \mapsto \Phi(g).$$

Since $\mathbb{D}(\mathscr{G}_x)$ is a Dieudonné module υ acting on $\mathbb{D}(\mathscr{G}_x)$ has non-negative weights and induces a minuscule cocharacter of $\mathrm{GL}(\mathbb{D}(\mathscr{G}_x))$. If $g \in X_\upsilon(b)$, then $g \cdot \mathbb{D}(\mathscr{G}_x)$ is stable under Frobenius and satisfies the axioms of a Dieudonné module (cf. (1.1.9)). Hence $g \cdot \mathbb{D}(\mathscr{G}_x)$ corresponds to a p-divisible group $\mathscr{G}_{gx}$ which is naturally equipped with a quasi-isogeny $\mathscr{G}_x \to \mathscr{G}_{gx}$, corresponding to the natural isomorphism $g \cdot \mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that

$$s_{\alpha,0,x} = g(s_{\alpha,0,x}) \in (g\mathbb{D}(\mathscr{G}_x))^{\otimes} = \mathbb{D}(\mathscr{G}_{gx})^{\otimes}.$$

We denote by $\mathcal{A}_{gx}$ the corresponding abelian variety, which is isogenous to $\mathcal{A}_x$, and canonically equipped with a K'-level structure, induced by that on $\mathcal{A}_x$.

Since $G \subset \mathrm{GSp}$, the weak polarization on $\mathcal{A}_x$ induces a weak polarization λ_{gx} on $\mathcal{A}_{gx}$. (The induced symplectic forms on $\mathbb{D}(\mathscr{G}_x)$ and $\mathbb{D}(\mathscr{G}_{gx})$ differ by a scalar). Thus we obtain a map

$$X_\upsilon(b) \to \mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})(\bar{\mathbb{F}}_p); \quad g \mapsto (\mathcal{A}_{gx}, \lambda_{gx}). \tag{1.4.3}$$

We again denote by $\Phi = \Phi_r$ the geometric r-Frobenius on $\mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})(\bar{\mathbb{F}}_p)$ and $\mathscr{S}_{\mathsf{K}}(G,X)(\bar{\mathbb{F}}_p)$.

Proposition (1.4.4). *There is a unique lifting of (1.4.3) to a map*

$$\iota_x : X_\upsilon(b) \to \mathscr{S}_{\mathsf{K}}(G,X)(\bar{\mathbb{F}}_p)$$

such that

$$s_{\alpha,0,x} = s_{\alpha,0,\iota_x(g)} \in \mathbb{D}(\mathscr{G}_{gx})^{\otimes}.$$

Moreover we have $\Phi \circ \iota_x = \iota_x \circ \Phi$.

Proof. The uniqueness follows from (1.3.11).

For $g \in X_\upsilon(b)$ we have

$$\varphi^r(\mathbb{D}(\mathscr{G}_{gx})) = (b\sigma)^r(g\mathbb{D}(\mathscr{G}_x)) = (b\sigma)^r(g)\sigma^r(\mathbb{D}(\mathscr{G}_x)) \\ = (b\sigma)^r(g)\mathbb{D}(\mathscr{G}_x) = \mathbb{D}(\mathscr{G}_{\Phi(x)}),$$

so (1.4.3) commutes with Φ. Hence, if ι_x exists, and we set $\iota'_x = \Phi \circ \iota_x \circ \Phi^{-1}$, then for $g \in X_\upsilon(b)$ we have

$$s_{\alpha,0,\iota'_x(g)} = \varphi^r(s_{\alpha,0,\iota_x(\Phi^{-1}(g))}) = \varphi^r(s_{\alpha,0,x}) = s_{\alpha,0,x}.$$

Thus $\iota'_x(g) = \iota_x(g)$ and ι_x commutes with Φ.

It remains to prove the existence, which will occupy us for most of the rest of this section.

(1.4.5) Let R be a smooth, integral $\bar{\mathbb{F}}_p$-algebra and $\mathscr{R}$ a frame for R. As in §1.2, we will denote by σ the absolute Frobenius on R and its lift to $\mathscr{R}$. For $n \geq 1$ we write $\mathscr{R}_n$ for the ring $\mathscr{R}$ considered as an $\mathscr{R}$-algebra via $\sigma^n : \mathscr{R} \to \mathscr{R}$. Similarly, we write R_n for the R-algebra $\mathscr{R}_n/p\mathscr{R}_n$.

Lemma (1.4.6). *Let $g \in G(\mathscr{R}_L)$ such that $g^{-1}b\sigma(g) \in G(\mathscr{R})p^\upsilon G(\mathscr{R})$. Then there exists $n \geq 1$, a p-divisible group $\mathscr{G}_{gx}$ over R_n, and a quasi-isogeny $\tau_g : \mathscr{G}_{gx} \to \mathscr{G}_x \otimes_{\mathcal{O}_L} R_n$ over R_n, such that*

1. *τ_g identifies $\mathbb{D}(\mathscr{G}_{gx})(\mathscr{R}_n)$ with $g \cdot (\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} \mathscr{R}_n)$.*
2. *There is a cocharacter μ' of gGg^{-1}, defined over $\mathscr{R}_n$ and conjugate to μ, such that the filtration on $\mathbb{D}(\mathscr{G}_{gx})(R_n)$ is induced by μ'^{-1}.*

Proof. We remark that in (1) the element g is viewed in $G(\mathscr{R}_{n,L})$ via the structure of $\mathscr{R}_L$-algebra on $\mathscr{R}_{n,L}$. Thus, if $\mathscr{R}_n$ is identified with $\mathscr{R}$ it coincides with $\sigma^n(g)$.

Write $g^{-1}b\sigma(g) = g_1\upsilon(p)g_2$ with $g_1, g_2 \in G(\mathscr{R})$. Since υ is a minuscule cocharacter of $\mathrm{GL}(\mathbb{D}(\mathscr{G}_x))$ whose weights on $\mathbb{D}(\mathscr{G}_x)$ are non-negative, we have

$$pg\mathbb{D}(\mathscr{G}_x) \subset b\sigma(g\mathbb{D}(\mathscr{G}_x)) = gg_1p^\upsilon g_2\mathbb{D}(\mathscr{G}_x) \subset g\mathbb{D}(\mathscr{G}_x).$$

Hence $g\mathbb{D}(\mathscr{G}_x)(\mathscr{R}) \subset \mathbb{D}(\mathscr{G}_x)\otimes_{\mathcal{O}_L}\mathscr{R}_L$. is stable under the Frobenius and Verschiebung.

Note that action of σ on $\Omega^1_{\mathscr{R}/\mathcal{O}_L}$ is trivial mod p, and hence is topologically nilpotent. In particular, $\sigma^n(g^{-1}dg) \in \mathrm{Lie}\, G \otimes_{\mathbb{Z}_p} \Omega^1_{\mathscr{R}/\mathcal{O}_L}$ for n sufficiently large, so

$$g^{-1}dg \in \mathrm{Lie}\, G \otimes_{\mathbb{Z}_p} \Omega^1_{\mathscr{R}_n/\mathcal{O}_L} \subset \mathrm{End}\,\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} \Omega^1_{\mathscr{R}_n/\mathcal{O}_L}.$$

Consider the connection $\nabla = 1 \otimes d$ on $\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} \mathscr{R}_{n,L}$. Then

$$\nabla(g \cdot \mathbb{D}(\mathscr{G}_x)) = dg \cdot \mathbb{D}(\mathscr{G}_x) = g(g^{-1}dg \cdot \mathbb{D}(\mathscr{G}_x)) \subset g \cdot \mathbb{D}(\mathscr{G}_x),$$

where we have written $\mathbb{D}(\mathscr{G}_x)$ for $\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} \mathscr{R}_n$. Hence $g\mathbb{D}(\mathscr{G}_x)$ is stable under ∇ and gives rise to a Dieudonné crystal on R_n [deJ, 2.3.4], which corresponds to a p-divisible group $\mathscr{G}_{gx}$ since the crystalline Dieudonné functor is an equivalence over R_n [deJ, 4.1.1]. The isogeny $g\mathbb{D}(\mathscr{G}_x) \to \mathbb{D}(\mathscr{G}_x)$ corresponds to an isogeny $\mathscr{G}_{gx} \to \mathscr{G}_x$ over R_n. This proves (1).

To show (2) we may assume $n \geq 1$, and we set $\mu' = g\sigma^{-1}(g_2)^{-1}\mu\sigma^{-1}(g_2)g^{-1}$ so that μ' is a cocharacter of gGg^{-1}. Note that this makes sense since g_2 is in the image of σ when considered as an element of $\mathscr{R}_n$. Then

$$\begin{aligned} b\sigma(\mu'(p)g\mathbb{D}(\mathscr{G}_x)) &= b\sigma(g\sigma^{-1}(g_2)^{-1}\mu(p)\sigma^{-1}(g_2)\mathbb{D}(\mathscr{G}_x)) \\ &= (gg_1\upsilon(p)g_2)(g_2^{-1}\upsilon(p^{-1})g_2)\mathbb{D}(\mathscr{G}_x) = gg_1g_2\mathbb{D}(\mathscr{G}_x) = g\mathbb{D}(\mathscr{G}_x). \end{aligned}$$

Thus the filtration on $\mathbb{D}(\mathscr{G}_{gx})(R_n) = g\mathbb{D}(\mathscr{G}_x) \otimes_{\mathcal{O}_L} \bar{\mathbb{F}}_p$ is given by μ'^{-1}. $\square$

(1.4.7) Now suppose that $g \in G(\mathscr{R}_L)$ with $g^{-1}b\sigma(g) \in G(\mathscr{R})p^{\upsilon}G(\mathscr{R})$, and that $g\mathbb{D}(\mathscr{G}_x)$ corresponds to a p-divisible group $\mathscr{G}_{gx}$ equipped with a quasi-isogeny $\mathscr{G}_{gx} \to \mathscr{G}_x \otimes_{\mathcal{O}_L} \mathscr{R}$. We also assume there is a cocharacter μ' of gGg^{-1}, defined over $\mathscr{R}$ and conjugate to μ, such that the filtration on $\mathbb{D}(\mathscr{G}_{gx})(R)$ is induced by μ'^{-1}.

Then we have φ-invariant tensors

$$s_{\alpha,0,x} = g(s_{\alpha,0,x}) \in \mathbb{D}(\mathscr{G}_{gx})^{\otimes}(\mathscr{R}).$$

We may think of these as morphisms of crystals $\mathbf{1} \to \mathbb{D}(\mathscr{G}_{gx})^{\otimes}$, which are also morphisms of φ-isocrystals.

As in (1.4.2), $\mathscr{G}_{gx}$ corresponds to an abelian scheme $\mathcal{A}_{gx}$ over R, equipped with a weak polarization λ_{gx}. Hence we obtain a map

$$\text{Spec } R \to \mathscr{S}_{\mathsf{K}'}(\text{GSp}, S^{\pm}) \tag{1.4.8}$$

such that $\mathscr{G}_{gx}$ is canonically identified with the p-divisible group of the pullback of $\mathcal{A}$ to Spec R.

Proposition (1.4.9). *Suppose that the image of x in $\mathscr{S}_{\mathsf{K}}(G,X)(\bar{\mathbb{F}}_p)$ lifts to a point $x_R \in (\text{Spec } R)(\bar{\mathbb{F}}_p)$ with $x_R^*(g) = 1$. Then there is a unique lifting of (1.4.8) to a map*

$$\iota_R : \text{Spec } R \to \mathscr{S}_{\mathsf{K}}(G,X)$$

such that

$$\iota_R^*(s_{\alpha,0}) = s_{\alpha,0,x} \in \mathbb{D}(\mathscr{G}_{gx})^{\otimes}.$$

Proof. As in (1.4.4), the uniqueness is a consequence of (1.3.11).

To show the existence, let

$$\text{Fil}^1(\mathbb{D}(\mathscr{G}_{gx})) \subset p\mu'(p)g\mathbb{D}(\mathscr{G}_x) \subset \mathbb{D}(\mathscr{G}_{gx})$$

denote the filtration induced by μ'^{-1}. By Grothendieck-Messing theory, this filtration corresponds to a p-divisible group $\tilde{\mathscr{G}}_{gx}$ over $\mathscr{R}$, which lifts $\mathscr{G}_{gx}$.

Since μ' is a G-valued and hence GSp-valued cocharacter, the principle polarization on $\mathscr{G}_{gx}$ lifts to a principle polarization of $\tilde{\mathscr{G}}_{gx}$. In particular, $\tilde{\mathscr{G}}_{gx}$ corresponds to an abelian scheme $\tilde{\mathcal{A}}_{gx}$ over $\mathscr{R}$, equipped with a weak polarization, lifting $(\mathcal{A}_{gx}, \lambda_{gx})$, and (1.4.8) lifts to a map

$$\iota_{\mathscr{R}}^- : \text{Spec } \mathscr{R} \to \mathscr{S}_{\mathsf{K}'}(\text{GSp}, S^{\pm}).$$

We claim that $\iota_{\mathscr{R}}^-$ factors through $\mathscr{S}_{\mathsf{K}}(G,X)^-$, and that the mod p reduction, ι_R, of the unique lift

$$\iota_{\mathscr{R}} : \text{Spec } \mathscr{R} \to \mathscr{S}_{\mathsf{K}}(G,X)$$

of $\iota^-_{\mathscr{R}}$, satisfies the conditions of the Proposition. Note that, assuming the existence of $\iota^-_{\mathscr{R}}$, a lift $\iota_{\mathscr{R}}$ exists, as $\mathscr{R}$ is formally smooth, and hence normal.

Let $\widehat{\mathscr{R}}_x$ denote the completion of $\mathscr{R}$ at the image of x_R. Since R is integral, Spec R is geometrically connected, so it suffices to prove the claim with $\widehat{\mathscr{R}}_x$ in place of $\mathscr{R}$.

Now let $\tilde{x} \in \mathscr{S}_{\mathsf{K}}(G,X)(\mathcal{O}_L)$ be a point lifting x. Let $K \subset \bar{L}$ be a finite extension of L with uniformizer π. We use the notation of (1.1.5), and denote by S the $W(\bar{\mathbb{F}}_p)$-algebra introduced there. Let $y : \widehat{\mathscr{R}}_x \to \mathcal{O}_K$ be an $\mathcal{O}_K$-valued point. By smoothness, y lifts to a map $\tilde{y} : \widehat{\mathscr{R}}_x \to S$. As $s_{\alpha,0,x}$ induces a map of φ-isocrystals $\mathbf{1} \to \mathbb{D}(\mathscr{G}_{gx})^{\otimes}$, we see that $\tilde{s}_\alpha := \tilde{y}^*(s_{\alpha,0,x})$ satisfy the conditions of (1.1.8). Hence the composite

$$\mathrm{Spec}\ \mathcal{O}_K \xrightarrow{y} \mathrm{Spec}\ \widehat{\mathscr{R}}_x \xrightarrow{\iota^-_{\widehat{\mathscr{R}}_x}} \mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})$$

factors through the component of $\widehat{U}_x$ containing $\tilde{x}$ by (1.3.9)(3). Here $\iota^-_{\widehat{\mathscr{R}}_x}$ denotes the map induced by $\iota^-_{\mathscr{R}}$. Since this holds for any y, it follows that $\iota^-_{\widehat{\mathscr{R}}_x}$ factors through the same component of $\widehat{U}_x$, and hence through $\mathscr{S}_{\mathsf{K}}(G,X)$.

This shows the existence of $\iota^-_{\mathscr{R}}$ and $\iota_{\mathscr{R}}$, and the fact that ι_R satisfies the condition of the proposition now follows from (1.3.9)(2), and the fact that $\iota_R^*(s_{\alpha,0})$ and $s_{\alpha,0,x}$ are both morphisms of crystals, so they agree on Spec R if they agree at x. $\square$

(1.4.10) *End of proof of (1.4.4)* As in the proof of the uniqueness, (1.3.11) shows that there is maximal subset $X_v(b)^\circ \subset X_v(b)$ such that (1.4.3) lifts to a map

$$\iota_x : X_v(b)^\circ \to \mathscr{S}_{\mathsf{K}}(G,X)(\bar{\mathbb{F}}_p)$$

with

$$s_{\alpha,0,x} = s_{\alpha,0,\iota_x(g)} \in \mathbb{D}(\mathscr{G}_{gx})^{\otimes}.$$

for $g \in X_v(b)^\circ$.

We first show that $X_v(b)^\circ$ is a union of connected components of $X_v(b)$. Let R be a smooth $\bar{\mathbb{F}}_p$-algebra with connected spectrum and a given frame $\mathscr{R}$, and $g \in X_v(b)(\mathscr{R})$. We have to show that if $X_v(b)^\circ$ contains the image of some point of (Spec $R)(\bar{\mathbb{F}}_p)$, then it contains the image of every such point. By (1.2.8), after replacing R be an étale covering, we may assume that $g^{-1}b\sigma(g) \in G(\mathscr{R})p^v G(\mathscr{R})$. After replacing $\mathscr{R}$ by $\mathscr{R}_n$, we may assume that the conclusion of (1.4.6) holds with $\mathscr{R}$ in place of $\mathscr{R}_n$. The result now follows from (1.4.9).

Let $K \subset \bar{L}$ be a finite extension of L, and $\tilde{\mathscr{G}}_x$ a G_W-adapted lifting of $\mathscr{G}_x$ to $\mathcal{O}_K$. By (1.3.9), $\tilde{\mathscr{G}}_x$ corresponds to a point $\tilde{x} \in \mathscr{S}_{\mathsf{K}}(G,X)(K)$. Fix an isomorphism $T_p\tilde{\mathscr{G}}_x \xrightarrow{\sim} V_{\mathbb{Z}_p}$ taking $s_{\alpha,p,\tilde{x}}$ to s_α. Since we have already identified $\mathbb{D}(\mathscr{G}_x)$ with $V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$ in (1.4.1), we obtain an isomorphism

$$T_p\tilde{\mathscr{G}}_x^*(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_L \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x)$$

which takes $s_{\alpha,p,\tilde{x}}$ to $s_{\alpha,0,x}$. This allows us to apply the construction of (1.2.16).

Let $g \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, and $g_0 \in X_v(b)$ the element corresponding to g in (1.2.18). After replacing K by a finite extension, we may assume that $g^{-1} \cdot T_p\mathscr{G}_{\tilde{x}}$ corresponds to a p-divisible group $\tilde{\mathscr{G}}'$ over $\mathcal{O}_K$, equipped with a quasi-isogeny $\mathscr{G}_{\tilde{x}} \to \tilde{\mathscr{G}}'$, which identifies $\mathbb{D}(\tilde{\mathscr{G}}' \otimes \bar{\mathbb{F}}_p)$ with $g_0\mathbb{D}(\mathscr{G}_x)$. This corresponds to a prime to p-isogeny $\mathcal{A}_{\tilde{x}} \to$

$\mathcal{A}'$, and $\mathcal{A}'$ corresponds to a point $g\tilde{x} \in \mathrm{Sh}(G,X)(K)$. Let gx denote the image of $g\tilde{x}$ in $\mathrm{Sh}(G,X)(\bar{\mathbb{F}}_p)$. Setting $\iota_x(g_0) = gx$ shows that $g_0 \in X_v(b)^\circ$.

By (1.2.23) we may choose $\tilde{\mathscr{G}}_x$ so that $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \pi_0(X_v(b))$ is surjective. Then $X_v(b)^\circ$ meets every component of $X_v(b)$, and hence $X_v(b)^\circ = X_v(b)$ by what we saw above. $\square$

(1.4.11) Let $(h, g_p, g^p) \in X \times G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ and $\mathcal{A}$ the abelian variety up to prime to p isogeny attached to (h, g_p, g^p). Then $\mathcal{A}$ is equipped with an isomorphism $H_1(\mathcal{A}(\mathbb{C}), \mathbb{Z}_{(p)}) \xrightarrow{\sim} g_p \cdot V_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Q}}$.

Fix an embedding of $\bar{\mathbb{Q}}$-algebras $\bar{L} \hookrightarrow \mathbb{C}$. Let $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$, and $\tilde{x} \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\mathcal{O}_{\bar{L}})$ a point lifting x. By [Ki 2, 2.2.6] (cf. also [Sa, Cor. 3.5(b)]) there is a an element $(h, 1, g^p_{\tilde{x}}) \in X \times G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ which maps to $\tilde{x} \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\mathbb{C})$. Attached to $(h, 1, g^p_{\tilde{x}})$, we have an isomorphism $H_1(\mathcal{A}(\mathbb{C}), \mathbb{Z}_{(p)}) \xrightarrow{\sim} V_{\mathbb{Z}_{(p)}}$, and hence an isomorphism $V_{\mathbb{Z}_p} \xrightarrow{\sim} T_p\mathscr{G}_{\tilde{x}}$ which takes s_α to $s_{\alpha,p,\tilde{x}}$. Then as in (1.4.10) above, we can apply the construction of (1.2.16), and obtain a map

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to X_v(b); \quad g_p \mapsto g_{p,0}.$$

Corollary (1.4.12). *With the above notation we have a commutative, $G(\mathbb{A}_f^p)$-equivariant diagram*

$$\begin{array}{ccc} h \times G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p) & \longrightarrow & \mathscr{S}_{\mathsf{K}_p}(G,X)(\mathcal{O}_{\bar{L}}) \\ \downarrow{\scriptstyle g_p \mapsto g_{p,0}} & & \downarrow \\ X_v(b) \times G(\mathbb{A}_f^p) & \xrightarrow{\iota_x} & \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) \end{array}$$

where the map on the left is given by

$$(h, g_p, g^p) \mapsto (g_{p,0}, (g^p_{\tilde{x}})^{-1} g^p).$$

Proof. By (1.3.5), the map ι_x in (1.4.4), extends to a $G(\mathbb{A}_f^p)$-equivariant map

$$\iota_x : X_v(b) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p).$$

The commutativity of the diagram follows from the definitions and (1.3.11). $\square$

Corollary (1.4.13). *The map*

$$\iota_x : X_v(b) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p),$$

is $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$-equivariant, where $\langle\Phi\rangle$ denotes the group generated by Φ.

Proof. The $\langle\Phi\rangle$-equivariance is part of (1.4.4), and the map is $G(\mathbb{A}_f^p)$-equivariant by construction. The $Z_G(\mathbb{Q}_p)$-equivariance follows (for example) from (1.4.12) together with the observation that if $z \in Z_G(\mathbb{Q}_p)$ and $g \in G(\mathbb{Q}_p)$, then $(zg)_0 = zg_0$ (cf. the proof of (1.2.21)). $\square$

(1.4.14) We call the image of ι_x in (1.4.13), the *isogeny class* of x.

Recall that attached to x we have a triple $(\mathcal{A}_x, \lambda_x, \varepsilon^p_x)$ consisting of a weakly polarized abelian variety $(\mathcal{A}_x, \lambda_x)$ over $\bar{\mathbb{F}}_p$ and an isomorphism

$$\varepsilon^p_x : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}}$$

which takes s_α to $(s_{\alpha,l,x})_{l\neq p}$.

Here we have passed to the limit over K^p in the construction of (1.3.6), and we are using the fact that $\bar{\mathbb{F}}_p$ is separably closed.

Proposition (1.4.15). *Let $x, x' \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$. Then x, x' lie in the same isogeny class if and only if there is a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ respecting weak polarizations such that the induced maps $\mathbb{D}(\mathscr{G}_x) \to \mathbb{D}(\mathscr{G}_{x'})$ and $\widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \to \widehat{V}^p(\mathcal{A}_{x'})_{\mathbb{Q}}$ take $s_{\alpha,0,x}$ to $s_{\alpha,0,x'}$, and $(s_{\alpha,l,x})_{l\neq p}$ to $(s_{\alpha,l,x'})_{l\neq p}$ respectively.*

Proof. First consider the composite

$$V_{\mathbb{A}_f^p} \xrightarrow[\varepsilon_x^p]{\sim} \widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \xrightarrow{\sim} \widehat{V}^p(\mathcal{A}_{x'})_{\mathbb{Q}} \xrightarrow[(\varepsilon_{x'}^p)^{-1}]{\sim} V_{\mathbb{A}_f^p}$$

This fixes s_α, and hence corresponds to an element of $G(\mathbb{A}_f^p)$. Hence replacing x' by another point in its isogeny class, we may assume that $\mathcal{A}_x \to \mathcal{A}_{x'}$ is compatible with ε_x^p and $\varepsilon_{x'}^p$.

Next note that, by (1.3.7)(2), there are isomorphisms

$$\mathbb{D}(\mathscr{G}_{x'}) \xrightarrow{\sim} V_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x)$$

where the first map takes $s_{\alpha,0,x'}$ to s_α and the second map takes s_α to $s_{\alpha,0,x}$. As in the proof of (1.1.19), this implies that $\mathbb{D}(\mathscr{G}_{x'}) = g \cdot \mathbb{D}(\mathscr{G}_x)$ for some $g \in X_v(b)$, where $b \in G(L)$ is associated to x as in (1.4.1).

Hence, after replacing x by an element in its isogeny class, we may assume that $\mathcal{A}_x \to \mathcal{A}_{x'}$ is a prime to p-isogeny compatible with ε_x^p and $\varepsilon_{x'}^p$. It follows by (1.3.11), and the moduli theoretic description of $\mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp}, S^{\pm})$, that $x = x'$. □

§2 Construction of CM points.

(2.1) Tate's theorem with Hodge cycles: The aim of this section is to prove that every isogeny class in $\mathscr{S}_{\mathsf{K}}(G,X)(\bar{\mathbb{F}}_p)$ contains a point which lifts to a special point on $\mathscr{S}_{\mathsf{K}}(G,X)$. For PEL type Shimura varieties a proof this, based on deformation theory, was sketched in a letter from Langlands to Rapoport. At the time, it did not seem possible to complete Langlands' sketch, and later authors used Honda-Tate theory instead [Mi 1], [Zi 2], [Ko 4]. Here we use an argument similar to that of Langlands (it is also possible to give an argument using Honda-Tate theory). This is now possible because of the theorem, which enters via the proof of (1.1.19), that a crystalline representation with all Hodge-Tate weights 0 or 1 arises from a p-divisible group.

We begin with an analogue of the main theorem of [Ta 2] and show that if $x \in \mathscr{S}_{\mathsf{K}}(G,X)(k)$ then $\mathcal{A}_x$ has sufficiently many automorphisms (in the isogeny category) which respect the Hodge cycles s_α.

We continue to use the notation introduced in §1, so that (G,X) is of Hodge type, and K_p is hyperspecial, but we now assume that $k \subset \bar{\mathbb{F}}_p$ is a finite extension of k_E, and we write r for the degree of k (rather than k_E) over $\mathbb{F}_p$, so that $q = p^r = |k|$. As before we set $W = W(k)$, and $K_0 = W[1/p]$.

(2.1.1) Let $l \neq p$ be a prime. As in (1.3.6), $R^1 h_{\text{ét}*}\mathbb{Q}_l$ extends to an l-adic local system on $\mathscr{S}_{\mathsf{K}}(G,X)$ (for example by (1.3.5)), and the Hodge cycles s_α give rise to sections $s_{\alpha,l}$ of $R^1 h_{\text{ét}*}\mathbb{Q}_l$ via the isomorphisms ε_{K} of (1.3.4).

For any $x \in \mathscr{S}_{\mathsf{K}}(G,X)(k)$, let $\bar{x}$ denote the $\bar{\mathbb{F}}_p$ point induced by x. The tensors $s_{\alpha,l,x} \in H^1_{\text{ét}}(\mathcal{A}_{\bar{x}}, \mathbb{Q}_l)$ are Frobenius invariant, and define a subgroup $G_l \subset \mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_{\bar{x}}, \mathbb{Q}_l))$ which may be identified with $G_{\mathbb{Q}_l}$. This identification is canonical up to conjugation by elements in the image of $\mathsf{K}^p \to G(\mathbb{Q}_l)$.

(2.1.2) Let $\gamma_l \in G(\mathbb{Q}_l)$ be the geometric q-Frobenius in $\mathrm{Gal}(\bar{\mathbb{F}}_p/k)$ acting on $H^1_{\text{ét}}(\mathcal{A}_{\bar{x}}, \mathbb{Q}_l)$, and let $I_{l/k} = I_{l,x/k} \subset G_{\mathbb{Q}_l}$ denote the centralizer of γ_l. If the integers are ordered multiplicatively, then the Zariski closures in $G_{\mathbb{Q}_l}$ of the subgroups generated by γ_l^n form a decreasing sequence of subgroups, which must stabilize as G is Noetherian. Hence the centralizers of $\gamma_l^n \subset G_{\mathbb{Q}_l}$ form an increasing sequence of subgroups $I_{l,n}$, which stabilizes. We denote by $I_l = I_{l,x}$ the subgroup $I_{l,n}$ for n sufficiently large.

As in (1.3.6), there exists an isomorphism

$$\mathbb{D}(\mathscr{G}_x) \xrightarrow{\sim} V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W$$

which takes $s_{\alpha,0,x}$ to s_α. We fix such an isomorphism. In particular, we then have identifications

$$G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} K_0 = G_{K_0} \xrightarrow{\sim} G(s_{\alpha,0,x}) \subset \mathrm{GL}(\mathbb{D}(\mathscr{G}_x)_{K_0}) \xrightarrow{\sim} \mathrm{GL}(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)_{K_0}),$$

where $G(s_{\alpha,0,x}) \subset \mathrm{GL}(\mathbb{D}(\mathscr{G}_x)_{K_0})$ denotes the subgroup defined by the $s_{\alpha,0,x}$.

Then the Frobenius on $\mathbb{D}(\mathscr{G}_x)$ has the form $\delta\sigma$ with $\delta \in G(K_0)$. The element δ is independent of the choice made in (1.4.1) up to σ-conjugacy by elements of $G(W)$. We set $\gamma_p = \delta\sigma(\delta)\dots\sigma^{r-1}(\delta) \in G(K_0)$. Then the conjugacy class of γ_p is independent of choices.

We denote by $I_{p/k} = I_{p,x/k}$ the algebraic group over $\mathbb{Q}_p$ whose points in a $\mathbb{Q}_p$-algebra R are given by

$$I_{p/k}(R) = \{\alpha \in G(W \otimes_{\mathbb{Z}_p} R) : \quad \delta\sigma(\alpha) = \alpha\delta\}.$$

We obviously have $I_{p/k} \subset J_\delta$ where J_δ is the group defined in (1.2.12), and $\gamma_p \in I_{p/k}(\mathbb{Q}_p)$. Moreover $I_{p/k} \otimes_{\mathbb{Q}_p} K_0$ is canonically isomorphic to the centralizer of γ_p in G_{K_0} [Ko 6, Lem. 5.4].

For n a positive integer, let $k_n \subset \bar{\mathbb{F}}_p$ denote the extension of k of degree n. Define $I_{p,n}$ in the same way as $I_{p/k}$ but with $W(k_n)$ in place of W. Then the $I_{p,n}$ form an increasing sequence of subgroups of J_δ, with $I_{p,n} \otimes_{W(k_n)} L$ isomorphic to the centralizer of γ_p^n in G_L. In particular, $I_{p,n}$ does not depend on n if n is sufficiently large, and we denote this subgroup of J_δ by $I_p = I_{p,x}$.

Finally, let $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ denote the $\mathbb{Q}$-group whose points in a $\mathbb{Q}$-algebra R are given by

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(R) = (\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x) \otimes_{\mathbb{Q}} R)^\times,$$

where $\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x)$ denote the endomorphisms of $\mathcal{A}_x$ viewed as an abelian variety up to isogeny.

We denote by $I_{/k} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ the closed subgroup, such that $I_{/k}(R)$ consists of those points of $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(R)$ which fix the tensors $s_{\alpha,l,x}$ (or equivalently map to an element of $G_l(R)$) for all primes $l \neq p$, and the tensors $s_{\alpha,0,x}$. Let $\gamma \in I_{/k}$ be the q-Frobenius. For each l, we have a natural map $I_{/k} \to I_{l/k}$ over $\mathbb{Q}_l$ which takes γ to γ_l.

Similarly, we write $I \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x \otimes_k \bar{\mathbb{F}}_p)$ for the subgroup whose points consist of the elements fixing the tensors $s_{\alpha,0,x}$ and $s_{\alpha,l,x}$ for all $l \neq p$. For each l, we have a natural map $I \to I_l$ over $\mathbb{Q}_l$. We remark that for $l = p$ this map sends $i \in I$ to the automorphism $\mathbb{D}(i^{-1})$ of $\mathbb{D}(\mathscr{G}_x)_{K_0}$.

Proposition (2.1.3). *The map ι_x of (1.4.12)) induces an injective map*

$$\iota_x : I(\mathbb{Q})\backslash X_v(\delta) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p). \tag{2.1.4}$$

Proof. Attached to any point $x' \in \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$ we have the data of an abelian variety $\mathcal{A}_{x'}$ up to prime to p-isogeny, the tensors $(s_{\alpha,0,x'}) \in \mathbb{D}(\mathscr{G}_{x'})^{\otimes}$ and the isomorphism $\varepsilon^p_{x'} : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \widehat{V}^p(\mathcal{A}_{x'})_{\mathbb{Q}}$. If $g^p \in G(\mathbb{A}_f^p)$, one sees from the definitions, that $\varepsilon^p_{gx} = \varepsilon^p_x \circ g$ is the isomorphism attached to $\mathcal{A}_{g^p \cdot x}$

Now let $g_1 = (g_{1,0}, g_1^p)$ and $g_2 = (g_{2,0}, g_2^p)$ be in $X_v(\delta) \times G(\mathbb{A}_f^p)$, and write $x_1 = \iota_x(g_1)$, and $x_2 = \iota_x(g_2)$. If $x_1 = x_2$ then there is a prime to p-isogeny $\theta : \mathcal{A}_{x_1} \xrightarrow{\sim} \mathcal{A}_{x_2}$ compatible[10] with $\varepsilon^p_{x_1}$ and $\varepsilon^p_{x_2}$, and taking $(s_{\alpha,0,x_1})$ to $(s_{\alpha,0,x_2})$. Conversely if a prime to p-isogeny with these properties exists, then x_1 and x_2 have the same image in $\mathscr{S}_{\mathsf{K}_p}(G, X)^-$ by the moduli theoretic description of $\mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp}, S^{\pm})$, and hence $x_1 = x_2$ by (1.3.11).

Such a prime to p-isogeny induces a quasi-isogeny $j : \mathcal{A}_x \to \mathcal{A}_x$ which identifies $\mathbb{D}(\mathscr{G}_{x_1})$ with $\mathbb{D}(\mathscr{G}_{x_2})$. Since θ takes $(s_{\alpha,0,x_1})$ to $(s_{\alpha,0,x_2})$, j fixes the $(s_{\alpha,0,x})$ so that $j \in I_p(\mathbb{Q}_p)$. Since θ is compatible with $\varepsilon^p_{x_1}$ and $\varepsilon^p_{x_2}$, j fixes $s_{\alpha,l,x}$ for $l \neq p$ and so $j \in I_l(\mathbb{Q}_l)$. Hence $j \in I(\mathbb{Q})$, and $g_2 = jg_1$.

Conversely, if $g_2 = jg_1$, with $j \in I(\mathbb{Q})$, then one sees in the same way that the isogeny j induces a prime to p isogeny $\mathcal{A}_{x_1} \xrightarrow{\sim} \mathcal{A}_{x_2}$, compatible with $\varepsilon^p_{x_1}$ and $\varepsilon^p_{x_2}$ and taking $(s_{\alpha,0,x_1})$ to $(s_{\alpha,0,x_2})$. Hence $x_1 = x_2$. □

Proposition (2.1.5). *Let $H^p = \prod_{l\neq p} I_{l/k}(\mathbb{Q}_l) \cap \mathsf{K}^p$, and $H_p = I_{p/k}(\mathbb{Q}_p) \cap G(W)$. Then the map (2.1.4) induces an injective map*

$$I_{/k}(\mathbb{Q})\backslash \prod_l I_{l/k}(\mathbb{Q}_l)/H_p \times H^p \to \mathscr{S}_{\mathsf{K}}(G, X)(k) \tag{2.1.6}$$

In particular, the left hand side of (2.1.6) is finite.

Proof. Since $x \in \mathscr{S}_{\mathsf{K}}(G, X)(k)$ the pair $(\mathcal{A}_x, \varepsilon^p_{\mathsf{K},x})$ is invariant under pullback by σ^{-r} on $\bar{\mathbb{F}}_p$. If follows that $(\gamma_l)_{l\neq p} \in H^p \subset \mathsf{K}^p$.

Let $g = (g_0, g^p) \in I_{p/k}(\mathbb{Q}_p) \times \prod_{l\neq p} I_{l/k}(\mathbb{Q}_l)$. Then

$$\begin{aligned}\Phi_r(g) = (\Phi_r(g_0), g^p) &= ((\delta\sigma)^r(g_0), g^p)) = (\gamma_p g_0, g^p) \\ &= \gamma(g_0, (\gamma_l^{-1})_{l\neq p} g^p)) = \gamma(g_0, g^p(\gamma_l^{-1})_{l\neq p}) \in I_{/k}(\mathbb{Q}) \cdot (g_0, g^p)H^p.\end{aligned}$$

Hence $\Phi_r(\iota_x(g)) = \iota_x(g)$, and (2.1.4) induces a well defined map as in (2.1.6).

To see that this map is injective, suppose that $g' = (g'_0, g'^p)$ in $I_{p/k}(\mathbb{Q}_p) \times \prod_{l\neq p} I_{l/k}(\mathbb{Q}_l)$ has the same image as g in $\mathscr{S}_{\mathsf{K}}(G, X)(k)$. By (2.1.6) we have $g' = i_0 g h^p$ with $h^p \in \mathsf{K}^p$ and $i_0 \in I(\mathbb{Q})$. Conjugating this equality by γ, we obtain

$$i_0^\gamma g h^{p\gamma} = g' = i_0 g h^p.$$

Hence we have

$$[i_0^{-1}, \gamma] = g^p[h^p, \gamma]g^{p-1} \in g^p \mathsf{K}^p g^{p-1}.$$

Similarly, we have $[i_0^{-1}, \gamma] \in gG(W)g^{-1}$. This implies that $[i_0^{-1}, \gamma]$ is an automorphism of the pair $(\mathcal{A}_{gx}, \varepsilon^p_{\mathsf{K},gx})$, and hence trivial, by our assumptions on K^p (see (1.3.4)). Hence $i_0 \in I_{/k}(\mathbb{Q})$ and therefore $h^p \in H^p$. This shows that (2.1.6) is injective. □

[10] More precisely, for each K^p there is a θ_{K^p} which is compatible with $\varepsilon^p_{x_1}$ and $\varepsilon^p_{x_2}$ mod K^p. For K^p sufficiently small θ_{K^p} is unique, and hence independent of K^p.

Corollary (2.1.7). *For some prime $l \neq p$, $I_{\mathbb{Q}_l} = I \otimes_{\mathbb{Q}} \mathbb{Q}_l$ contains the connected component of the identity in I_l. In particular, the ranks of I and G are equal.*

Proof. Let $l \neq p$ be a prime such that G is split at l, and such that γ_l acting on $H^1_{\text{ét}}(\mathcal{A}_{\bar{x}}, \mathbb{Q}_l)$ has all its eigenvalues in $\mathbb{Q}_l$. Then the connected component of the Zariski closure of $\langle\gamma_l\rangle$ is a split torus in $G_{\mathbb{Q}_l}$, with centralizer I_l. In particular, the Levi subgroup $I_l \subset G_{\mathbb{Q}_l}$ is a connected, split group.

Now by (2.1.5), applied to an suitable extension of k, the quotient $I(\mathbb{Q}_l)\backslash I_l(\mathbb{Q}_l)$ is compact. Hence by [BT, Prop. 9.3], $I_{\mathbb{Q}_l} = I \otimes_{\mathbb{Q}} \mathbb{Q}_l$ contains a Borel subgroup of I_l. It follows that $I_l/I_{\mathbb{Q}_l}$ is proper over $\mathbb{Q}_l$.

Now I contains the subgroup of scalars $\mathbb{G}_m \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ and $(I/\mathbb{G}_m)(\mathbb{R})$ is compact by the positivity of the Rosati involution. Hence I is reductive, which implies that $I_l/I_{\mathbb{Q}_l}$ is affine. Since I_l is geometrically connected, it follows that $I_l/I_{\mathbb{Q}_l} = \mathrm{Spec}\ \mathbb{Q}_l$, and $I_{\mathbb{Q}_l} = I_l$. In particular,

$$\mathrm{rk}\, I = \mathrm{rk}\, I_{\mathbb{Q}_l} = \mathrm{rk}\, I_l = \mathrm{rk}\, G.$$

□

(2.2) Lifting to CM points: We continue to use the notation of the previous subsection.

(2.2.1) If $\epsilon : \mathbb{G}_m \to G$ is a cocharacter defined over some finite extension of $\mathbb{Q}_p$, which commutes with all its Galois conjugates, then we denote by $\bar{\epsilon} = \bar{\epsilon}^G$ the fractional G-valued cocharacter (see [De 2, 1.3.4]) which is the average of the Galois conjugates of ϵ. If ϵ factors through a torus $T \subset G$ defined over $\mathbb{Q}_p$, then $\bar{\epsilon}$ may be viewed as a cocharacter $\bar{\epsilon}^T \in X_*(T)_{\mathbb{Q}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall that we have the cocharacter $\nu_\delta : \mathbb{G}_m \to G$ which is central in J_δ and hence in $I_p \subset J_\delta$.

Lemma (2.2.2). *Let $T \subset I_p$ be a maximal torus (defined over $\mathbb{Q}_p$). Then there exists a cocharacter $\mu_T \in X_*(T)$ defined over $\bar{\mathbb{Q}}_p$ such that*

(1) *As a G_{K_0}-valued cocharacter, μ_T is conjugate to μ.*
(2) $\bar{\mu}^T_T = \nu_\delta^{-1}$.

Proof. This is proved in [LR, Lem. 5.11]. We recall the argument for the convenience of the reader.

First consider any cocharacter $\epsilon \in X_*(T)$ defined over $\mathbb{Q}_p$. Then ϵ may be regarded as a cocharacter of G such that $\sigma(\epsilon) = \delta^{-1}\epsilon\delta$. Choose $g \in G(K_0)$ such that $\epsilon' = g\epsilon g^{-1}$ is dominant for some Borel subgroup of G defined over $\mathbb{Q}_p$. Since the conjugacy class of ϵ' is σ-stable, ϵ' is defined over $\mathbb{Q}_p$ by [Ko 1, 1.1.3(a)]. That is, $\sigma(g^{-1})\epsilon'\sigma(g) = \sigma(g^{-1}\epsilon' g) = \delta^{-1}g^{-1}\epsilon' g\delta$, which implies that $g\delta\sigma(g^{-1})$ commutes with ϵ'. Modifying, the isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes_W K_0 \xrightarrow{\sim} V_{K_0}$ by g, we may assume that δ commutes with ϵ.

Now let $T' \subset T$ be the maximal $\mathbb{Q}_p$-split sub-torus, and choose ϵ so that the centralizer of ϵ, $M_{K_0} \subset G_{K_0}$ is equal to that of T'. Applying the argument above for this ϵ shows that we may assume that T' commutes with $\delta \in M_{K_0}(K_0)$. The condition $\delta\sigma(\alpha) = \alpha\delta$ for $\alpha \in T'$ implies $\sigma(\alpha) = \alpha$. Hence T' may also be viewed as a subgroup of $G_{\mathbb{Q}_p}$, and M_{K_0} arises from the centralizer $M \subset G_{\mathbb{Q}_p}$ of T'.

Now let $T'_2 \supset T'$ be a maximal split torus in $G_{\mathbb{Q}_p}$, and $T_2 \subset G_{\mathbb{Q}_p}$ its centralizer. Then $T_2 \subset M$. Let $P \subset G$ be a parabolic subgroup with unipotent radical N, and such that $P = MN$. Let $B \subset G_{\mathbb{Q}_p}$, be a Borel subgroup contained in P.

Let $g \in X_\upsilon(\delta)$. By the Iwasawa decomposition, we may assume that $g = nm$ with $n \in N(L)$ and $m \in M(L)$. Then $g^{-1}\delta\sigma(g) = m^{-1}\delta\sigma(m)n'$ with $n' \in N(L)$. Let $\upsilon_2 \in X_*(T_2)$ with

$$m^{-1}\delta\sigma(m) \in (M(L)\cap G(\mathcal{O}_L))p^{\upsilon_2}(M(L)\cap G(\mathcal{O}_L)).$$

It follows by [RR, Thm. 4.2(ii)], that ν_δ and $\bar{\upsilon}_2^M$ have the same image in $\pi_1(M)_{\mathbb{Q}} = \pi_1(M)\otimes_{\mathbb{Z}}\mathbb{Q}$.

We claim that υ_2 is conjugate to υ in G. This is proved in [LR] using the Satake transform [Ko 1, 2.3.7]. One can also argue as follows: Let $\chi \in X_*(Z_M)$ be a cocharacter such that $\chi(\alpha) > 0$ for every root $\alpha \in \operatorname{Lie} N$. Then $n'(s) := \chi(s)n'\chi(s)^{-1} \in N(L\langle s, s^{-1}\rangle)$ is contained in $N(L\langle s\rangle)$, and $n'(0) = 1$. Here $L\langle s\rangle$ denotes the Tate algebra. For $s_0 \in \bar{\mathbb{F}}_p^\times$ let $[s_0] \in L$ be the Teichmüller representative. Then

$$m^{-1}\delta\sigma(m)n'([s_0]) = \chi([s_0])m^{-1}\delta\sigma(m)n'\chi([s_0])^{-1} \in G(\mathcal{O}_L)p^\upsilon G(\mathcal{O}_L)$$

so that $m^{-1}\delta\sigma(m) = m^{-1}\delta\sigma(m)n'(0) \in G(\mathcal{O}_L)p^\upsilon G(\mathcal{O}_L)$ by (1.2.6), which proves the claim.

Now let $\mu_T \in X_*(T)$ be a cocharacter which is conjugate to $\sigma^{-1}(\upsilon_2^{-1})$ in M. By what we just saw, μ_T is conjugate to $\sigma^{-1}(\upsilon^{-1}) = \mu$ in G. Moreover, the image of ν_δ^{-1} in $\pi_1(M)_{\mathbb{Q}}$ is equal to that of $\bar{\upsilon}_2^{-1M}$, and hence to that of $\bar{\mu}_T^M$.

Note that $\bar{\mu}_T^T$ is computed with respect to the $\mathbb{Q}_p$-structure on T, and $\bar{\mu}_T^M$ with respect to that on M. The action of $\operatorname{Gal}(K_0/\mathbb{Q}_p)$ on the source and target of $T_{K_0} \to M_{K_0}$ differs by conjugation by δ. Since conjugate cocharacters have the same image in $\pi_1(M)$, one sees that $\bar{\mu}_T^T$ and $\bar{\mu}_T^M$ have the same image in $\pi_1(M)_{\mathbb{Q}}$.

Finally, ν_δ^{-1} and $\bar{\mu}_T^T$ have the same image in $\pi_1(M)_{\mathbb{Q}}$. Note that $\bar{\mu}_T^T \in X_*(T')$ as $T' \subset T$ is the maximal $\mathbb{Q}_p$-split torus, so ν_δ^{-1} and $\bar{\mu}_T^T$ lie in $X_*(T')_{\mathbb{Q}}$. Since $T' \subset M$ is central, we have $\nu_\delta^{-1} = \bar{\mu}_T^T$ in $X_*(T')_{\mathbb{Q}}$. □

Theorem (2.2.3). *The isogeny class $\iota_x(X_\upsilon(\delta)\times G(\mathbb{A}_f^p))$ contains a point which is the reduction of a special point on $\mathrm{Sh}_{\mathsf{K}}(G,X)$.*

Proof. Let $K \subset \bar{\mathbb{Q}}_p$ be the field of definition of the cocharacter μ_T constructed in (2.2.2), thought of as G_{K_0}-valued cocharacter. By [RZ, Prop. 1.21] μ_T^{-1} induces an admissible filtration on $\mathbb{D}(\mathscr{G}_x)_K$. We apply (1.1.19) to μ_T. We may replace x by $\iota_x(g)$, for $g \in X_\upsilon(\delta)$ the element constructed in (1.1.19), and assume that $\mathscr{G}_x$ has a G_W-admissible deformation $\tilde{\mathscr{G}}_x$ such that the filtration on $\mathbb{D}(\tilde{\mathscr{G}}_x)_K$ is given by μ_T^{-1}. By (1.3.9)(3), we have $\tilde{\mathscr{G}}_x = \mathscr{G}_{\tilde{x}}$ for some $\tilde{x} \in \mathrm{Sh}_{\mathsf{K}}(G,X)(K)$ lifting x.

Above $T \subset I_p$ was any maximal torus defined over $\mathbb{Q}_p$. Since I and I_p have the same rank by (2.1.7), we may assume that T is induced by a maximal torus in the $\mathbb{Q}$-group I, which we again denote by T. Now we have $T \hookrightarrow I \hookrightarrow \operatorname{Aut}_{\mathbb{Q}}\mathcal{A}_x$, and the induced action of T on $\mathbb{D}(\mathscr{G}_{\tilde{x}})_K$ respects filtrations. Hence the action of T on $\mathcal{A}_x$ (in the isogeny category) lifts to $\mathcal{A}_{\tilde{x}}$.

Since T fixes the tensors $s_{\alpha,0,x}$, it fixes $s_{\alpha,p,\tilde{x}}$ and hence $s_{\alpha,B,\tilde{x}}$. Thus T is naturally a subgroup of G, and hence a maximal torus in G. The Mumford-Tate group of $\mathcal{A}_{\tilde{x}}$, is a subgroup of G which commutes with T, and hence a subgroup of T. Hence $\tilde{x} \in \mathrm{Sh}_{\mathsf{K}}(G,X)$ is a special point. □

(2.2.4) In fact we have proved the following more precise statement

Corollary (2.2.5). *For any maximal torus $T \subset I$, and any $\mu_T \in X_*(T)$ which is conjugate to μ in G, and satisfies $\bar{\mu}_T^T = \nu_\delta^{-1}$, there exists a point $x' \in \iota_x(X_v(\delta) \times G(\mathbb{A}_f^p))$ and a lifting of x' to a special point $\tilde{x}' \in \mathscr{S}_{\mathsf{K}}(G,X)(K')$, with $K' \subset \bar{\mathbb{Q}}_p$ a finite extension of K_0, such that the filtration on $\mathbb{D}(\mathscr{G}_{x'})_{K'} \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_x)_{K'}$, corresponding to $\tilde{x}'$, is given by μ_T.*

In particular, the inclusion $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{x'}$ lifts to $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}'}$, and the Mumford-Tate group of $A_{\tilde{x}'}$ is contained in T.

(2.3) Frobenii and centralizers: We will use (2.2.3) to deduce some consequence about the conjugacy class of Frobenius and its centralizer in G. We continue to use the notation introduced above.

Corollary (2.3.1). *There exists an element $\gamma_0 \in G(\mathbb{Q})$ such that*

1. *For $l \neq p$, γ_0 is conjugate to $\gamma_l \in G(\mathbb{Q}_l)$.*
2. *γ_0 is stably conjugate to γ_p in $G(\bar{\mathbb{Q}}_p)$.*
3. *The image of γ_0 in $G(\mathbb{R})$ is elliptic.*

Proof. Let $x' \in \iota_x(X_v(\delta) \times G(\mathbb{A}_f^p))$ and $\tilde{x}'$ be as in (2.2.5). Let $k' \supset k$ and $K' \supset K_0$ be fields of definition for x' and $\tilde{x}'$ respectively. Let $\mathcal{A}_{\bar{x}} \to \mathcal{A}_{x'}$ be the canonical quasi-isogeny, defined over k'. We may regard the q-Frobenius

$$\gamma \in T(\mathbb{Q}) \subset I(\mathbb{Q}) \subset \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_x$$

as an element of $\mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{x'}$.

By (2.2.5) γ lifts to an element of $\tilde{\gamma} \in \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}'}$. Fix an embedding $K' \hookrightarrow \mathbb{C}$. The action of $\tilde{\gamma}$ on the Betti cohomology $H^1_B(\mathcal{A}_{\tilde{x}'}(\mathbb{C}),\mathbb{Q})$ produces an element $\gamma_0 \in G(\mathbb{Q})$, which is conjugate to γ_l by the isomorphism between Betti and l-adic cohomology. Similarly, γ_0 and γ_p are conjugate in $G(\mathbb{C})$, by the isomorphisms between crystalline, de Rham and Betti cohomology. Hence γ_0 and γ_p are conjugate over $\bar{K}$.

Finally, as in the proof of (2.1.7), $T(\mathbb{R})/w_h(\mathbb{R}^\times)$ is compact, so $\gamma_0 \in T(\mathbb{Q})$ is elliptic. □

Corollary (2.3.2). *For every finite prime (including $l = p$) the natural maps*

$$I_{/k,\mathbb{Q}_l} = I_{/k} \otimes_{\mathbb{Q}} \mathbb{Q}_l \to I_{l/k} \tag{2.3.3}$$

and

$$I_{\mathbb{Q}_l} = I \otimes_{\mathbb{Q}} \mathbb{Q}_l \to I_l \tag{2.3.4}$$

are isomorphisms.

Proof. We prove the second claim first. For $l \neq p$, I_l is a Levi subgroup of $G_{\mathbb{Q}_l}$ and hence connected. Similarly $I_p \otimes_{\mathbb{Q}_p} K_0$ is connected, and hence so is I_p. By (2.3.1), the subgroups $I_l \subset G_{\mathbb{Q}_l}$ and I_p all have dimension equal to that of the centralizer of $\gamma_0^n \in G(\mathbb{Q})$ for n sufficiently large. By (2.1.7), this dimension must be equal to that of I.

Now (2.3.3) follows by taking the centralizer of γ_l on both sides of (2.3.4). This is clear for $l \neq p$, and for $l = p$, it follows from the remark in (2.1.2) that $I_{p/k} \otimes_{\mathbb{Q}_p} K_0$ is the centralizer of γ_p in G_{K_0}, and hence in $I_p \otimes_{\mathbb{Q}_p} K_0$. □

Corollary (2.3.5). *Let $\gamma_0 \in G(\mathbb{Q})$ be as in (2.3.1), and let $I_0 \subset G$ be the centralizer of γ_0. Then there exists an isomorphism $I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ which makes I an inner form of I_0, and such that for any prime l the diagram*

$$\begin{array}{ccc} I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l & \xrightarrow{\sim} & I_l \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l \\ \| & & \uparrow\sim \\ I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l & \xrightarrow{\sim} & I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \end{array}$$

commutes up to inner automorphisms.

Proof. Choose T and x' as in (2.3.1). It suffices to prove the Corollary for $\gamma_0 \in T(\mathbb{Q})$ the q-Frobenius in $\mathrm{Aut}\mathcal{A}_{x'}$. More precisely, we may assume that $\tilde{x}'$ is the image of a point $(h_T, 1) \in X \times G(\mathbb{A}_f)$, where $h_T : \mathbb{S} \to T$. This gives rise to an identification of $H_1(\mathcal{A}_{\tilde{x}'}(\mathbb{C}), \mathbb{Q})$ with $V_{\mathbb{Q}}$ and of $V_{\mathbb{A}_f^p}$ with $\widehat{V}^p(\mathcal{A}_{\tilde{x}'})_{\mathbb{Q}}$, so that T acting on $H_1(\mathcal{A}_{\tilde{x}'}, \mathbb{Q})$ is naturally a subgroup of $I_0 \subset G$. We also choose an isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}} \xrightarrow{\sim} V^*_{\mathbb{Q}_p^{\mathrm{ur}}}$ which takes $s_{\alpha,0,x}$ to s_α, and is compatible with the action of T. This is possible, for example by Steinberg's theorem.

For each l (including $l = p$), the composite isomorphism

$$I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \xrightarrow{\sim} I_l \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l, \tag{2.3.6}$$

maps T to T via the identity. Here the first isomorphism is induced by the identification $V_{\mathbb{A}_f^p}$ with $\widehat{V}^p(\mathcal{A}_{\tilde{x}'})_{\mathbb{Q}}$ for $l \neq p$ and by our chosen isomorphism $\mathbb{D}(\mathscr{G}_x) \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}} \xrightarrow{\sim} V^*_{\mathbb{Q}_p^{\mathrm{ur}}}$ for $l = p$.

Now for two finite primes l, l' choose an isomorphism $\bar{\mathbb{Q}}_l \xrightarrow{\sim} \bar{\mathbb{Q}}_{l'}$. Composing (2.3.6) for l with its inverse for l' gives an automorphism of $I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l$ which fixes T. Such an automorphism fixes the roots of T acting on $\mathrm{Lie}\, I_0$. Hence it acts trivially on the Dynkin diagram of I_0, and is inner. This shows that the isomorphisms (2.3.6) for different l are in the same inner class.

We choose an isomorphism $I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ in this inner class. The argument above shows that this isomorphism may be chosen so that it is the identity on T. Then the Galois action on $I_0(\bar{\mathbb{Q}})$ and $I(\bar{\mathbb{Q}})$ differs by a cocycle $(c_\sigma) \in Z^1(\mathbb{Q}, \mathrm{Aut}(I)(\bar{\mathbb{Q}}))$, with c_σ acting trivially on T. As above, this implies that c_σ is inner, given by conjugation by an element of $T(\bar{\mathbb{Q}})$. □

§3 The Langlands-Rapoport conjecture

(3.1) The quasi-motivic Galois gerb: Our aim in this subsection is to recall the definition and properties of the quasi-motivic Galois gerb introduced[11] in [Re, App. B] (following partly [Mi 2], cf. also [Ra, §8]) generalizing Langlands-Rapoport [LR].

(3.1.1) Let k be a field of characteristic 0, $\bar{k}$ an algebraic closure, and $k \subset k' \subset \bar{k}$ a Galois extension of k. A k'/k-Galois gerb is the data of a linear algebraic group G over k', together with an extension of topological groups

$$1 \to G(k') \to \mathfrak{G} \xrightarrow{q} \mathrm{Gal}(k'/k) \to 1$$

[11] In fact Reimann and some other authors prefer to work with the more geometric notion of a groupoid. This is partly a matter of taste. We prefer the original formulation of Langlands-Rapoport in terms of Galois gerbs, since it is more transparently related to the calculations in Galois cohomology which lie at the heart of the subject.

where $G(k')$ has the discrete topology, such that

(1) For any $\tau \in \mathrm{Gal}(k'/k)$ and $g_\tau \in \mathfrak{G}$ a lift of τ, conjugation by g_τ is induced by an algebraic, τ-linear automorphism g_τ^{alg} of G. (That is an isomorphism $\tau^*G \to G$ over k'.)

(2) There is a finite extension K/k in k', and a continuous group theoretic section

$$\mathrm{Gal}(k'/K) \to \mathfrak{G}; \quad \tau \mapsto g_\tau$$

such that conjugation by g_τ^{alg} induces a K-structure on G, and the map

$$q^{-1}(\mathrm{Gal}(k'/K)) \to G(k') \rtimes \mathrm{Gal}(k'/K); \quad g \mapsto (gg_{q(g)}^{-1}, q(g))$$

is a topological isomorphism.

If we pull back the product of the Zariski and profinite topologies on $G(k') \rtimes \mathrm{Gal}(k'/K)$ via the map in (2), we get a topology on the finite index subgroup $q^{-1}(\mathrm{Gal}(k'/K)) \subset \mathfrak{G}$, and hence a topology on $\mathfrak{G}$. This topology does not depend on the choice of section g_τ. We refer to it as the Zariski topology on $\mathfrak{G}$.

We call G the kernel of $\mathfrak{G}$ and write $G = \mathfrak{G}^\Delta$. A morphism of k'/k-Galois gerbs is a continuous morphism $f : \mathfrak{G} \to \mathfrak{G}'$ inducing the identity on $\mathrm{Gal}(k'/k)$, together with a map of algebraic groups $f^{\mathrm{alg}} : G \to G'$ such that f and f^{alg} define the same map $G(k') \to G'(k')$.

We say that two morphisms $f_1, f_2 : \mathfrak{G} \to \mathfrak{G}'$ are *conjugate*[12] if they differ by conjugation by an element of $G'(k')$. The set of such elements is naturally the k-points of a k-scheme $\underline{\mathrm{Isom}}(f_1, f_2)$. If R is a k-algebra, and $\mathfrak{G}_R$ denotes the pushout of $\mathfrak{G}$ by $G(k') \to G(k' \otimes_k R)$, then we may regard f_1, f_2 as maps $\mathfrak{G}_R \to \mathfrak{G}'_R$, and

$$\underline{\mathrm{Isom}}(f_1, f_2)(R) = \{g \in \mathfrak{G}'^\Delta(k' \otimes_k R); \mathrm{Int}(g) \circ f_1 = f_2\},$$

In particular, if $f_2 = f_1$, this scheme is a k-group which we denote $I_{f_1} = \mathrm{Aut}(f_1)$.

We will often work with projective limits of k'/k-Galois gerbs. Two morphisms f_1, f_2 between such projective limits are said to be conjugate if they are the projective limits of sequences of morphisms $\{f_{1,i}\}_{i \geq 1}, \{f_{2,i}\}_{i \geq 1}$ with $f_{1,i}$ conjugate to $f_{2,i}$.

We call a $\bar{k}/k$-Galois gerb, simply a Galois gerb over k. In this case we sometimes abuse notation and write $\mathfrak{G}^\Delta$ for $G(\bar{k})$. A k'/k-Galois gerb gives rise to a Galois gerb by pulling back by $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(k'/k)$, and pushing out by $G(k') \to G(\bar{k})$. Conversely, any Galois gerb arises from a k'/k-Galois gerb for some finite Galois extension k'/k contained in k.

If G is a linear algebraic group defined over k, then the semi-direct product $\mathfrak{G}_G := G(k') \rtimes \mathrm{Gal}(k'/k)$ is called the *neutral k'/k-gerb* attached to G..

Lemma (3.1.2). *Let G be a linear algebraic group over k and $f : \mathfrak{G}' \to \mathfrak{G}_G$ a map of k'/k-Galois gerbs. Then*

(1) *$I_{f,k'}$ may be naturally identified with the centralizer of $f^{\mathrm{alg}}(\mathfrak{G}'^\Delta)$ in $G_{k'}$. For a k'-algebra R, the identification on R-points is given by*

$$I_{f,k'}(R) \hookrightarrow G(k' \otimes_k R) \to G(R),$$

[12] This is what is usually called *equivalent* in the literature. We avoid this terminology since we will have occasion to consider morphisms which are conjugate without identifying them.

the final map being the natural one.

(2) *The set of maps* $f' : \mathfrak{G}' \to \mathfrak{G}_G$ *with* $f'^{\mathrm{alg}} = f^{\mathrm{alg}}$ *is in bijection with the set of continuous cocycles* $Z^1(\mathrm{Gal}(k'/k), I_f(k'))$*. Such a map* f' *is conjugate to* f *if and only if the class of the corresponding cocycle in* $H^1(\mathrm{Gal}(k'/k), I_f(k'))$ *is trivial.*

Proof. This is standard (for example cf. [Mi 2, A.18], but we give the construction as we will make repeated use of it later.

Write $G' = \mathfrak{G}'^{\Delta}$. Let $I_0 \subset G_{k'}$ denote the centralized of $f^{\mathrm{alg}}(G')$, and fix a continuous section $\tau \mapsto \varrho_\tau$ of $\mathfrak{G}' \to \mathrm{Gal}(k'/k)$. For R a k-algebra, and $i \in G(k' \otimes_k R)$ we have $i \in I_f(R)$ if and only if $i \in I_0(k' \otimes_k R)$ and $f(\varrho_\tau) i f(\varrho_\tau)^{-1} = i$ for $\tau \in \mathrm{Gal}(k'/k)$.

Now if $i \in I_0(k' \otimes_k R)$ and $g' \in G'(k' \otimes_k R)$ we have

$$f(g')f(\varrho_\tau)if(\varrho_\tau)^{-1}f(g')^{-1} \\ = f(\varrho_\tau)f(\varrho_\tau^{-1}g'\varrho_\tau)if(\varrho_\tau^{-1}g'\varrho_\tau)^{-1}f(\varrho_\tau)^{-1} = f(\varrho_\tau)if(\varrho_\tau)^{-1}.$$

Hence conjugation by $f(\varrho_\tau)$ induces an algebraic τ-semilinear (by (3.1.1)(1)) automorphism of I_0. For $\sigma, \tau \in \mathrm{Gal}(k'/k)$, we have $\varrho_{\sigma\tau}\varrho_\tau^{-1}\varrho_\sigma^{-1} =: g'_{\sigma,\tau} \in G'(k')$. Since $f(g'_{\sigma,\tau})$ commutes with I_0, conjugation by $f(\varrho_\tau)$ defines a k-structure on I_0, and one sees from the definition that I_f is the resulting k-group.

If we now identify $I_{f,k'}$ with $I_0 \subset G_{k'}$, then the map in (1) is the composite

$$I_f(R) \to I_f(k' \otimes_k R) \to I_f(R),$$

which is obviously the identity.

To show (2), suppose that $f'^{\mathrm{alg}} = f^{\mathrm{alg}}$, and write $f'(\varrho_\tau) = g_\tau f(\varrho_\tau)$ for $\tau \in \mathrm{Gal}(k'/k)$ and $g_\tau \in G(k')$. We claim that $g_\tau \in I_0(k')$. To check this we may replace $\mathfrak{G}'$ and $\mathfrak{G}$ by the corresponding $\bar{k}/k$-gerbs, and assume that $k' = \bar{k}$. Since $f'^{\mathrm{alg}} = f^{\mathrm{alg}}$, for $g' \in G'(\bar{k})$, we have

$$f(\varrho_\tau g' \varrho_\tau^{-1}) = f'(\varrho_\tau g' \varrho_\tau^{-1}) = g_\tau f(\varrho_\tau g' \varrho_\tau^{-1}) g_\tau^{-1}$$

so that $g_\tau \in I_0(\bar{k})$.

Now if $h_\sigma, h_\tau \in G'(k')$ then

$$g_\sigma f(\varrho_\sigma h_\sigma) g_\tau f(\varrho_\tau h_\tau) = [g_\sigma f(\varrho_\sigma h_\sigma) g_\tau f(\varrho_\sigma h_\sigma)^{-1}] f(\varrho_\sigma h_\sigma \varrho_\tau h_\tau).$$

As any element of $\mathfrak{G}'$ can be uniquely written as $\varrho_\tau h_\tau$ with $h_\tau \in G'(k')$, and some τ, the map $\mathfrak{G}' \to \mathfrak{G}_G$ given by $\varrho_\tau h_\tau \mapsto g_\tau f(\varrho_\tau h_\tau)$, is a homomorphism if and only if g_τ is a $I_f(k')$-valued cocycle.

If f' is conjugate to f, then since $f'^{\mathrm{alg}} = f^{\mathrm{alg}}$, it must be conjugate by an element $i \in I_0(k')$ such that

$$f'(\varrho_\tau) = if(\varrho_\tau)i^{-1} = [if(\varrho_\tau)i^{-1}f(\varrho_\tau)^{-1}]f(\varrho_\tau).$$

Hence g_τ is a coboundary corresponding to i^{-1}. Conversely, if the cohomology class of g_τ is trivial, then writing it as a coboundary one sees that f and f' are conjugate. □

(3.1.3) Recall that we have fixed embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$ in (1.3), and that for $L \subset \bar{\mathbb{Q}}$ a finite extension of $\mathbb{Q}$, we denote by L_v the closure of L in $\bar{\mathbb{Q}}_v$. We set $L(v) = L \cap \mathbb{Q}_v$.

For k'/k as above, and G a linear algebraic group over k', we will write $R_{k'/k}G$ for the restriction of scalars $\mathrm{Res}_{k'/k}G$.

For $L \subset \bar{\mathbb{Q}}$ a finite Galois extension of $\mathbb{Q}$ set

$$Q^L = (R_{L(\infty)/\mathbb{Q}}\mathbb{G}_m \times R_{L(p)/\mathbb{Q}}\mathbb{G}_m)/\mathbb{G}_m$$

where the quotient is by the diagonal embedding of $\mathbb{G}_m$. Let

$$\psi^L : Q^L \to R_{L/\mathbb{Q}}\mathbb{G}_m$$

be the map[13] induced by the natural inclusion $R_{L(p)/\mathbb{Q}}\mathbb{G}_m \to R_{L/\mathbb{Q}}\mathbb{G}_m$ and the inverse of the natural inclusion $R_{L(\infty)/\mathbb{Q}}\mathbb{G}_m \to R_{L/\mathbb{Q}}\mathbb{G}_m$.

For $v = p, \infty$ we denote by $\nu(v)^L$ the cocharacter of Q^L, defined over $\mathbb{Q}_v$, obtained by composing the cocharacter of $R_{L(v)/\mathbb{Q}}\mathbb{G}_m$ corresponding to the inclusion $L(v) \hookrightarrow \mathbb{Q}_v$, and the natural map $R_{L(v)/\mathbb{Q}}\mathbb{G}_m \to Q^L$.

We have [Re, B2.2]

Lemma (3.1.4). *The triple $(Q^L, \nu(\infty)^L, \nu(p)^L)$ is an initial object in the category of triples (T, ν_∞, ν_p) consisting of a $\mathbb{Q}$-torus T which splits over L, and cocharacters ν_∞, ν_p defined over $\mathbb{R}$ and $\mathbb{Q}_p$ respectively, and satisfying*

$$\sum_{v \in \{\infty, p\}} [L_v : \mathbb{Q}_v]^{-1} \mathrm{tr}_{L/\mathbb{Q}}(\nu_v) = 0.$$

(3.1.5) If $L'/\mathbb{Q}$ is a Galois extension containing L, there is a map $\omega'_{L'/L} : Q^{L'} \to Q^L$ induced by the maps

$$N_{L'(v)/L(v)}^{[L'_v : L_v]} : R_{L'(v)/\mathbb{Q}}\mathbb{G}_m \to R_{L(v)/\mathbb{Q}}\mathbb{G}_m.$$

for $v = p, \infty$ [Re, p117]. We denote by Q the pro-torus $\varprojlim_L Q^L$ where L runs over Galois extensions $L/\mathbb{Q}$ contained in $\bar{\mathbb{Q}}$, and the transition maps are given by $\omega_{L'/L}$. The maps ψ^L induce a map of pro-tori

$$\psi^\Delta : Q \to R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m := \varprojlim_L R_{L/\mathbb{Q}}\mathbb{G}_m,$$

where the transition maps in the inverse limit on the right are given by the norm.

We have for $v = \infty, p$

$$(\omega_{L'/L})_*(\nu(v)^{L'}) = [L'_v : L_v]\nu(v)^L \in X_*(Q^L).$$

Hence we have a cocharacter $\nu(\infty) : \mathbb{G}_{m,\mathbb{R}} \to Q_\mathbb{R}$, induced by the cocharacters $\nu(\infty)^L$ with $L_\infty = \mathbb{C}$.

[13]This agrees with Reimann's convention, and is the inverse of the one in [LR].

Let $\mathbb{D}$ be the pro-torus over $\mathbb{Q}_p$ with cocharacter group $\mathbb{Q}$. So $\mathbb{D} = \varprojlim D_n$ where $D_n = \mathbb{G}_{m,\mathbb{Q}_p}$ the inverse limit runs over non-negative integers ordered multiplicatively, and the transition maps are given by $x \mapsto x^{n'/n}$ for $n|n'$. If $n = [L_p : \mathbb{Q}_p]$, then the maps $\nu(p)^L : D_n \to Q^L$ induce a map of pro-tori $\nu(p) : \mathbb{D} \to Q_{\mathbb{Q}_p}$.

(3.1.6) We now define a Galois gerb $\mathfrak{G}_v$ over $\mathbb{Q}_v$ for each place v of $\mathbb{Q}$. For $v \neq p, \infty$, we set $\mathfrak{G}_v$ equal to the trivial Galois gerb $\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$. We set $\mathfrak{G}_\infty$ equal to the extension

$$1 \to \mathbb{C}^\times \to \mathfrak{G}_\infty \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

corresponding to the fundamental class in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$.

Finally, for $v = p$, we have for each Galois extension $L_p/\mathbb{Q}_p$ in $\bar{\mathbb{Q}}_p$ the $L_p/\mathbb{Q}_p$-Galois gerb

$$1 \to L_p^\times \to \mathfrak{G}_p^{L_p} \to \mathrm{Gal}(L_p/\mathbb{Q}_p) \to 1$$

corresponding to the fundamental class in $H^2(\mathrm{Gal}(L_p/\mathbb{Q}_p), L_p^\times)$. Let $\tilde{\mathfrak{G}}_p^{L_p}$ denote the Galois gerb over $\mathbb{Q}_p$ induced by $\mathfrak{G}_p^{L_p}$. The properties of the fundamental class imply that that for $L_p' \subset \bar{\mathbb{Q}}_p$ a Galois extension over $\mathbb{Q}_p$ containing L_p, there is a map of Galois gerbs $\tilde{\mathfrak{G}}_p^{L_p'} \to \tilde{\mathfrak{G}}_p^{L_p}$ with the induced map

$$\mathbb{G}_m = \mathfrak{G}_p^{L_p',\Delta} \to \mathfrak{G}_p^{L_p,\Delta} = \mathbb{G}_m$$

equal to $[L_p' : L_p]$.

Passing to the limit over L_p, we obtain a pro-Galois gerb $\mathfrak{G}_p$ over $\mathbb{Q}_p$ with $\mathfrak{G}_p^\Delta = \mathbb{D}$.

Let $\mathbb{Q}_p^{\mathrm{ur}} \subset \bar{\mathbb{Q}}_p$ denote the maximal unramified extension of $\mathbb{Q}_p$. We could have also made the above construction using only unramified extensions $L_p/\mathbb{Q}_p$. We denote by $\mathfrak{D}_n$ the $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ Galois gerb induced by $\mathfrak{G}_p^{\mathbb{Q}_{p^n}}$. Passing to the limit, as before, this yields a $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ pro-Galois gerb $\mathfrak{D}$, with $\mathfrak{D}^\Delta = \mathbb{D}$, whose induced Galois gerb over $\mathbb{Q}_p$ is canonically isomorphic to $\mathfrak{G}_p$.

(3.1.7) For a Galois gerb $\mathfrak{G}$ over $\mathbb{Q}$, and v a place of $\mathbb{Q}$ we denote by $\mathfrak{G}(v)$ the induced Galois gerb over $\mathbb{Q}_v$, obtained by pullback by $\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and pushout by $\mathfrak{G}^\Delta(\bar{\mathbb{Q}}) \to \mathfrak{G}^\Delta(\bar{\mathbb{Q}}_v)$. We use analogous notation for maps of Galois gerbs.

We have the following definition [Re, B2.2]: A quasi-motivic Galois gerb is the data of a pro-Galois gerb $\mathfrak{Q}$ over $\mathbb{Q}$ together with morphisms $\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$ for every place v of $\mathbb{Q}$ such that

(1) $(\mathfrak{Q}^\Delta, \zeta_\infty^\Delta, \zeta_p^\Delta) = (Q, \nu(\infty), \nu(p))$, the identification $\mathfrak{Q}^\Delta = Q(\bar{\mathbb{Q}})$ being compatible[14] with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

(2) For every Galois extension $L/\mathbb{Q}$ in $\bar{\mathbb{Q}}$, let $\mathfrak{Q}^L$ be the Galois gerb obtained from $\mathfrak{Q}$ via pushout by $Q \to Q^L$. For some positive integer n, the torus Q^L arises from a torus over $\mathcal{O}_{\bar{\mathbb{Q}}}[1/n]$ which we denote by the same symbol. After replacing n by a larger integer, we may assume that the continuous

[14]This compatibility seems to have been omitted in [Re] and elsewhere in the literature. It is, however clearly satisfied by Reimann's construction, and it is used in the proof of uniqueness (see (3.1.9) below), which uses the fact that the a quasi-motivic Galois gerb corresponds to an element of $H^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), Q)$.

2-cocycle of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with values in $Q^L(\bar{\mathbb{Q}})$ which defines $\mathfrak{Q}^L$ takes values in $\mathcal{O}_{\bar{\mathbb{Q}}}[1/n]$, and hence defines an extension

$$1 \to Q^L(\mathcal{O}_{\bar{\mathbb{Q}}}[1/n]) \to \mathfrak{Q}^L(\mathcal{O}_{\bar{\mathbb{Q}}}[1/n]) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

Hence for any prime $l \nmid n$ we obtain an extension

$$1 \to Q^L(\mathcal{O}_{\bar{\mathbb{Q}}_l}) \to \mathfrak{Q}^L(\mathcal{O}_{\bar{\mathbb{Q}}_l}) \to \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l) \to 1. \tag{3.1.8}$$

which induces $\mathfrak{Q}^L(l)$ by pushout. We require that for almost all l ζ_l induces a section of (3.1.8).

We have the following [Re, B 2.8]

Theorem (3.1.9). *A quasi-motivic Galois gerb $(\mathfrak{Q}, \zeta_v)$ exists. If $(\tilde{\mathfrak{Q}}, \tilde{\zeta}_v)$ is another quasi-motivic Galois gerb then there is an isomorphism $\alpha : \mathfrak{Q} \to \tilde{\mathfrak{Q}}$ such that ζ_v is conjugate to $\alpha \circ \zeta_v$ for all v. Such an isomorphism α is unique up to conjugacy.*

Moreover, there is a morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}$ whose kernel is the map ψ^Δ defined in (3.1.5), and which is unique up to conjugacy.

(3.1.10) From now on we fix a choice of quasi-motivic Galois gerb $\mathfrak{Q}$.

Consider a torus T over $\mathbb{Q}$ equipped with a cocharacter μ defined over some Galois extension $L/\mathbb{Q}$ contained in $\bar{\mathbb{Q}}$. The characters of $R_{L/\mathbb{Q}}\mathbb{G}_m$ may be identified with functions $\mathrm{Gal}(L/\mathbb{Q}) \to \mathbb{Z}$, and $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ with the group ring $\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$. We have a map $R_{L/\mathbb{Q}}\mathbb{G}_m \to T$ corresponding to the map on cocharacters given by

$$X_*(R_{L/\mathbb{Q}}\mathbb{G}_m) \to X_*(T); \quad \sum a_\tau \tau \mapsto \sum a_\tau \tau(\mu).$$

This gives rise to a map of Galois gerbs

$$\psi_\mu : \mathfrak{Q} \xrightarrow{\psi} \mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_T.$$

Write $\nu(p)_n$ (resp. $\nu(p)^L_n$) for $\nu(p)$ (resp. $\nu(p)^L$) viewed as a cocharacter with source $D_n = \mathbb{G}_m$. If $n = [L_p : \mathbb{Q}_p]$, then we have

$$\psi^\Delta_\mu \circ \nu(p)_n = \sum_{\tau \in \mathrm{Gal}(L_p/\mathbb{Q}_p)} \tau(\mu).$$

Hence if we think of $\nu(p)$ as a fractional cocharacter with source $D_1 = \mathbb{G}_m$, then

$$\psi^\Delta_\mu \circ \nu(p) = [L_p : \mathbb{Q}_p]^{-1} \sum_{\tau \in \mathrm{Gal}(L_p/\mathbb{Q}_p)} \tau(\mu). \tag{3.1.11}$$

Similarly, if $L_\infty = \mathbb{C}$,

$$\psi^\Delta_\mu \circ \nu(\infty) = -\sum_{\tau \in \mathrm{Gal}(L_\infty/\mathbb{R})} \tau(\mu). \tag{3.1.12}$$

(3.2) Strictly monoidal categories: We recall some constructions involving strictly monoidal categories which will be necessary in the formulation of the Langlands-Rapoport conjecture and the proof of results about it.

(3.2.1) Recall that a *crossed module* [Mi 2, App. B], is a homomorphism of groups $\tilde{H} \to H$ together with an action of H on $\tilde{H}$ which lifts the action of H on itself by conjugation, and such that the induced action of $\tilde{H}$ on itself is by conjugation. Given this, we can form a category $H/\tilde{H}$ whose objects are the same as those of H, and for which the morphisms $\mathrm{Hom}(h_1, h_2) = \{\tilde{h} \in \tilde{H} : h_2 = \tilde{h}h_1\}$

The category $H/\tilde{H}$ is strictly monoidal: The composition

$$H/\tilde{H} \times H/\tilde{H} \to H/\tilde{H}$$

is given by composition in H on objects. If $\tilde{h} : h_1 \to h_2$ and $\tilde{h}' : h_1' \to h_2'$ are morphisms then the composition sends $(\tilde{h}, \tilde{h}')$ to $\tilde{h}\tilde{h}'^{h_1}$. If $h \in H$, then conjugation by h induces an auto-equivalence of $H/\tilde{H}$. If h is the image of an element of $\tilde{H}$, then this auto-equivalence is isomorphic to the identity.

If H is a group we may think of $H = H/\{1\}$ as a strictly monoidal category with the only morphisms being the identities.

By a morphism of strictly monoidal categories we mean a functor which strictly respects the monoidal structures. We say two such functors $\phi_1, \phi_2 : C_1 \to C_2$ are isomorphic, if there is an isomorphism of functors $\phi_1 \xrightarrow{\sim} \phi_2$ respecting monoidal structures. If $\phi_1, \phi_2 : C \to H/\tilde{H}$ are two such morphisms we say that ϕ_1, ϕ_2 are conjugate if ϕ_2 is obtained from ϕ_1 by composing with the auto-equivalence of $H/\tilde{H}$ obtained by conjugation by some $h \in H$. We say ϕ_1, ϕ_2 are *conjugate-isomorphic* if ϕ_1 is conjugate to a functor which is isomorphic to ϕ_2.

(3.2.2) Let k be a field of characteristic 0 with algebraic closure $\bar{k}$, and G a connected, reductive algebraic group over k. Let $\tilde{G}$ denote the simply connected cover of G^{der}. Then the commutator map $G \times G \to G$ factors via $G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G$. Applying this with $\tilde{G}$, one obtains [De 2, 2.0.2] a map $G \times G \to \tilde{G}$. In particular, the action of $\tilde{G}$ on itself by conjugation extends to an action of G. Hence we may apply the discussion above and consider the strictly monoidal categories $G(\bar{k})/\tilde{G}(\bar{k})$ and $\mathfrak{G}_G/\tilde{G}(\bar{k})$. We will write $G/\tilde{G}(\bar{k})$ for the former category, and $\mathfrak{G}_{G/\tilde{G}}$ for the latter one.

Let $X_*(G/\tilde{G})$ denote the set of isomorphism class of morphisms $\mathbb{G}_m(\bar{k}) \to G/\tilde{G}(\bar{k})$, which are induced by an algebraic morphism $\mathbb{G}_m \to G$ over $\bar{k}$. The category $G/\tilde{G}(\bar{k})$ may also be thought of as the $\bar{k}$-points of the Picard stack $G/\tilde{G}$. In Lemma (3.2.4), we show that $X_*(G/\tilde{G})$, as defined above is in bijection with the set of morphisms of Picard stacks $\mathbb{G}_m \to G/\tilde{G}$. Although we make no further use of this point of view it may be helpful in motivating some of the arguments.

Lemma (3.2.3). *Let $\phi_1, \phi_2 : C \to G/\tilde{G}(\bar{k})$ be morphisms from a strictly monoidal category C. Then ϕ_1, ϕ_2 are conjugate-isomorphic if and only if they are isomorphic*

Proof. This may be seen either by using the fact that the commutator map factors through $\tilde{G}$, as in (3.2.2), or by writing an element $g \in G(\bar{k})$ as a product $z \cdot \tilde{g}$ with $z \in Z_G(\bar{k})$ and $\tilde{g} \in \tilde{G}(\bar{k})$. ☐

Lemma (3.2.4). *The map which assigns to a G-valued cocharacter its class in $\pi_1(G)$ induces an isomorphism*

$$X_*(G/\tilde{G}) \xrightarrow{\sim} \pi_1(G).$$

Moreover $X_*(G/\tilde{G})$ is in bijection with the set of (equivalence classes of) morphisms of Picard stacks $\mathbb{G}_m \to G/\tilde{G}$ over $\bar{k}$.

Proof. The map is clearly surjective. We have to show that two cocharacter $\mu_1, \mu_2 : \mathbb{G}_m \to G$ have the same image in $\pi_1(G)$ if and only if they induce isomorphic functors $\mathbb{G}_m(\bar{k}) \to G/\tilde{G}(\bar{k})$.

Let $T \subset G$ be a maximal torus. By (3.2.3), we may replace μ_1, μ_2 by conjugate cocharacters and assume that $\mu_1, \mu_2 \in X_*(T)$. Let $\tilde{T}$ be the preimage of T in $\tilde{G}$. Then μ_1 and μ_2 are isomorphic if and only if $\mu_1\mu_2^{-1} \in X_*(\tilde{T})$, which is equivalent to asking that the image of $\mu_1\mu_2^{-1}$ in $\pi_1(G)$ is trivial.

To see the final claim, we have to show that any morphism $\phi : \mathbb{G}_m \to G/\tilde{G}$ lifts to G. Pulling back $G \to G/\tilde{G}$ by ϕ gives rise to a group scheme $G' \subset G$ over $\bar{k}$, which is an extension of $\mathbb{G}_m$ by $\tilde{G}$. In particular, G' is a connected, reductive group over $\bar{k}$. Now for any maximal torus $T \subset G'$, the map $T \to G' \to \mathbb{G}_m$ admits a splitting. In particular ϕ lifts to G. □

(3.2.5) We recall the construction of the fibre product for categories.

If C_1, C_2, C are categories equipped with functors $F_1 : C_1 \to C$ and $F_2 : C_2 \to C$, then the 2-fibre product $C_1 \times_C C_2$ is the category whose objects are triples (c_1, c_2, α) where c_1 and c_2 are objects of C_1 and C_2 respectively, and α is an isomorphism $\alpha : F_1(c_1) \xrightarrow{\sim} F_2(c_2)$. A morphism $(c_1, c_2, \alpha) \to (c_1', c_2', \alpha')$ consists of morphisms $\delta_i : c_i \to c_i'$ for $i = 1, 2$ such that $\alpha' \circ F_1(\delta_1) = F_2(\delta_2) \circ \alpha$.

One checks immediately, that for any category D to give a functor $G : D \to C_1 \times_C C_2$ is equivalent to giving functors $G_i : D \to C_i$ for $i = 1, 2$ and an isomorphism $F_1 \circ G_1 \xrightarrow{\sim} F_2 \circ G_2$.

If C_1, C_2, C and D are strictly monoidal, and F_1, F_2 are morphisms of strictly monoidal categories then G is strictly monoidal if and only if G_1, G_2 are strictly monoidal, and $F_1 \circ G_1 \xrightarrow{\sim} F_2 \circ G_2$ is an isomorphism of monoidal functors.

Lemma (3.2.6). *The morphisms*

$$G(\bar{\mathbb{Q}}) \to G^{\mathrm{ad}}(\bar{\mathbb{Q}}) \times_{G^{\mathrm{ad}}/\tilde{G}(\bar{\mathbb{Q}})} G/\tilde{G}(\bar{\mathbb{Q}}).$$

and

$$\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ad}}} \times_{\mathfrak{G}_{G^{\mathrm{ad}}/\tilde{G}}} \mathfrak{G}_{G/\tilde{G}}$$

are equivalences of strictly monoidal categories.

Proof. This follows easily from the definitions. □

(3.2.7) We end this subsection by recalling some facts about Galois cohomology of crossed modules [Bo, §3].

Let $\tilde{H} \to H$ be a crossed module, and Δ a profinite group. An action of Δ on $\tilde{H} \to H$ is the data of a compatible, continuous (for the discrete topology) action of Δ on $\tilde{H}, H$ such that for $\tilde{h} \in \tilde{H}$, $h \in H$ and $\tau \in \Delta$, we have $\tau(\tilde{h}^h) = \tau(\tilde{h})^{\tau(h)}$.

For $i = -1, 0, 1$ there exist pointed sets $H^i(\Delta, \tilde{H} \to H)$ having the following properties:

(1) For $i = 0, 1$, $H^i(\Delta, 1 \to H) = H^i(\Delta, H)$, the usual Δ-cohomology of H.
(2) If $(\tilde{H} \to H) \to (\tilde{H}' \to H')$ is a morphism of crossed modules which is a quasi-isomorphism (i.e it induces isomorphisms on kernels and cokernels), then there is an induced isomorphism

$$H^i(\Delta, \tilde{H} \to H) \xrightarrow{\sim} H^i(\Delta, \tilde{H}' \to H')$$

(3) There is a long exact sequence of pointed sets

$$1 \to H^{-1}(\Delta, \tilde{H} \to H) \to H^0(\Delta, \tilde{H}) \to H^0(\Delta, H)$$
$$\to H^0(\Delta, \tilde{H} \to H) \to H^1(\Delta, \tilde{H}) \to H^1(\Delta, H) \to H^1(\Delta, \tilde{H} \to H).$$

(4) If the cokernel of $\tilde{H} \to H$ is abelian, then $H^i(\Delta, \tilde{H} \to H)$ has a canonical structure of abelian group.

It follows from (1) and (2) that if $\tilde{H} \subset H$ then $H^i(\Delta, \tilde{H} \to H) = H^i(\Delta, H/\tilde{H})$ for $i = 0, 1$.

Suppose $\bar{k}/k$ is as above, and $\tilde{\mathcal{H}} \to \mathcal{H}$ is a *crossed module in algebraic groups over* k. That is, a morphism of algebraic groups over k, together with an action of $\mathcal{H}$ on $\tilde{\mathcal{H}}$ which lifts the action of $\mathcal{H}$ on itself by conjugation, and such that the induced action of $\tilde{\mathcal{H}}$ on itself is by conjugation. For $i = 0, 1$, we will write[15] $H^i(k, \tilde{\mathcal{H}} \to \mathcal{H})$ for $H^i(\mathrm{Gal}(\bar{k}/k), \tilde{\mathcal{H}}(\bar{k}) \to \mathcal{H}(\bar{k}))$. When $\tilde{\mathcal{H}} = 1$, we use the analogous notation $H^i(k, \mathcal{H})$ (resp. $Z^i(k, \mathcal{H})$) to denote the cohomology of (resp. cocycles valued in) $\mathcal{H}(\bar{k})$.

(3.3) The Langlands-Rapoport Conjecture: In this subsection we formulate the Langlands-Rapoport conjecture, including its extension to the case of non-simply-connected derived group.

(3.3.1) Let G be a reductive group over $\mathbb{Q}$, and μ a cocharacter of G. The construction of (3.1.10) produces a morphism $\psi_{\mu_{\mathrm{ab}}} : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ab}}}$. When G^{der} is not simply connected, we will need a refinement of this, which we now explain.

The composite

$$\mu_{\widetilde{\mathrm{ab}}} : \mathbb{G}_m(\bar{\mathbb{Q}}) \xrightarrow{\mu} G(\bar{\mathbb{Q}}) \to G/\tilde{G}(\bar{\mathbb{Q}}).$$

is a morphism of strictly monoidal categories. The remark regarding commutators in (3.2.2) shows that conjugate cocharacters μ give rise to isomorphic morphisms.

We can now apply the construction of (3.1.10), to obtain a morphism

(3.3.2)
$$R_{L/\mathbb{Q}}\mathbb{G}_m(\bar{\mathbb{Q}}) = X_*(R_{L/\mathbb{Q}}\mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{G}_m(\bar{\mathbb{Q}}) \to G/\tilde{G}(\bar{\mathbb{Q}}); \sum a_\tau \tau \otimes x \mapsto \sum a_\tau \mu^\tau_{\widetilde{\mathrm{ab}}}(x).$$

Finally, taking the semi-direct product with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on both sides of (3.3.2), we obtain a morphism

$$\psi_{\mu_{\widetilde{\mathrm{ab}}}} : \mathfrak{Q} \xrightarrow{\psi} \mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m} \xrightarrow{(3.3.2)} \mathfrak{G}_{G/\tilde{G}}.$$

(3.3.3) Let (G, μ) be as above, and suppose that $G_{\mathbb{Q}_p}$ has a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. We also suppose that μ is defined over a number field E in which p is unramified, and induces a cocharacter of $G_{\mathbb{Z}_p}$. We now define the set responsible for p-power isogenies in the Langlands-Rapoport conjecture.

Let $\mathbb{Q}_{p^n} \subset \bar{\mathbb{Q}}_p$ be the unramified extension of $\mathbb{Q}_p$ of degree n. If $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ denotes the Frobenius, then there is an element $d_\sigma \in \mathfrak{D}$ with image σ and such that for every $n \geq 1$, the image of d_σ^n in $\mathfrak{G}_p^{\mathbb{Q}_{p^n}}$ is $p^{-1} \in \mathfrak{G}_p^{\mathbb{Q}_{p^n}\,\Delta}$.

Consider a morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G(p)$, and let $\mathfrak{G}_G^{\mathrm{ur}}(p)$ denote the neutral $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-Galois gerb attached to G. Denote by $X_p(\theta)$ the set of $g \in G(\bar{\mathbb{Q}}_p)/G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$ such

[15]Note that $H^0(k, \tilde{\mathcal{H}} \to \mathcal{H})$ is denoted $H^1(k, \tilde{\mathcal{H}} \to \mathcal{H})$ in [De 1, 2.4.3]. This pointed set is isomorphic to the set of isomorphism classes of $\tilde{\mathcal{H}}$-torsors equipped with a trivialization of the induced $\mathcal{H}$-torsor.

that $\theta_g = \mathrm{Int}(g^{-1}) \circ \theta$ is induced by a morphism of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs $\theta_g^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}(p)$ with $\theta_g^{\mathrm{ur}}(d_\sigma) = b_g \rtimes \sigma$ and $b_g \in G(\mathbb{Z}_p^{\mathrm{ur}})p^{-\mu}G(\mathbb{Z}_p^{\mathrm{ur}})$. Here $\mathbb{Z}_p^{\mathrm{ur}}$ denotes the ring of integers of $\mathbb{Q}_p^{\mathrm{ur}}$. Since $\mathfrak{G}_p$ is induced by the $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-gerb $\mathfrak{D}$, we may regard $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$ as a subgroup of $\mathfrak{G}_p$. The condition that $\mathrm{Int}(g^{-1}) \circ \theta_g$ is induced by a morphism of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs, is equivalent to $\mathrm{Int}(g^{-1}) \circ \theta(\tau) = 1 \rtimes \tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$. To see this note that the latter condition implies that $\mathrm{Int}(g^{-1}) \circ \theta(\mathfrak{G}_p^{\Delta}(\mathbb{Q}_p^{\mathrm{ur}})) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$.

If $E_p = \mathbb{Q}_{p^r}$, we equip $X_p(\theta)$ with a p^r-Frobenius $\Phi = \Phi_r$ defined by the formula

$$\theta_{\Phi(g)}^{\mathrm{ur}} = \mathrm{Int}(1 \rtimes \sigma^r) \circ \theta_g^{\mathrm{ur}}.$$

Then $b_{\Phi(g)} = \sigma^r(b_g) \in G(\mathbb{Z}_p^{\mathrm{ur}})p^{-\mu}G(\mathbb{Z}_p^{\mathrm{ur}})$, and $\Phi(g) = gb_g\sigma(b_g)\dots\sigma^{r-1}(b_g)$.

It will be useful to compare $X_p(\theta)$ to a somewhat more explicit set used to parametrize p-power isogenies in [LR] and [Re]. By [LR, §5, p56], there exists $g_0 \in G(\bar{\mathbb{Q}}_p)$ such that $\mathrm{Int}(g_0^{-1})\circ\theta$ is induced by a map of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}(p)$. Let $\theta^{\mathrm{ur}}(d_\sigma) = b_{g_0} \rtimes \sigma$.

Lemma (3.3.4). *The map $g \mapsto g_0 g$ induces a bijection*

$$X_{-\mu}(b_{g_0}) \xrightarrow{\sim} X_p(\theta)$$

which is compatible with the action of Φ.

Proof. Let $g \in G(\bar{\mathbb{Q}}_p)$. If $g \in X_{-\mu}(b_{g_0})$ then $g_0 g \in X_p(\theta)$. Conversely, suppose that $g_0 g \in X_p(\theta)$. Then for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$ we have

$$1 \rtimes \tau = \mathrm{Int}((g_0 g)^{-1}) \circ \theta(\tau) = \mathrm{Int}(g^{-1})(1 \rtimes \tau) = g^{-1}\tau(g) \rtimes \tau.$$

Hence $g \in G(\mathbb{Q}_p^{\mathrm{ur}})$. As $b_{g_0 g} = g^{-1}b_{g_0}\sigma(g)$, we see that $g \in X_{-\mu}(b_{g_0})$.

Finally computing the action of Φ on $X_p(\theta)$ we have

$$\Phi(g_0 g) = g_0 g b_{g_0 g}\sigma(b_{g_0 g})\dots\sigma^{r-1}(b_{g_0 g}) = g_0 b_{g_0}\sigma(b_{g_0})\dots\sigma^{r-1}(b_{g_0})\sigma^r(g).$$

The final claim in the lemma follows. □

(3.3.5) Suppose that (G, X) is a Shimura datum and $h \in X$. Choose $w \in \mathfrak{G}_\infty$ whose image is complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and such that $w^2 = -1 \in \mathfrak{G}_\infty^{\Delta}(\mathbb{C}) = \mathbb{C}^\times$. Then there is a map $\xi_\infty : \mathfrak{G}_\infty \to \mathfrak{G}_G(\infty)$ given by w_h on $\mathfrak{G}_\infty^{\Delta}$ and $\xi_\infty(w) = \mu_h(-1) \rtimes c$. (cf. [Re, §B3]). We remark that since conjugation by $\mu_h(-1)$ is a Cartan involution of G, and w_h is central, $I_{\xi_\infty} = G_{\mathbb{R}}^*$, the (inner) form of $G_{\mathbb{R}}$ with compact adjoint group.

For $v \neq \infty, p$ we write $\xi_v : \mathfrak{G}_v \to \mathfrak{G}_G(v)$ for the morphisms $\tau \mapsto 1 \rtimes \tau$.

(3.3.6) We now assume that (G, X) is a Shimura datum, with reflex field $E \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ and that $G_{\mathbb{Q}_p}$ has a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. By [Ki 2, 2.3.2] $G_{\mathbb{Z}_p}$ arises from a reductive model $G_{\mathbb{Z}_{(p)}}$ of G over $\mathbb{Z}_{(p)}$. The existence of $G_{\mathbb{Z}_p}$ implies that p is unramified in E. In particular, E_p is an unramified extension $\mathbb{Q}_{p^r}$ of $\mathbb{Q}_p$. Then μ_h is conjugate to a cocharacter μ which is defined over E_p, and induces a cocharacter of $G_{\mathbb{Z}_p}$.

For any morphism ϕ of Galois gerbs with target $\mathfrak{G}_G$, we denote by $\phi_{\widetilde{\mathrm{ab}}}$, ϕ_{ab} and ϕ^{ad} the composite of ϕ with the projections $\mathfrak{G}_G \to \mathfrak{G}_{G/\tilde{G}}$, $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ab}}}$ and $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ad}}}$ respectively.

A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is called *admissible* if

(1) The composite

$$\phi_{\widetilde{\mathrm{ab}}} : \mathfrak{Q} \xrightarrow{\phi} \mathfrak{G}_G \to \mathfrak{G}_{G/\tilde{G}}$$

is conjugate-isomorphic to $\psi_{\mu_{\widetilde{\mathrm{ab}}}}$.

(2) For $v \neq p$, the composite $\phi(v) \circ \zeta_v : \mathfrak{G}_v \to \mathfrak{G}_G(v)$ is conjugate to ξ_v. We set

$$X^p(\phi) = \{(g_l)_{l \neq \infty, p} \in G(\bar{\mathbb{A}}_f^p) : \mathrm{Int}(g_l) \circ \xi_l = \phi(l) \circ \zeta_l\}$$

where $\bar{\mathbb{A}}_f^p$ denotes the restricted product $\prod'_{l \neq p} \bar{\mathbb{Q}}_l$. The condition (3.1.7)(2) implies that $X^p(\phi)$ is non-empty [Re, B3.6], and hence a $G(\mathbb{A}_f^p)$-torsor.

(3) We require that $X_p(\phi) := X_p(\phi(p) \circ \zeta_p)$ is non-empty. Let $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}(p)$ be a morphism such that the induced map of $\bar{\mathbb{Q}}_p/\mathbb{Q}_p$-Galois gerbs is conjugate to $\phi(p) \circ \zeta_p$, and set $\theta^{\mathrm{ur}}(d_\sigma) = b \rtimes \sigma$. By (3.3.4), and the result of Wintenberger [Wi 2] mentioned in (1.2.10), $X_p(\phi)$ is non-empty[16] if and only if $b \in B(G, -\mu)$.

Finally note that $I_\phi = \mathrm{Aut}(\phi)$, acts on $X(\phi) := X_p(\phi) \times X^p(\phi)$ by left multiplication, and write

$$S(\phi) = \varprojlim_{\mathsf{K}^p} I_\phi(\mathbb{Q}) \backslash X_p(\phi) \times X^p(\phi)/\mathsf{K}^p$$

where K^p runs over compact open subgroups of $G(\mathbb{A}_f^p)$. Then $S(\phi)$ is equipped with an action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, and a commuting action of the Frobenius Φ.

We have the following conjecture, which is a slight extension of the Langlands-Rapoport conjecture for hyperspecial level.

Conjecture (3.3.7). *Let (G, X) be a Shimura datum, and $\mathsf{K}_p \subset G(\mathbb{Q}_p)$ a hyperspecial subgroup. Then the Shimura variety $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ admits a smooth canonical model*[17] *$\mathscr{S}_{\mathsf{K}_p}(G, X)$ over $\mathcal{O}_{(p)}$. If $\bar{\mathbb{F}}_p$ denotes the residue field of $\bar{\mathbb{Q}}_p$, then there is a bijection*

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} S(\phi) \tag{3.3.8}$$

compatible with the action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, and the operator Φ, which acts on $\mathscr{S}_{\mathsf{K}_p}(G, X)$ as the geometric r-Frobenius. Here ϕ runs over representatives for the distinct conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$.

(3.3.9) Although, we have limited ourselves to the case of hyperspecial level structure, this is not essential. One can for example extend the case of the conjecture formulated in [Ra, §9] to the case of non-simply connected derived group by imposing (3.3.6)(1) in place of the condition a) in Def. 9.1 of *loc. cit.* (The latter is the analogue of (3.3.6)(1) but with $\mathfrak{G}_{G^{\mathrm{ab}}}$ in place of $\mathfrak{G}_{G/\tilde{G}}$.)

(3.4) Lifting admissible morphisms: In this subsection we give an alternate expression for the part of the right hand side of (3.3.8) which corresponds to admissible ϕ which induce a fixed admissible morphism $\mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$. This will eventually

[16] As mentioned above, this definition of $X_p(\phi)$ differs slightly from those given in [LR] and [Re], which define this set as $X_{-\mu}(b)$. More precisely, [LR] defines it as $X_\mu(\theta^{\mathrm{ur}})$ due to the change in the normalization of the map ψ mentioned above.

[17] That is a smooth $G(\mathbb{A}_f^p)$-equivariant extension of $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ which satisfies Milne's extension property [Mi 2, §2]

allow us to show in (3.8.12) below that the truth of (3.3.7) depends only on the adjoint Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and G^{der}.

The following lemma shows that the condition (2) in (3.3.6) is consistent with the condition (1).

Lemma (3.4.1). *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism satisfying (3.3.6)(1). Then for $v \neq p$, the morphism $(\phi(v) \circ \zeta_v)_{\widetilde{\mathrm{ab}}} = \phi_{\widetilde{\mathrm{ab}}}(v) \circ \zeta_v$ is conjugate-isomorphic to $\xi_{v,\widetilde{\mathrm{ab}}}$.*

Proof. Let $L/\mathbb{Q}$ be a finite Galois extension in $\bar{\mathbb{Q}}$, such that $\psi_{\mu_{\widetilde{\mathrm{ab}}}}$ factors through $\mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}$, and $L_\infty = \mathbb{C}$. Let

$$\mu_0 \in \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})] = X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$$

be the cocharacter corresponding to the identity, and $w_0 = \sum_{\tau \in \mathrm{Gal}(L_\infty/\mathbb{R})} \tau(\mu_0)^{-1}$. Let $\xi_{\infty,0} : \mathfrak{G}_\infty \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}$ be the morphism given by $\xi_{\infty,0}^\Delta = w_0$ and $\xi_{\infty,0}(w) = \mu_0(-1) \rtimes c$. For $v \neq p, \infty$ we write $\xi_{v,0}$ for the morphism $\tau \mapsto 1 \rtimes \tau$.

We claim that for $v \neq p$ the morphisms $\psi(v) \circ \zeta_v$ is conjugate to $\xi_{v,0}$. To see this, note first that the two morphisms have the equal kernels. This is trivial for $v \neq \infty$ and it follows from (3.1.12) for $v = \infty$.

We now apply (3.1.2) with $f = \xi_{v,0}$. The action of $\tau \in \mathrm{Gal}(k'/k)$ on the points of the group I_f of (3.1.2) is given by conjugation by $f(\varrho_\tau)$. Since $R_{L/\mathbb{Q}}\mathbb{G}_m$ is commutative, this implies we have $I_f = I_0 = R_{L/\mathbb{Q}}\mathbb{G}_m$. Since an induced torus has trivial cohomology, (3.1.2) implies that $\psi(v) \circ \zeta_v$ is conjugate to $\xi_{v,0}$.

Pushing these two morphisms forward by the map $\mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_{G/\tilde{G}}$ of (3.3.2), we find that $\psi_{\mu_{\widetilde{\mathrm{ab}}}}(v) \circ \zeta_v$ is conjugate to $\xi_{v,\widetilde{\mathrm{ab}}}$, and the lemma follows since $\psi_{\mu_{\widetilde{\mathrm{ab}}}}$ is conjugate-isomorphic to $\phi_{\widetilde{\mathrm{ab}}}$ by (3.3.6)(1). $\square$

Lemma (3.4.2). *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism of Galois gerbs. If ϕ satisfies (3.3.6)(1) and $X_p(\phi^{\mathrm{ad}})$ is non-empty, then $X_p(\phi)$ is non-empty.*

Proof. As in (3.3.3), let $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}(p)$ be a morphism of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-Galois gerbs such that the induced morphism of $\bar{\mathbb{Q}}_p/\mathbb{Q}_p$-Galois gerbs θ is conjugate to $\phi(p) \circ \zeta_p$. Let $\theta^{\mathrm{ur}}(d_\sigma) = b \rtimes \sigma$.

Fix a Borel $B \subset G_{\mathbb{Q}_p}$ and a maximal torus $T \subset B$. Let $\mu^* \in X_*(T)$ be conjugate to μ, with $-\mu^*$ dominant. As remarked in (1.2.10), $X_p(\phi) = X_{-\mu}(b)$ is non-empty if and only if

$$\kappa_G(b) = -\mu^\natural \in \pi_1(G)_\Gamma, \text{ and } \bar{\nu}_b \leqslant -\bar{\mu}^*, \tag{3.4.3}$$

where $\Gamma = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$.

Since $X_p(\phi^{\mathrm{ad}})$ is non-empty, we must have $\bar{\nu}_b + \alpha \leqslant -\bar{\mu}^*$ for some $\alpha \in X_*(Z_G)_{\mathbb{Q}}$. The definition of the element $d_\sigma \in \mathfrak{D}$ implies that $\theta^{\mathrm{ur},\Delta} = -\nu_b$. Since $\theta^{\mathrm{ur},\Delta}$ is conjugate to $\phi(p)^\Delta \circ \nu(p)$ and ϕ_{ab} is conjugate to the map induced by ψ_μ, it follows from (3.1.11) that ν_b and $-\bar{\mu}^*$ have the same image in $X_*(G^{\mathrm{ab}})$. This implies that $\alpha = 0$, and establishes the second condition in (3.4.3).

For the first condition let $L/\mathbb{Q}$ be a finite Galois extension such that ψ_μ factors through $\mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}$. Since G is unramified, μ is defined over an unramified extension of $\mathbb{Q}_p$, and we may assume that L is unramified. Let $\tilde{d}_\sigma \in \mathfrak{G}_p$ be a lift of d_σ, and write the image of $\psi(p)\circ\zeta_p(\tilde{d}_\sigma)$ in $\mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}(p)$ as $b'\rtimes\sigma$. Using [RR, 4.2(ii)] and (3.1.11) we have

$$\kappa_{R_{L/\mathbb{Q}}\mathbb{G}_m}(b') = -\psi^L(p)\circ\nu(p) = -[L_p:\mathbb{Q}_p]^{-1}\sum_{\tau\in\mathrm{Gal}(L_p/\mathbb{Q}_p)}\tau(\mu_0) \tag{3.4.4}$$

as elements of $X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)_\Gamma$ (with the same notation as in (3.4.1)).

If $T\subset G$ is a torus through which μ factors, then we have the composite

$$X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)\underset{(3.1.10)}{\to} X_*(T)\to X_*(G/\tilde{G})\underset{(3.2.4)}{\xrightarrow{\sim}}\pi_1(G).$$

Applying this map to (3.4.4) shows that $\kappa_G(b) = -\mu^\natural$ in $\pi_1(G)_\Gamma$. □

(3.4.5) For a connected reductive group H, we denote by $\tilde{H}$ the simply connected cover of H^{der}. We will need the following result of Borovoi [Bo, Thm. 5.12].

Theorem (3.4.6). *Let H be a connected reductive group over $\mathbb{Q}$. The diagram*

$$\begin{array}{ccc} H^1(\mathbb{Q},H) & \longrightarrow & H^1(\mathbb{Q},\tilde{H}\to H) \\ \downarrow & & \downarrow \\ H^1(\mathbb{R},H) & \longrightarrow & H^1(\mathbb{R},\tilde{H}\to H) \end{array}$$

is cartesian and all the maps are surjective.

(3.4.7) For any linear algebraic group H over $\mathbb{Q}$ we denote by $H(\mathbb{R})^+$ the connected component of the identity of $H(\mathbb{R})$. For any subfield $R\subset\mathbb{R}$, we write $H(R)^+ = H(R)\cap H(\mathbb{R})^+$.

Write $G(\bar{\mathbb{Q}})^\natural$ (resp. $G(\bar{\mathbb{Q}})^\natural_+$) for the preimage in $G(\bar{\mathbb{Q}})$ of $G^{\mathrm{ad}}(\mathbb{Q})$ (resp. $G^{\mathrm{ad}}(\mathbb{Q})^+$). As in (3.2.2), the commutator map on $G^\natural(\bar{\mathbb{Q}})\times G^\natural(\bar{\mathbb{Q}})$ factors through $G^{\mathrm{ad}}(\mathbb{Q})\times G^{\mathrm{ad}}(\mathbb{Q})$ and takes values in $\tilde{G}(\mathbb{Q})$. Hence $G^\natural(\bar{\mathbb{Q}})/G(\mathbb{Q})$ is an abelian group. Similarly $G(\bar{\mathbb{Q}})^\natural_+/G(\mathbb{Q})_+$ is an abelian group.

Lemma (3.4.8). *The map*

$$G(\bar{\mathbb{Q}})\to Z^1(\mathbb{Q},Z_G);\quad g\mapsto(\tau\mapsto g^{-1}\tau(g)) \tag{3.4.9}$$

identifies $G(\bar{\mathbb{Q}})^\natural_+/G(\mathbb{Q})_+$ with the set of cocycles in $Z^1(\mathbb{Q},Z_G)$ such that

$$[z]\in\ker\left(H^1(\mathbb{Q},Z_G)\to H^1(\mathbb{R},Z_G)\right)\cap\mathrm{Im}\left(H^1(\mathbb{Q},Z_{\tilde{G}})\to H^1(\mathbb{Q},Z_G)\right) \tag{3.4.10}$$

Proof. Let z be a cocycle in the image of $G(\bar{\mathbb{Q}})^\natural_+$ under (3.4.9). Since the image of $G(\mathbb{R})\to G^{\mathrm{ad}}(\mathbb{R})$ contains $G^{\mathrm{ad}}(\mathbb{R})^+$, z has trivial image in $H^1(\mathbb{R},Z_G)$. As the map $\tilde{G}(\bar{\mathbb{Q}})\to G^{\mathrm{ad}}(\bar{\mathbb{Q}})$ is surjective, $[z]$ is in the image of $H^1(\mathbb{Q},Z_{\tilde{G}})$. Hence $[z]$ satisfies (3.4.10).

Conversely, the expression in (3.4.10) is equal to

$$\ker\left(H^1(\mathbb{Q},Z_G)\to H^1(\mathbb{R},Z_G)\oplus H^1(\mathbb{Q},\tilde{G}\to G)\right)$$

by (3.2.7)(2) and (3). In particular, if $[z]$ satisfies (3.4.10), then its image in $H^1(\mathbb{Q},G)$ is trivial, by (3.4.6), and there exists $g\in G(\bar{\mathbb{Q}})$ such that $z_\tau = g^{-1}\tau(g)$. Since the image of $[z]$ in $H^1(\mathbb{R},Z_G)$ is trivial, the image of g in $G^{\mathrm{ad}}(\mathbb{Q})$ lifts to an element of $G(\mathbb{R})$. By the real approximation theory, we may modify g by an element of $G(\mathbb{Q})$, so the image of g in $G^{\mathrm{ad}}(\mathbb{Q})$ lies in $G^{\mathrm{ad}}(\mathbb{Q})^+$. Then $g\in G(\bar{\mathbb{Q}})^\natural_+$, which proves the lemma. □

Proposition (3.4.11). *Let $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ be an admissible morphism (for the Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$). The set of admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0 is naturally a $G(\bar{\mathbb{Q}})^{\natural}_{+}/G(\mathbb{Q})_{+}$-torsor, and hence non-empty.*

If $I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}$ denotes the preimage of $I_{\phi_0}(\mathbb{Q}) \subset G^{\mathrm{ad}}(\bar{\mathbb{Q}})$ in $G(\bar{\mathbb{Q}})$, and ϕ is an admissible lifting of ϕ_0, then the set of admissible liftings of ϕ_0 conjugate to ϕ is a $I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}/I_{\phi}(\mathbb{Q})$-torsor.

Proof. As ϕ_0 satisfies (3.3.6)(1) the monoidal functors

$$\phi^{\mathrm{ad}}_{0,\widetilde{\mathrm{ab}}} : \mathfrak{Q} \xrightarrow{\phi_0} \mathfrak{G}_{G^{\mathrm{ad}}} \to \mathfrak{G}_{G^{\mathrm{ad}}/\tilde{G}}$$

and

$$\psi^{\mathrm{ad}}_{\mu_{\widetilde{\mathrm{ab}}}} : \mathfrak{Q} \xrightarrow{\psi_{\mu_{\widetilde{\mathrm{ab}}}}} \mathfrak{G}_{G/\tilde{G}} \to \mathfrak{G}_{G^{\mathrm{ad}}/\tilde{G}}$$

are conjugate-isomorphic. Let $\psi'_{\mu_{\widetilde{\mathrm{ab}}}}$ be a conjugate of $\psi_{\mu_{\widetilde{\mathrm{ab}}}}$ such that $\psi'^{\mathrm{ad}}_{\mu_{\widetilde{\mathrm{ab}}}}$ is isomorphic to $\phi^{\mathrm{ad}}_{0,\widetilde{\mathrm{ab}}}$. Each such isomorphism corresponds to a monoidal functor

$$\phi : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}} \times_{\mathfrak{G}_{G^{\mathrm{ad}}/\tilde{G}}} \mathfrak{G}_{G/\tilde{G}}$$

which specializes to ϕ_0 and $\psi'_{\mu_{\widetilde{\mathrm{ab}}}}$, and hence, via (3.2.6), to a $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0 and satisfying (3.3.6)(1). Note that, by (3.4.2), any such ϕ also satisfies (3.3.6)(3).

Now fix such a lifting ϕ, and a continuous section $\varrho_\tau : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathfrak{Q}$. Note that $\phi^{\Delta} : Q \to G$ is determined by its composite with the isogeny $G \to G^{\mathrm{ad}} \times G^{\mathrm{ab}}$, and hence is independent of the choice of ϕ satisfying (3.3.6)(1). Hence, as in the proof of (3.1.2), any other such lift ϕ' has the form $\phi' = z \cdot \phi$ where $z \in Z^1(\mathbb{Q}, Z_G)$, and $z \cdot \phi$ is given by $(z \cdot \phi)(\varrho_\tau h) = z_\tau \phi(\varrho_\tau h)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $h \in Q(\bar{\mathbb{Q}})$.

Since ϕ' satisfies (3.3.6)(1), the composite

$$\phi'_{\widetilde{\mathrm{ab}}} : \mathfrak{Q} \xrightarrow{\phi'} \mathfrak{G}_G \to \mathfrak{G}_{G/\tilde{G}}$$

is isomorphic to the conjugate of $\phi_{\widetilde{\mathrm{ab}}}$ by some $g \in G(\bar{\mathbb{Q}})$. Writing $g = z' \cdot \tilde{g}$ with $z' \in Z_G(\bar{\mathbb{Q}})$ and $\tilde{g} \in \tilde{G}(\bar{\mathbb{Q}})$, one sees that $\phi'_{\widetilde{\mathrm{ab}}}$ is isomorphic to the conjugate of $\phi_{\widetilde{\mathrm{ab}}}$ by z'. Using the description of the monoidal structure on $\mathfrak{G}_{G^{\mathrm{ad}}/\tilde{G}}$ given in (3.2.1), this means that there is a cocycle $\tilde{z} \in Z^1(\mathfrak{Q}, Z_{\tilde{G}}(\bar{\mathbb{Q}}))$ such that

$$z_\tau = \tilde{z}_\tau \cdot (z' \tau(z')^{-1})$$

for $\tau \in \mathfrak{Q}$. Here $\mathfrak{Q}$ acts on Z_G and $Z_{\tilde{G}}$ via $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and we think of $z \in Z^1(\mathfrak{Q}, Z_G(\bar{\mathbb{Q}}))$. The restriction of $\tilde{z}$ to the torus $\mathfrak{Q}^{\Delta} = Q(\bar{\mathbb{Q}})$ is a homomorphism $Q(\bar{\mathbb{Q}}) \to Z_{\tilde{G}}(\bar{\mathbb{Q}})$, and hence trivial, so that $\tilde{z} \in Z^1(\mathbb{Q}, Z_{\tilde{G}})$. It follows that $\phi' = z \cdot \phi$ with

$$[z] \in \mathrm{Im}\,(H^1(\mathbb{Q}, Z_{\tilde{G}}) \to H^1(\mathbb{Q}, Z_G)). \tag{3.4.12}$$

Conversely, for any z such that (3.4.12) holds, $z \cdot \phi$ satisfies (3.3.6)(1). One checks easily that this bijection makes the set of such liftings a torsor under the group of cocycles satisfying (3.4.12).[18]

[18] One can also obtain this description directly from (3.4.4) by showing the set of liftings is in bijection with $\mathrm{Aut}(\phi_{\mu^{\tilde{\mathrm{ab}}}, G^{\mathrm{ad}}/\tilde{G}})/\mathrm{Aut}(\phi_{\mu^{\tilde{\mathrm{ab}}}})$ and identifying each of these automorphism groups with a space of cocycles.

For $v \neq p$, the same argument as above, shows that $\phi(v) \circ \zeta_v$ differs from ξ_v by an element of $Z^1(\mathbb{Q}_v, Z_G)$ whose class in $H^1(\mathbb{Q}_v, Z_G)$ is in Im $(H^1(\mathbb{Q}_v, Z_{\tilde{G}}) \to H^1(\mathbb{Q}_v, Z_G))$. These two morphisms are conjugate if and only if the image of this cocycle is trivial in $H^1(\mathbb{Q}_v, G)$ if $v \neq \infty$ and in $H^1(\mathbb{R}, G^*_{\mathbb{R}})$, where $G^*_{\mathbb{R}}$ denotes the inner form of $G_{\mathbb{R}}$ with compact adjoint group, as in (3.3.5). Since $H^1(\mathbb{Q}_v, \tilde{G}) = 0$ for v a finite prime, this condition is automatic if $v \neq \infty$. Now the natural map $H^1(\mathbb{Q}, Z_{\tilde{G}}) \to H^1(\mathbb{R}, Z_{\tilde{G}})$ is surjective. To see this embed $Z_{\tilde{G}}$ in an induced torus T, and apply the real approximation theorem to $T/Z_{\tilde{G}}$. Hence, after replacing ϕ by $z \cdot \phi$ for a suitable z in the image of $Z^1(\mathbb{Q}, Z_{\tilde{G}})$, we may assume that ϕ satisfies (3.3.6)(2).

The set of lifts satisfying (1)-(3) in (3.3.6) is thus a torsor under the group of $z \in Z^1(\mathbb{Q}, Z_G)$ such that

(3.4.13) $$[z] \in \ker(H^1(\mathbb{Q}, Z_G) \to H^1(\mathbb{R}, G^*_{\mathbb{R}})) \cap \mathrm{Im}\ (H^1(\mathbb{Q}, Z_{\tilde{G}}) \to H^1(\mathbb{Q}, Z_G))$$

Since $G^{*\mathrm{ad}}(\mathbb{R})$ is connected, the map $H^1(\mathbb{R}, Z_G) \to H^1(\mathbb{R}, G^*_{\mathbb{R}})$ is injective and (3.4.13) is equivalent to

(3.4.14) $$[z] \in \ker(H^1(\mathbb{Q}, Z_G) \to H^1(\mathbb{R}, Z_G)) \cap \mathrm{Im}\ (H^1(\mathbb{Q}, Z_{\tilde{G}}) \to H^1(\mathbb{Q}, Z_G)).$$

By (3.4.8) the map

$$G(\bar{\mathbb{Q}})^{\natural}_{+} \to Z^1(\mathbb{Q}, Z_G); \quad g \mapsto (\tau \mapsto g^{-1}\tau(g))$$

identifies $G(\bar{\mathbb{Q}})^{\natural}_{+}/G(\mathbb{Q})_{+}$ with the subgroup of cocycles in $Z^1(\mathbb{Q}, Z_G)$ satisfying (3.4.14).

Finally, if ϕ and $z \cdot \phi$ are two admissible liftings of ϕ_0, which are conjugate by some $g \in G(\bar{\mathbb{Q}})$, then $g\phi(\varrho_\tau h)g^{-1} = z_\tau \phi(\varrho_\tau h)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $h \in Q(\bar{\mathbb{Q}})$. This implies $g \in I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}$. □

(3.4.15) Fix ϕ_0 as in (3.4.11). Then $I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}$ acts by left multiplication on the disjoint union $\coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)$, where $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ runs over admissible morphisms lifting ϕ_0. If $i \in I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}$, then left multiplication by i maps $X(\phi)$ bijectively onto $X(\mathrm{Int}(i) \circ \phi)$. We set

$$S(G, \phi_0) = \varprojlim_{\mathsf{K}^p} I_{\phi_0}(\bar{\mathbb{Q}})^{\natural} \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)/\mathsf{K}^p$$

where K^p runs over compact open subgroups of $G(\mathbb{A}^p_f)$.

We will need a variant of the above definitions. Let $\mathsf{\tau} \in I^{\mathrm{ad}}_{\phi}(\mathbb{A}_f) = I^{\mathrm{ad}}_{\phi_0}(\mathbb{A}_f)$. Set

$$S_{\mathsf{\tau}}(\phi) = \varprojlim_{\mathsf{K}^p} I_{\phi}(\mathbb{Q}) \backslash X(\phi)/\mathsf{K}^p$$

and

$$S_{\mathsf{\tau}}(G, \phi_0) = \varprojlim_{\mathsf{K}^p} I_{\phi_0}(\bar{\mathbb{Q}})^{\natural} \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)/\mathsf{K}^p$$

where the quotient is taken with respect to the action of $I_{\phi}(\mathbb{Q})$ (resp. $I_{\phi_0}(\bar{\mathbb{Q}})^{\natural}$) on $X(\phi)$ (resp. $\coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)$) obtained by conjugating the natural action by $\mathsf{\tau}$.

Corollary (3.4.16). *The natural map*

$$\coprod_{[\phi],\phi^{\mathrm{ad}}=\phi_0} S_{\mathsf{T}}(\phi) \to S_{\mathsf{T}}(G,\phi_0)$$

is a bijection, where on the left hand side, $[\phi]$ runs over a set of representatives for the distinct conjugacy classes of admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0.

Proof. Using (3.4.11) one finds that

$$S_{\mathsf{T}}(G,\phi_0) = I_{\phi_0}(\bar{\mathbb{Q}})^\natural \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} S_{\mathsf{T}}(\phi) = \coprod_{[\phi],\phi^{\mathrm{ad}}=\phi_0} S_{\mathsf{T}}(\phi)$$

□

(3.5) Special morphisms: In this subsection we show that every admissible morphism is conjugate to a special morphism. We do this by reducing to the case of simply connected derived group proved by Langlands-Rapoport [LR, Satz 5.3].

(3.5.1) We begin by recalling the construction of the Weil torus [LR, 3.1], and its characterization as a quotient of the torus Q constructed in (3.1.3) [Re, Lem. B.2.3].

Let $L \subset \bar{\mathbb{Q}}$ be a CM-field, and $L_0 \subset L$ its maximal totally real subfield. Let $m \geq 1$, and $q = p^m$. By a q-Weil number in L we mean an element $\pi \in L$ such that, for some $\iota \in \mathbb{Z}$, we have $|\pi|_v = q^{\iota/2}$ for every place $v|\infty$ of L, π is a unit outside p, and $\log_q |\prod_{\tau \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} \tau(\pi)|_v \in \mathbb{Z}$. We denote by $P^L(m)$ the torus whose character group consists of q-Weil numbers modulo the subgroup of roots of unity in L. For $\pi \in L$ a q-Weil number, and $m|m'$, $\pi^{m'/m}$ is a $q^{m'/m}$-Weil number. This induces a map $P^L(m') \to P^L(m)$. We set $P^L = \varprojlim P^L(m)$ where the limit is over positive integers ordered by divisibility.

Let $\chi_\pi \in X^*(P^L)$ denote the character corresponding to a q-Weil number π. Note that P^L is equipped with cocharacters $\nu(\infty)^L$ and $\nu(p)^L$ characterized by

$$\langle \nu(v)^L, \chi_\pi \rangle = \log_q |\prod_{\tau \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} \tau(\pi)|_v \tag{3.5.2}$$

for $v = \infty, p$. The map

$$X^*(P^L) \to \mathbb{Z}^{\mathrm{Gal}(L/\mathbb{Q})} : \chi \mapsto \{\langle \tau(\nu(p)^L), \chi \rangle\}_\tau$$

is injective. In particular $X^*(P^L)$ is finitely generated, and P^L is a torus.

We have the following [Re, B.2.3].

Lemma (3.5.3). *Let L be a CM field, and $L_0 \subset L$ its maximal totally real subfield. The triple $(P^L, \nu(\infty)^L, \nu(p)^L)$ is the initial object in the category of triples (T, ν_∞, ν_p) consisting of a torus T which splits over L, and cocharacters $\nu_\infty, \nu_p \in X_*(T)$, defined over $\mathbb{Q}$ and $L(p)$ respectively, such that*

$$[L_p : \mathbb{Q}_p]^{-1} \mathrm{tr}_{L/L_0} \nu_p + \nu_\infty = 0.$$

In particular, there is a surjection $Q^L \to P^L$ taking the cocharacters $\nu(\infty)^L, \nu(p)^L$ of Q^L to the corresponding cocharacters of P^L.

Corollary (3.5.4). *For $n \in \mathbb{N}^+$ sufficiently large (ordered by divisibility) there exist elements $\delta_n \in P^L(\mathbb{Q})$, which are unique modulo $P^L(\mathbb{Q})_{\mathrm{tors}}$, such that*

1. *If $\pi \in L$ is a $q = p^m$-Weil number with $m|n$ then $\chi_\pi(\delta_n) = \pi^{n/m}$*
2. *For $n|n'$ $\delta_{n'} = \delta_n^{n'/n}$ modulo $P^L(\mathbb{Q})_{\mathrm{tors}}$.*
3. *If $L \subset L' \subset \bar{\mathbb{Q}}$ are CM fields, then the natural map $P^{L'} \to P^L$ preserves the elements δ_n.*

Proof. This is proved in [LR, p 31]. □

(3.5.5) We set $P = \varprojlim_L P^L$, with the transition maps induced by sending a q-Weil number π to itself. As in (3.1.5), if $L \subset L'$ is an inclusion of CM-fields, then for $v = \infty, p$, $\nu(v)^{L'}$ pushes forward to $[L'_v : L_v]\nu(v)^L$. Hence we obtain a cocharacter $\nu(\infty) : \mathbb{G}_m \to P$ induced by ν^L with $L \neq L_0$, and $\nu(p) : \mathbb{D} \to P$ induced by $\nu(p)^L$. These coincide with the cocharacters induced by the cocharacter $\nu(\infty), \nu(p)$ of Q. We denote by $\mathfrak{P}$ the Galois gerb obtained from $\mathfrak{Q}$ by pushing out by $Q \to P$.

(3.5.6) Let T be a torus over $\mathbb{Q}$, and $\eta \in X_*(T)$. Recall that η is said to satisfy the *Serre condition* if

$$(\tau - 1)(\iota + 1)(\eta) = (\iota + 1)(\tau - 1)(\eta) = 0,$$

for $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ the complex conjugation and all $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We say T satisfies the Serre condition if every $\eta \in X_*(T)$ satisfies the Serre condition. This is equivalent to requiring that T is isogenous to a $\mathbb{Q}$-torus T_1 which has the form $T_1^+ \times T_1^-$, where T_1^+ is split over $\mathbb{Q}$, and $T_1^-(\mathbb{R})$ is compact. Any such torus splits over a CM-field.

Lemma (3.5.7). *Suppose the connected component of the identity $Z_G^0 \subset Z_G$ splits over a CM field, and w_h is rational. Then every admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ factors through $\mathfrak{P}$.*

In particular, this holds if Z_G^0 satisfies the Serre condition.

Proof. The first statement follows from [Re, B.3.9]. If Z_G^0 satisfies the Serre condition, then it splits over a CM field, and $w_h = (\iota + 1)\mu_h^{-1} \in X_*(Z_G)$ is defined over $\mathbb{Q}$. □

Lemma (3.5.8). *Let T be a $\mathbb{Q}$-torus, $h_T : \mathbb{S} \to T$ and $i : T \hookrightarrow G$ a map of $\mathbb{Q}$-groups such that $i \circ h_T \in X$. Then*

$$i \circ \psi_{\mu_{h_T}} : \mathfrak{Q} \overset{\psi_{\mu_{h_T}}}{\to} \mathfrak{G}_T \to \mathfrak{G}_G$$

is an admissible morphism.

Proof. This is proved in [LR, Lem. 5.2]. Since our setting is more general than that of *loc. cit* we recall the argument.

Clearly $i \circ \psi_{\mu_{h_T}}$ satisfies (3.3.6)(1). Now let $L/\mathbb{Q}$ be a Galois extension such that T splits over L, and $L_\infty = \mathbb{C}$. Then $\psi_{\mu_{h_T}}$ is the composite

$$\mathfrak{Q} \overset{\psi}{\to} \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_T.$$

constructed in (3.1.10). For any Galois gerb $\mathfrak{G}$, the conjugacy classes of maps $f : \mathfrak{G} \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}$ with f^Δ equal to some fixed map, are classified by $H^1(\mathbb{Q}, R_{L/\mathbb{Q}}\mathbb{G}_m) =$

0 by (3.1.2)(2). Hence any two such maps are conjugate. The same remark applies to $\mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}(v)$ for any v.

Suppose that $v \neq p$ is a finite prime. Then the above remark shows that $\psi(v) \circ \zeta_v$ is conjugate to $\tau \mapsto 1 \rtimes \tau$. Hence $i \circ \psi_{\mu_{h_T}}(v) \circ \zeta_v$ is conjugate to ξ_v.

Now let $\mu_0 \in X_*(R_{L/\mathbb{Q}}\mathbb{G}_m) = \mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$ correspond to the identity. If $w \in \mathfrak{G}_\infty$ is as in (3.3.5), then (3.1.12) (applied with $T = R_{L/\mathbb{Q}}\mathbb{G}_m$) shows that there is a well defined map $\xi^L_\infty : \mathfrak{G}_\infty \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}(\infty)$ given by $\psi^\Delta \circ \nu(\infty)$ on $\mathfrak{G}^\Delta_\infty$ and sending w to $\mu_0(-1) \rtimes c$. This map is conjugate to $\psi(\infty) \circ \zeta_\infty$. It follows that ξ_∞ is conjugate to $i \circ \psi_{\mu_{h_T}}(\infty) \circ \zeta_\infty$. Thus $i \circ \psi_{\mu_{h_T}}$ satisfies (3.3.6)(2).

To check (3.3.6)(3), note that the map $\psi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}(p)$ is conjugate to a map of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs $\mathfrak{D} \to \mathfrak{G}^{\mathrm{ur}}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}$. Then $\psi_{\mu_{h_T}}(p) \circ \zeta_p$ is conjugate to a morphism induced by a map of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs, $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}^{\mathrm{ur}}_T(p)$. As in (3.3.6), we write $\theta^{\mathrm{ur}}(d_\sigma) = b \rtimes \sigma$, with $b \in T(\mathbb{Q}_p^{\mathrm{ur}})$.

We have $\nu_b = -\theta^{\mathrm{ur}} \circ \nu(p)$ as in the proof of (3.4.2). Let $T' \subset G_{\mathbb{Q}_p}$ be the centralizer of a maximal split torus containing the image of $i \circ \nu_b$. After conjugating i, we may assume that T' arises from a torus in $G_{\mathbb{Z}_p}$. Fix a Borel $B \supset T'$ of $G_{\mathbb{Z}_p}$, and let $\mu^* \in X_*(T')$ be conjugate to μ, with $-\mu^*$ dominant. Let M be the centralizer of $i \circ \nu_b$. It suffices to show that $X^M_{-\mu}(i(b))$ is non-empty. To do this we check the two conditions in (3.4.3) (applied to M). The condition $\nu_b \leqslant -\bar{\mu}^*$ holds as $-\bar{\mu}^*$ is dominant, and ν_b is central in M. An argument, as in the proof of (3.4.2) shows that $\kappa_M(i(b)) = [i \circ -\mu_{h_T}] = [-\mu] \in \pi_1(M)_\Gamma$. □

(3.5.9) Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. We say that ϕ is *special* if there exists a $\mathbb{Q}$-torus T, an $h_T : \mathbb{S} \to T$, and an inclusion $i : T \hookrightarrow G$ over $\mathbb{Q}$ such that $i \circ h_T \in X$, and ϕ is conjugate to $i \circ \psi_{\mu_{h_T}}$.

Lemma (3.5.10). *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, and $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ the morphism induced by ϕ. Then ϕ is special if and only if ϕ_0 is special.*

Proof. If ϕ special, then clearly ϕ_0 is special.

Now suppose that ϕ_0 is special. After conjugating ϕ_0 by an element of $G^{\mathrm{ad}}(\bar{\mathbb{Q}})$ we may assume that $\phi_0 = i \circ \psi_{\mu_{h_T}}$ for some torus $i : T \hookrightarrow G^{\mathrm{ad}}$, and $h_T : \mathbb{S} \to T$, with $i \circ h_T \in X^{\mathrm{ad}}$. By the real approximation theorem, we may conjugate i and ϕ_0 by an element of $G^{\mathrm{ad}}(\mathbb{Q})$ and assume that h_T lifts to an element $h \in X$.

Let $\tilde{T}$ be the preimage of T in G, so that $h : \mathbb{S} \to \tilde{T}$ lifts h_T. Let $\tilde{i} : \tilde{T} \to G$ denote the inclusion. By (3.5.8) $\tilde{i} \circ \psi_{\mu_h}$ is an admissible morphism. Now let $g \in G(\bar{\mathbb{Q}})^\natural_+$, and set $z_\tau = g^{-1}\tau(g)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then for $t \in T(\bar{\mathbb{Q}})$ and $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have

$$g(z_\tau \tilde{i}(t), \tau)g^{-1} = (g\tilde{i}(t)g^{-1}, \tau) = (g\tilde{i}g^{-1}(t), \tau)$$

in $\mathfrak{G}_G$. Thus $z \cdot (\tilde{i} \circ \psi_{\mu_h})$ is conjugate to the morphism $(g\tilde{i}g^{-1}) \circ \psi_{\mu_h}$, and hence is special. It follows from (3.4.11) that every admissible lifting $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ of ϕ_0 is special. □

Theorem (3.5.11). *Every admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is special.*

Proof. Consider a Shimura datum (H, Y) with H an adjoint group. We first show that there is a Shimura datum (G, X) with $(H, Y) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$, G^{der} simply connected, and such that $Z_{G,\mathbb{R}}$ is a compact torus, so in particular Z_G satisfies the Serre condition.

By [De 2, 2.3.4(b)] the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Dynkin diagram of H factors through $\mathrm{Gal}(L/\mathbb{Q})$ with L a CM-field. In particular, $Z_{\tilde{H}}$ can be embedded in a sum of copies of $R_{L/\mathbb{Q}}\mathbb{G}_m$, and hence in a sum of copies of

$$S_L = \ker(R_{L/\mathbb{Q}}\mathbb{G}_m \overset{N_{L/L_0}}{\to} R_{L_0/\mathbb{Q}}\mathbb{G}_m)$$

where L_0 denotes the totally real subfield of L. Fix such an embedding $Z_{\tilde{H}} \hookrightarrow S_L^n$, and let $G = (\tilde{H} \times S_L^n)/Z_{\tilde{H}}$. Then $Z_{G,\mathbb{R}} = S_{L,\mathbb{R}}^n$ is compact.

Now any $h \in Y$ factors as $h : \mathbb{S}/\mathbb{S}_0 \to H = G^{\mathrm{ad}}$. Since Z_G is a compact torus, h lifts to a map $\tilde{h} : \mathbb{S}/\mathbb{S}_0 \to G$, and we may take X to be the $G(\mathbb{R})$-conjugacy class of $\tilde{h}$.

The theorem for (G, X) follows from [LR, Satz 5.3], and (3.5.7). Hence the lemma holds for any Shimura datum (H, Y) with H adjoint (3.4.11). Finally, the theorem holds in general by (3.5.10). $\square$

(3.6) Connected components: In this subsection, we develop a theory of connected components for the sets $S_{\mathrm{T}}(G, \phi_0)$. We will use special morphisms to show that the set parameterizing these components is in canonical bijection with the set of (geometric) connected components of $\mathrm{Sh}_{\mathrm{K}_p}(G, X)$.

(3.6.1) Let $\phi : \mathfrak{Q} \to \mathfrak{G}_{G/\tilde{G}}$ be a morphism. We define sets $X_p(\phi)$ and $X^p(\phi)$ analogously to (3.3.6). First let $\tilde{X}^p(\phi)$ denote the set of pairs $(g_l, \epsilon_l)_{l\neq\infty,p}$ where $(g_l)_{l\neq\infty,p} \in G(\bar{\mathbb{A}}_f^p)$ and ϵ_l is an isomorphism $\mathrm{Int}(g_l) \circ \xi_l \xrightarrow{\sim} \phi(l) \circ \zeta_l$. Here we again denote by ξ_l the morphism $\mathfrak{G}_l \overset{\xi_l}{\to} \mathfrak{G}_G(l) \to \mathfrak{G}_{G/\tilde{G}}(l)$. More explicitly, ϵ_l is given by a cocycle $\epsilon_l \in Z^1(\mathbb{Q}_l, \tilde{G}(\bar{\mathbb{Q}}_l))$, such that

$$\epsilon_l(\tau)\mathrm{Int}(g_l) \circ \xi_l(\tau) = \phi(l) \circ \zeta_l(\tau)$$

in $G(\bar{\mathbb{Q}}_l)$. Here, we think of $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ as acting on $\tilde{G}(\bar{\mathbb{Q}}_l)$ via $\mathrm{Int}(g_l) \circ \xi_l$ and conjugation. Explcitly, this action is given by $\tau(h) = g_l\tau(g_l)^{-1}\tau(h)\tau(g_l)g_l^{-1}$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ and $h \in \tilde{G}(\bar{\mathbb{Q}}_l)$. We require that $(\epsilon_l(\tau_l))_{l\neq p,\infty} \in \tilde{G}(\bar{\mathbb{A}}_f^p)$ for any $(\tau_l)_{l\neq p} \in \prod \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$.

The group $\tilde{G}(\bar{\mathbb{A}}_f^p)$ acts on $\tilde{X}^p(\phi)$ by

$$h \cdot (g_l, \epsilon_l)_{l\neq\infty,p} = (hg_l, \tau \mapsto \epsilon_l \cdot \tau(h_l)h_l^{-1})_{l\neq\infty,p},$$

where $\tau(h_l)$ refers to the twisted Galois action just mentioned. We set

$$X^p(\phi) = \tilde{G}(\bar{\mathbb{A}}_f^p)\backslash\tilde{X}^p(\phi).$$

We define $X_p(\phi)$ as the set of $g \in G(\bar{\mathbb{Q}}_p)/(\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}}) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}}))$ such that $\theta_g = \mathrm{Int}(g^{-1}) \circ \phi(p) \circ \zeta_p$ is induced by a morphism[19]

$$\theta_g^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_{G/\tilde{G}}^{\mathrm{ur}}(p) := \mathfrak{G}_G^{\mathrm{ur}}/\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}})$$

[19]Note that this condition on a morphism $\mathfrak{G}_p \to \mathfrak{G}_{G/\tilde{G}}$ is not stable under isomorphism of monoidal functors.

with $\theta_g^{\mathrm{ur}}(d_\sigma) = b_g \rtimes \sigma$ and $\tilde{\kappa}_G(b_g) = [-\mu] \in \pi_1(G)$. As above, we set $X(\phi) = X_p(\phi) \times X^p(\phi)$.

We remark that $X^p(\phi)$ and $X_p(\phi)$ depend on ϕ and not just on its isomorphism class. More precisely, if $\phi, \phi' : \mathfrak{Q} \to \mathfrak{G}_{G/\tilde{G}}$ are isomorphic, then $X(\phi)$ and $X(\phi')$ can be identified, if one chooses an isomorphism of functors $\phi \xrightarrow{\sim} \phi'$. This choice is canonical up to the group $\mathrm{Aut}(\phi)$, which in general acts non-trivially on $X(\phi)$. We will apply this definition when ϕ is induced by a morphism $\mathfrak{Q} \to \mathfrak{G}_G$, in which case $X(\phi)$ should be viewed as depending on the latter morphism (cf. also (3.6.5) below).

When $G^{\mathrm{der}} = \tilde{G}$, the set $X^p(\phi)$ has a simpler description: Let $X^{p\prime}(\phi)$ denote the set of $g \in G^{\mathrm{ab}}(\bar{\mathbb{A}}_f^p)$ such that $\mathrm{Int}(g_l) \circ \xi_l = \phi(l) \circ \zeta_l$ for all l. Then one checks easily that the map $X^p(\phi) \to X^{p\prime}(\phi)$ which sends $(g_l, \epsilon_l)_{l \neq \infty, p}$ to the image of $(g_l)_{l \neq \infty, p}$ is a bijection.

Fix an admissible $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ as before and let

$$\tilde{\pi}(G, \phi_0) = \coprod_{\phi^{\mathrm{ad}} = \phi_0} X(\phi_{\widetilde{\mathrm{ab}}}),$$

where the disjoint union runs over admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0.

If $H \subset G(\mathbb{A}_f)$ is a subgroup, we denote by H^- its closure in $G(\mathbb{A}_f)$.

Lemma (3.6.2).

(1) *Right multiplication induces a well defined action of* $G(\bar{\mathbb{Q}})_+^\natural$ *on* $\tilde{\pi}(G, \phi_0)$.

(2) *For any admissible* $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ *lifting* ϕ_0, *right multiplication induces an action of* $G(\mathbb{A}_f)/\tilde{G}(\mathbb{A}_f)$ *on* $X(\phi_{\widetilde{\mathrm{ab}}})$. *If* $g_0 \in X(\phi_{\widetilde{\mathrm{ab}}})$ *then the natural map*

$$G(\mathbb{A}_f)/\tilde{G}(\mathbb{A}_f) G_{\mathbb{Z}_p}(\mathbb{Z}_p) \xrightarrow{g \mapsto g_0 g} X(\phi_{\widetilde{\mathrm{ab}}})$$

is a bijection.

(3) *If* $g_0 \in \tilde{\pi}(G, \phi_0)$, *then the natural map*

$$\pi(G) := G(\mathbb{Q})_+^- \backslash G(\mathbb{A}_f)/G_{\mathbb{Z}_p}(\mathbb{Z}_p) \xrightarrow{g \mapsto g_0 \cdot g} \varprojlim_{\mathsf{K}^p} \tilde{\pi}(G, \phi_0)/G(\bar{\mathbb{Q}})_+^\natural \cdot \mathsf{K}^p =: \pi(G, \phi_0)$$

is a bijection. Here K^p *runs over compact open subgroups of* $G(\mathbb{A}_f^p)$. *Thus* $\pi(G, \phi_0)$ *is a* $\pi(G)$*-torsor.*

Proof. Let $g \in G(\bar{\mathbb{Q}})_+^\natural$, $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ an admissible morphism lifting ϕ_0, and write $z_\tau = g^{-1}\tau(g)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, so that $(z_\tau)_\tau \in Z^1(\mathbb{Q}, Z_G)$.

Let $(g_l, \epsilon_l)_{l \neq p} \in \tilde{X}^p(\phi_{\widetilde{\mathrm{ab}}})$. By right multiplication in (1) we mean the map induced by right multiplication on elements of $X_p(\phi)$ and sending $(g_l, \epsilon_l)_{l \neq p}$ in $\tilde{X}^p(\phi_{\widetilde{\mathrm{ab}}})$ to $(g_l g, \epsilon_l)_{l \neq p}$. For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ we have

$$\mathrm{Int}(g_l g) \circ \xi_l(\tau) = \mathrm{Int}(g_l)(z_\tau^{-1} \cdot \xi_l(\tau)) = z_\tau^{-1} \mathrm{Int}(g_l) \circ \xi_l(\tau) \xrightarrow[\epsilon_l]{\sim} (z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})(l) \circ \zeta_l(\tau).$$

So $(g_l g, \epsilon_l)_{l \neq p} \in \tilde{X}^p(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$. Thus g maps $\tilde{X}^p(\phi_{\widetilde{\mathrm{ab}}})$ bijectively onto $\tilde{X}^p(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$, and this bijection commutes with the action of $\tilde{G}(\bar{\mathbb{A}}_f^p)$, inducing a bijection of $X^p(\phi_{\widetilde{\mathrm{ab}}})$ onto $X^p(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$

Now let $g_0 \in X_p(\phi_{\widetilde{\mathrm{ab}}})$. The image of $\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}}) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$ in $G(\bar{\mathbb{Q}}_p)$ is stable under conjugation by g. Hence if $h \in \tilde{G}(\mathbb{Q}_p^{\mathrm{ur}}) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$, then $g_0hg = g_0g(g^{-1}hg)$, so g_0g is a well defined element of $G(\bar{\mathbb{Q}}_p)/\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}}) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$. We have

$$\theta_{g_0g} := \mathrm{Int}(g^{-1}g_0^{-1})(z^{-1} \cdot \phi)(p) \circ \zeta_p = z^{-1} \cdot (\mathrm{Int}(g^{-1}) \circ \theta_{g_0}).$$

As in (3.3.3), to show that θ_{g_0g} is induced by an unramified morphism $\theta^{\mathrm{ur}}_{g_0g} : \mathfrak{D} \to \mathfrak{G}_{G^{\mathrm{ur}}/\tilde{G}}(p)$, we have to check that $\theta_{g_0g}(1 \rtimes \tau) = 1 \rtimes \tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$. To see this, note that

$$\mathrm{Int}(g^{-1}) \circ \theta_{g_0}(1 \rtimes \tau) = g^{-1}(1 \rtimes \tau)g = g^{-1}\tau(g) \rtimes \tau = z_\tau \cdot 1 \rtimes \tau.$$

Since $\tilde{\kappa}_G(g^{-1}b_{g_0}g) = \tilde{\kappa}_G(b_{g_0}) = [-\mu]$, it follows that $g_0g \in X_p(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$. This shows that right multiplication by $G(\bar{\mathbb{Q}})^{\natural}_{+}$ gives a well defined action on $\tilde{\pi}(G, \phi_0)$, and proves (1).

To see (2), note that if $g_0 \in X_p(\phi_{\widetilde{\mathrm{ab}}})$, and $g \in G(\mathbb{Q}_p)$, then $\mathrm{Int}(g^{-1}) \circ \theta^{\mathrm{ur}}_{g_0}(d_\sigma) = g^{-1}b_{g_0}g \rtimes \sigma$, so $g_0g \in X_p(\phi_{\widetilde{\mathrm{ab}}})$. Since $G(\mathbb{A}_f^p)/\tilde{G}(\mathbb{A}_f^p)$ clearly acts on $X^p(\phi_{\widetilde{\mathrm{ab}}})$, one sees that right multiplication induces an action of $G(\mathbb{A}_f)/\tilde{G}(\mathbb{A}_f)$ on $X(\phi_{\widetilde{\mathrm{ab}}})$. We have to show this action is simple and transitive.

Let $g_0 = (g_{0,l}, \epsilon_l)_{l \neq p} \in X^p(\phi)$. Since the class of $\epsilon_l \in H^1(\mathbb{Q}_l, \tilde{G})$ is trivial, we have $\epsilon_l = h_l\tau(h_l)^{-1}$ for some $h_l \in \tilde{G}(\bar{\mathbb{Q}}_l)$, again using the twisted Galois action on $\tilde{G}$, for which ϵ_l is a cocycle. For some integer N, G admits a reductive model $G_{\mathbb{Z}[1/N]}$ over $\mathbb{Z}[1/N]$. Then for almost all l, ϵ_l takes values in $G_{\mathbb{Z}_l}(\bar{\mathbb{Z}}_l)$. In this case we can take $h_l \in G_{\mathbb{Z}_l}(\bar{\mathbb{Z}}_l)$.[20] Thus $h = (h_l)_{l \neq p} \in G(\bar{\mathbb{A}}_f^p)$. As $g_0 = h \cdot g_0 \in X^p(\phi)$, we may assume that $(\epsilon_l)_{l \neq p} = \mathbf{1}$, the trivial cocycle.

The set $G(\mathbb{A}_f^p)/\tilde{G}(\mathbb{A}_f^p)$ clearly acts on the set of such g_0. Suppose that $g = (g_l)_{l \neq p} \in G(\bar{\mathbb{A}}_f^p)$ with $(g_{0,l}g, \mathbf{1})_{l \neq p} \in X^p(\phi)$. Then $\mathrm{Int}(g_{0,l}g_l) \circ \xi_l = \phi(l) \circ \zeta_l$ for each l, and $\mathrm{Int}(g_l) \circ \xi_l = \xi_l$. For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, $\mathrm{Int}(g_l) \circ \xi_l(\tau) = g_l\tau(g_l)^{-1} \rtimes \tau$, so we find that $g_l \in G(\mathbb{Q}_l)$, and $g \in G(\mathbb{A}_f^p)$, which shows the action of $G(\mathbb{A}_f^p)/\tilde{G}(\mathbb{A}_f^p)$ is transitive.

Suppose that $g \in G(\mathbb{A}_f^p)$, with $g_0g = g_0$ in $X^p(\phi)$, so that $(g_0, \mathbf{1}) = ((g_0gg_0^{-1}) \cdot g_0, \mathbf{1}) \in X^p(\phi)$. Then there exists a $h_l \in \tilde{G}(\bar{\mathbb{Q}}_l)$ such that $h_l\tau(h_l)^{-1} = 1$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, with the twisted action, and $h_l = g_{0,l}g_lg_{0,l}^{-1}$ in $G(\bar{\mathbb{Q}}_l)$. This implies that $g_{0,l}^{-1}h_lg_{0,l} \in \tilde{G}(\mathbb{Q}_l)$, is a lift of g_l. Thus $G(\mathbb{A}_f^p)/\tilde{G}(\mathbb{A}_f^p)$ acts simply.

Next suppose $g_0 \in X_p(\phi_{\widetilde{\mathrm{ab}}})$, and $g \in G(\bar{\mathbb{Q}}_p)$ with $g_0g \in X_p(\phi_{\widetilde{\mathrm{ab}}})$. Then a simple computation shows that $g^{-1}\tau(g) = 1$ for all $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$, so $g \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Likewise if $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a lift of Frobenius, then the image of $g^{-1}b_{g_0}\tau(g)$ in $\pi_1(G)$ is equal $[-\mu]$, and hence coincides with the image of b_{g_0}. Hence the image of g lies in $\pi_1(G)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$, and the class of g in $G(\mathbb{Q}_p^{\mathrm{ur}})/\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}})G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$ contains an element in $G(\mathbb{Q}_p)$ by (1.2.3).

This shows that $G(\mathbb{Q}_p)/\tilde{G}(\mathbb{Q}_p) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ acts transitively on $X_p(\phi_{\widetilde{\mathrm{ab}}})$. The Cartan decomposition implies that

$$\begin{aligned}\mathrm{Im}(\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}}) \to G(\mathbb{Q}_p^{\mathrm{ur}})) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}}) \cap G(\mathbb{Q}_p) \\ = \mathrm{Im}(\tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)) \cdot G_{\mathbb{Z}_p}(\mathbb{Z}_p) = \ker \tilde{\kappa}_G|_{G(\mathbb{Q}_p)},\end{aligned}$$

[20]Indeed, in this case triviality of the class of ϵ_l is less deep, being a consequence of Lang's Lemma.

which shows that the action is simple. This completes the proof of (2).

Now, using (2) and (3.4.11), one sees that $\pi(G,\phi_0)$ is in bijection with

$$\varprojlim_{\mathsf{K}^p} G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/\tilde{G}(\mathbb{A}_f)G(\mathbb{Z}_p)\mathsf{K}^p \xrightarrow{\sim} \pi(G,\phi_0). \tag{3.6.3}$$

Since $G(\mathbb{Q})_+^-$ contains $\tilde{G}(\mathbb{A}_f)$, by the strong approximation theorem, and the closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}_f)/G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ coincides with the image of $G(\mathbb{Q})_+^-$, as $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is compact, the left side of (3.6.3) is equal to $G(\mathbb{Q})_+^-\backslash G(\mathbb{A}_f)/G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. $\square$

Corollary (3.6.4). *For any admissible morphism $\phi:\mathfrak{Q}\to\mathfrak{G}_G$,*

$$\pi(G,\phi) := \varprojlim X(\phi_{\widetilde{\mathrm{ab}}})/G(\mathbb{Q})_+\cdot\mathsf{K}^p$$

is a $\pi(G)$-torsor. Here K^p runs over compact open subgroups of $G(\mathbb{A}_f^p)$.

If $\phi,\phi':\mathfrak{Q}\to\mathfrak{G}$ are admissible morphisms such that ϕ^{ad} and ϕ'^{ad} are conjugate. Then the $\pi(G)$-torsors $\pi(G,\phi)$ and $\pi(G,\phi')$ are canonically isomorphic.

Proof. The first claim follows from (3.6.2)(2). For the second claim, suppose ϕ^{ad} and ϕ'^{ad} are conjugate by an element $g^{\mathrm{ad}}\in G^{\mathrm{ad}}(\bar{\mathbb{Q}})$. Lift g^{ad} to $\tilde{g}\in\tilde{G}(\bar{\mathbb{Q}})$. Left multiplication by $\tilde{g}$ induces a bijection between $X(\phi_{\widetilde{\mathrm{ab}}})$ and $X(\tilde{g}\phi_{\widetilde{\mathrm{ab}}}\tilde{g}^{-1})$, and a bijection between $\pi(G,\phi)$ and $\pi(G,\tilde{g}\phi\tilde{g}^{-1})$ as $\pi(G)$-torsors. Here on $X^p(\phi_{\widetilde{\mathrm{ab}}})$, left multiplication means the map sending $(g_l,\epsilon_l)_{l\neq p,\infty}$ to $(\tilde{g}g_l,\tilde{g}\epsilon_l\tilde{g}^{-1})_{l\neq p,\infty}$. Since $\tilde{g}\phi^{\mathrm{ad}}\tilde{g}^{-1}=\phi'^{\mathrm{ad}}$, we can then identify $\pi(G,\tilde{g}\phi\tilde{g}^{-1})$ and $\pi(G,\phi')$ using (3.6.2)(3). The second claim now follows once we check that the bijection between $\pi(G,\phi)$ and $\pi(G,\phi')$ just constructed does not depend on the choice of $\tilde{g}$.

To see this, let $I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural$ and $\tilde{G}(\bar{\mathbb{Q}})_+^\natural$ denote the preimages of $I_{\phi_0}(\bar{\mathbb{Q}})^\natural$ and $G(\bar{\mathbb{Q}})_+^\natural$ in $\tilde{G}(\bar{\mathbb{Q}})$. Two choices of $\tilde{g}$ differ by an element $i\in I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural$. It suffices to show that there exists $g\in\tilde{G}(\bar{\mathbb{Q}})_+^\natural$ such that $ihg^{-1}=h$ for all $h\in X(\phi_{\widetilde{\mathrm{ab}}})$. (Here right multiplication has the same meaning as in (3.6.2)).

Let $z_\tau=i^{-1}\tau(i)$. Then the class of z_τ in $H^1(\mathbb{R},Z_{\tilde{G}})$ is trivial, as $(I_{\tilde{\phi}_0}/Z_{\tilde{G}})(\mathbb{R})$ is compact, and hence connected. In particular, the image of z_τ in $H^1(\mathbb{R},\tilde{G})$ is trivial, and hence the image of z_τ in $H^1(\mathbb{Q},\tilde{G})$ is trivial by the Hasse principle. Hence there exists $g\in\tilde{G}(\bar{\mathbb{Q}})_+^\natural$ such that $g^{-1}\tau(g)=z_\tau$.

Right multiplication by g induces a bijection between $X(\phi_{\widetilde{\mathrm{ab}}})$ and $X(z^{-1}\cdot\phi_{\widetilde{\mathrm{ab}}})$, as does left multiplication by i (via the natural action of $I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural$). If $h\in X(\phi_{\widetilde{\mathrm{ab}}})$, then $ihg^{-1}=h[h^{-1}ihg^{-1}]\in X(\phi_{\widetilde{\mathrm{ab}}})$. Since $h^{-1}ihg^{-1}\in\tilde{G}(\bar{\mathbb{A}}_f)$, it follows from (3.6.2)(2) that $ihg^{-1}=h$ in $X(\phi_{\widetilde{\mathrm{ab}}})$. $\square$

(3.6.5) We use (3.6.4) to identify the sets $\pi(G,\phi)$ with ϕ^{ad} conjugate to a given morphism $\phi_0:\mathfrak{Q}\to\mathfrak{G}_{G^{\mathrm{ad}}}$, and denote the resulting set simply by $\pi(G,\phi_0)$, as in (3.6.2).

The reader should note the following subtlety: Suppose that $G^{\mathrm{der}}=\tilde{G}$, and $\phi,\phi':\mathfrak{Q}\to\mathfrak{G}_G$, are two admissible morphisms with $\phi_{\widetilde{\mathrm{ab}}}=\phi'_{\widetilde{\mathrm{ab}}}$. Then, by definition, the underlying sets of $\pi(G,\phi)$ and $\pi(G,\phi')$ coincide. However, the map identifying these two sets in (3.6.4) is in general not the identity.

If we no longer assume that $G^{\mathrm{der}}=\tilde{G}$, and assume that $\phi_{\widetilde{\mathrm{ab}}}$ and $\phi'_{\widetilde{\mathrm{ab}}}$ are equivalent, there is an analogous canonical, “tautological” identification of $\pi(G,\phi)$ and

$\pi(G,\phi')$ (corresponding to the identity map when $G^{\mathrm{der}} = \tilde{G}$), but its definition is somewhat less obvious, and we will not need it.

(3.6.6) Let T be a $\mathbb{Q}$-torus, and $\mu_T \in X_*(T)$. We now make the analogues of the definitions in (3.6.1) in this setting. Recall that we have the map $\psi_{\mu_T} : \mathfrak{Q} \to \mathfrak{G}_T$. We define

$$X^p(\psi_{\mu_T}) = \{(g_l)_{l\neq\infty,p} \in T(\bar{\mathbb{A}}_f^p) : \mathrm{Int}(g_l)\circ\xi_l = \psi_{\mu_T}(l)\circ\zeta_l\}.$$

To define $X_p(\psi_{\mu_T})$, recall that for any connected reductive group H over $\mathbb{Q}_p$, Kottwitz [Ko 5, §7] has defined a surjective map $\tilde{\kappa}_H : H(\mathbb{Q}_p^{\mathrm{ur}}) \to \pi_1(H)_I$, generalizing the map $\tilde{\kappa}_G$ introduced in (1.2.1) above. Here $I \subset \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ denotes the inertia subgroup. In particular, we may apply this to the torus T, and obtain a map $\tilde{\kappa}_T : T(\mathbb{Q}_p^{\mathrm{ur}}) \to X_*(T)_I$. If $\mathcal{T}$ denotes the Néron model of T over $\mathbb{Z}_p$, and $\mathcal{T}^\circ \subset \mathcal{T}$ denotes the connected component of the identity, then $\ker\tilde{\kappa}_T = \mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$ [Ra, Rem. 2.2].

We define $X_p(\psi_{\mu_T})$ as the set of $t \in T(\bar{\mathbb{Q}}_p)/\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$ such that $\theta_t = \mathrm{Int}(t^{-1})\circ\psi_{\mu_T}(p)\circ\zeta_p$ is induced by a morphism

$$\theta_t^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_T^{\mathrm{ur}}$$

with $\theta_t^{\mathrm{ur}}(d_\sigma) = b_t \rtimes \sigma$ and $\tilde{\kappa}_T(b_t) = [-\mu_T] \in X_*(T)_I$.

Finally, we set $X(\psi_{\mu_T}) = X_p(\psi_{\mu_T}) \times X^p(\psi_{\mu_T})$, and

$$S(\psi_{\mu_T}) = T(\mathbb{Q})^-\backslash X(\psi_{\mu_T}).$$

It will be convenient to extend these definitions to the case when T is a protorus. If $T = \varprojlim T_i$ where each T_i is a torus, and $\mu_T \in X_*(T)$, we denote by $\mu_{T_i} \in X_*(T_i)$ the image of μ_T, and we define

$$X(\psi_{\mu_T}) = \varprojlim X(\psi_{\mu_{T_i}})$$

and similarly for $S(\psi_{\mu_T})$.

Proposition (3.6.7). *The set $X(\psi_{\mu_T})$ is a $T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p)$-torsor. Moreover, there is a canonical isomorphism*

$$S(\psi_{\mu_T}) \xrightarrow{\sim} T(\mathbb{Q})^-\backslash T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p).$$

In particular, this isomorphism is functorial in the pair (T,μ_T).

Proof. We begin by checking that $X^p(\psi_{\mu_T})$ is non-empty, and that $\psi_{\mu_T}(p)\circ\zeta_p$ is conjugate to a morphism $\mathfrak{D} \to \mathfrak{G}_T^{\mathrm{ur}}$. It suffices to do this in the universal case where $(T,\mu_T) = (R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m, \mu_0)$ where $\mu_0 \in X_*(R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m)$ corresponds to $1 \in \mathbb{Z}[\![\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]\!]$. Then $\psi_{\mu_0} = \psi$.

Note that, by construction ψ^Δ is defined over $\mathbb{Q}$. Since $H^1(\mathbb{Q}_l, R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m) = \{1\}$, for $l \neq p$, (3.1.2)(2) implies that $\psi(l)$ is conjugate to the morphism of neutral Galois gerbs $\mathfrak{Q}(l) \to \mathfrak{G}_T(l)$ given by $\psi^\Delta(l)\rtimes 1$. This implies $X^p(\psi)$ is non-empty. Similarly, since $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}}), R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m) = \{1\}$, we have

$$H^2(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p), R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m) \subset H^2(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m).$$

As the pushout of $\mathfrak{G}_p$ by $(\psi(p)\circ\zeta_p)^\Delta$ is a split extension, this implies the pushout of $\mathfrak{D}$ by the $\mathbb{Q}_p^{\mathrm{ur}}$-points of the same map is also split. It follows that $(\psi(p)\circ\zeta_p)^\Delta$ is induced by a map of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerbs $\mathfrak{D}\to\mathfrak{G}_T^{\mathrm{ur}}$. Since, $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), R_{\bar{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m)=\{1\}$, the induced map of Galois gerbs must be conjugate to $\psi(p)\circ\zeta_p$.

The argument that $X_p(\psi_{\mu_T})$ is non-empty is analogous to (and in fact technically simpler than) the proof of (3.4.2). It suffices to consider T of the form $R_{L/\mathbb{Q}}\mathbb{G}_m$ for some Galois extension $L/\mathbb{Q}$. Let $\theta^{\mathrm{ur}}:\mathfrak{D}\to\mathfrak{G}_T^{\mathrm{ur}}$ be a map such that the induced map of Galois gerbs is conjugate to $\psi_{\mu_T}(p)\circ\zeta_p$. Write $\theta^{\mathrm{ur}}(d_\sigma)=b\rtimes\sigma$. Then $\nu_b=-\psi_{\mu_T}^\Delta\circ\nu(p)=-\bar{\mu}_T$, where $\bar{\mu}_T\in X_*(T)^\Gamma\otimes\mathbb{Q}$ denotes the average of the Galois orbit of μ_T. Hence by [RR, Thm. 4.2(ii)], b and $-\mu_T$ have the same image in $X_*(T)_\Gamma\otimes\mathbb{Q}$ and hence in $X_*(T)_\Gamma$, which is torsion free. In particular, if $b'\in T(\mathbb{Q}_p^{\mathrm{ur}})$ satisfies $\tilde{\kappa}_T(b')=-\mu_T$, then b and b' have the same image in $X_*(T)_\Gamma$, and hence b is σ-conjugate to b', as b and b' are basic.

This shows that $X(\psi_{\mu_T})$ is non-empty. One checks immediately that $X^p(\psi_{\mu_T})$ is a $T(\mathbb{A}_f^p)$-torsor. If $t_1,t_2\in X_p(\psi_{\mu_T})$, then $t=t_1t_2^{-1}\in T(\mathbb{Q}_p^{\mathrm{ur}})$, and satisfies $\tilde{\kappa}_T(\sigma(t)t^{-1})=1$, and hence $\sigma(t)t^{-1}\in\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$. Hence by Lang's Lemma, there exists $t'\in\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$ with $tt'^{-1}\in T(\mathbb{Q}_p)$. This shows that $t_1,t_2\in T(\bar{\mathbb{Q}}_p)/\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$ differ by an element of $T(\mathbb{Q}_p)$. It follows that $X(\psi_{\mu_T})$ is a $T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p)$-torsor, and hence that $S(\psi_{\mu_T})$ is a $T(\mathbb{Q})^-\backslash T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p)$-torsor.

To show that $S(\psi_{\mu_T})$ has a canonical trivialization, it suffices to show that $S(\psi)$ consists of a single element. For this consider the maps of inverse limits

$$\begin{aligned}&\varprojlim_L (L^\times)^-\backslash(L\otimes_{\mathbb{Q}}\mathbb{A})^\times/(L\otimes_{\mathbb{Q}}\mathbb{R})^{\times,+}\\ &\quad=\varprojlim_L (L^{\times,+})^-\backslash(L\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times\to\varprojlim(L^\times)^-\backslash(L\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times/(\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times:=C,\end{aligned}$$

where $L\subset\bar{\mathbb{Q}}$ runs over all finite extension of $\mathbb{Q}$, and as usual, a superscript "+" means that we take the intersection of the subgroup with the connected component of the identity in $(L\otimes_{\mathbb{Q}}\mathbb{R})^\times$, and "−" means that we take the closure of $L^\times$ or $L^{\times,+}$ in the indicated topological group. The inverse limit on the left is trivial, by class field theory. The map of inverse limits is surjective, because $L^{\times,+}$ has finite index in $L^\times$, and $(\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$ is compact. Hence C is trivial. Let $\mu_{0,L}\in X_*(R_{L/\mathbb{Q}}\mathbb{G}_m)$ denote the image of μ_0. As C is an inverse limit of profinite groups, $S(\psi)=\varprojlim_L S(\psi_{\mu_{0,L}})$ is a torsor under C, and we have $S(\psi)=\{1\}$. □

(3.6.8) We now consider the category $\mathscr{SH}_p$ whose objects consist of a reductive group $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, and a Shimura datum (G,X), where $G=G_{\mathbb{Z}_{(p)}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Q}$. Morphisms consist of morphisms of reductive groups $G_{\mathbb{Z}_{(p)}}\to G'_{\mathbb{Z}_{(p)}}$ inducing a morphism of Shimura data. For any $(G_{\mathbb{Z}_{(p)}},X)$ in $\mathscr{SH}_p$, We have the associated adjoint object $(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}},X^{\mathrm{ad}})$, and we denote by $\mathscr{SH}_p(G^{\mathrm{ad}},X^{\mathrm{ad}})\subset\mathscr{SH}_p$ the subcategory consisting of objects whose adjoint object is $(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}},X^{\mathrm{ad}})$. We will usually write (G,X) for an object in $\mathscr{SH}_p$, so that the reductive group $G_{\mathbb{Z}_{(p)}}$ is understood.

As above, we set $\mathsf{K}_p=G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, and we write

$$\pi(G,X)=\pi_0(\mathscr{S}_{\mathsf{K}_p}(G,X)(\mathbb{C}))$$

for the set of connected components of $\mathscr{S}_{\mathsf{K}_p}(G,X)(\mathbb{C})$ so that $\pi(G,X)$ is torsor under $\pi(G)$ [De 2, 2.1.3].

Let T be a torus over $\mathbb{Q}$, equipped with a cocharacter $h_T : \mathbb{S} \to T_{\mathbb{R}}$, and let $\mu_T = \mu_{h_T} \in X_*(T)$ be defined as in (1.3.1). Suppose $i : T \to G$ is a morphism such that $i \circ h_T \in X$. We obtain a morphism

$$T(\mathbb{Q})^- \backslash T(\mathbb{A}_f) \overset{t \mapsto h \times t}{\longrightarrow} \mathscr{S}_{\mathsf{K}_p}(G,X)(\mathbb{C}) \to \pi(G,X)$$

which induces a map

$$T(\mathbb{Q})^- \backslash T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p) \to \pi(G,X). \tag{3.6.9}$$

Indeed, i sends $\mathcal{T}^\circ(\mathbb{Z}_p)$, to the kernel of $\tilde{\kappa}_G|_{G(\mathbb{Q}_p)}$, which is equal to $\tilde{G}(\mathbb{Q}_p)G(\mathbb{Z}_p)$.

Similarly, from the definitions, we have a map

$$S(\psi_{\mu_T}) \to \pi(G, i \circ \psi_{\mu_T}) = \pi(G, (i \circ \psi_{\mu_T})_0).$$

Again, this uses that i maps $\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$, to the kernel of $\tilde{\kappa}_G$, which is equal to $\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}})G(\mathbb{Z}_p^{\mathrm{ur}})$.

Proposition (3.6.10). *Let (G,X) be in $\mathscr{SH}_p$, and $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ an admissible morphism. There exists a unique system of isomorphisms of $\pi(G)$ torsors*

$$\vartheta_G : \pi(G, \phi_0) \xrightarrow{\sim} \pi(G,X)$$

with the following properties:

1. *ϑ is functorial for morphisms in $\mathscr{SH}_p(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.*
2. *For any triple (T,h,i) as above such that $h_T \in X$, and $(i \circ \psi_{\mu_T})_0$ is conjugate to ϕ_0, the diagram*

$$\begin{array}{ccc} S(\psi_{\mu_T}) & \xrightarrow{(3.6.7)} & T(\mathbb{Q})^- \backslash T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p) \\ \downarrow & & \downarrow {\scriptstyle (3.6.9)} \\ \pi(G,\phi_0) & \xrightarrow{\vartheta_G} & \pi(G,X) \end{array} \tag{3.6.11}$$

commutes.

Proof. By (3.5.11), for any admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0, there exists a (T,h,i) as above, such that $i \circ \psi_{\mu_T}$ is conjugate to ϕ. There exists a unique map of $\pi(G)$-torsors, ϑ_G so that (3.6.11) commutes for the chosen triples (T,h,i). We will show that (3.6.11) also commutes for any other triple (T',h',i') such that $\psi_{\mu_{T'}}$ is conjugate to an admissible lift of ϕ_0. Assuming this, it follows that ϑ_G satisfies (1), because the other three maps in (3.6.11) are functorial with respect to maps in $\mathscr{SH}_p(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

Let $G_1 \to G$ be a z-extension, induced by a map of reductive groups $G_{1,\mathbb{Z}_{(p)}} \to G_{\mathbb{Z}_{(p)}}$ [Ko 6, §1]. Lift X to a $G'(\mathbb{R})$-conjugacy class X_1. Then there is a map of $\mathbb{Q}$-tori $T_1 \to T$, a $h_1 : \mathbb{S} \to T_{1,\mathbb{R}}$, lifting h and a $i_1 : T_1 \to G_1$ over $\mathbb{Q}$ lifting i. Similarly, any (T',h',i') as above lifts to a triple (T_1', h_1', i_1'). As above, we define a map ϑ_{G_1} using (T_1, h_1, i_1). Then ϑ_{G_1} induces ϑ_G via $G_1 \to G$. Thus, if (3.6.11) commutes for (T_1', h_1', i_1') (with ϑ_{G_1} as the map on the bottom), then it commutes for (T', h', i'). It follows that it suffices to prove that ϑ_G satisfies (2) when $G^{\mathrm{der}} = \tilde{G}$, and we assume this from now on.

As remarked in (3.6.1), when $G^{\mathrm{der}} = \tilde{G}$, the set $X(\phi_{\widetilde{\mathrm{ab}}})$ has a more concrete description, which we will make use of here. Let $e_T \in \pi(G, \psi_{\mu_T})$ denote the image of the composite

$$S(\psi) \to S(\psi_{\mu_T}) \to \pi(G, \psi_{\mu_T}),$$

and similarly for $e_{T'} \in \pi(G, \psi_{\mu_{T'}})$. We also denote by $e_T, e_{T'}$ the images of these elements in $\pi(G, \phi_0)$. Note that the morphisms $\psi^{\mathrm{ab}}_{\mu_T}, \psi^{\mathrm{ab}}_{\mu_{T'}} : \mathfrak{Q} \to \mathfrak{G}^{\mathrm{ab}}_G$ are equal, since they are both equal to the composite of ψ and map $\mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}}\mathbb{G}_m \to G^{\mathrm{ab}}$ induced by μ_h for any $h \in X$. In particular, this means the underlying sets $\pi(G, \psi_{\mu_T})$ and $\pi(G, \psi_{\mu_{T'}})$ coincide. This is the "tautological" identification of these sets mentioned in (3.6.5). From the constructions, one sees that under this identification, the elements e_T and $e_{T'}$ coincide. We now compare this with how these sets are identified by (3.6.4).

By (3.4.11), there exists $g \in G(\bar{\mathbb{Q}})$ be such that $g\psi_{\mu_T}g^{-1} = z \cdot \psi_{\mu_{T'}}$ where $z \in Z^1(\mathbb{Q}, Z_{\tilde{G}})$, is a cocycle of the form $g_+\tau(g_+)^{-1}$ for some $g_+ \in \tilde{G}(\bar{\mathbb{Q}})^\natural_+ = G(\bar{\mathbb{Q}})^\natural_+ \cap \tilde{G}(\bar{\mathbb{Q}})$. Since $\psi^{\mathrm{ab}}_{\mu_T} = \psi^{\mathrm{ab}}_{\mu_{T'}}$, we find $\tau(g)g^{-1} \in \tilde{G}(\mathbb{Q})$ for all $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, so the image of g, g^{ab}, lies in $G^{\mathrm{ab}}(\mathbb{Q})$. By the constructions in (3.6.2) and (3.6.4), the isomorphism from $\pi(G, \psi_{\mu_T})$ and $\pi(G, \psi_{\mu_{T'}})$ in (3.6.4) is given by the tautological identification followed by left multiplication by g. Thus we have $e_{T'} = (g^{\mathrm{ab}})^{-1}e_T$ in $\pi(G, \phi_0)$.

Now let f_T denote the image of 1 under (3.6.9), and similarly for $f_{T'}$. Let $g_1 \in G(\mathbb{R})$ be such that $g_1 h g_1^{-1} = h'$. Under the isomorphism

$$G(\mathbb{Q})^- \backslash \pi_0(X) \times G(\mathbb{A}_f) \xrightarrow{\sim} G(\mathbb{Q})^-_+ \backslash G(\mathbb{A}_f)$$

the image of $g_1 \times 1$ is equal to g_2^{-1} where $g_2 \in G(\mathbb{Q})$, is any element which maps to the image of g_1 under $G(\mathbb{Q}) \to G(\mathbb{R})/G(\mathbb{R})_+$. We have $f_{T'} = g_2^{-1}f_T$. We claim that g and g_1 have the same image in $G^{\mathrm{ab}}(\mathbb{R})/Z_G(\mathbb{R})$. Assume this for a moment. Then g^{ab} lifts to an element of $G(\mathbb{R})$, and hence by the Hasse principle to an element $g' \in G(\mathbb{Q})$, as $G^{\mathrm{der}} = \tilde{G}$ is simply connected. As $G(\mathbb{R})_+ \supset Z_G(\mathbb{R})\tilde{G}(\mathbb{R})$, our assumption implies that g' and g_1 have the same image in $G(\mathbb{R})/G(\mathbb{R})_+$. Thus we can take $g_2 = g'$ above, and we see that

$$f_{T'} = g'^{-1}f_T = (g^{\mathrm{ab}})^{-1}f_T.$$

As ϑ_G takes e_T to f_T, by construction, it takes $e_{T'}$ to $f_{T'}$ for any (T', h', i'), and it follows that (3.6.11) commutes for any (T', h', i').

It remains to show the claim. Let $L \subset \bar{\mathbb{Q}}$ be a number field, such that T and T' split over L. Let μ_0 and $\xi_{\infty,0} : \mathfrak{G}_\infty \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m}$ be as in (3.4.1), and write ψ_L for the composite of ψ and the projection $\mathfrak{G}_{R_{\bar{\mathbb{Q}}/\mathbb{Q}}}\mathbb{G}_m \to \mathfrak{G}_{R_{L/\mathbb{Q}}}\mathbb{G}_m$. Then $\xi_{\infty,0} = t\psi_L \circ \zeta_\infty t^{-1}$ for some $t \in \mathfrak{G}_{R_{L/\mathbb{Q}}}\mathbb{G}_m(\bar{\mathbb{Q}})$ by (3.1.2)(2). Let $\xi_{\infty,T} : \mathfrak{G}_\infty \to \mathfrak{G}_T \to \mathfrak{G}_G$ be the composite map induced by $\xi_{\infty,0}$, and similarly for $\xi_{\infty,T'}$. The map $\pi_T : R_{L/\mathbb{Q}}\mathbb{G}_m \to T$ takes μ_0 to μ_T by definition. Hence $g_1\xi_{\infty,T}g_1^{-1} = \xi_{\infty,T'}$. Thus we have

$$\begin{aligned}\psi_{\mu_{T'}} \circ \zeta_\infty = \pi_{T'}(t)^{-1}\xi_{\infty,T'}\pi_{T'}(t) &= \pi_{T'}(t)^{-1}g_1\xi_{\infty,T}g_1^{-1}\pi_{T'}(t)\\ &= \pi_{T'}(t)^{-1}g_1\pi_T(t)\psi_{\mu_T} \circ \zeta_\infty\pi_T(t)^{-1}g_1^{-1}\pi_{T'}(t).\end{aligned}$$

Let $g_1' = \pi_{T'}(t)^{-1} g_1 \pi_T(t)$. As the image of $(\psi_{\mu_T} \circ \zeta_\infty)^\Delta$ lies in $Z_G(\mathbb{R})$,

$$g_1' \psi_{\mu_T} \circ \zeta_\infty g_1'^{-1} = \psi_{\mu_{T'}} \circ \zeta_\infty = (gg_+)\psi_{\mu_T} \circ \zeta_\infty (gg_+)^{-1}.$$

Hence $g_1'(gg_+)^{-1} \in I_{\psi_{\mu_T} \circ \zeta_\infty} \xrightarrow{\sim} G^*_{\mathbb{R}}$. In particular, the images of g_1' and gg_+ are in the same connected component in $G^{\mathrm{ab}}(\mathbb{R})$, and hence have the same image in $G^{\mathrm{ab}}(\mathbb{R})/Z_G(\mathbb{R})$. Since $\pi_T(t)$ and $\pi_{T'}(t)$ have the same image in $G^{\mathrm{ab}}(\bar{\mathbb{Q}})$, g_1 and g_1' have the same image in $G^{\mathrm{ab}}(\mathbb{R})/Z_G(\mathbb{R})$. As the image of g_+ in $G^{\mathrm{ab}}(\bar{\mathbb{Q}})$, is trivial, this proves the claim. □

(3.7) The refined conjecture: In this subsection we formulate a refinement of the Langlands-Rapoport conjecture in the style of Pfau [Pf, §3]. This refinement implies (3.3.7) but is better suited to passing between different Shimura data with the same adjoint datum. In particular it will allow us to reduce the case of abelian type Shimura data to the case of Hodge type.

(3.7.1) Let $\bar{\mathbb{Z}}_p$ denote the ring of integers of $\bar{\mathbb{Q}}_p$. We slightly abuse notation and denote by $G(\bar{\mathbb{Z}}_{(p)})_+^\natural$ the preimage of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ = G^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Z}_{(p)}) \cap G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\bar{\mathbb{Q}})$. We keep the notation of (3.4), and in particular we fix an admissible morphism $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$.

Lemma (3.7.2). *There is an action of $G(\bar{\mathbb{Z}}_{(p)})_+^\natural$ on $\coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)$ given by multiplication on the right, where the disjoint union is over admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0. This action induces a right action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $S_{\mathrm{K}}(G, \phi_0)$.*

Proof. Let $g \in G(\bar{\mathbb{Z}}_{(p)})_+^\natural$, and $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ an admissible morphism lifting ϕ_0. For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ write $z_\tau = g^{-1}\tau(g)$, so that $(z_\tau)_\tau \in Z^1(\mathbb{Q}, Z_G)$. The same argument as in the proof of (3.6.2) shows that right multiplication by g induces a bijection of $X(\phi)$ onto $X(z^{-1} \cdot \phi)$.

Now let $g_0 \in X_p(\phi)$. As in the proof of (3.6.2) one sees that $g_0 g$ is a well defined element of $G(\bar{\mathbb{Q}}_p)/G(\mathbb{Z}_p^{\mathrm{ur}})$, and that

$$\theta_{g_0 g} := \mathrm{Int}(g^{-1} g_0^{-1})(z^{-1} \cdot \phi)(p) \circ \zeta_p = z^{-1} \cdot (\mathrm{Int}(g^{-1}) \circ \theta_{g_0})$$

is induced by an unramified morphism $\theta^{\mathrm{ur}}_{g_0 g} : \mathfrak{D} \to \mathfrak{G}^{\mathrm{ur}}_G(p)$.

Finally, note that for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ a lift of σ,

$$b_{g_0 g} = z_\tau^{-1} g^{-1} b_{g_0} \tau(g) = z_\tau^{-1} g^{-1} b_{g_0} g z_\tau = g^{-1} b_{g_0} g = g'^{-1} b_{g_0} g',$$

where $g' \in G(\bar{\mathbb{Z}}_p)$ has the same image as g in $G^{\mathrm{ad}}(\mathbb{Z}_p)$. Hence

$$b_{g_0 g} \in G(\bar{\mathbb{Z}}_p) p^{-\mu} G(\bar{\mathbb{Z}}_p) \cap G(\mathbb{Q}_p^{\mathrm{ur}}) = G(\mathbb{Z}_p^{\mathrm{ur}}) p^{-\mu} G(\mathbb{Z}_p^{\mathrm{ur}}).$$

This proves the first claim of the lemma. The second claim follows, as $Z_G(\bar{\mathbb{Q}}) \subset I_{\phi_0}(\bar{\mathbb{Q}})^\natural$. □

(3.7.3) We recall the notation of [De 2, 2.0]. Let H be a group equipped with an action of a group Δ, and $\Gamma \subset H$ a Δ-stable subgroup. Suppose given a Δ-equivariant map $\varphi : \Gamma \to \Delta$ where Δ acts on itself by conjugation, and suppose that for $\gamma \in \Gamma$, $\varphi(\gamma)$ acts on H as conjugation by γ. Then the elements of the form

$(\gamma, \varphi(\gamma)^{-1})$ form a normal subgroup of the semi-direct product $H \rtimes \Delta$. We denote by $H *_\Gamma \Delta$ the quotient of $H \rtimes \Delta$ by this subgroup.

Combining the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ given by (3.7.2) and the natural action of $G(\mathbb{A}_f^p)$ on $S_\tau(G, \phi_0)$, we obtain an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p) *_{G(\mathbb{Z}_{(p)})_+} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $S_\tau(G, \phi_0)$, where $G(\mathbb{Z}_{(p)})_+ = G_{\mathbb{Z}_p}(\mathbb{Z}_p) \cap G(\mathbb{Q})_+$. Write $G(\mathbb{Z}_{(p)})_+^-$ for the closure of $G(\mathbb{Z}_{(p)})_+$ in $G(\mathbb{A}_f^p)$, and set $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ = G(\mathbb{Z}_{(p)})_+^- *_{G(\mathbb{Z}_{(p)})_+} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$.

Note that there is natural projection $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \to \pi(G)$ sending (h, g) to the class of h. In particular, there is a natural action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ on $\pi(G, \phi_0)$ via the $\pi(G)$-torsor structure on $\pi(G, \phi_0)$ given by (3.6.2)(3).

Lemma (3.7.4). *The natural projections $X(\phi) \to X(\phi_{\widetilde{\mathrm{ab}}})$ induce a $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant, surjective map*

$$c_G : S_\tau(G, \phi_0) \to \pi(G, \phi_0).$$

Proof. We use the notation of (3.6.4). We saw in that lemma that for $i \in I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural$, there exists $g \in \tilde{G}(\bar{\mathbb{Q}})_+^\natural$ such that $g^{-1}\tau(g) = z_\tau := i^{-1}\tau(i)$, and that right multiplication by g and left multiplication by i induce the same bijection between $X(\phi_{\widetilde{\mathrm{ab}}})$ and $X(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$. Similarly, if $h \in X(\phi_{\widetilde{\mathrm{ab}}})$, then $\tau i \tau^{-1} h g^{-1} = h$ in $X(\phi_{\widetilde{\mathrm{ab}}})$. Thus $\tau i \tau^{-1}$ and g induce the same bijection, and there is a natural map

$$I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural \backslash \coprod_{\phi^{\mathrm{ad}} = \phi_0} X(\phi) \to \coprod_{\phi^{\mathrm{ad}} = \phi_0} X(\phi_{\widetilde{\mathrm{ab}}}) / \tilde{G}(\bar{\mathbb{Q}})_+^\natural$$

($I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural$ acting via the conjugate of the natural action by τ).

Since $I_{\phi_0}(\bar{\mathbb{Q}})^\natural = I_{\tilde{\phi}_0}(\bar{\mathbb{Q}})^\natural \cdot Z_G(\bar{\mathbb{Q}})$, the map in the lemma is well defined. By [Ki 2, 3.3.3] the natural map

$$G(\mathbb{Z}_{(p)})_+^- \backslash G(\mathbb{A}_f^p) \to G(\mathbb{Q})_+^- \backslash G(\mathbb{A}_f) / G_{\mathbb{Z}_p}(\mathbb{Z}_p) \tag{3.7.5}$$

is a bijection. Hence the surjectivity follows from the $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariance. $\square$

(3.7.6) By [De 2, §2.7], $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ is equipped with an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ extending the natural action of $G(\mathbb{A}_f^p)$, and the map

$$\mathrm{Sh}_{\mathsf{K}_p}(G, X)(\mathbb{C}) \to \pi(G, X),$$

taking a point to its connected component is $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant, with $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ acting on $\pi(G, X)$ via the projection $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \to \pi(G)$.

If $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ admits an integral canonical model $\mathscr{S}_{\mathsf{K}_p}(G, X)$ in the sense of [Mi 2], then we obtain an induced $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant map

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p) \to \pi(G, X) \tag{3.7.7}$$

identifying the geometrically connected components of $\mathscr{S}_{\mathsf{K}_p}(G, X)$ with $\pi(G, X)$. The results of [Ki 2] imply that $\mathscr{S}_{\mathsf{K}_p}(G, X)$ exists if (G, X) is of abelian type and $p > 2$.

As in (1.4.2) we denote by Φ the geometric r-Frobenius on $\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$. Recall the isomorphism $\vartheta_G : \pi(G, \phi_0) \xrightarrow{\sim} \pi(G, X)$ constructed in (3.6.10).

Conjecture (3.7.8). *Let (G, X) be a Shimura datum, and $\mathsf{K}_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ a hyperspecial subgroup. Then there exists a $Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant commutative diagram*

$$\begin{array}{ccc} \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) & \xrightarrow{\sim} & \coprod_{[\phi_0]} S(G,\phi_0) \\ \downarrow & & \downarrow \\ \pi(G,X) & \xrightarrow[\vartheta_G]{} & \coprod_{[\phi_0]} \pi(G,\phi_0) \end{array} \tag{3.7.9}$$

compatible with the action of the operator Φ. Here ϕ_0 runs over representatives for the distinct conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$.

(3.7.10) Remarks:

(1) That (3.7.8) implies (3.3.7) follows immediately from (3.4.16).

(2) To prove the conjecture, it suffices to construct a diagram which is $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant and compatible with Φ. To see this note that $Z_G(\mathbb{Q})$ (embedded diagonally in $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$) and $Z_G(\mathbb{Z}_p) = Z_{G_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_p)$ act trivially on both sides of the conjectured isomorphism. Thus the $Z_G(\mathbb{Q}_p)$-equivariance follows from the $G(\mathbb{A}_f^p)$-equivariance once we check that $Z_G(\mathbb{Z}_p) \cdot Z_G(\mathbb{Q}) = Z_G(\mathbb{Q}_p)$.

Now $Z_G^0(\mathbb{Z}_p) \cdot Z_G^0(\mathbb{Q}) = Z_G^0(\mathbb{Q}_p)$ by [CSu, Prop. 2.1,2.2]. On the other hand, the maps

$$Z_G(\mathbb{Z}_p) \to (Z_G/Z_G^0)(\mathbb{Z}_p) \to (Z_G/Z_G^0)(\mathbb{Q}_p)$$

are surjective, the first since $H^1(\mathbb{Z}_p, Z_G^0) = 0$ by Lang's lemma, and the second because Z/Z^0 is a finite flat group scheme over $\mathbb{Z}_p$. The equality $Z_G(\mathbb{Z}_p) \cdot Z_G(\mathbb{Q}) = Z_G(\mathbb{Q}_p)$ follows.

(3) We will prove some cases of the conjecture with $S_\tau(G, \phi_0)$ in place of $S(G, \phi_0)$, in §4.

(3.8) Connected components again: Let $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ be an admissible morphism. In this subsection we study the fibres of the map $S_\tau(G, \phi_0) \to \pi(G, \phi_0)$. These correspond via (3.7.8) to connected components of Shimura varieties. In analogy with the theory of Shimura varieties, we show that these subsets depend only on the adjoint Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and the derived group G^{der}, and that they can be used to reconstruct $S_\tau(G, \phi_0)$.

(3.8.1) For the remainder of the paper, we fix a connected component $X^+ \subset X^{\mathrm{ad}}$. We denote by $\mathscr{SH}_p(G^{\mathrm{ad}}, X^+)$ the full subcategory of $\mathscr{SH}_p(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ consisting of objects (H, Y) such that $X^+ \subset Y$. Suppose (G, X) is in $\mathscr{SH}_p(G^{\mathrm{ad}}, X^+)$. Then under the composite map

$$\pi_0(X) \to \pi(G, X) \xrightarrow[\vartheta_G^{-1}]{\sim} \pi(G, \phi_0)$$

X^+ maps to a point $y \in \pi(G, \phi_0)$. We set $S_\tau(G, \phi_0)^+ = c_G^{-1}(y)$.

Suppose that $h : G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}}$ is a surjective map of reductive groups over $\mathbb{Z}_{(p)}$, which induces an isomorphism on derived groups. Write $G_2 = G_{2,\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$. We denote by μ_2 the image of μ in $X_*(G_2)$, and $y_2 = h(y) \in \pi(G_2, \phi_0)$. More generally, we denote by a subscript "2" the analogue for G_2 of a construction for G - e.g $I_{2,\phi_0}(\bar{\mathbb{Q}})^\natural$.

Lemma (3.8.2). *The natural map* $S_{\tau}(G,\phi_0)^+ \to S_{\tau}(G_2,\phi_0)^+$ *is a bijection.*

Proof. Let $\mathscr{A}(G,G_2) = \ker(\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \to \mathscr{A}(G_{2,\mathbb{Z}_{(p)}}))$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{A}(G_{\mathbb{Z}_{(p)}})^{\circ} & \longrightarrow & \mathscr{A}(G_{\mathbb{Z}_{(p)}}) & \longrightarrow & G(\mathbb{A}_f^p)/G(\mathbb{Z}_{(p)})_+^- & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^{\circ} & \longrightarrow & \mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) & \longrightarrow & G_2(\mathbb{A}_f^p)/G_2(\mathbb{Z}_{(p)})_+^- & \longrightarrow & 0
\end{array}$$

Since $G^{\mathrm{der}} = G_2^{\mathrm{der}}$, the map on the left is a bijection [Ki 2, 3.3.2], [De 2, 2.0.12]. Hence, the natural map

$$\mathscr{A}(G,G_2) \to \ker(G(\mathbb{A}_f^p)/G(\mathbb{Z}_{(p)})_+^- \to G_2(\mathbb{A}_f^p)/G_2(\mathbb{Z}_{(p)})_+^-)$$

is a bijection, and $\mathscr{A}(G,G_2)$ acts simply transitively on $\ker(\pi(G,\phi_0) \to \pi(G_2,\phi_0))$ by (3.6.2)(3) and (3.7.5).

We now check that the map

$$h^{-1}(S_{\tau}(G_2,\phi_0)^+) \to S_{\tau}(G_2,\phi_0)^+ \tag{3.8.3}$$

is surjective and that $\mathscr{A}(G,G_2)$ acts transitively on its fibres. Assuming this, the lemma follows easily.

We begin by checking the statement regarding transitivity. For $i = 1,2$ let $\phi_i : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism lifting ϕ_0, and let $x_i \in X(\phi_i)$. Let $h_*(\phi_i) : \mathfrak{Q} \to \mathfrak{G}_{G_2}$ denote the morphism induced by ϕ_i. Suppose that x_1 and x_2 map to the same element of $S_{\tau}(G_2,\phi_0)^+$. We want to show their images in $S_{\tau}(G,\phi_0)$ differ by an element of $\mathscr{A}(G,G_2)$. Since the map $I_{\phi_0}(\bar{\mathbb{Q}})^{\natural} \to I_{2,\phi_0}(\bar{\mathbb{Q}})^{\natural}$ is surjective, we may assume that $h_*(\phi_1) = h_*(\phi_2)$, and that x_1 and x_2 have the same image in $X(h_*(\phi_1))$.

Let $Z = \ker(G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}})$. By (3.4.11), we have $\phi_2 = z \cdot \phi_1$, where $z = (z_{\tau}) \in Z^1(\mathbb{Q}, Z)$ and $z_{\tau} = g^{-1}\tau(g)$ for some $g \in G(\bar{\mathbb{Q}})_+^{\natural}$ with $h(g) \in G_2(\mathbb{Q})_+$. Write $x_1 = g_0 \times (g_l)_{l \neq p} \in G(\bar{\mathbb{A}}_f)$ and $x_2 = g_0 z_0 \times (g_l z_l)_{l\neq p}$, where $(z_l)_{l \neq p} \in Z(\bar{\mathbb{A}}_f)$ and $z_0 \in G(\bar{\mathbb{Q}}_p)$ is in the preimage of $G_2(\mathbb{Z}_p^{\mathrm{ur}})$. Then, as in (3.3.3), the condition (3.3.6)(3) applied to ϕ_1, ϕ_2, implies that for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$, we have

$$g_0^{-1}\phi_1 \circ \zeta_p(1 \rtimes \tau)\tau(g_0) = z_0^{-1}g_0^{-1}z_{\tau}\phi_1 \circ \zeta_p(1\rtimes\tau)\tau(g_0 z_0) = 1,$$

and hence that $z_0^{-1}z_{\tau}\tau(z_0) = 1$.

Similarly, applying (3.3.6)(3) again, for any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, the image of $z_0^{-1}z_{\tau}\tau(z_0)$ in $G^{\mathrm{ab}}(\bar{\mathbb{Q}}_p)$ lies in $G^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$. Since $z_0^{-1}z_{\tau}\tau(z_0)$ also maps into $G_2(\mathbb{Z}_p^{\mathrm{ur}})$, we have $z_0^{-1}z_{\tau}\tau(z_0) \in G(\mathbb{Z}_p^{\mathrm{ur}})$. Hence $z'_{\tau} = z_0^{-1}z_{\tau}\tau(z_0) \in Z^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p), Z(\mathbb{Z}_p^{\mathrm{ur}}))$. By étale descent, the cocycle z'_{τ} corresponds to an étale Z-torsor. As the induced $G_{\mathbb{Z}_p}$-torsor is necessarily trivial, there exists $z_1 \in G(\mathbb{Z}_p^{\mathrm{ur}})$ such that $z_1^{-1}\tau(z_1) = z'_{\tau}$. Replacing z_0 by $z_0 z_1^{-1}$ (which does not change the class of $g_0 z_0$ in $X_p(\phi_2)$), we may assume that $z_{\tau} = z_0^{-1}\tau(z_0)$.

It follows that $z_0^{-1}g \in G(\mathbb{Q}_p)$. Since $G(\mathbb{Z}_p)\cdot G(\mathbb{Q})^+ = G(\mathbb{Q}_p)$ [Ki 2, 2.2.6] (cf. [Sa, Cor. 3.5]), after multiplying g on the right by an element of $G(\mathbb{Q})^+$, we may assume that $z_0^{-1}g \in G(\mathbb{Z}_p)$.

Write $g^{\mathrm{ad}} \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ for the image of g. For $l \neq p$, (3.3.6)(2) implies that $z_l\tau(z_l)^{-1} = z_\tau = \tau(z_l)^{-1}z_l$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$. Hence $z_lg^{-1} \in G(\mathbb{Q}_l)$. It follows that $\eta = (z_lg^{-1})_{l\neq p} \rtimes g^{\mathrm{ad}} \in \mathscr{A}(G_{\mathbb{Z}_{(p)}})$ lies in $\mathscr{A}(G, G_2)$, and under the action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ on $S_\mathrm{\tau}(G, \phi_0)$ defined in (3.7.3), η sends x_1 to x_2. This proves the claim regarding transitivity.

Finally we check that (3.8.3) is surjective. By (3.4.11), any admissible morphism $\mathfrak{Q} \to \mathfrak{G}_{G_2}$ lifting ϕ_0, has the form $h_*(\phi)$ for some admissible $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0. Suppose $x_2 \in X(h_*(\phi))$ satisfies $c_{G_2}(x_2) = y_2$, and let $\bar{x}_2$ denote the image of x_2 in $X(h_*(\phi)_{\widetilde{\mathrm{ab}}})$.

Let $\bar{x} \in X(\phi_{\widetilde{\mathrm{ab}}})$ be an element which maps to y. By (3.6.2), there is a $g_2 \in G_2(\mathbb{Q})_+^-$ such that $h(\bar{x}) \cdot g_2 = \bar{x}_2$. By [De 2, 2.0.13] we may write $g_2 = j_2g_2'z_2$ where $g_2' \in G_2(\mathbb{Q})_+$, $j_2 \in G_2^{\mathrm{der}}(\mathbb{Q})_+^- = G^{\mathrm{der}}(\mathbb{Q})_+^-$ and $z_2 \in Z_{G_2}(\mathbb{Q})^-$. We may replace $\bar{x}$ by $\bar{x}j_2$, and assume that $j_2 = 1$. Then $h(\bar{x}) \cdot g_2' = z_2^{-1}\bar{x}_2$ in $X(\phi_{\widetilde{\mathrm{ab}}})$. Since $x_2 = z_2^{-1}x_2$ as elements of $S_\mathrm{\tau}(h_*(\phi))$, we may assume that $g_2 = g_2' \in G_2(\mathbb{Q})_+$.

Let $g \in G(\bar{\mathbb{Q}})_+^\natural$ be a lift of g_2, and $z = (g^{-1}\tau(g)) \in Z^1(\mathbb{Q}, Z)$ the cocycle associated with g. Then multiplication by g induces a bijection between $X(\phi_{\widetilde{\mathrm{ab}}})$ and $X(z^{-1} \cdot \phi_{\widetilde{\mathrm{ab}}})$. Hence, if we replace ϕ by $z^{-1}\phi$, which does not change $h_*(\phi)$, then we may assume that $h(\bar{x}) = \bar{x}_2$.

Now since $G^{\mathrm{der}} = G_2^{\mathrm{der}}$ we have $G = G_2 \times_{G_2^{\mathrm{ab}}} G^{\mathrm{ab}}$. Note that any $G^{\mathrm{der}}_{2,\mathbb{Z}_p}$-torsor over $\mathbb{Z}_p^{\mathrm{ur}}$ is trivial, so the map $G_2(\mathbb{Z}_p^{\mathrm{ur}}) \to G_2^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$ is surjective. Using this one checks easily that the map

$$X_p(\phi) \to X_p(h_*(\phi)) \times_{X_p(h_*(\phi)_{\mathrm{ab}})} X_p(\phi_{\mathrm{ab}})$$

is a bijection. Similarly

$$X^p(\phi) \to X^p(h_*(\phi)) \times_{X^p(h_*(\phi)_{\mathrm{ab}})} X^p(\phi_{\mathrm{ab}})$$

is a bijection. Thus the pair $(x_2, \bar{x}) \in X(h_*(\phi)) \times_{X(h_*(\phi)_{\mathrm{ab}})} X(\phi_{\mathrm{ab}})$ corresponds to an element $x \in X(\phi)$ which maps to x_2. This proves the surjectivity. □

(3.8.4) Recall that E is the the reflex field of (G, X), and μ is defined over $E_p = \mathbb{Q}_{p^r} \subset \mathbb{Q}_p^{\mathrm{ur}}$. Let $\langle\Phi\rangle \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^r})$ denote the subgroup generated by the geometric Frobenius $\Phi = \Phi_r$. Using the action of the r-Frobenius on $S_\mathrm{\tau}(\phi)$, defined as in (3.3.6), we obtain an action of $\langle\Phi\rangle$ on $S_\mathrm{\tau}(\phi)$ for any admissible $\phi : \mathfrak{Q} \to \mathfrak{G}_G$.

Let $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ an admissible morphism, and $y \in \pi(G, \phi_0)$. Let

$$\mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0) \subset \mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$$

denote the stabilizer of $S_\mathrm{\tau}(G, \phi_0)^+ \subset S_\mathrm{\tau}(G, \phi_0)$.

Lemma (3.8.5). *$\mathscr{E}_p^r(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}) = \mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0)$ is an extension of $\langle\Phi\rangle$ by $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$ and depends only on $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$, X^+, and the integer r. In particular, this extension does not depend on $\mathrm{\tau}$ or ϕ_0.*

Proof. We begin by checking that $\mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0)$ does not depend on $\mathrm{\tau}$ or ϕ_0. Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism lifting ϕ_0 and $x = (x_p, x^p) \in X(\phi)$. Suppose that $c_G(x) = y$. Then using the notation of (3.3.6), the image of b_{x_p} in $\tilde{G}(\mathbb{Q}_p^{\mathrm{ur}})\backslash G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathbb{Z}_p^{\mathrm{ur}}}(\mathbb{Z}_p) = \pi_1(G)$ is $[-\mu]$. Hence for $n \in \mathbb{Z}$,

$$c_G(\Phi^n(x)) = c_G(x)N_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}([-\mu])^n \in \pi(G, \phi_0)$$

where we regard $N_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}([-\mu])$ as an element of

$$\tilde{G}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subset G(\mathbb{Q})_+^-\backslash G(\mathbb{A}_f)/G_{\mathbb{Z}_p}(\mathbb{Z}_p) = \pi(G).$$

Thus $(h, \Phi^n) \in \mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$ is in $\mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0)$ if and only if the image of h in $\pi(G)$ is $N_{\mathbb{Q}_{p^r}/\mathbb{Q}_p}([-\mu])^{-n}$. This condition does not depend on τ or ϕ_0. Moreover, the isomorphism (3.7.5) shows that $\mathscr{E}_p^r(G_{\mathbb{Z}_{(p)}}) = \mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0)$ is an extension of $\langle\Phi\rangle$ by $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$.

Now suppose that $G_{2,\mathbb{Z}_{(p)}} \to G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ is a morphism which induces an isomorphism on adjoint groups and an identification $G^{\mathrm{der}}_{2,\mathbb{Z}_{(p)}} = G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$, and μ_2 a cocharacter of G_2 which is defined over $\mathbb{Q}_{p^r}$, and such that the cocharacters of G^{ad} induced by μ_2 and μ are conjugate.

To show that $\mathscr{E}_p^r(G_{\mathbb{Z}_{(p)}}) = \mathscr{E}_p^r(G_{2,\mathbb{Z}_{(p)}})$ suppose first that there is a surjection $G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}}$ inducing $G_{2,\mathbb{Z}_{(p)}} \to G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$. Then the bijection of (3.8.2) implies that there is a natural map $\mathscr{E}_p^r(G_{\mathbb{Z}_{(p)}}) \to \mathscr{E}_p^r(G_{2,\mathbb{Z}_{(p)}})$, and since $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$ depends only on $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$ [Ki 2, 3.3.7], this map is an isomorphism.

In general let $G_{3,\mathbb{Z}_{(p)}} = G_{\mathbb{Z}_{(p)}} \times_{G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}} G_{2,\mathbb{Z}_{(p)}}$. Applying the observation of the previous paragraph shows that

$$\mathscr{E}_p^r(G_{\mathbb{Z}_{(p)}}) = \mathscr{E}_p^r(G_{3,\mathbb{Z}_{(p)}}) = \mathscr{E}_p^r(G_{2,\mathbb{Z}_{(p)}}).$$

□

Corollary (3.8.6). *The pair $(S_\tau(G, \phi_0)^+, \mathscr{E}_p(G, \phi_0))$ considered as a set equipped with an action of a group depends only on τ, ϕ_0, $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$, and X^+. More precisely, any morphism in $\mathscr{SH}_p(G^{\mathrm{ad}}, X^+)$, which induces an isomorphism on derived groups, induces a functorial isomorphism between the corresponding pairs.*

Proof. This can be seen as in the proof of (3.8.5), using (3.8.2) □

(3.8.7) Write $S_\tau(G^{\mathrm{der}}, \phi_0)^+ = S_\tau(G, \phi_0)^+$ equipped with its action of $\mathscr{E}_p^r(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}})$.

Lemma (3.8.8).

(1) *There is a natural isomorphism*

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}_p^r(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}) \xrightarrow{\sim} \mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$$

(2) *The actions of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ and $\mathscr{E}_p^r(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}})$ on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_\tau(G^{\mathrm{der}}, \phi_0)^+$ given by $(a, s)a' = (aa', s)$ and $(a, s)e = (\bar{e}^{-1}a\bar{e}, se)$ respectively (here $\bar{e}$ denotes the image of e in $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$) induce an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$ on $[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_\tau(G^{\mathrm{der}}, \phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$.*

(3) *There is a natural isomorphism of sets with $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle\Phi\rangle$-action*

$$S_\tau(G, \phi_0) \xrightarrow{\sim} [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_\tau(G^{\mathrm{der}}, \phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ. \tag{3.8.9}$$

Proof. (1) is a formal consequence of (3.8.5). The actions of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ and $\mathscr{E}^r_p(G^{\rm der}_{\mathbb{Z}_{(p)}})$ defined in (2) induce an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}^r_p(G^{\rm der}_{\mathbb{Z}_{(p)}})$ on $[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_{\rm T}(G^{\rm der},\phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$ and hence an action of $\langle\Phi\rangle \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})$ via the isomorphism in (1).

The natural inclusion $S_{\rm T}(G^{\rm der},\phi_0)^+ \subset S_{\rm T}(G,\phi_0)$ is $\mathscr{E}^r_p(G^{\rm der}_{\mathbb{Z}_{(p)}})$-equivariant by definition, and has a unique extension to a $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}^r_p(G^{\rm der}_{\mathbb{Z}_{(p)}})$-equivariant map

$$[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_{\rm T}(G^{\rm der},\phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \to S_{\rm T}(G,\phi_0).$$

Since $\mathscr{A}(G_{\mathbb{Z}_{(p)}})/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \xrightarrow{\sim} \pi(G)$ this map is a bijection. One checks easily that it is compatible with the action of $\langle\Phi\rangle$. □

Lemma (3.8.10). *Let $G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}}$ be a surjective map of reductive groups over $\mathbb{Z}_{(p)}$ with kernel $Z \subset Z_G$. We regard G_2, as equipped with the cocharacter μ_2, induced by μ. Suppose that Z is an induced torus. Then*

(1) *There is natural isomorphism*

$$S_{\rm T}(G_2,\phi_0) \xrightarrow{\sim} S_{\rm T}(G,\phi_0)/Z(\mathbb{A}^p_f) = [\mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \times S_{\rm T}(G,\phi_0)]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})$$

(2) *There is a natural isomorphism*

$$\mathscr{E}^r_p(G^{\rm der}_{2,\mathbb{Z}_{(p)}}) \xrightarrow{\sim} \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}^r_p(G^{\rm der}_{\mathbb{Z}_{(p)}}).$$

and a natural isomorphism of sets with $\mathscr{E}^r_p(G_{2,\mathbb{Z}_{(p)}})$-action

$$S_{\rm T}(G_2^{\rm der},\phi_0)^+ = [\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ \times S_{\rm T}(G^{\rm der},\phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ.$$

Proof. Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism lifting ϕ_0, and $\phi_2 : \mathfrak{Q} \to \mathfrak{G}_{G_2}$ the morphism induced by ϕ. We begin by showing that $X_p(\phi_2) \xrightarrow{\sim} X_p(\phi)/Z(\mathbb{Q}_p)$.

Let $g_0 \in X_p(\phi)$, and $g_{0,2}$ its image in $X_p(\phi_2)$. By (3.3.4), any element of $X_p(\phi_2)$ has the form $g_{0,2}h_2$ for some h_2 in $G_2(\mathbb{Q}_p^{\rm ur})$ which satisfies $h_2^{-1}b_{g_{0,2}}\sigma(h_2) \in G_2(\mathbb{Z}_p^{\rm ur})p^{-\mu_2}G_2(\mathbb{Z}_p^{\rm ur})$. Let η_2 be a cocharacter of G_2, defined over $\mathbb{Q}_p^{\rm ur}$, such that h_2 is in the η_2 part of the Cartan decomposition. Let η be a cocharacter of G, defined over $\mathbb{Q}_p^{\rm ur}$ and lifting η_2. Note that by modifying η by an element of $X_*(Z)$ we may choose η so that its image in $\pi_1(G)$ is any given element which maps to the image of η_2 in $\pi_1(G_2)$.

Since Z is an induced torus, the ${\rm Gal}(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p)$-module $X_*(Z)$ has trivial cohomology. Hence, the map

$$\pi_1(G)^{{\rm Gal}(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p)} \to \pi_1(G_2)^{{\rm Gal}(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p)}$$

is surjective, so we may assume that the image of η lies in $\pi_1(G)^{{\rm Gal}(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p)}$. Since any element of $G_2(\mathbb{Z}_p^{\rm ur})$ lifts to $G(\mathbb{Z}_p^{\rm ur})$, this shows that there exists $h \in G(\mathbb{Q}_p^{\rm ur})$ lifting h_2 and such that $\tilde\kappa_G(h) \in \pi_1(G)^{{\rm Gal}(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p)}$. Since $b_{g_0} \in G(\mathbb{Z}_p^{\rm ur})p^{-\mu}G(\mathbb{Z}_p^{\rm ur})$, this implies $\tilde\kappa_G(h^{-1}b_{g_0}\sigma(h)) = [-\mu]$, and hence $h^{-1}b_{g_0}\sigma(h) \in G(\mathbb{Z}_p^{\rm ur})p^{-\mu}G(\mathbb{Z}_p^{\rm ur})$ as h lifts h_2.

This shows that $X_p(\phi) \to X_p(\phi_2)$ is surjective. Two elements of $X_p(\phi)$ lie in the same fibre of this map if and only if they differ by an element of

$$(Z(\mathbb{Q}_p^{\mathrm{ur}})/Z(\mathbb{Z}_p^{\mathrm{ur}}))^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} = Z(\mathbb{Q}_p)/Z(\mathbb{Z}_p).$$

This shows that $X_p(\phi_2) \xrightarrow{\sim} X_p(\phi)/Z(\mathbb{Q}_p)$.

Since Z is an induced torus, we obviously have $X^p(\phi_2) = X_p(\phi)/Z(\mathbb{A}_f^p)$. It follows that the map $I_\phi(\mathbb{Q})\backslash X(\phi)/Z(\mathbb{A}_f^p) \to I_{\phi_2}(\mathbb{Q})\backslash X(\phi_2)$ is surjective, and it is injective since $Z(\mathbb{Q}) \subset I_\phi(\mathbb{Q})$ is dense in $Z(\mathbb{Q}_p)$ and the map $I_\phi(\mathbb{Q}) \to I_{\phi_2}(\mathbb{Q})$ surjective. This implies that $S_\tau(\phi)/Z(\mathbb{A}_f^p) \xrightarrow{\sim} S_\tau(\phi_2)$.

Again using the fact that Z is an induced torus, and (3.4.11), one sees easily that the map $\phi \mapsto \phi_2$ induces a bijection between conjugacy classes of admissible $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0 and conjugacy classes of admissible $\phi : \mathfrak{Q} \to \mathfrak{G}_{G_2}$ lifting ϕ_0. Hence, by (3.4.16), we have a bijection $S_\tau(G_2, \phi_0) \xrightarrow{\sim} S_\tau(G, \phi_0)/Z(\mathbb{A}_f^p)$. This proves (1).

Now let $y \in \pi(G, \phi_0)$ and $y_2 \in \pi(G_2, \phi_0)$ the image of y. There is a natural map $\mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0) \to \mathscr{E}_p(G_{2,\mathbb{Z}_{(p)}}, \phi_0)$, which induces a map

$$\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0) \to \mathscr{E}_p(G_{2,\mathbb{Z}_{(p)}}, \phi_0),$$

for example using the description of $\mathscr{E}_p(G_{\mathbb{Z}_{(p)}}, \phi_0)$ in the proof of (3.8.5). This map is an isomorphism since both sides are extension of $\langle\Phi\rangle$ by $\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ$. This proves (2).

Similarly, we obtain a $\mathscr{E}_p(G_{2,\mathbb{Z}_{(p)}}, \phi_0)$-equivariant map

$$[\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ \times S_\tau(G, \phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \to S_\tau(G_2, \phi_0)^+. \tag{3.8.11}$$

Since $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \subset \mathscr{A}(G_{\mathbb{Z}_{(p)}})$ is the stabilizer of $S_\tau(G, \phi_0)^+ \subset S_\tau(G, \phi_0)$, (1) implies that (3.8.11) is injective. To see that it is surjective, let $g_2 \in c_{G_2}^{-1}(y_2)$. By what we saw above, there exists $g \in S_\tau(G, \phi_0)$ mapping to g_2, and $h \in G(\mathbb{A}_f^p)$ such that $c_G(g \cdot h) = y$. If h_2 is the image of h in $G_2(\mathbb{A}_f^p)$, then $h_2 \in G_2(\mathbb{Z}_{(p)})_+^-$ by (3.6.2) and (3.7.5), and (h_2^{-1}, gh) maps to g_2 under (3.8.11). □

Corollary (3.8.12). *Let $G_{2,\mathbb{Z}_{(p)}}$ be a reductive group over $\mathbb{Z}_{(p)}$ with generic fibre G_2, and (G_2, X_2) a Shimura datum. Set $r_2 = [E(G_2, X_2)_p : \mathbb{Q}_p]$.*

Suppose that $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \to G^{\mathrm{der}}_{2,\mathbb{Z}_{(p)}}$ is a central isogeny which induces an isomorphism of adjoint Shimura data $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and let r' be an integer divisible by r, r_2. Then there is an isomorphism

$$\mathscr{E}_p^{r'}(G^{\mathrm{der}}_{2,\mathbb{Z}_{(p)}}) \xrightarrow{\sim} \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ *_{\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ} \mathscr{E}_p^{r'}(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}),$$

an isomorphism of sets with $\mathscr{E}_p^{r'}(G^{\mathrm{der}}_{2,\mathbb{Z}_{(p)}})$-action

$$S_\tau(G_2^{\mathrm{der}}, \phi_0)^+ = [\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})^\circ \times S_\tau(G^{\mathrm{der}}, \phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ,$$

and an isomorphism of sets with $\mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \times \langle\Phi_{r'}\rangle$-action

$$S_\tau(G_2, \phi_0) = [\mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \times S_\tau(G^{\mathrm{der}}, \phi_0)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ. \tag{3.8.13}$$

Proof. We begin by proving the first two statements. Let $G_{3,\mathbb{Z}_{(p)}} = G_{\mathbb{Z}_{(p)}} \times_{G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}} G_{2,\mathbb{Z}_{(p)}}$. Then $G^{\mathrm{der}}_{3,\mathbb{Z}_{(p)}} = G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$. By (3.8.5) and (3.8.6) we may replace G by G_3, and assume that the isomorphism $G_2^{\mathrm{ad}} \xrightarrow{\sim} G^{\mathrm{ad}}$ is induced by a map of reductive groups $G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}}$.

Let $Z = \ker(G_{\mathbb{Z}_{(p)}} \to G_{2,\mathbb{Z}_{(p)}})$, and embed Z in an induced torus Z'. Using (3.8.5) and (3.8.6) again, we may replace $G_{\mathbb{Z}_{(p)}}$ by $(G_{\mathbb{Z}_{(p)}} \times Z')/Z$, and assume that Z is an induced torus. In this case the first two statements follow from (3.8.10).

The final statement now follows from (3.8.8). □

§4 Counting points mod p

(4.1) Twisting abelian varieties: We recall, and slightly extend some of the constructions of [Ki 2, §3.1].

(4.1.1) Let Z be any affine $\mathbb{Q}$-group, and $\mathcal{P}$ a right Z-torsor. Let $\mathcal{O}_{\mathcal{P}}$ be the affine ring of $\mathcal{P}$. We give $\mathcal{O}_{\mathcal{P}}$ a left action of Z via $(zf)(x) = f(xz)$, for z a point of Z, and $f \in \mathcal{O}_{\mathcal{P}}$. For any $\mathbb{Q}$-vector space W equipped with an action of Z, we will write $W^{\mathcal{P}} = (W \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}})^Z$. We have

Lemma (4.1.2).

(1) *The functor $W \mapsto W^{\mathcal{P}}$ is exact, and compatible with tensor products and duals. The natural map*

$$W^{\mathcal{P}} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} \to W \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} \tag{4.1.3}$$

is an isomorphism.

(2) *Let $F/\mathbb{Q}$ be a finite Galois extension and $\tilde{\omega} \in \mathcal{P}(F)$. Specializing (4.1.3) by $\tilde{\omega}$ gives rise to an isomorphism*

$$\vartheta_{\tilde{\omega}} : W^{\mathcal{P}} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} W \otimes_{\mathbb{Q}} F.$$

For $\tau \in \mathrm{Gal}(F/\mathbb{Q})$ we have

$$\tau \circ \vartheta_{\tilde{\omega}} \circ \tau^{-1} = c_{\tilde{\omega}}(\tau)^{-1} \vartheta_{\tilde{\omega}},$$

where $c_{\tilde{\omega}}(\tau) = \tilde{\omega}^{-1}\tau(\tilde{\omega}) \in Z(F)$.

Proof. The first claim follows from [Ki 2, 3.1.1]. To see the claim in (2) let

$$\sum w_i \otimes f_i \in W^{\mathcal{P}} \subset W \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}}.$$

Then for z a section of Z and x a section of $\mathcal{P}$, we have $\sum_i w_i \otimes f_i(x) = \sum_i zw_i \otimes f_i(xz)$ so that

$$\sum_i w_i \otimes f_i(xz) = \sum_i z^{-1} w_i \otimes f_i(x). \tag{4.1.4}$$

In particular,

$$\begin{aligned} \tau \circ \vartheta_{\tilde{\omega}} \circ \tau^{-1}(\sum w_i \otimes f_i(\tilde{\omega})) &= \sum_i w_i \otimes f_i(\tau(\tilde{\omega})) \\ &= \sum_i w_i \otimes f_i(\tilde{\omega} c_{\tilde{\omega}}(\tau)) = \sum_i c_{\tilde{\omega}}(\tau)^{-1} w_i \otimes f_i(\tilde{\omega}) \end{aligned}$$

(cf. [Ki 2, 3.2.5][21]). The compatibility with duals and tensor products follows easily from (2). □

[21] Note that in [Ki 2], $c_{\tilde{\omega}}(\tau)$ is defined to be $\tau(\tilde{\omega})\tilde{\omega}^{-1}$. However in the situation of that paper $\tilde{\omega} \in G(F)$ with G a $\mathbb{Q}$-group with center Z, so that we have $\tau(\tilde{\omega})\tilde{\omega}^{-1} = \tilde{\omega}^{-1}\tau(\tilde{\omega})$.

Lemma (4.1.5). *Let R be a $\mathbb{Q}$-algebra, and W a finite free R-module equipped with an action of Z. Let $(s_\alpha) \subset W^{\otimes}$ be a collection of Z-invariant tensors defining an R-subgroup $G_R \subset \mathrm{GL}(W)$.*

Then the s_α are naturally elements of $W^{\mathcal{P}\otimes}$, and the R-scheme of isomorphisms $W^{\mathcal{P}} \xrightarrow{\sim} W$ respecting the s_α is isomorphic to the G_R-torsor $(\mathcal{P} \times G_R)/Z$ where Z acts on G_R by $(z, g) \mapsto z^{-1}g$.

Proof. The compatibility with tensor products and duals in (4.1.2) implies that we have $W^{\mathcal{P}\otimes} \xrightarrow{\sim} W^{\otimes\mathcal{P}}$(The tensor product over R is a suitable quotient of the tensor product over $\mathbb{Q}$.) Since the s_α are Z-invariant, the first claim follows.

Let $\mathcal{P}_{G_R}$ denote the R-scheme of isomorphisms $W^{\mathcal{P}} \xrightarrow{\sim} W$ respecting the s_α. By (4.1.2)(1) $\mathcal{P}_{G_R}$ is a right G_R-torsor, with G_R acting via $g(h) = g^{-1} \circ h$, and (4.1.3) induces a map $\eta : \mathcal{P}_R \to \mathcal{P}_{G_R}$, where η is Z-equivariant by (4.1.4).

We extend η to a map $\mathcal{P}_R \times G_R \to \mathcal{P}_{G_R}$ by G_R-equivariance. This last map clearly factors through $(\mathcal{P}_R \times G_R)/Z$, and the resulting map is necessarily an isomorphism. □

(4.1.6) Let S be a scheme, and $\mathcal{A}/S$ an abelian S-scheme. We consider $\mathcal{A}$ as an abelian scheme up to isogeny, and for T an S-scheme we set $\mathcal{A}(T) = \mathrm{Mor}_S(T, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose that $\mathcal{A}$ is equipped with an action of Z. The functor on S-schemes given by

$$T \mapsto (\mathcal{A}(T) \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}})^Z$$

is isomorphic to $\mathcal{A}^{\mathcal{P}}$, for an abelian S-scheme $\mathcal{A}^{\mathcal{P}}$ (well defined up to isogeny). The natural map

$$\mathcal{A}^{\mathcal{P}} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}}$$

is an isomorphism [Ki 2, 3.1.3].

If $\mathcal{A}^*$ denotes the dual of $\mathcal{A}$, there is a natural isomorphism $\mathcal{A}^{\mathcal{P}*} \xrightarrow{\sim} \mathcal{A}^{*\mathcal{P}}$. If $\mathcal{A}$ is equipped with a weak polarization $\lambda : \mathcal{A} \to \mathcal{A}^*$ then there is a unique weak polarization $\lambda^{\mathcal{P}} : \mathcal{A}^{\mathcal{P}} \to \mathcal{A}^{\mathcal{P}*}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}^{\mathcal{P}} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} & \xrightarrow{\lambda^{\mathcal{P}}\otimes 1} & \mathcal{A}^{\mathcal{P}*} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} & \xrightarrow{\lambda\otimes 1} & \mathcal{A}^* \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}} \end{array}$$

commutes up to an element of $\mathcal{O}_{\mathcal{P}}^\times$ [Ki 2, 3.1.5].

Lemma (4.1.7). *Let $\mathcal{A} \mapsto H(\mathcal{A})$ be an exact, additive functor from the category of abelian S-schemes up to isogeny to the category of $\mathbb{Q}$-vector spaces. If $\mathcal{A}$ is an abelian S-scheme equipped with an action of Z, we have a canonical isomorphism $H(\mathcal{A}^{\mathcal{P}}) \xrightarrow{\sim} H(\mathcal{A})^{\mathcal{P}}$.*

Proof. Let W, F and $\tilde{\omega}$ be as in (4.1.2). Then by (4.1.2)(2) we have

$$W^{\mathcal{P}} = \{w \in W \otimes_{\mathbb{Q}} F : c_{\tilde{\omega}}(\tau)\tau(w) = w, \quad \tau \in \mathrm{Gal}(F/\mathbb{Q})\}$$

Hence $W^{\mathcal{P}}$ is the mutual kernel of $c_{\tilde{\omega}}(\tau)\tau - 1$ acting on $W \otimes_{\mathbb{Q}} F$.

The lemma follows by applying this description with $W = \mathcal{A}(T)$ for an S-scheme T, and $H(\mathcal{A})$, and using the exactness of $H(\mathcal{A})$. □

(4.1.8) We explain how to twist level structures (cf. [Ki 2, 3.2.5, 3.4.3]). Let $\mathcal{A}/S$ be an abelian scheme equipped with an action of Z, as above, and V a $\mathbb{Q}$-vector space of dimension $2\dim_S\mathcal{A}$. We also assume that V is equipped with a faithful action of Z, so that $Z \subset \mathrm{GL}(V)$.

Suppose first that S is a $\mathbb{Q}$-scheme, and set

$$\widehat{V}(\mathcal{A}) = \varprojlim_n \mathcal{A}[n],$$

where the inverse limit runs over positive integers n ordered multiplicatively. This is an étale local system on S, and We set $\widehat{V}(\mathcal{A})_{\mathbb{Q}} = \widehat{V}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $C_Z \subset \mathrm{GL}(V)$ be the commutant of Z, and let $\mathsf{K} \subset C_Z(\mathbb{A}_f)$ be a compact (but not necessarily open) subgroup, and

$$\varepsilon_{\mathsf{K}} \in \Gamma(S, \underline{\mathrm{Isom}}_Z(V_{\mathbb{A}_f}, \widehat{V}(\mathcal{A})_{\mathbb{Q}})/\mathsf{K}).$$

Here $\underline{\mathrm{Isom}}_Z(V_{\mathbb{A}_f}, \widehat{V}(\mathcal{A})_{\mathbb{Q}})$ denotes the (pro-)étale sheaf of isomorphisms respecting the action of Z.

Let $\mathcal{P}$ be a Z-torsor. Since the $\mathrm{GL}(V)$-torsor induced by $\mathcal{P}$ is necessarily trivial, we may identify $\mathcal{P}$ with the fibre of $\mathrm{GL}(V) \to \mathrm{GL}(V)/Z$ for some $\omega \in (\mathrm{GL}(V)/Z)(\mathbb{Q})$. Let $\tilde{\omega} \in \mathrm{GL}(V)(F)$ be a lift of ω with $F/\mathbb{Q}$ a finite Galois extension.

Using (4.1.2), one sees that the isomorphism (4.1.3) identifies $V^{\mathcal{P}}$ with $\tilde{\omega}^{-1}V \subset V \otimes_{\mathbb{Q}} F$, and (4.1.4) implies that the induced isomorphism $\tilde{\omega}^{-1} : V \xrightarrow{\sim} V^{\mathcal{P}}$ depends only on ω and not on $\tilde{\omega}$. We set $\mathsf{K}^{\omega} = \omega\mathsf{K}\omega^{-1}$, and define

$$\varepsilon^{\omega}_{\mathsf{K}^{\omega}} \in \Gamma(S, \underline{\mathrm{Isom}}_Z(V_{\mathbb{A}_f}, \widehat{V}(\mathcal{A}^{\mathcal{P}})_{\mathbb{Q}})/\mathsf{K}^{\omega})$$

as the section given étale locally by

$$V_{\mathbb{A}_f} \xrightarrow{\tilde{\omega}^{-1}\cdot} V^{\mathcal{P}}_{\mathbb{A}_f} \xrightarrow{\varepsilon_{\mathsf{K}}} \widehat{V}(\mathcal{A})^{\mathcal{P}}_{\mathbb{Q}} \xrightarrow{\sim} \widehat{V}(\mathcal{A}^{\mathcal{P}})_{\mathbb{Q}}.$$

Similarly, if S is a $\mathbb{Z}_{(p)}$-scheme, $\mathsf{K}^p \subset C_Z(\mathbb{A}^p_f)$ is a compact subgroup and

$$\varepsilon_{\mathsf{K}^p} \in \Gamma(S, \underline{\mathrm{Isom}}_Z(V_{\mathbb{A}^p_f}, \widehat{V}^p(\mathcal{A})_{\mathbb{Q}})/\mathsf{K}^p),$$

then choosing $\tilde{\omega}$ as above we obtain a section

$$\varepsilon^{\omega}_{\mathsf{K}^{p\omega}} \in \Gamma(S, \underline{\mathrm{Isom}}_Z(V_{\mathbb{A}^p_f}, \widehat{V}^p(\mathcal{A}^{\mathcal{P}})_{\mathbb{Q}})/\mathsf{K}^{p\omega}).$$

(4.2) Twisting and Shimura varieties: The aim of this subsection is to generalize the construction of [Ki 2, §3], which gives a moduli theoretic description of the action of a subgroup of $G^{\mathrm{ad}}(\mathbb{Q})$ in $\mathrm{Sh}(G, X)$.

(4.2.1) Suppose that G is an algebraic group over $\mathbb{Q}$, and $G' \subset G$ a closed subgroup. Let $\omega \in (G/Z_{G'})(\mathbb{Q})$. Then we have the "conjugate of G' by ω," $G'^{\omega} \subset G$: By descent, it suffices to define this subgroup étale locally, when we can lift ω to a section $\tilde{\omega}$ of G, and we set $G'^{\omega} = \tilde{\omega}G'\tilde{\omega}^{-1}$. Note that we have an isomorphism of algebraic groups over $\mathbb{Q}$

$$G' \xrightarrow{\sim} G'^{\omega}; \quad g \mapsto \tilde{\omega}g\tilde{\omega}^{-1}.$$

Concretely, choose $\tilde{\omega} \in G(F)$ lifting ω, where F is a finite Galois extension of $\mathbb{Q}$. Then $c_{\tilde{\omega}}(\tau) = \tilde{\omega}^{-1}\tau(\tilde{\omega}) \in Z_{G'}(F)$ for $\tau \in \mathrm{Gal}(F/\mathbb{Q})$, and $c_{\tilde{\omega}}$ is a cocycle with values in $Z_{G'}(F)$. The subgroup $\tilde{\omega}G'\tilde{\omega}^{-1}$, defined over F, is stable under $\mathrm{Gal}(F/\mathbb{Q})$ and hence descends to a subgroup $G'^{\omega} \subset G$ over $\mathbb{Q}$, which depends only on ω and not on $\tilde{\omega}$. The fibre of $G \to G/Z_{G'}$ over ω is a $Z_{G'}$-torsor, whose class in $H^1(\mathbb{Q}, Z_{G'})$ is equal to that of $c_{\tilde{\omega}}$.

(4.2.2) Now suppose that $(G', X') \hookrightarrow (G, X)$ is an embedding of Shimura data. Let $\mathsf{K}_\infty \subset G_{\mathbb{R}}$ be the centralizer of some $h'_0 \in X'$, viewed as an algebraic group over $\mathbb{R}$. Let $(G/Z_{G'})(\mathbb{Q})_1 \subset (G/Z_{G'})(\mathbb{Q})$ be the kernel of the composite

$$(G/Z_{G'})(\mathbb{Q}) \to H^1(\mathbb{Q}, Z_{G'}) \to H^1(\mathbb{R}, Z_{G'}) \to H^1(\mathbb{R}, \mathsf{K}_\infty).$$

Let $\omega \in (G/Z_{G'})(\mathbb{Q})_1$ and $\tilde{\omega} \in G(F)$, as above. If $h' \in X'$, then $\tilde{\omega}h'\tilde{\omega}^{-1} \in X$: By assumption there exists $g_\infty \in \mathsf{K}_\infty(\mathbb{C})$ with $g_\infty^{-1}\tau(g_\infty) = \tilde{\omega}^{-1}\tau(\tilde{\omega})$ for $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Hence $\alpha = \tilde{\omega}g_\infty^{-1} \in G(\mathbb{R})$, and

$$\tilde{\omega}h'\tilde{\omega}^{-1} = \alpha h' \alpha^{-1} \in X.$$

Since $h' \in X'$, $h'^{\omega} := \tilde{\omega}h'\tilde{\omega}^{-1}$ depends only on ω and not on $\tilde{\omega}$. Set $X'^{\omega} = \{h'^{\omega},\ h' \in X'\}$. There are maps of Shimura data

$$(G', X') \xrightarrow{\sim} (G'^{\omega}, X'^{\omega}) \hookrightarrow (G, X),$$

the first map being given by conjugation by $\tilde{\omega}$, and hence maps of Shimura varieties

$$\mathrm{Sh}(G', X') \xrightarrow{\sim} \mathrm{Sh}(G'^{\omega}, X'^{\omega}) \hookrightarrow \mathrm{Sh}(G, X).$$

If $\mathsf{K}' \subset G'(\mathbb{A}_f)$ is a compact open subgroup with $\mathsf{K}' \subset \mathsf{K}$, then $\mathsf{K}'^{\omega} = \tilde{\omega}\mathsf{K}'\tilde{\omega}^{-1}$ is a compact open subgroup of $G'^{\omega}(\mathbb{A}_f)$, and there is an induced map on complex points of Shimura varieties

$$\mathfrak{i}_\omega^{\mathrm{an}} : \mathrm{Sh}_{\mathsf{K}'}(G', X')(\mathbb{C}) \xrightarrow{\sim} \mathrm{Sh}_{\mathsf{K}'^{\omega}}(G'^{\omega}, X'^{\omega})(\mathbb{C}) \hookrightarrow \mathrm{Sh}_{\mathsf{K}_\omega}(G, X)(\mathbb{C})$$

where $\mathsf{K}_\omega \subset G(\mathbb{A}_f)$ is a compact open subgroup such that the final map is defined and a closed embedding.

If $G' = G$, then the above construction describes the action of $G^{\mathrm{ad}}(\mathbb{Q})_1$ on $\mathrm{Sh}(G, X)$ defined in [De 2]. We will apply this construction when $G' = Z_{G'}$ is a torus in G.

(4.2.3) Now let V be as in (1.3.2), and let

$$(G', X') \hookrightarrow (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$$

be embeddings of Shimura data. As above, we have a collection of tensors $(s_\alpha) \in V^{\otimes}$ defining $G \subset \mathrm{GL}(V)$. Fix compact open subgroups $\mathsf{K}' \subset G'(\mathbb{A}_f)$ and $\mathsf{K} \subset G(\mathbb{A}_f)$ such that there is an embedding

$$\mathrm{Sh}_{\mathsf{K}'}(G', X') \hookrightarrow \mathrm{Sh}_{\mathsf{K}}(G, X).$$

Let E' be the reflex field of (G', X'), T an E'-scheme, and $x \in \mathrm{Sh}_{\mathsf{K}'}(G', X')(T)$. Attached to x we have a tuple $(\mathcal{A}, \lambda, \varepsilon_{\mathsf{K}'})$ consisting of an abelian scheme up to isogeny $\mathcal{A}$ over T, equipped with a weak polarization λ and a level structure [Ki 2, 3.2.4]

$$\varepsilon_{\mathsf{K}'} \in \Gamma(T, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f}, \widehat{V}(\mathcal{A})_{\mathbb{Q}})/\mathsf{K}')$$

which respects polarizations up to a scalar, and takes s_α to the section $(s_{\alpha,l,x})_l \in \widehat{V}(\mathcal{A})_{\mathbb{Q}}^{\otimes}$ defined in (1.3.6).

By [Ki 2, 3.2.2] $\mathcal{A}$ is equipped with an action of $Z_{G'}$.

Lemma (4.2.4). *The isomorphism $\varepsilon_{\mathsf{K}'}$ is compatible with the action of $Z_{G'}$ on V and $\mathcal{A}$.*

Proof. It suffices to check this on complex points, so we may assume $T = \mathrm{Spec}\ \mathbb{C}$. Then there is an isomorphism $H_1(\mathcal{A}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} V$, which is canonical up to multiplication by elements of $G'(\mathbb{Q})$. By definition of the action of $Z_{G'}$ on $\mathcal{A}$, this isomorphism commutes with the action of $Z_{G'}$. If $x = (h', g')$ with $h' \in X'$ and $g' \in G'(\mathbb{A}_f)$, then $\varepsilon_{\mathsf{K}'}$ is induced by the composite

$$V_{\mathbb{A}_f} \stackrel{g'\cdot}{\to} V_{\mathbb{A}_f} \xrightarrow{\sim} H_1(\mathcal{A}(\mathbb{C}), \mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(\mathcal{A})_{\mathbb{Q}}.$$

Since g' commutes with $Z_{G'}$, this composite is compatible with the action of $Z_{G'}$ on the two sides. $\square$

(4.2.5) Now let ω and $\tilde{\omega}$ be as in (4.2.2), and let $\mathcal{P}$ be the fibre of $G \to G/Z_{G'}$ over ω. Applying the construction of §4.1, we obtain a triple $(\mathcal{A}^{\mathcal{P}}, \lambda^{\mathcal{P}}, \varepsilon^{\omega}_{\mathsf{K}'^{\omega}})$, and hence a morphism

$$\mathfrak{i}_\omega : \mathrm{Sh}_{\mathsf{K}'}(G', X') \to \mathrm{Sh}_{\mathsf{K}^{\psi}_{\omega}}(\mathrm{GSp}, S^{\pm})$$

for some compact open subgroup $\mathsf{K}^{\psi}_{\omega} \subset \mathrm{GSp}(\mathbb{A}_f)$.

Proposition (4.2.6). *The map $\mathfrak{i}_\omega$ factors through $\mathrm{Sh}_{\mathsf{K}_\omega}(G, X)$ and induces $\mathfrak{i}^{\mathrm{an}}_\omega$ on complex points.*

Moreover, the tensor $\mathfrak{i}^*_\omega(s_{\alpha,B})$ coincides at each $x \in \mathrm{Sh}_{\mathsf{K}'}(G', X')(\mathbb{C})$, with the tensors $s_{\alpha,B,x}$ viewed in $H^1_B(\mathcal{A}_x(\mathbb{C}), \mathbb{Q})^{\otimes}$ via (4.1.5) and the isomorphism of (4.1.7).

Proof. It suffices to check the proposition on complex points.

Choose $\mathsf{K}^{\psi} \subset \mathrm{GSp}(\mathbb{A}_f)$ compact open so that we have an embedding

$$\mathrm{Sh}_{\mathsf{K}}(G, X) \hookrightarrow \mathrm{Sh}_{\mathsf{K}^{\psi}}(\mathrm{GSp}, S^{\pm}).$$

Let $x \in \mathrm{Sh}_{\mathsf{K}}(G', X')(\mathbb{C})$ and $(\mathcal{A}, \lambda, \varepsilon_{\mathsf{K}'})$ the tuple attached to x. To compute the image of x in $\mathrm{Sh}_{\mathsf{K}^{\psi}}(\mathrm{GSp}, S^{\pm})$, choose an isomorphism

$$\tau : H_1(\mathcal{A}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} V_{\mathbb{Q}}$$

compatible with polarizations up to a scalar. The Hodge structure on $H_1(\mathcal{A}(\mathbb{C}), \mathbb{C})$ is given by a $\mathrm{GSp}(\mathbb{R})$-valued cocharacter h, and we have an element $g \in \mathrm{GSp}(\mathbb{A}_f)$ given by the composite

$$V_{\mathbb{A}_f} \xrightarrow{\varepsilon_{\mathsf{K}'}} \widehat{V}(\mathcal{A})_{\mathbb{Q}} \xrightarrow[\tau^{-1}]{\sim} V_{\mathbb{A}_f}.$$

The pair $(h, g) \in S^{\pm} \times \mathrm{GSp}(\mathbb{A}_f)/\mathsf{K}^{\psi}$ is independent of τ up to multiplication by $\mathrm{GSp}(\mathbb{Q})$ on the left. Since $x \in \mathrm{Sh}_{\mathsf{K}}(G, X)(\mathbb{C})$, we may choose $\tau : H_1(\mathcal{A}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} V_{\mathbb{Q}}$ taking $s_{\alpha,B,x}$ to s_α. Then $(h, g) \in X \times G(\mathbb{A}_f)$.

As in (4.1.8), multiplication by $\tilde{\omega}$ identifies $V^{\mathcal{P}}_{\mathbb{Q}} \subset V_{\mathbb{Q}} \otimes_{\mathbb{Q}} F$ with $V_{\mathbb{Q}}$. We compute the image of $\mathfrak{i}_\omega(x)$ in $\mathrm{Sh}_{\mathsf{K}^{\psi}}(\mathrm{GSp}, S^{\pm})$ using the isomorphism

$$H_1(\mathcal{A}^{\mathcal{P}}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} H_1(\mathcal{A}(\mathbb{C}), \mathbb{Q})^{\mathcal{P}} \xrightarrow[\tau]{\sim} V^{\mathcal{P}}_{\mathbb{Q}} \xrightarrow[\tilde{\omega}]{\sim} V_{\mathbb{Q}}. \tag{4.2.7}$$

Then the Hodge structure on $H_1(\mathcal{A}^{\mathcal{P}}(\mathbb{C}), \mathbb{Q})$ is given by the cocharacter h^{ω}, and $\varepsilon^{\omega}_{\mathsf{K}^{\omega}}$ corresponds to $\tilde{\omega} g \tilde{\omega}^{-1}$.

Since (4.2.7) takes $s_{\alpha,B,x}$ to s_α the final claim of the proposition also follows. $\square$

(4.2.8) Suppose we are in the situation of (1.3.3), so that $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with $\mathsf{K}_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ hyperspecial, and $G \hookrightarrow \mathrm{GSp}$ is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. Applying the construction above with $(G', X') = (G, X)$ and $\omega \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ (the notation being as in (3.7.1)), we obtain a map

$$\mathfrak{i}_\omega : \mathrm{Sh}_{\mathsf{K}}(G, X) \to \mathrm{Sh}_{\mathsf{K}^\omega}(G, X)$$

and hence, by the extension property, a map of $\mathcal{O}_{(p)}$-schemes

$$\mathfrak{i}_\omega : \mathscr{S}_{\mathsf{K}}(G, X) \to \mathscr{S}_{\mathsf{K}^\omega}(G, X),$$

where $\mathsf{K}^\omega = \mathsf{K}_p\mathsf{K}^{p\omega}$. If we convert this left action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ into a right action (by letting ω act via $\mathfrak{i}_{\omega^{-1}}$), this equips the scheme $\mathscr{S}_{\mathsf{K}_p}(G, X)$ with an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p) *_{G(\mathbb{Z}_{(p)})_+} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ (cf. [Ki 2, §3.3]).

Now let T be an $\mathcal{O}_{(p)}$-scheme, and $x \in \mathscr{S}_{\mathsf{K}}(G, X)(T)$. Let $\mathcal{P}_{\mathbb{Z}_{(p)}}$ be the fibre of $G_{\mathbb{Z}_{(p)}} \to G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ over ω, so that $\mathcal{P}_{\mathbb{Z}_{(p)}}$ is a $Z_{G_{\mathbb{Z}_{(p)}}}$-torsor. Then $\mathcal{A}_{\mathfrak{i}_\omega(x)}$ is an abelian scheme up to prime to p-isogeny. When viewed as an abelian scheme up to isogeny it is canonically identified with $\mathcal{A}_x^{\mathcal{P}}$, where $\mathcal{P} = \mathcal{P}_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$. To see this, it suffices to check with $T = \mathscr{S}_{\mathsf{K}}(G, X)$ and x the tautological point. Over $T \otimes \mathbb{Q}$ such an identification exists by the definition of $\mathfrak{i}_\omega$, and it extends over all of T by [FC, 2.7].

Lemma (4.2.9). *Let $F/\mathbb{Q}$ be a finite Galois extension, and $\mathcal{O}_{F,(p)} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let $\tilde{\omega} \in G(\mathcal{O}_{F,(p)})$ be a point lifting ω. Then the composite*

$$\mathcal{A}_{\mathfrak{i}_\omega(x)} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}_x^{\mathcal{P}} \otimes_{\mathbb{Q}} F \underset{\vartheta_{\tilde{\omega}}}{\to} \mathcal{A}_x \otimes_{\mathbb{Q}} F$$

induces an isomorphism of abelian schemes up to prime to p isogeny

$$\mathcal{A}_{\mathfrak{i}_\omega(x)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)} \xrightarrow{\sim} \mathcal{A}_x \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}.$$

Proof. By the Néron mapping property it suffices to prove this for the universal abelian scheme over $\mathrm{Sh}_{\mathsf{K}}(G, X)$, which follows from the case $T = \mathrm{Spec}\, \mathbb{C}$.

By construction we have a commutative diagram

$$\begin{array}{ccc} V_{\mathbb{A}_f} \otimes F & \xrightarrow{\tilde{\omega}^{-1}\cdot} & V_{\mathbb{A}_f} \otimes F \\ \downarrow{\scriptstyle \varepsilon^\omega_{\mathsf{K}^\omega}} & & \downarrow{\scriptstyle \varepsilon_{\mathsf{K}}} \\ \widehat{V}(\mathcal{A}^{\mathcal{P}})_{\mathbb{Q}} \otimes F & \xrightarrow{\vartheta_{\tilde{\omega}}} & \widehat{V}(\mathcal{A})_{\mathbb{Q}} \otimes F. \end{array}$$

By the definition of the abelian schemes up to prime to p isogeny associated to $(\mathcal{A}, \varepsilon_{\mathsf{K}})$ and $(\mathcal{A}^{\mathcal{P}}, \varepsilon^\omega_{\mathsf{K}^\omega})$, the vertical maps in the above diagram induce isomorphisms $V_{\mathbb{Z}_p} \xrightarrow{\sim} T_p(\mathcal{A}^{\mathcal{P}})$ and $V_{\mathbb{Z}_p} \xrightarrow{\sim} T_p(\mathcal{A})$. Hence it suffices to show that multiplication by $\tilde{\omega}^{-1}$ induces an automorphism of $V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}$, which follows from $\tilde{\omega} \in G(\mathcal{O}_{F,(p)})$. $\square$

(4.3) Kottwitz triples: We now return to the assumptions in force in §3, so that (G, X) is a Shimura datum, and the group G is equipped with a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, which is sometimes again denoted by G. We also assume in this section that Z_G^0 satisfies the Serre condition. We continue to use the embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$ fixed in (1.3).

(4.3.1) Let $r \geq 1$ be a non-negative integer, and set $K_0 = \mathrm{Fr}\,(W(k))$ where $k = \mathbb{F}_{p^r}$. A Kottwitz triple $\mathfrak{k} = (\gamma_0, (\gamma_l)_{l \neq p}, \delta)$ of level r consists of

1. An element $\gamma_0 \in G(\mathbb{Q})$ defined up to conjugacy in $G(\bar{\mathbb{Q}})$.
2. An element $(\gamma_l)_{l \neq p} \in G(\mathbb{A}_f^p)$.
3. An element $\delta \in G(K_0)$ defined up to Frobenius conjugacy by elements of $G(W(k))$.

This data is required to satisfy the following conditions:

(i) γ_0 is conjugate to $(\gamma_l)_{l \neq p}$ in $G(\bar{\mathbb{A}}_f^p)$, where $\bar{\mathbb{A}}_f^p$ is as in (3.3.6).

(ii) γ_0 is conjugate to $\gamma_p = \delta\sigma(\delta)\dots\sigma^{r-1}(\delta)$ in $G(\bar{\mathbb{Q}}_p)$.

(iii) The image of γ_0 in $G(\mathbb{R})$ is elliptic.

We also require a further condition described in *(iv)* below.

For the triples which are important in applications, one can choose a representative of γ_0 in its $G(\bar{\mathbb{Q}})$-conjugacy class so that γ_0 is conjugate to $(\gamma_l)_{l \neq p}$ in $G(\mathbb{A}_f^p)$. However, it will be convenient not to assume this immediately.

For any such $\mathfrak{k}$ and l a finite prime we define groups $I_{l/k}$ as in (2.1.1): If $l \neq p$, then $I_{l/k}$ is the centralizer of γ_l in $G_{\mathbb{Q}_l}$, while for a $\mathbb{Q}_p$-algebra R we set

$$I_{p/k}(R) = \{\alpha \in G(W(k) \otimes_{\mathbb{Z}_p} R) : \quad \delta\sigma(\alpha) = \alpha\delta\}.$$

Let $I_{0/k}$ denote the centralizer of γ_0 in G. Then $I_{l/k}$ is an inner form of $I_{0/k} \otimes_{\mathbb{Q}} \mathbb{Q}_l$ for $l \neq p$, and, as in (2.1), $I_{p/k}$ is an inner form of $I_{0/k} \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Similarly we obtain groups I_0, I_l and I_p, by replacing γ_0, γ_l and $W(k)$ by γ_0^n, γ_l^n, and $W(k_n)$ for n sufficiently large.

If $(\gamma_0, (\gamma_l)_{l \neq p}, \delta)$ is a Kottwitz triple of level r, then for any positive integer m, $(\gamma_0^m, (\gamma_l^m)_{l \neq p}, \delta)$ is a Kottwitz triple of level rm. We consider the smallest equivalence relation on the set of all Kottwitz triples of some level $r \geq 1$ under which $(\gamma_0, (\gamma_l)_{l \neq p}, \delta)$ is equivalent to $(\gamma_0^m, (\gamma_l^m)_{l \neq p}, \delta)$ for all $m \geq 1$, where the latter is a Kottwitz triple of level rm. It is clear that I_0, I_l and I_p, depend only on the equivalence class of $\mathfrak{k}$. An equivalence of Kottwitz triples of some level $r \geq 1$ will be called a Kottwitz triple. This is not to be confused with the equivalence relation on the collection of Kottwitz triples defined in (4.3.2) below.

Recall that an inner twisting of I_0 is a group I over $\mathbb{Q}$ together with an isomorphism $I_0 \otimes \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes \bar{\mathbb{Q}}$, given up to inner automorphisms of $I \otimes \bar{\mathbb{Q}}$, which realizes I as an inner form of I_0. We assume that $\mathfrak{k}$ also satisfies the following condition

(iv) There exists an inner twisting I of I_0, such that $I \otimes_{\mathbb{Q}} \mathbb{R}$ is anisotropic mod center, and $I \otimes_{\mathbb{Q}} \mathbb{Q}_l$ is isomorphic to I_l an inner twist of $I_0 \otimes \mathbb{Q}_l$, for all finite primes l. That is, there exists an isomorphism $I \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} I_l$ such that the diagram

$$\begin{array}{ccc} I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l & \xrightarrow{\sim} & I_l \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l \\ \| & & \uparrow \wr \\ I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l & \xrightarrow{\sim} & I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \end{array}$$

commutes up to conjugation by an element of $I_0(\bar{\mathbb{Q}}_l)$.

We remark that by the Hasse principle for adjoint groups, I is uniquely determined as an inner twisting of I_0 by these conditions. More precisely, if I and I' are two such groups, then there is an isomorphism $I \xrightarrow{\sim} I'$ which is canonical up to conjugation by elements of $I^{\mathrm{ad}}(\mathbb{Q})$, and intertwines the isomorphisms with I_0.

Let $I_{\mathbb{A}_f^p} = I_{\mathbb{A}_f^p}(\mathfrak{k})$ denote the centralizer of $(\gamma_l^n)_{l\neq p}$ in $G_{\mathbb{A}_f^p}$ for n sufficiently large, and set $I_{\mathbb{A}_f} = I_p \times I_{\mathbb{A}_f^p}$. Since I_l is unramified for l sufficiently large, condition *(iv)* above implies that there exists an isomorphism $\iota : I \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f}$ as inner forms of I_0. We call the tuple $\tilde{\mathfrak{k}} = (\gamma_0, (\gamma_l)_{l\neq p}, \delta, \iota)$ a refined Kottwitz triple, and we say that $\tilde{\mathfrak{k}}$ is a refinement of $\mathfrak{k}$.

We set

$$S(\tilde{\mathfrak{k}}) = I(\mathbb{Q})\backslash X_\upsilon(\delta) \times G(\mathbb{A}_f^p)$$

which is a set equipped with an action of $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$.

(4.3.2) Let $\mathfrak{k}, \mathfrak{k}'$ be Kottwitz triples. We say that $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent, and write $\mathfrak{k} \sim \mathfrak{k}'$, if there exist representatives $\mathfrak{k} = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$ and $\mathfrak{k}' = (\gamma_0', (\gamma_l')_{l\neq p}, \delta')$ of the same level r, such that

(1) $(\gamma_l)_{l\neq p}$ and $(\gamma_l')_{l\neq p}$ are conjugate in $G(\mathbb{A}_f^p)$.

(2) δ and δ' are σ-conjugate in $G(K_0)$, where $K_0 = \mathrm{Fr}\, W(\mathbb{F}_{p^r})$, as before.

Suppose that $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent Kottwitz triples, with $\delta' = g_0\delta\sigma(g_0)^{-1}$ and $(\gamma_l')_{l\neq p} = g^p\gamma_l g^{p-1}$ with $g^p \in G(\mathbb{A}_f^p)$ and $g_0 \in G(K_0)$. If $\tilde{\mathfrak{k}}$ is a refinement of $\mathfrak{k}$, we can define a refinement $\tilde{\mathfrak{k}}'$ of $\mathfrak{k}'$ as follows: Recall the group $I_{\mathbb{A}_f}$ defined above. If $I'_{\mathbb{A}_f}$ denotes the corresponding group attached to $\mathfrak{k}'$, then as subgroups of $G(L)\times G(\mathbb{A}_f^p)$, $I'_{\mathbb{A}_f} = (g_0, g^p) I_{\mathbb{A}_f} (g_0, g^p)^{-1}$. We take ι' to be the map ι composed with conjugation by (g_0, g^p) and we set $\tilde{\mathfrak{k}}' = (\gamma_0', (\gamma_l')_{l\neq p}, \delta', \iota')$. Note that this depends on the choice of (g_0, g^p).

Left multiplication by (g_0, g^p) induces a bijection

$$X_\upsilon(\delta) \times G(\mathbb{A}_f^p) \to X_\upsilon(\delta') \times G(\mathbb{A}_f^p) \tag{4.3.3}$$

which is I-equivariant when I acts via ι and ι' on the left and right side of (4.3.3) respectively. This induces an isomorphism

$$S(\tilde{\mathfrak{k}}) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}').$$

(4.3.4) Denote by $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^I$ the kernel of

$$G^{\mathrm{ad}}(\mathbb{Z}_{(p)}) \to H^1(\mathbb{Q}, Z_G) \to H^1(\mathbb{Q}, I),$$

and let $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ be the intersection of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^I$ and $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$.

We will show that the action of $G(\mathbb{A}_f^p)$ on $S(\tilde{\mathfrak{k}})$ extends to an action of

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I = G(\mathbb{A}_f^p) *_{G(\mathbb{Z}_{(p)})_+} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$$

which commutes with the action of $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p)$.

Let $G(\bar{\mathbb{Z}}_{(p)})_+^{\natural,I}$ denote the preimage of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ in $G(\bar{\mathbb{Q}})$. For $h \in G(\bar{\mathbb{Z}}_{(p)})_+^{\natural,I}$, the class of the cocycle $(\tau \mapsto z_\tau = \tau(h)h^{-1}) \in Z^1(\mathbb{Q}, Z_G)$ becomes trivial in $H^1(\mathbb{Q}, I)$. Hence there exists $i_h \in I(\bar{\mathbb{Q}})$ such that $h^{-1}\tau(h) = i_h^{-1}\tau(i_h)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Here $\tau(i_h)$ is computed using the structure of $\mathbb{Q}$-group on I.

The isomorphisms $I \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f}$ allow us to regard I as a subgroup of G over $\mathbb{A}_f^p$ and over $\mathbb{Q}_p^{\mathrm{ur}}$. In particular, we may regard i_h as an element of $G(\bar{\mathbb{A}}_f^p)$.

Lemma (4.3.5). *Let $g = (g_0, g^p) \in X_\upsilon(\delta) \times G(\mathbb{A}_f^p)$, and $h \in G(\bar{\mathbb{Z}}_{(p)})_+^{\natural, I}$. Then*

(1) *The element*

$$i_h^{-1} gh \in G(\bar{\mathbb{Q}}_p)/G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}}) \times G(\bar{\mathbb{A}}_f^p)$$

lies in $X_\upsilon(\delta) \times G(\mathbb{A}_f^p)$.

(2) *The image of $i_h^{-1}gh$ in $S(\tilde{\mathfrak{k}})$, does not depend on the choice of i_h, and $g \mapsto h(g) := i_h^{-1}gh$ induces an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$ on $S(\tilde{\mathfrak{k}})$ which commutes with $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p)$.*

Proof. If $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ with $l \neq p$, then

$$\tau(i_h^{-1} g_l h) = z_\tau^{-1} i_h^{-1} g_l h z_\tau = i_h^{-1} g_l h$$

so $i_h^{-1} g^p h \in G(\mathbb{A}_f^p)$. A similar calculation shows that $i_h^{-1} g_0 h \in G(\mathbb{Q}_p^{\mathrm{ur}})$, where we again denote by g_0 a representative of g_0 in $G(\mathbb{Q}_p^{\mathrm{ur}})$.

To check that $i_h^{-1} g_0 h \in X_\upsilon(\delta)$, we may modify h and i_h by an element of $Z_G(\bar{\mathbb{Q}})$ and assume that $h \in G(\bar{\mathbb{Z}}_p)$. Let $\tilde{\sigma} \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ be a lift of σ, which we think of as acting diagonally on $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$. We have

$$\begin{aligned} h^{-1} g_0^{-1} i_h \delta \tilde{\sigma}(i_h)^{-1} \sigma(g_0) \tilde{\sigma}(h) &= h^{-1} g_0^{-1} i_h \tilde{\sigma}_I(i_h^{-1}) \delta \sigma(g_0) \tilde{\sigma}(h) \\ &= z_{\tilde{\sigma}}^{-1} h^{-1} g_0^{-1} \delta \sigma(g_0) \tilde{\sigma}(h) \in G(\bar{\mathbb{Z}}_p) p^\upsilon G(\bar{\mathbb{Z}}_p). \end{aligned}$$

Here we write $\tilde{\sigma}_I(\cdot)$ for the action of $\tilde{\sigma}$ computed with respect to the structure of $\mathbb{Q}$-group on I. This proves (1).

Replacing g by ig for $i \in I(\mathbb{Q})$ does not change i_h, and we have $i_h^{-1} igh = [i_h^{-1} i i_h] i_h^{-1} gh$. Since $i_h^{-1} i i_h \in I(\mathbb{Q})$, $g \mapsto i_h^{-1} gh$ induces a well defined map $S(\tilde{\mathfrak{k}}) \to S(\tilde{\mathfrak{k}})$.

The choice of i_h is unique up to left multiplication by elements of $I(\mathbb{Q})$. If $z \in Z_G(\bar{\mathbb{Q}})$, then we have $i_{zh} = z i_h$ modulo $I(\mathbb{Q})$. Hence the image of $i_h^{-1} gh$ in $S(\tilde{\mathfrak{k}})$ depends only on the image of h in $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ and not on the choice of h or i_h.

If $h, h' \in G(\bar{\mathbb{Z}}_{(p)})_+^{\natural, I}$, then

$$(hh')^{-1} \tau(hh') = h'^{-1} h^{-1} \tau(h) h' h'^{-1} \tau(h') = h^{-1} \tau(h) h'^{-1} \tau(h').$$

A similar calculation with $i_h, i_{h'}$ shows that $i_{hh'} = i_h i_{h'} \bmod I(\mathbb{Q})$, so that the above induces a well defined action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ on $S(\tilde{\mathfrak{k}})$.

If $h \in G(\mathbb{Q})_+$, then we may take $i_h = 1$, and the action of h is on $S(\tilde{\mathfrak{k}})$ is via right multiplication. Hence, combining the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ on $S(\tilde{\mathfrak{k}})$ with that of $G(\mathbb{A}_f^p)$ we obtain an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$ on $S(\tilde{\mathfrak{k}})$. This action clearly commutes with $Z_G(\mathbb{Q}_p)$. To see that it commutes with Φ we compute, as above,

$$\begin{aligned} (\delta\sigma)^r (i_h^{-1} gh) = (\delta\tilde{\sigma})^r (i_h^{-1} gh) &= \tilde{\sigma}_I^r (i_h)^{-1} (\delta\tilde{\sigma})^r (g) \tilde{\sigma}^r(h) \\ &= z_{\tilde{\sigma}^r}^{-1} i_h^{-1} \Phi(g) h z_{\tilde{\sigma}^r} = i_h^{-1} \Phi(g) h. \end{aligned}$$

□

(4.3.6) We now define the twist of an equivalence class of Kottwitz triples by certain elements of $H^1(\mathbb{Q}, I)$. Let $\mathfrak{k} = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$ be a Kottwitz triple (up to equivalence). Suppose $\beta \in H^1(\mathbb{Q}, I)$ is an element whose image in $H^1(\mathbb{R}, I)$ is trivial, and such that for every prime $l \neq p$, the image of β under the composite

$$H^1(\mathbb{Q}, I) \to H^1(\mathbb{Q}_l, I) \xrightarrow{\sim} H^1(\mathbb{Q}_l, I_l) \to H^1(\mathbb{Q}_l, G)$$

is trivial.

Fix a cocycle in $Z^1(\mathbb{Q}, I)$ representing β, which we again denote by β, and an isomorphism $\mathfrak{i} : I \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f}$ as in (4.3.1). For any prime l, we regard β as taking values in $I_l(\bar{\mathbb{Q}}_l)$ via $\mathfrak{i}$. For $l \neq p$ there exists $g_l \in G(\bar{\mathbb{Q}}_l)$ with $g_l^{-1}\tau(g_l) = \beta(\tau)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$. We set $\gamma_l^\beta = g_l\gamma_l g_l^{-1}$. Then

$$\tau(\gamma_l^\beta) = \tau(g_l)\gamma_l\tau(g_l)^{-1} = g_l\beta(\tau)\gamma_l\beta(\tau)^{-1}g_l^{-1} = \gamma_l^\beta$$

since $\beta(\tau) \in I_l(\bar{\mathbb{Q}}_l)$ commutes with γ_l. Hence $\gamma_l^\beta \in G(\mathbb{Q}_l)$. The element g_l is independent of the choice of cocycle representing β modulo multiplication by elements in $G(\mathbb{Q}_l)$ and $I(\bar{\mathbb{Q}}_l)$. Hence γ_l^β is well determined up to conjugacy in $G(\mathbb{Q}_l)$. In particular, since $I^{\mathrm{ad}}(\mathbb{Q}_l)$ acts trivially on $H^1(\mathbb{Q}_l, I)$, the $G(\mathbb{Q}_l)$-conjugacy class of γ_l^β does not depend on the choice of $\mathfrak{i}$. For some integer N, G admits a reductive model $G_{\mathbb{Z}[1/N]}$ over $\mathbb{Z}[1/N]$. For almost all l the image of (the chosen representative of) β in $Z^1(\mathbb{Q}_l, G)$ takes values in $G_{\mathbb{Z}_l}(\bar{\mathbb{Z}}_l)$, and we may take $g_l \in G_{\mathbb{Z}_l}(\bar{\mathbb{Z}}_l)$ by Lang's Lemma. Thus $(\gamma_l^\beta)_{l\neq p}$ is conjugate to γ_0 in $G(\bar{\mathbb{A}}_f^p)$.

After increasing the finite field k, we may view I_p as a subgroup of $R_{K_0/\mathbb{Q}_p}G$. By Steinberg's theorem, after increasing k further, we may assume that the image of β under the composite

$$H^1(\mathbb{Q}, I) \to H^1(\mathbb{Q}_p, I) \xrightarrow{\sim} H^1(\mathbb{Q}_p, I_p) \to H^1(\mathbb{Q}_p, R_{K_0/\mathbb{Q}_p}G)$$

is trivial. Thus, there exists $g_0 \in G(K_0 \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)$ such that $g_0^{-1}\tau(g_0) = \beta(\tau)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. We set $\delta^\beta = g_0\delta\sigma(g_0)^{-1}$, where, of course, σ denotes the Frobenius on K_0. Then for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$\tau(\delta^\beta) = \tau(g_0)\delta(\sigma\otimes\tau)(g_0)^{-1} = g_0\beta(\tau)\delta\sigma(\beta(\tau))^{-1}\sigma(g_0)^{-1} = \delta^\beta.$$

Hence $\delta^\beta \in G(K_0)$.

The element g_0 is independent of the choice of cocycle representing β modulo multiplication by elements in $G(K_0)$ and $I(\bar{\mathbb{Q}}_p)$. Hence δ^β is well determined up to σ-conjugacy in $G(K_0)$. Note that $\delta^\beta\sigma(\delta^\beta)\dots\sigma^{r-1}(\delta^\beta) = g_0\gamma_p g_0^{-1}$.

We set $\mathfrak{k}^\beta = (\gamma_0, (\gamma_l^\beta)_{l\neq p}, \delta^\beta)$. Then $\mathfrak{k}^\beta$ satisfies conditions *(i)-(iii)* by what we saw above. By construction, for any finite prime l the group I_l^β attached to $\mathfrak{k}^\beta$ is the inner form of I_l attached to β. Hence $\mathfrak{k}^\beta$ satisfies *(iv)* with the inner form of I corresponding to β.

From the definitions one easily obtains

Lemma (4.3.7). $\mathfrak{k}^\beta \sim \mathfrak{k}$ *if and only if*

$$\beta \in \ker\left(H^1(\mathbb{Q}, I) \to \prod_l H^1(\mathbb{Q}_l, I)\right)$$

where the product runs over all finite primes.

(4.3.8) Suppose that $\tilde{\mathfrak{k}} = (\gamma_0, (\gamma_l)_{l\neq p}, \delta, \iota)$ is a refinement of $\mathfrak{k}$. If β is the image of an element $h \in G^{\mathrm{ad}}(\mathbb{Q})^+$, we can lift the above construction and define a refined triple $\tilde{\mathfrak{k}}^\beta$ giving rise to $\mathfrak{k}^\beta$.

If $\beta = [h]$, then the image of β in $H^1(\mathbb{Q}, I^{\mathrm{ad}})$ is trivial, so I is also the inner twisting of I_0 attached to $\mathfrak{k}^\beta$. Let $\tilde{h}$ be any lifting of h to $G(\bar{\mathbb{Q}})$. The calculations above show that $\mathfrak{k}^\beta = (\gamma_0, \tilde{h}\gamma_l\tilde{h}^{-1}, \tilde{h}\delta\tilde{h}^{-1})$. Note that $\mathfrak{k}^\beta$ is independent of the choice of $\tilde{h}$ as a Kottwitz triple, not just up to equivalence.

We have $I_{\mathbb{A}_f}(\mathfrak{k}^\beta) = \tilde{h}I_{\mathbb{A}_f}\tilde{h}^{-1} \subset G(L)\times G(\mathbb{A}_f^p)$, and we take the isomorphism $I\otimes_{\mathbb{Q}}\mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f}(\mathfrak{k}^\beta)$ to be ι composed with conjugation by $\tilde{h}$.

(4.3.9) Let T be a $\mathbb{Q}$-torus, $h_T : \mathbb{S} \to T_{\mathbb{R}}$ and $i : T \hookrightarrow G$ a map of $\mathbb{Q}$-groups such that $i\circ h_T \in X$. Then $T/(Z_G\cap T)$ satisfies the Serre condition, and our assumption on Z_G implies that T satisfies the Serre condition. By (3.5.7), any admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ factors through $\mathfrak{P}$. In particular, $\psi_{\mu_{h_T}}$ factors through $\mathfrak{P}$, and we again denote by $\psi_{\mu_{h_T}} : \mathfrak{P} \to \mathfrak{G}_G$ the induced morphism. We give two constructions which associate an equivalence class of Kottwitz triples to (T, h_T, i).

Let $L/\mathbb{Q}$ be a finite, CM Galois extension contained in $\bar{\mathbb{Q}}$ such that $\mu_{h_T}\in X_*(T)$ is defined over L. Recall the elements $\delta_n \in P^L(\mathbb{Q})$ constructed in (3.5.4). By (3.5.4), and the definition of $\nu(p)$ we have for any $q = p^m$ Weil number $\pi \in L$, and any integer n with $[L_p : \mathbb{Q}_p]|n$,

$$-\frac{1}{n}\langle \nu(p)^L_n, \chi_\pi\rangle = -\langle \nu(p), \chi_\pi\rangle = \frac{1}{m}v_p(\pi) = \frac{1}{n}\langle \nu_{\delta_n}, \chi_\pi\rangle$$

and so $\nu(p)^L_n = -\nu_{\delta_n}$. Thus δ_n and $\nu(p)^L_n(p^{-1})$ have the same Newton cocharacter, and after increasing n, we may assume that they are σ^n-conjugate in $P^L(\widehat{\mathbb{Q}_p^{\mathrm{ur}}})$.

Let $c \in P^L(\widehat{\mathbb{Q}_p^{\mathrm{ur}}})$ be such that $c\nu(p)^L_n(p^{-1})\sigma^n(c)^{-1} = \delta_n$, Since $\sigma^n(c)c^{-1} = \delta_n^{-1}\nu(p)^L_n(p^{-1}) \in P^L(\mathbb{Q}_p)$, one sees that $\sigma(c)c^{-1}$ is invariant under σ^n, so $\sigma(c)c^{-1} \in P^L(\mathbb{Q}_{p^n})$. In particular, this implies that $\sigma^{n'}(c)c^{-1} = (\sigma(c)^nc^{-1})^{n'/n}$ for $n|n'$, so the element c can be chosen independently of n.

Let $\mathfrak{P}^L(p)^{\mathrm{ur}}$ denotes the pushout of $\mathfrak{D}_n$ by $\nu(p)^L_n : D_n(\mathbb{Q}_p^{\mathrm{ur}}) \to P^L(p)(\mathbb{Q}_p^{\mathrm{ur}})$. Then ζ_p induces a morphism $\mathfrak{D} \to \mathfrak{P}^L(p)^{\mathrm{ur}}$, and we again write $d_\sigma \in \mathfrak{P}^L(p)^{\mathrm{ur}}$ for the image of d_σ under this map. Since $\sigma(c)c^{-1}$ is invariant under σ^n, $cd_\sigma c^{-1}$, which is a priori contained in the pushout of $\mathfrak{D}_n$ by $D_n(\mathbb{Q}_p^{\mathrm{ur}}) \to P^L(\widehat{\mathbb{Q}_p^{\mathrm{ur}}})$, is in fact an element of $\mathfrak{P}^L(p)^{\mathrm{ur}}$. Choose $t\in T(\bar{\mathbb{Q}}_p)$ as in (3.6.6) (such a t exists by (3.6.7)). Then, in particular, θ_t is induced by a morphism $\theta_t^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_T^{\mathrm{ur}}$, which then induces a morphism $\theta_t^{\mathrm{ur}} : \mathfrak{P}^L(p)^{\mathrm{ur}} \to \mathfrak{G}_T^{\mathrm{ur}}$. After increasing n, we may assume that θ_t^{ur} is induced by a morphisms of $\mathbb{Q}_{p^n}/\mathbb{Q}_p$-gerbs.

Set $\gamma_0 = \psi^\Delta_{\mu_{h_T}}(\delta_n)$, and write

$$\theta_t^{\mathrm{ur}}(cd_\sigma c^{-1}) = \delta \rtimes \sigma \in T(\mathbb{Q}_p^{\mathrm{ur}}) \rtimes \sigma.$$

Since

$$cd_\sigma^n c^{-1} = c\nu(p)^L_n(p^{-1}) \rtimes \sigma^n c^{-1} = c\nu(p)^L_n(p^{-1})\sigma^n(c)^{-1} \rtimes \sigma^n$$

we have

$$\delta\sigma(\delta)\dots\sigma^{n-1}(\delta) = t\gamma_0t^{-1} = \gamma_0.$$

Since c and t are unique up to multiplication by elements of $P^L(\mathbb{Q}_p^{\mathrm{ur}})$ and $T(\mathbb{Q}_p^{\mathrm{ur}})$ respectively, δ is well defined up to σ-conjugacy in $T(\mathbb{Q}_p^{\mathrm{ur}})$.

Let $\mathfrak{k}(T, h_T) = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$, where $\gamma_l = \gamma_0$ for $l \neq p$. Then $i_*(\mathfrak{k}(T, h_T))$ is a Kottwitz triple. Indeed this triple satisfies *(i)* and *(ii)* (4.3.1). It satisfies *(iii)* because $i(\gamma_0) \in i(T)$ and $i \circ h_T \in X$, so $(i(T)/Z_G)(\mathbb{R})$ is compact. It satisfies *(iv)* because $\phi = i \circ \psi_{\mu_{h_T}}$ is admissible by (3.5.8), so we may take $I = I_\phi$.

We call a Kottwitz triple $\mathfrak{k}$ *special* if there exists (T, h_T, i) as above such that $i_*(\mathfrak{k}(T, h_T)) \sim \mathfrak{k}$.

We now give a second construction of a Kottwitz triple. Consider the composite (cf. [De 2, 2.2])

$$N(\mu_{h_T}) : R_{L/\mathbb{Q}}(\mathbb{G}_m) \overset{R_{L/\mathbb{Q}}(\mu_{h_T})}{\to} R_{L/\mathbb{Q}}(T) \overset{N_{L/\mathbb{Q}}}{\to} T \tag{4.3.10}$$

which is the map considered in (3.1.10).

Let $\pi \in L_p$ be a uniformizer. Since T satisfies the Serre condition, it is isogenous to a product $T_1^+ \times T_1^-$, where $T_1^+(\mathbb{R})$ is compact, and T_1^- splits over $\mathbb{Q}$. Hence there exists a compact open subgroup $H \subset T(\mathbb{A}_f)$ such that $T(\mathbb{Q}) \cap H = \{1\}$. Moreover, $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/H$ is finite, so there exists a positive integer s such that $N(\mu_{h_T})(\pi^{-1})^s = \gamma_0' \bmod H$, for some uniquely determined $\gamma_0' \in T(\mathbb{Q})$.

We may assume that H is small enough that $N(\mu_{h_T})(\pi^{-1})^s \gamma_0'^{-1}$ is contained in $\mathcal{T}^\circ(\mathbb{Z}_p)T(\mathbb{A}_f^p)$, where $\mathcal{T}^\circ$ denotes the connected component of the Néron model of T over $\mathbb{Z}_p$. Then by Lang's Lemma, there exists $c' \in T(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})$ such that

$$c' N_{L_p/\mathbb{Q}_p}(\mu_{h_T}(\pi^{-1}))^s \sigma^n(c')^{-1} = \gamma_0',$$

where $n = s[L_{p,0} : \mathbb{Q}_p]$, with $L_{p,0}$ the maximal unramified subfield of L_p. Let $\delta' = c' N_{L_p/L_{p,0}}(\mu_{h_T}(\pi^{-1}))\sigma(c')^{-1}$. Then

$$\delta'\sigma(\delta') \dots \sigma^{n-1}(\delta') = c' N_{L_p/\mathbb{Q}_p}(\mu_{h_T}(\pi^{-1}))^s \sigma^n(c')^{-1} = \gamma_0'.$$

Since $\sigma(\gamma_0') = \gamma_0'$, $\sigma^n(\delta') = \delta'$ so that $\delta' \in T(\mathbb{Q}_{p^n})$. Set $\mathfrak{k}'(T, h_T) = (\gamma_0', (\gamma_l')_{l\neq p}, \delta')$ $\gamma_l' = \gamma_0'$ for $l \neq p$.

Lemma (4.3.11). *If T satisfies the Serre condition, then for n sufficiently large (with the same choice of n in each construction) we have $\gamma_0 = \gamma_0'$ and δ and δ' are σ-conjugate in $T(\mathbb{Q}_{p^n})$. In particular, $i_*(\mathfrak{k}'(T, h_T))$ is a Kottwitz triple and is equivalent to $i_*(\mathfrak{k}(T, h_T))$.*

Proof. Using the functoriality of the assignment $b \mapsto \nu_b$, and (3.1.11), we have

$$\nu_{\gamma_0'} = s\nu_{N_{L_p/\mathbb{Q}_p}(\mu_{h_T}(\pi^{-1}))} = -\frac{s}{[L_p : L_{p,0}]} \sum_{\tau \in \mathrm{Gal}(L_p/\mathbb{Q}_p)} \tau(\mu_{h_T}) = -\psi^{\Delta}_{\mu_{h_T}} \circ \nu(p)_n = \nu_{\gamma_0}.$$

Since $\gamma_0, \gamma_0' \in T(\mathbb{Q})$ are units outside p, and T satisfies the Serre condition, this implies that $\gamma_0'\gamma_0^{-1} \in T(\mathbb{Q})_{\mathrm{tors}}$. Hence after increasing n we have $\gamma_0 = \gamma_0'$.

By the definition of t, the image of δ in $X_*(T)_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is equal to the image of $-\mu$. It follows by [Ko 2, 2.5] that δ and δ' have the same image in $X_*(T)_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$. Hence, by *loc. cit* they are σ conjugate by an element $a \in T(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})$. Since

$$\delta\sigma(\delta) \dots \sigma^{n-1}(\delta) = \delta'\sigma(\delta') \dots \sigma^{n-1}(\delta')$$

we find $a = \sigma^n(a)$, so $a \in T(\mathbb{Q}_{p^n})$. $\square$

(4.4) Isogeny classes and triples: In this subsection we compare the isogeny classes studied in §1,2 with the set $S(\tilde{\mathfrak{k}})$ associated to a refined triple in §4.3.

(4.4.1) For $H' \to H$ a crossed module in algebraic groups over $\mathbb{Q}$ (cf. (3.2.7)) we write

$$Ш^\infty(\mathbb{Q}, H' \to H) = \ker\left(H^1(\mathbb{Q}, H' \to H) \to H^1(\mathbb{R}, H' \to H)\right)$$

and we use the analogous notation for an algebraic group over $\mathbb{Q}$.

Lemma (4.4.2). *Let H be a connected reductive group over $\mathbb{Q}$. Then the natural map*

$$Ш^\infty(\mathbb{Q}, H) \to Ш^\infty(\mathbb{Q}, \tilde{H} \to H)$$

is a bijection. In particular, $Ш^\infty(\mathbb{Q}, H)$ is naturally an abelian group.

Proof. The first claim follows from the cartesian diagram in (3.4.6), and the second claim then follows from (3.2.7)(4).

Lemma (4.4.3). *Let H, Q, Q' be connected reductive groups over $\mathbb{Q}$ and $Q \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} Q' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ an inner twisting. Let $i : Q \to H$ and $i' : Q' \to H$ be embeddings of $\mathbb{Q}$-groups, which are conjugate over $\bar{\mathbb{Q}}$, when we identify Q and Q' via the given inner twisting.*

1. *There is a canonical isomorphism $Ш^\infty(\mathbb{Q}, Q) \xrightarrow{\sim} Ш^\infty(\mathbb{Q}, Q')$.*
2. *The maps*

$$i, i' : Ш^\infty(\mathbb{Q}, Q) \overset{(1)}{=} Ш^\infty(\mathbb{Q}, Q') \to Ш^\infty(\mathbb{Q}, H) \tag{4.4.4}$$

induced by i and i' are equal.

Proof. For (1) note that by (4.4.2) and (3.2.7)(2)

$$Ш^\infty(\mathbb{Q}, Q) \xrightarrow{\sim} Ш^\infty(\mathbb{Q}, \tilde{Q} \to Q) \xrightarrow{\sim} Ш^\infty(\mathbb{Q}, Z_{\tilde{Q}} \to Z_Q) \xrightarrow{\sim} Ш^\infty(\mathbb{Q}, \tilde{Q}' \to Q') \xrightarrow{\sim} Ш^\infty(\mathbb{Q}, Q').$$

In view of the above, to prove (2) it suffices to show that the natural maps

$$H^1(\mathbb{Q}, Z_{\tilde{Q}} \to Z_Q) \to H^1(\mathbb{Q}, \tilde{H} \to H)$$

induced by i and i' are equal. This follows easily from the isomorphism

$$H^1(\mathbb{Q}, Z_{\tilde{H}} \to Z_H) \xrightarrow{\sim} H^1(\mathbb{Q}, \tilde{H} \to H).$$

$\square$

Lemma (4.4.5). *Let $H' \subset H$ be connected reductive groups over $\mathbb{R}$, such that $H^{\mathrm{ad}}(\mathbb{R})$ is compact and H' contains the maximal $\mathbb{R}$-split torus $Z \subset Z_H$. If $c \in H^1(\mathbb{R}, H')$ has trivial image in $H^1(\mathbb{R}, H)$ then $c = 0$.*

Proof. By Hilbert's theorem 90 it suffices to show that a class in $H^1(\mathbb{R}, H'/Z)$ which has trivial image in $H^1(\mathbb{R}, H/Z)$ is trivial. Hence, we may replace H and H' by H/Z and H'/Z respectively, and assume that $H(\mathbb{R})$ is compact.

By [Se, III Thm. 6] the map of pointed sets

$$H^1(\mathbb{R}, H') \to H^1(\mathbb{R}, H)$$

may be identified with

$$H'[2]/H' \to H[2]/H$$

where H' and H acts by conjugation on the subsets of 2-torsion elements $H'[2]$ and $H[2]$ respectively. Under the latter map the preimage of the identity in $H[2]/H$ is clearly the identity. □

(4.4.6) We now return to the notation of §2.1. In particular, (G, X) is of Hodge type, equipped with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$, the group G is equipped with a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, $\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, and $k \subset \bar{\mathbb{F}}_p$ is a finite extension of k_E with $|k| = p^r$.

Let $x \in \mathscr{S}_{\mathsf{K}_p}(G, X)(k)$. By (2.1.2) and (2.3.1) attached to x we have a Kottwitz triple $\mathfrak{k}(x) = (\gamma_0, (\gamma_l)_{l \neq p}, \delta)$. That this triple satisfies condition *(iv)* of (4.3.1) follows from (2.3.5), which shows that we may take I to be the subgroup of $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ which fixes $s_{\alpha,0,x}$ and $s_{\alpha,l,x}$ for all $l \neq p$. Indeed, (2.3.5) shows that (for this choice of I) $\mathfrak{k}(x)$ has a natural refinement $\tilde{\mathfrak{k}}(x)$.

Write $\mathscr{I}_x = \mathrm{Im}(\iota_x) \subset \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$ where ι_x is the map of (2.1.3). The equivalence class of $\mathfrak{k}(x)$ depends only on $\mathscr{I}_x$ and not on x, and we will denote it by $\mathfrak{k}(\mathscr{I}_x)$.

(4.4.7) Let $Ш^{\infty}_G(\mathbb{Q}, I)$ denote the kernel of the composite

$$Ш^{\infty}(\mathbb{Q}, I) \xrightarrow[(4.4.3)]{\sim} Ш^{\infty}(\mathbb{Q}, I_0) \to Ш^{\infty}(\mathbb{Q}, G).$$

Then $Ш^{\infty}_G(\mathbb{Q}, I)$ is independent of the choice of γ_0 by (4.4.3)(2). Given $\beta \in Ш^{\infty}_G(\mathbb{Q}, I)$ we will construct an isogeny class $\mathscr{I}^{\beta}_x \subset \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$.

Fix a maximal torus $T \subset I$. By (2.2.5), there exists an $x' \in \mathscr{I}_x$ and a lifting $\tilde{x}' \in \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{Q}}_p)$ such that $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{x'}$ lifts to $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}'}$. Then $\tilde{x}'$ is necessarily defined over $\bar{\mathbb{Q}}$, and the Mumford-Tate group of $\mathcal{A}_{\tilde{x}'}$ is contained in T. In particular, we have a natural embedding $T \hookrightarrow G$ of algebraic groups over $\mathbb{Q}$, and $\tilde{x}'$ is in the image of

$$T(\mathbb{Q})\backslash T(\mathbb{R}) \times G(\mathbb{A}_f) \to \mathscr{S}_{\mathsf{K}_p}(G, X)(\mathbb{C}).$$

Hence, by (1.4.12), x' and $\tilde{x}'$ may be chosen so that $\tilde{x}'$ is in the image of

$$\mathrm{Sh}(T, h_T) \to \mathrm{Sh}_{\mathsf{K}_p}(G, X).$$

Here $h_T : \mathbb{S} \to T$ denotes the cocharacter giving the Hodge structure on $\mathcal{A}_{\tilde{x}'}$. We may assume that $\gamma_0 \in T(\mathbb{Q})$, when the map $T \hookrightarrow I_0 \xrightarrow{\sim} I$ over $\bar{\mathbb{Q}}$, is conjugate to the inclusion $T \subset I$.

By [Bo, Thm. 5.11] T may be chosen so that β is the image of an element $\tilde{\beta} \in H^1(\mathbb{Q}, T)$, and $\tilde{\beta} \in \text{Ш}^\infty(\mathbb{Q}, T)$ by (4.4.5). Since the image of $\tilde{\beta}$ in $H^1(\mathbb{R}, G)$ and $H^1(\mathbb{Q}, \tilde{G} \to G)$ is trivial, its image in $H^1(\mathbb{Q}, G)$ is trivial by (4.4.2), and $\tilde{\beta}$ is the image of an element $\omega \in (G/T)(\mathbb{Q})_1$.

Finally we apply the construction of §4.2, with $(G', X') = (T, h_T)$. Using the notation of that section, we obtain a point $\mathfrak{i}_\omega(\tilde{x}') \in \mathrm{Sh}_{\mathrm{K}_p^\omega}(G, X)(\bar{\mathbb{Q}}_p)$. Choose a lift of $\mathfrak{i}_\omega(\tilde{x}')$ in $\mathrm{Sh}_{\mathrm{K}_p^\omega \cap \mathrm{K}_p}(G, X)(\bar{\mathbb{Q}}_p)$ and again denote by $\mathfrak{i}_\omega(\tilde{x}')$ the image of this lift in $\mathrm{Sh}_{\mathrm{K}_p}(G, X)(\bar{\mathbb{Q}}_p)$. Note that the isogeny class of this image is independent of choice of lift.

If $\mathcal{P}_\omega$ is the fibre of $G \to G/T$ over ω, then

$$\mathcal{A}_{\tilde{x}'}^{\mathcal{P}_\omega} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}_{\tilde{x}'} \otimes_{\mathbb{Q}} F$$

for some finite extension $F/\mathbb{Q}$, so $\mathcal{A}_{\tilde{x}'}^{\mathcal{P}_\omega}$ has good reduction, as $\mathcal{A}_{\tilde{x}'}$ does. Hence $\mathfrak{i}_\omega(\tilde{x}')$ admits a reduction $\mathfrak{i}_\omega(x') \in \mathscr{S}_{\mathrm{K}_p}(G, X)(\bar{\mathbb{F}}_p)$. We set $\mathscr{I}_x^\beta = \mathscr{I}_{\mathfrak{i}_\omega(x')}$.

Proposition (4.4.8). *The isogeny class $\mathscr{I}_x^\beta$ depends only on β, and not on T or ω. The assignment $\beta \mapsto \mathscr{I}_x^\beta$ defines an inclusion of $\text{Ш}_G^\infty(\mathbb{Q}, I)$ into the set of isogeny classes in $\mathscr{S}_{\mathrm{K}_p}(G, X)(\bar{\mathbb{F}}_p)$.*

Proof. Recall that by (4.1.7) and (4.1.5), the tensors $s_{\alpha,0,x'}$ (resp. $s_{\alpha,l,x}$ for $l \neq p$), may be viewed in $H^1_{\mathrm{cris}}(\mathcal{A}_{x'}^{\mathcal{P}_\omega}/\mathcal{O}_L)_L^\otimes$ (resp. $H^1_{\text{ét}}(\mathcal{A}_{x'}^{\mathcal{P}_\omega}, \mathbb{Q}_l)^\otimes$). Let $y = \mathfrak{i}_\omega(x')$. By (4.2.6), the weakly polarized, abelian variety up to isogeny attached to y is naturally identified with $(\mathcal{A}_{x'}^{\mathcal{P}_\omega}, \lambda_{x'}^{\mathcal{P}_\omega})$, and under this identification the tensors $s_{\alpha,0,x'}$ (resp. $s_{\alpha,l,x'}$ for $l \neq p$), are identified with $s_{\alpha,0,y}$ (resp. $s_{\alpha,l,y}$).

Now let $\mathcal{Q}$ be the $\mathbb{Q}$-scheme of isomorphisms $\mathcal{A}_{x'}^{\mathcal{P}_\omega} = \mathcal{A}_y \to \mathcal{A}_{x'}$ which are compatible with weak polarizations and fix $s_{\alpha,0,x'}$ and $s_{\alpha,l,x'}$ for $l \neq p$. As in the proof of (4.1.5), using the canonical isomorphism

$$\mathcal{A}_{x'}^{\mathcal{P}_\omega} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}_\omega} \xrightarrow{\sim} \mathcal{A}_{x'} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}_\omega}$$

one sees that $\mathcal{Q}$ is an I-torsor, which admits a T-equivariant map $\mathcal{P}_\omega \to \mathcal{Q}$. Hence $\mathcal{Q}$ is isomorphic to the I-torsor $\mathcal{P}_\beta$ induced by $\mathcal{P}_\omega$, and associated to the class $\beta \in H^1(\mathbb{Q}, I)$.

It follows from (1.4.15) that the isogeny class of y depends only on β, and that for $\beta, \beta' \in \text{Ш}_G^\infty(\mathbb{Q}, I)$, $\mathscr{I}_x^\beta = \mathscr{I}_x^{\beta'}$ if and only if $\beta = \beta'$. $\square$

(4.4.9) If $\beta \in \text{Ш}_G^\infty(\mathbb{Q}, I)$, then for $l \neq p$ the image of β in $H^1(\mathbb{Q}_l, G)$ is trivial, as $H^1(\mathbb{Q}_l, \tilde{G}) = 0$. So β satisfies the conditions imposed in (4.3.6) on an element used to define the twist $\mathfrak{k}^\beta$ of a Kottwitz triple $\mathfrak{k}$. We will show that the fibre over $\mathfrak{k}(\mathscr{I}_x)$ of the map

$$\mathfrak{k} : \{\mathscr{I}_y : y \in \mathscr{S}_{\mathrm{K}_p}(G, X)(\bar{\mathbb{F}}_p)\} \to \{\text{triples } \mathfrak{k}\}/\sim; \quad \mathscr{I}_y \mapsto \mathfrak{k}(\mathscr{I}_y) \tag{4.4.10}$$

is naturally a torsor under the abelian group

$$\text{Ш}_G(\mathbb{Q}, I) = \ker(\text{Ш}_G^\infty(\mathbb{Q}, I) \to \prod_v H^1(\mathbb{Q}_v, I))$$

where v runs over finite places of $\mathbb{Q}$.

Write $\mathfrak{k}(\mathscr{I}_x) = (\gamma_0(\mathscr{I}_x), (\gamma_l(\mathscr{I}_x))_{l \neq p}, \delta(\mathscr{I}_x))$.

Lemma (4.4.11). *For $\beta \in Ш^{\infty}_G(\mathbb{Q}, I)$ we have $\mathfrak{k}(\mathscr{I}_x^\beta) \sim \mathfrak{k}(\mathscr{I}_x)^\beta$.*

Proof. For $l \neq p$, fix isomorphisms $V^*_{\mathbb{Q}_l} \xrightarrow{\sim} H^1(\mathcal{A}_x, \mathbb{Q}_l)$ and $V^*_{\mathbb{Q}_l} \xrightarrow{\sim} H^1(\mathcal{A}_x^{\mathcal{P}_\beta}, \mathbb{Q}_l)$ taking s_α to $s_{\alpha,l,x}$, where $\mathcal{P}_\beta$ is an I-torsor with class β. By (4.1.7), the isomorphism (4.1.3) gives rise to an isomorphism

$$V^*_{\mathbb{Q}_l} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} V^*_{\mathbb{Q}_l} \otimes_{\mathbb{Q}} F \tag{4.4.12}$$

which intertwines the $G(\mathbb{Q}_l)$-conjugacy classes of $\gamma_l(\mathscr{I}_x^\beta)$ and $\gamma_l(\mathscr{I}_x)$. By (4.1.2)(2), (4.4.12) is given by an element $g_l^{-1} \in G(\mathbb{Q}_l \otimes F)$, such that $(g_l^{-1}\tau(g_l))_\tau$ is an $I(F)$-valued cocycle whose class is equal to β by (4.1.2). Hence $\gamma_l(\mathscr{I}_x^\beta) = g_l\gamma_l(\mathscr{I}_x)g_l^{-1}$, where we again write g_l for the image of g_l in $G(\bar{\mathbb{Q}}_l)$.

Similarly, choosing isomorphisms $V^*_{K_0} \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_x/W(k))_{K_0}$ and $V^*_{K_0} \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_x^\beta/W(k))_{K_0}$ taking s_α to $s_{\alpha,0,x}$, and using (4.1.2), we find an isomorphism

$$V^*_{K_0} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} V^*_{K_0} \otimes_{\mathbb{Q}} F$$

which intertwines the σ-conjugacy classes of $\delta(\mathscr{I}_x^\beta)$ and $\delta(\mathscr{I}_x)$, and is given by an element $g_0^{-1} \in G(K_0 \otimes_{\mathbb{Q}} F)$, such that $g_0^{-1}\tau(g_0)_\tau$ is an $I(F)$-valued cocycle whose class is equal to β. Hence $\delta(\mathscr{I}_x^\beta) = g_0\delta(\mathscr{I}_x)\sigma(g_0)^{-1}$, where we again write g_0 for the image of g_0 in $G(K_0 \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)$.

Comparing with the definition of $\mathfrak{k}(\mathscr{I}_x)^\beta$ in (4.3.6), proves the lemma. □

Proposition (4.4.13). *If $\beta \in Ш_G(\mathbb{Q}, I)$, we have $\mathfrak{k}(\mathscr{I}_x) \sim \mathfrak{k}(\mathscr{I}_x^\beta)$, and the assignment $\mathscr{I}_y \mapsto \mathscr{I}_y^\beta$ makes the fibre of $\mathfrak{k}$ over $\mathfrak{k}(\mathscr{I}_x)$ into a $Ш_G(\mathbb{Q}, I)$-torsor.*

Proof. Since the image of β is trivial at all finite places, we have $\mathfrak{k}(\mathscr{I}_x) \sim \mathfrak{k}(\mathscr{I}_x)^\beta \sim \mathfrak{k}(\mathscr{I}_x^\beta)$ by (4.3.7) and (4.4.11), so that $Ш_G(\mathbb{Q}, I)$ acts on the fibre of $\mathfrak{k}$ over $\mathfrak{k}(\mathscr{I}_x)$.

In view of (4.4.8), it remains to show that $Ш_G(\mathbb{Q}, I)$ acts transitively on the fibre of $\mathfrak{k}$ over $\mathfrak{k}(\mathscr{I}_x)$. Let $x' \in \mathscr{S}(G,X)(\bar{\mathbb{F}}_p)$ be such that $\mathfrak{k}(\mathscr{I}_x) \sim \mathfrak{k}(\mathscr{I}_{x'})$. Write $I' \subset \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{x'}$ for the subgroup fixing $s_{\alpha,0,x}$ and $s_{\alpha,l,x}$ for $l \neq p$. By (2.3.5), I and I' may both be regarded as inner twistings of I_{γ_0}. Since $\mathfrak{k}(\mathscr{I}_x) \sim \mathfrak{k}(\mathscr{I}_{x'})$, we saw in (4.3.1), that there is an isomorphism $I \xrightarrow{\sim} I'$ as inner twists of I_{γ_0}.

Fix a maximal torus $T \subset I \xrightarrow{\sim} I'$. As in (4.4.7), using (2.2.5) we may replace x, x' by other points in their respective isogeny classes, and assume that x, x' admit liftings $\tilde{x}, \tilde{x}' \in \mathscr{S}_{\mathrm{K}_p}(G,X)(\bar{\mathbb{Q}})$ such that

1. (1) The maps $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_x$ and $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{x'}$ lift to $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}}$ and $T \hookrightarrow \mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{\tilde{x}'}$.
2. (2) The Hodge filtrations on $H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}})$ and $H^1_{\mathrm{dR}}(\mathcal{A}_{\tilde{x}'})$ are given by the same T-valued cocharacter μ_T.
3. (3) If $i, i' : T \hookrightarrow G$ are the embeddings obtained by regarding T as a subgroup of the Mumford-Tate groups of $\mathcal{A}_{\tilde{x}}$ and $\mathcal{A}_{\tilde{x}'}$ then $\tilde{x}, \tilde{x}'$ are in the images of
$$i, i' : \mathrm{Sh}(T, h_T) \to \mathrm{Sh}(G,X)$$
respectively.

By the reciprocity law, for r sufficiently large, the geometric $q = p^r$-Frobenius on l-adic or crystalline cohomology of $\mathcal{A}_x$ and $\mathcal{A}_{x'}$ is given by an element $\gamma \in T(\mathbb{Q})$. Our assumptions imply that $\gamma_0 := i(\gamma)$ and $\gamma_0' := i'(\gamma)$ are conjugate in $G(\bar{\mathbb{Q}})$. Let

$I_0, I_0' \subset G$ denote the centralizers of γ_0 and γ_0' respectively, and let $g \in G(\bar{\mathbb{Q}})$ be such that $g\gamma_0 g^{-1} = \gamma_0'$. We assume that r is large enough that I_0, I_0' have the same dimension as I.

Fix inner twistings $I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ and $I_0' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ as in (2.3.5). By definition, the diagram

$$\begin{array}{ccc} I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} & \xrightarrow{\sim} & I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \\ \downarrow{\scriptstyle \mathrm{Int}(g)} & & \downarrow{\scriptstyle \sim} \\ I_0' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} & \xrightarrow{\sim} & I' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \end{array}$$

commutes up to conjugation. The commutativity of the diagram in (4.3.1)*(iv)* implies that the composite

$$T \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \overset{i}{\hookrightarrow} I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

is conjugate to the natural inclusion $T \subset I$ by an element of $I(\bar{\mathbb{Q}})$. Since an analogous statement holds with i' and the inner twisting $I_0' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I' \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$, the composite

$$T \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \overset{g^{-1}i'g}{\hookrightarrow} I_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

is also conjugate to the natural inclusion. Hence i and $g^{-1}i'g$ are conjugate under $I_0(\bar{\mathbb{Q}})$, which implies that we may assume that g has been chosen so that $i' = gig^{-1}$. Since $i(T) \subset G$ is its own commutator, g is unique up to right multiplication by elements of $i(T)(\bar{\mathbb{Q}})$. In particular, for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $c_\tau = g^{-1}\tau(g) \in i(T)(\bar{\mathbb{Q}})$.

Now let $\mathsf{K}_\infty \subset G_{\mathbb{R}}$ be the centralizer of $i \circ h_T$. Since $i \circ h_T, i' \circ h_T \in X$, there is a $g_\infty \in G(\mathbb{R})$ such that

$$g_\infty(i \circ h_T)g_\infty^{-1} = i' \circ h_T = g(i \circ h_T)g^{-1}.$$

So $g_\infty^{-1}g \in \mathsf{K}_\infty(\mathbb{C})$, and $(g_\infty^{-1}g)^{-1}s(g_\infty^{-1}g) = g^{-1}s(g)$ for $s \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Hence the image of c in $H^1(\mathbb{R}, \mathsf{K}_\infty)$ is trivial.

It follows that c arises from a point $\omega \in (G/T)(\mathbb{Q})_1$. If $\mathcal{P}_\omega$ denotes the corresponding T-torsor, then by (4.2.6), there is an isomorphism $\mathcal{A}_{\tilde{x}'} \xrightarrow{\sim} \mathcal{A}_{\tilde{x}}^{\mathcal{P}_\omega}$ which respects the Hodge tensors s_α. It follows that $\mathscr{I}_{x'} = \mathscr{I}_x^\beta$, where $\beta \in H^1(\mathbb{Q}, I)$ denotes the image of c.

By (4.4.5) the image of c in $H^1(\mathbb{R}, T)$ is trivial, and hence so is its image in $H^1(\mathbb{R}, I)$. Since $\mathfrak{k}(\mathscr{I}_x) \sim \mathfrak{k}(\mathscr{I}_{x'}) \sim \mathfrak{k}(\mathscr{I}_x)^\beta$, the image of β in $H^1(\mathbb{Q}_l, I)$ is trivial for every finite prime l by (4.3.7), so $\beta \in \text{Ш}_G(\mathbb{Q}, I)$. $\square$

Proposition (4.4.14). *The subgroup $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I \subset \mathscr{A}(G_{\mathbb{Z}_{(p)}})$ is the stabilizer of $\mathscr{I}_x \subset \mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)$, and there is a $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant bijection*

$$\iota_x^{-1} : \mathscr{I}_x \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x))$$

taking x to 1.

Proof. The first claim follows from (4.4.8). By (1.4.13) and (2.1.3), there is $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$-equivariant bijection $\iota_x^{-1} : \mathscr{I}_x \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x)))$, which takes x to 1 by definition. We shall check that this bijection is $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$-equivariant.

Let $x' \in \mathscr{I}_x$, so that $(\mathcal{A}_{x'}, \varepsilon^p_{x'}) = (\mathcal{A}_{g_0x}, \varepsilon^p_x \circ g^p)$ with $g_0 \in X_\upsilon(\delta)$ and $g^p \in G(\mathbb{A}^p_f)$. Let $h \in G(\bar{\mathbb{Z}}_{(p)})^{\natural,I}_{+}$, and h^{ad} the image of h in $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$. We let $\mathcal{P}$ denote the Z_G-torsor associated to $h^{\mathrm{ad}-1}$, so $h^{-1} \in \mathcal{P}(F)$ for some Galois extension $F/\mathbb{Q}$. Then[22] $(\mathcal{A}_{x'}, \varepsilon^p_{x'}) \cdot h^{\mathrm{ad}} = (\mathcal{A}^{\mathcal{P}}_{x'}, \varepsilon^{p,h^{-1}}_{x'})$. Finally, as in (4.3.4) we choose $i_h \in I(\bar{\mathbb{Q}})$ such that $i_h^{-1}\tau(i_h) = h^{-1}\tau(h)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. By (4.3.5), we have to check that $\iota_x^{-1}(x' \cdot h) = i_h^{-1} g h$, where $g = (g_0, g^p)$.

The composite

$$\mathcal{A}^{\mathcal{P}}_{g_0x} \otimes_{\mathbb{Q}} F \overset{\vartheta_{h^{-1}}}{\to} \mathcal{A}_{g_0x} \otimes_{\mathbb{Q}} F \to \mathcal{A}_x \otimes_{\mathbb{Q}} F \overset{i_h^{-1}}{\to} \mathcal{A}_x \otimes_{\mathbb{Q}} F. \tag{4.4.15}$$

induces the bottom row of the commutative diagram

$$\begin{array}{ccccccc}
V_{\mathbb{A}^p_f} \otimes_{\mathbb{Q}} F & \xrightarrow{h} & V_{\mathbb{A}^p_f} \otimes_{\mathbb{Q}} F & \xrightarrow{g^p} & V_{\mathbb{A}^p_f} \otimes_{\mathbb{Q}} F & \xrightarrow{i_h^{-1}} & V_{\mathbb{A}^p_f} \otimes_{\mathbb{Q}} F \\
\downarrow{\scriptstyle \varepsilon^{p,h^{-1}}_{x'}} & & \downarrow{\scriptstyle \varepsilon^p_x \circ g^p} & & \downarrow{\scriptstyle \varepsilon^p_x} & & \downarrow{\scriptstyle \varepsilon^p_x} \\
\widehat{V}^p(\mathcal{A}^{\mathcal{P}}_{g_0x}) \otimes_{\mathbb{Q}} F & \xrightarrow{\vartheta_{h^{-1}}} & \widehat{V}^p(\mathcal{A}_{g_0x}) \otimes_{\mathbb{Q}} F & \xrightarrow{\sim} & \widehat{V}^p(\mathcal{A}_x) \otimes_{\mathbb{Q}} F & \xrightarrow{i_h^{-1}} & \widehat{V}^p(\mathcal{A}_x) \otimes_{\mathbb{Q}} F
\end{array}$$

By (4.1.2)

$$\tau(\vartheta_{h^{-1}})\vartheta^{-1}_{h^{-1}} = \tau(h)h^{-1} = \tau(i_h)i_h^{-1} \in Z_G(\bar{\mathbb{Q}})$$

which implies that (4.4.15) is $\mathrm{Gal}(F/\mathbb{Q})$-equivariant, and hence is induced by a quasi-isogeny $\mathcal{A}^{\mathcal{P}}_{g_0x} \to \mathcal{A}_x$. Thus, the commutative diagram above induces a commutative diagram

$$\begin{array}{ccc}
V_{\mathbb{A}^p_f} & \xrightarrow{i_h^{-1} g^p h} & V_{\mathbb{A}^p_f} \\
\downarrow{\scriptstyle \varepsilon^{p,h^{-1}}_{x'}} & & \downarrow{\scriptstyle \varepsilon^p_x} \\
\widehat{V}^p(\mathcal{A}^{\mathcal{P}}_{x'}) & \longrightarrow & \widehat{V}^p(\mathcal{A}_x)
\end{array}$$

It remains to show that (4.4.15) identifies $\mathbb{D}(\mathscr{G}^{\mathcal{P}}_{g_0x})$ with $(i_h^{-1}gh)\mathbb{D}(\mathscr{G}_x)$, where $\mathscr{G}^{\mathcal{P}}_{g_0x}$ is the p-divisible group of $\mathcal{A}^{\mathcal{P}}_{g_0x}$. It suffices to check the analogous statements for $\mathbb{D}(\mathscr{G}^{\mathcal{P}}_{g_0x}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}$ and $(i_h^{-1}gh)\mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}$. But (4.2.9) implies[23] that (4.4.15) identifies $\mathbb{D}(\mathscr{G}^{\mathcal{P}}_{g_0x}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}$ with

$$(i_h^{-1}g)(\mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}) = (i_h^{-1}gh)\mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{F,(p)}.$$

□

(4.4.16) Before stating the final result of this subsection, we need some notation. Suppose $h \in G^{\mathrm{ad}}(\mathbb{Q})^+$, let $\mathcal{P}$ be the corresponding Z_G-torsor, and let $\beta \in H^1(\mathbb{Q}, Z_G)$ be the class of $\mathcal{P}$. Let $\tilde{h} \in G^{\mathrm{ad}}(\bar{\mathbb{Q}})$ be a lift of h, so that $\tilde{h} \in \mathcal{P}(F)$ for some finite extension $F/\mathbb{Q}$. The cocycle $\tau \mapsto \tilde{h}^{-1}\tau(\tilde{h})$ represents β.

As in (2.1.2) fix an isomorphism

$$\mathbb{D}(\mathscr{G}_x) \xrightarrow{\sim} V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W$$

[22] Note that we are using the right action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{I,+}$ on $\mathscr{S}_{\mathsf{K}_p}(G, X)$ so h acts via $\mathfrak{i}_{h^{-1}}$.

[23] Keeping in mind that i_h^{-1} acts as $\mathbb{D}(i_h)$ on $\mathbb{D}(\mathscr{G}_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, as in (2.1.2).

which takes $s_{\alpha,0,x}$ to s_α. If $x' \in \mathscr{I}_x^\beta$, then the natural isomorphism induced by the choice of $\tilde{h}$

$$\mathcal{A}_{x'} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}_x \otimes_{\mathbb{Q}} F$$

identifies $\mathbb{D}(\mathcal{A}_{x'})$ with $g_0^{-1}\mathbb{D}(\mathcal{A}_x)$ for some $g_0 \in G(L \otimes_{\mathbb{Q}} F)$ such that $z_\tau = g_0^{-1}\tau(g)$. Similarly the isomorphism

$$V_{\mathbb{A}_f} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} V_{\mathbb{A}_f} \otimes_{\mathbb{Q}} F$$

induced by $\epsilon_{x'}$ and ϵ_x is given by multiplication by $(g^p)^{-1}$ on $V_{\mathbb{A}_f^p}$, where $g^p \in G(\mathbb{A}_f^p \otimes_{\mathbb{Q}} F)$ again gives rise to the cocycle z_τ. Set $g_{x'} = (g_0, g^p)$.

Proposition (4.4.17). *With the notation above, there exists a* $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$*-equivariant bijection*

$$\iota_x^{-1\beta} : \mathscr{I}_x^\beta \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x)^\beta); \quad x' \mapsto \tilde{h}g_{x'}^{-1},$$

which depends only on h and not on $\tilde{h}$.

Moreover, if Z_G is a torus then under the projection

$$c_G : \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) \to \pi(G,X)$$

we have $c_G(x') = (\tilde{h}g_{x'}^{-1})c_G(x)$.

Proof. Write $\mathfrak{k}(x) = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$. Then

$$\tilde{\mathfrak{k}}(x') = (h\gamma_0 h^{-1}, g^p(\gamma_l)_{l\neq p}g^{p-1}, g_0\delta\sigma(g_0)^{-1}, \iota'),$$

with the action of ι' given by ι and conjugation by (g_0, g^p). Applying (4.4.14) to x' we obtain a $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant bijection $\mathscr{I}_x^\beta \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x'))$. Composing with multiplication on the left by $\tilde{h}g_{x'}^{-1}$ as in (4.3.2), yields an the isomorphism $\mathscr{I}_x^\beta \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x)^\beta)$. One checks easily that this isomorphism is independent of the choice of $x' \in \mathscr{I}_x^\beta$, and it sends x' to $\tilde{h}g_{x'}^{-1}$ by construction. Modifying $\tilde{h}$ by an element of $Z_G(\bar{\mathbb{Q}})$ changes $g_{x'}$ by the same element, so the isomorphism depends only on h.

We now show the second claim. Since Z_G is a torus, and G is unramified, $\mathcal{P}$ splits over an unramified extension of $\mathbb{Q}_p$. Hence by [MB, Thm. 1.3], we may assume that F has been chosen so that p is unramified in F. Suppose $\tilde{x} \in \mathrm{Sh}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{Q}}_p)$ is a lift of x, and fix an isomorphism $H^1_{\text{ét}}(\mathcal{A}_{\tilde{x}}, \mathbb{Z}_p) \xrightarrow{\sim} V^*_{\mathbb{Z}_p}$ sending $s_{\alpha,p,\tilde{x}}$ to s_α. This allows us to promote $\tilde{x}$ to a point $\tilde{y} \in \mathrm{Sh}(G,X)(\bar{\mathbb{Q}}_p)$. Let $h\tilde{x}$ denote the image of $\iota_h(\tilde{y})$ in $\mathrm{Sh}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{Q}}_p)$. Then there is a natural isomorphism

$$\mathcal{A}_{h\tilde{x}} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}^\beta_{\tilde{x}} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}_{\tilde{x}} \otimes_{\mathbb{Q}} F$$

which identifies $H^1_{\text{ét}}(\mathcal{A}_{h\tilde{x}}, \mathbb{Z}_p)$ with $\tilde{h}^{-1}H^1_{\text{ét}}(\mathcal{A}_{\tilde{x}}, \mathbb{Z}_p)$.

Now let $\mathscr{I}^\beta_{\tilde{x}} \subset \mathrm{Sh}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{Q}}_p)$ be the isogeny class of $h\tilde{x}$. If $\tilde{x}' \in \mathscr{I}^\beta_{\tilde{x}}$ then, as above we have a natural isomorphism

$$\mathcal{A}_{\tilde{x}'} \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \mathcal{A}_{\tilde{x}} \otimes_{\mathbb{Q}} F,$$

which identifies $H^1_{\text{ét}}(\mathcal{A}_{h\tilde{x}}, \mathbb{Z}_p)$ with $g_p^{-1}H^1_{\text{ét}}(\mathcal{A}_{\tilde{x}}, \mathbb{Z}_p)$. and sends to $V_{\mathbb{A}_f^p}$ to $(g^p)^{-1}V_{\mathbb{A}_f^p}$, where $g = (g_p, g^p) \in G(\bar{\mathbb{A}}_f \otimes_{\mathbb{Q}} F)$ gives rise to the cocycle z_τ. In particular $\tilde{h}g^{-1} \in G(\mathbb{A}_f)$. Let $x' \in \mathscr{I}_x^\beta$ denote the reduction of $\tilde{x}'$. Since the action of $G(\mathbb{Q})^+$ on $\mathrm{Sh}(G, X)(\bar{\mathbb{Q}}_p)$ does not change the image of a point in $\pi(G, X)$, we have

$$c_G(x') = c_G(\tilde{x}') = \tilde{h}g^{-1}c_G(h\tilde{x}) = \tilde{h}g^{-1}c_G(\tilde{x}) = \tilde{h}g^{-1}c_G(x).$$

Write $g_{x'} = (g_0, g^p)$, as above. By (1.2.18), applied to the group[24] $R_{F/\mathbb{Q}}G$, g_p and g_0 have the same image in $\pi_1(R_{F/\mathbb{Q}}G)^\Gamma$. Hence $\tilde{h}g^{-1}$ and $\tilde{h}g_0^{-1}$ have the same image in

$$G(\mathbb{Q}_p)/\tilde{G}(\mathbb{Q}_p)G_{\mathbb{Z}_p}(\mathbb{Z}_p) = \pi_1(G)^\Gamma \subset \pi_1(R_{F/\mathbb{Q}}G)^\Gamma.$$

This proves the second claim of the lemma when x' is the reduction of a point $\tilde{x}' \in \mathscr{I}_{\tilde{x}}^\beta$. By (1.2.23) and (1.3.9), we may choose $\tilde{x}$ such that the image of $\mathscr{I}_{\tilde{x}}^\beta \to \mathscr{I}_x^\beta$ meets every connected component of $\mathscr{I}_x^\beta$. Hence the second claim follows from (1.2.6). $\square$

(4.5) Admissible morphisms and triples: We now return to the assumptions of (4.3). In particular we assume that Z_G^0 satisfies the Serre condition. We recall how to attach Kottwitz triples to admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ (cf. [LR, 5.25]). By (3.5.7), any such morphism factors through $\mathfrak{P}$, and we will again denote by $\phi : \mathfrak{P} \to \mathfrak{G}_G$ the induced morphism.

(4.5.1) Suppose $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is admissible, and let $g = (g_0, g^p) \in X_p(\phi) \times X^p(\phi)$. We attach a Kottwitz triple $\mathfrak{k}(g)$ to g as follows. Choose a CM number field L such that ϕ factors through $\mathfrak{P}^L$, and let n be a positive integer divisible by $[L_p : \mathbb{Q}_p]$. We set $(\gamma_l)_{l \neq p} = g^{p-1}\phi^\Delta(\delta_n)g^p$. Since $g^{p-1}\phi(l)g^p(1 \rtimes \tau) = 1 \rtimes \tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, $(\gamma_l)_{l \neq p} \in G(\mathbb{A}_f^p)$.

To lighten notation write $\phi(p)_{g_0} = g_0^{-1}\phi(p)g_0$, and similarly for $\phi^\Delta(p)_{g_0}$. Since $g_0 \in X_p(\phi)$ we have $\phi^\Delta(p)_{g_0}(1 \rtimes \tau) = 1 \rtimes \tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$, which implies $\phi^\Delta(p)_{g_0}(P^L(\mathbb{Q}_p^{\mathrm{ur}})) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$. Thus, with the notation of (4.3.9), $\phi(p)_{g_0}$ is induced by a morphism $\theta_{g_0}^{\mathrm{ur}} : \mathfrak{P}^L(p)^{\mathrm{ur}} \to \mathfrak{G}_G^{\mathrm{ur}}$. After increasing n we may assume that $\theta_{g_0}^{\mathrm{ur}}$ is induced by a map of $\mathbb{Q}_{p^n}/\mathbb{Q}_p$-gerbs.

Let $c \in P^L(\widehat{\mathbb{Q}_p^{\mathrm{ur}}})$ be as in (4.3.9). Choose $c' \in P^L(\mathbb{Q}_p^{\mathrm{ur}})$ sufficiently close to c that $\theta_{g_0}^{\mathrm{ur}}(c'c^{-1}) \subset G(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$. After increasing n we may assume that $c' \in P^L(\mathbb{Q}_{p^n})$. Write

$$\theta_{g_0}^{\mathrm{ur}}(c'^{-1}cd_\sigma c^{-1}c') = \delta \rtimes \sigma.$$

Using that $\sigma(c)c^{-1}$ is fixed by σ^n, we see as in (4.3.9) that $\delta \in G(\mathbb{Q}_{p^n})$, and δ is well defined up to σ-conjugacy by elements of $G(\mathbb{Z}_{p^n})$. As an element of $G(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})$ modulo σ-conjugacy by elements of $G(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$, $\delta = \delta'$, where $\delta' \rtimes \sigma = \theta_{g_0}^{\mathrm{ur}}(d_\sigma)$.

By [LR, Lem. 5.4] for n divisible by m, and sufficiently large, $\phi(\delta_n) \in G(\bar{\mathbb{Q}})$ is conjugate to an element $\gamma_0 \in G(\mathbb{Q})$.[25] Thus $\gamma_0 = \phi'(\delta_n)$ for some morphism ϕ', which is conjugate to ϕ. Hence $\gamma_0 \in I_{\phi'(\infty)\circ\zeta_\infty} \xrightarrow{\sim} G_{\mathbb{R}}^*$, so γ_0 is elliptic in $G(\mathbb{R})$.

[24] Here we use that p is unramified in F, so that $R_{F/\mathbb{Q}}G$ is unramified at p, and (1.2.18) applies. This is why we had to assume Z_G was connected. However (1.2.18) should be true without this assumption.

[25] Of course, we have already seen in §3.5 the stronger statement that ϕ is conjugate to a special morphism.

One sees immediately that $(\gamma_l)_{l\neq p}$ is conjugate to γ_0 in $G(\bar{\mathbb{A}}_f)$. Since

$$(c'^{-1}cd_\sigma c^{-1}c')^n = c'^{-1}(c\nu(p)^L_n(p^{-1})\sigma^n(c)^{-1} \rtimes \sigma^n)c' = c'^{-1}\delta_n c' \rtimes \sigma^n,$$

we have

$$\delta\sigma(\delta)\dots\sigma^{n-1}(\delta) \rtimes \sigma^n = \theta^{\mathrm{ur}}_{g_0}(c'^{-1}cd^n_\sigma c^{-1}c') = \theta^{\mathrm{ur}}_{g_0}(c')\theta^{\mathrm{ur}}_{g_0}(\delta_n)\theta^{\mathrm{ur}}_{g_0}(c')^{-1} \rtimes \sigma^n.$$

Thus $\delta\sigma(\delta)\dots\sigma^{n-1}(\delta)$ is conjugate to γ_0 in $G(\bar{\mathbb{Q}}_p)$.

We set $\mathfrak{k}(g) = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$. This is well defined as a Kottwitz triple, as the elements δ_n are well defined modulo $P^L(\mathbb{Q})_{\mathrm{tors}}$. The triple $\mathfrak{k}(g)$ satisfies the condition *(iv)* of (4.4.1) with $I = I_\phi$, and the natural isomorphism

$$I_{\gamma_0} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I_{\phi'} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \xrightarrow{\sim} I_\phi \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}.$$

where the second isomorphism depends on a choice of element in $G(\bar{\mathbb{Q}})$ conjugating ϕ' into ϕ.

One checks immediately that $\mathfrak{k}(g)$ depends on g only as an element of $S(\phi) = I_\phi(\mathbb{Q})\backslash X(\phi)$. The equivalence class of $\mathfrak{k}(g)$ depends only on ϕ, and we will denote it by $\mathfrak{k}(\phi)$.

Below we denote by y_ϕ the element $\phi^\Delta(p)(c^{-1}c') \times 1 \in G(\widehat{\mathbb{Z}}^{\mathrm{ur}}_p) \times G(\mathbb{A}^p_f)$.

Lemma (4.5.2). *For $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ an admissible morphism, and $g = (g_0, g^p) \in X(\phi)$, $\mathfrak{k}(g)$ satisfies the condition (iv) of (4.4.1) with $I = I_\phi$, and there exists a $(I(\mathbb{A}_f), \langle\Phi\rangle \times G(\mathbb{A}^p_f))$-bivariant bijection*

$$X(\phi) \xrightarrow{\sim} X_\upsilon(\delta) \times G(\mathbb{A}^p_f); \quad g' \mapsto g^{-1}y_\phi g'.$$

Proof. This follows from the definitions, keeping in mind (3.3.4). The factor y_ϕ arises because we are using $X_\upsilon(\delta)$ on the right side of the isomorphism. It disappears, if we identify the right side with $X_\upsilon(\delta')\times G(\mathbb{A}^p_f)$, via the natural isomorphism $X_\upsilon(\delta') \xrightarrow{\sim} X_\upsilon(\delta)$. □

(4.5.3) Suppose ϕ and g are as above. Since I_ϕ acts naturally on $X(\phi)$, there is a canonical refinement $\tilde{\mathfrak{k}}(g)$ of $\mathfrak{k}(g)$. Let $\tau \in I^{\mathrm{ad}}_\phi(\mathbb{A}_f)$. We denote by $\tilde{\mathfrak{k}}(g, \tau)$ the refinement of $\mathfrak{k}(g)$ obtained by composing the natural isomorphism $I_\phi \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f}(\mathfrak{k}(\phi))$ with conjugation by τ.

Corollary (4.5.4). *There exists a $\langle\Phi\rangle \times G(\mathbb{A}^p_f)$-equivariant bijection*

$$S_\tau(\phi) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g, \tau)) \quad g' \mapsto g^{-1}y_\phi g'.$$

Proof. We have produced an isomorphism $I_{\phi,\mathbb{Q}_l} \xrightarrow{\sim} I_l$ for each prime l and an $I(\mathbb{A}_f)$-equivariant isomorphism $X(\phi) \xrightarrow{\sim} X_\upsilon(\delta)\times G(\mathbb{A}^p_f)$ compatible with the action of $\langle\Phi\rangle \times G(\mathbb{A}^p_f)$. This induces an isomorphism $S(\tilde{\mathfrak{k}}(g, \tau)) \xrightarrow{\sim} I_\phi(\mathbb{Q})\backslash X(\phi)$, where the action of $I_\phi(\mathbb{Q})$ on $X(\phi)$ is via the conjugate of the natural action by τ. □

(4.5.5) By (3.1.2) the set of conjugacy classes of morphisms $\phi' : \mathfrak{Q} \to \mathfrak{G}_G$ with $\phi^\Delta = \phi'^\Delta$ is in bijection with $H^1(\mathbb{Q}, I)$, where $I = I_\phi$. For $\beta \in H^1(\mathbb{Q}, I)$ we denote by ϕ^β a representative of the corresponding conjugacy class of morphisms.

Lemma (4.5.6). *If $\beta \in Ш^{\infty}_G(\mathbb{Q}, I)$ and $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is admissible, then ϕ^β is admissible, and the map $\beta \mapsto \phi^\beta$ defines an inclusion of $Ш^{\infty}_G(\mathbb{Q}, I)$ into the set of conjugacy classes of admissible morphism $\mathfrak{Q} \to \mathfrak{G}_G$.*

Moreover we have $\mathfrak{k}(\phi^\beta) \sim \mathfrak{k}(\phi)^\beta$.

Proof. We begin by checking that ϕ^β is admissible. Let $I_{\tilde{G}}$ be as in (4.3.6). The image of β in

$$H^1(\mathbb{Q}, \tilde{G} \to G) \xrightarrow{\sim} H^1(\mathbb{Q}, Z_{\tilde{G}} \to Z_G) \xrightarrow{\sim} H^1(\mathbb{Q}, I_{\tilde{G}} \to I)$$

is trivial, and so the class of β is represented by the image of a cocycle $(z_\tau) \in Z^1(\mathbb{Q}, I_{\tilde{G}})$. Thus, we may assume that $\phi^\beta = z \cdot \phi$. Using the definition of the monoidal structure on $\mathfrak{G}_{G/\tilde{G}}$, and the inclusion $\tilde{I}(\bar{\mathbb{Q}}) \subset \tilde{G}(\bar{\mathbb{Q}})$, as in (3.1.2), one finds that $(z \cdot \phi)_{\widetilde{\mathrm{ab}}}$ is isomorphic to $\phi_{\widetilde{\mathrm{ab}}}$. Thus ϕ^β satisfies (3.3.6)(1).

Now let $(g_l)_{l \neq p} \in X^p(\phi)$. Then under the composite

$$H^1(\mathbb{Q}, I_\phi) \overset{\mathrm{Int}(g_l^{-1})}{\to} H^1(\mathbb{Q}_l, I_{\mathrm{Int}(g_l^{-1}) \circ \phi(l)}) \to H^1(\mathbb{Q}_l, G) \to H^1(\mathbb{Q}_l, \tilde{G} \to G).$$

the image of β is trivial, and hence its image in $H^1(\mathbb{Q}_l, G)$ is trivial. Choose $h_l \in G(\bar{\mathbb{Q}}_l)$ such that $g_l^{-1} z_\tau g_l = h_l^{-1} \tau(h_l)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$. Then

$$\mathrm{Int}(h_l g_l^{-1}) \circ (z \cdot \phi(l) \circ \zeta_l)(\tau) = \mathrm{Int}(h_l)(g_l^{-1} z_\tau g_l 1 \rtimes \tau)) = h_l g_l^{-1} z_\tau g_l \tau(h_l)^{-1} \rtimes \tau = 1 \rtimes \tau.$$

Hence $g_l h_l^{-1} \in X^p(z \cdot \phi)$. In particular, ϕ^β satisfies (3.3.6)(2) for $v = l$ and it satisfies (3.3.6)(2) for $v = \infty$, since β is trivial at ∞.

Let $g_0 \in X_p(\phi)$. Then $I_{\phi(p)_{g_0}}(\bar{\mathbb{Q}}_p) \subset G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)$ with $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acting on $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$ via the second factor. By Steinberg's theorem the image of β under the composite

$$H^1(\mathbb{Q}, I_\phi) \overset{\mathrm{Int}(g_0^{-1})}{\to} H^1(\mathbb{Q}_p, I_{\phi(p)_{g_0}}) \to H^1(\mathbb{Q}_p, G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p))$$

is trivial. Set $z'_\tau = g_0^{-1} z_\tau g_0$, and choose $h_0 \in G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)$ such that $z'_\tau = h_0^{-1}(1 \otimes \tau)(h_0)$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Denote by $\bar{h}_0$ the image of h_0 in $G(\bar{\mathbb{Q}}_p)$ under the map $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p) \to G(\bar{\mathbb{Q}}_p)$. To show that ϕ^β satisfies (3.3.6)(3), we first show that

$$(z \cdot \phi)(p)_{g_0 \bar{h}_0^{-1}} = \mathrm{Int}(\bar{h}_0 g_0^{-1}) \circ z \cdot \phi(p) = \bar{h}_0 z' \cdot \phi(p)_{g_0} \bar{h}_0^{-1}$$

is unramified. To do this we have to show that this morphism sends $1 \rtimes \tau$ to $1 \rtimes \tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$. For such a τ we have $\overline{\tau(h)} = \tau(\bar{h})$. This implies $z'_\tau = \bar{h}_0^{-1} \tau(\bar{h}_0)$. So the above morphism send $1 \rtimes \tau$ to

$$\bar{h}_0 z'_\tau \rtimes \tau \bar{h}_0^{-1} = \bar{h}_0 z_\tau \tau(\bar{h}_0)^{-1} \rtimes \tau = 1 \rtimes \tau.$$

If $\phi(p)_{g_0}(c d_\sigma c^{-1}) = \delta \rtimes \sigma$, as in (4.5.1), then one computes that

$$(z \cdot \phi(p))_{g_0 \bar{h}_0^{-1}}(c d_\sigma c^{-1}) = h_0 \delta (\sigma \otimes 1)(h_0)^{-1} \rtimes \sigma.$$

Note that

$$(1\otimes\tau)(h_0\delta(\sigma\otimes 1)(h_0)^{-1}) = h_0 z'_\tau\delta(\sigma\otimes 1)(z'^{-1}_\tau)(\sigma\otimes 1)(h_0)^{-1} = h_0\delta(\sigma\otimes 1)(h_0)^{-1}$$

since $z'_\tau \in I_{\phi(p)(g_0)}(\bar{\mathbb{Q}})$, so $h_0\delta(\sigma\otimes 1)(h_0)^{-1}\in G(\mathbb{Q}_p^{\mathrm{ur}})$. Since $X_{-\mu}(\delta)\neq 0$, δ satisfies the conditions (3.4.3), which implies that $h_0\delta(\sigma\otimes 1)(h_0)^{-1}$ does also. To see this we regard h_0 as an element of $R_{F/\mathbb{Q}_p}G(\mathbb{Q}_p^{\mathrm{ur}})$ for some finite Galois extension $F/\mathbb{Q}_p$ in $\bar{\mathbb{Q}}_p$. Then $\pi_1(R_{F/\mathbb{Q}_p}G) = \pi_1(G)\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q}_p)]$ and we consider the inclusion

$$\pi_1(G)_{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} \hookrightarrow \pi_1(R_{F/\mathbb{Q}_p}G)_{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} = \pi_1(G)_{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q}_p)].$$

The image of $h_0\delta(\sigma\otimes 1)(h_0)^{-1}$ in $\pi_1(R_{F/\mathbb{Q}_p}G)_{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is equal to that of δ, and hence $\kappa_G(h_0\delta(\sigma\otimes 1)(h_0)^{-1}) = \kappa_G(\delta)$. Likewise, we have $\bar{\nu}_\delta = \bar{\nu}_{h_0\delta(\sigma\otimes 1)(h_0)^{-1}}$. It follows that $X_{-\mu}(h_0\delta(\sigma\otimes 1)(h_0)^{-1})$ is non-empty. Hence ϕ^β satisfies (3.3.6)(3).

We have shown that ϕ^β is admissible. That $\beta\mapsto\phi^\beta$ is an inclusion follows from (3.1.2).

To check the last two statements write $\mathfrak{k}(\phi) = (\gamma_0, (\gamma_l)_{l\neq p}, \delta)$. Then the above calculations show that $\mathfrak{k}(\phi^\beta)\sim(\gamma_0, (h_l\gamma_l h_l^{-1})_{l\neq p}, h_0\delta(\sigma\otimes 1)(h_0)^{-1})$. Comparing with the definition in (4.3.6), one sees that this is equal to $\mathfrak{k}(\phi)^\beta$. □

Proposition (4.5.7). *Let $\mathfrak{k}_0$ be Kottwitz triple, and I the inner form of I_{γ_0} associated to $\mathfrak{k}_0$. If the fibre of*

$$\text{(4.5.8)}\qquad \{\text{Admissible } \phi:\mathfrak{Q}\to\mathfrak{G}_G\}/\text{conjugacy}\to\{\text{triples}\}/\sim;\quad \phi\mapsto\mathfrak{k}(\phi)$$

over $\mathfrak{k}_0$ is non-empty, then it is naturally a $\text{Ш}_G(\mathbb{Q}, I)$-torsor.

Proof. Suppose that $\phi:\mathfrak{Q}\to\mathfrak{G}_G$ is an admissible morphism with $\mathfrak{k}(\phi)\sim\mathfrak{k}_0$. For any $\beta\in\text{Ш}_G(\mathbb{Q}, I)$ we have $\mathfrak{k}(\phi^\beta)\sim\mathfrak{k}(\phi)^\beta\sim\mathfrak{k}_0$ by (4.5.6) and (4.3.7). We have to show that any admissible morphism ϕ' with $\mathfrak{k}(\phi')\sim\mathfrak{k}_0$ is conjugate to ϕ^β for some $\beta\in\text{Ш}_G(\mathbb{Q}, I)$.

Since the subgroup generated by δ_n is dense in P^L for any CM field L, any admissible morphism ϕ' with $\mathfrak{k}(\phi')\sim\mathfrak{k}_0$ is conjugate to a morphism whose restriction to $P^L(\bar{\mathbb{Q}})$ is equal to ϕ^Δ. Thus, it suffices to consider ϕ' with $\phi'^\Delta = \phi^\Delta$. By (3.1.2), any such ϕ' has the form $\phi' = z\cdot\phi$, where $z\in Z^1(\mathbb{Q}, I)$, and two such ϕ' are conjugate, if and only if the corresponding cocycles are equal in $H^1(\mathbb{Q}, I)$.

Since ϕ' satisfies (3.3.6)(2) for $v=\infty$, the image of z in $H^1(\mathbb{R}, G^*_{\mathbb{R}})$ (which is well defined as I/Z_G is compact at ∞), is trivial. By (4.4.5) this implies that the image of z in $H^1(\mathbb{R}, I)$ is trivial. Since ϕ' satisfies (3.3.6)(1), the image of z in $H^1(\mathbb{Q}, G)$ lies in the image of $H^1(\mathbb{Q}, \tilde{G})$ (cf. the proof of (3.4.11)). Hence the image of z in $H^1(\mathbb{Q}, \tilde{G}\to G)$ is trivial, and $[z]\in\text{Ш}_G^\infty(\mathbb{Q}, I)$ by (4.4.2).

Since $\mathfrak{k}_0\sim\mathfrak{k}(z\cdot\phi)\sim\mathfrak{k}_0^{[z]}$, we have $[z]\in\text{Ш}_G(\mathbb{Q}, I)$ by (4.3.7). □

Lemma (4.5.9). *Let $\phi:\mathfrak{Q}\to\mathfrak{G}_G$ be an admissible morphism, $I = I_\phi$, and $\phi_0:\mathfrak{Q}\to\mathfrak{G}_{G^{\mathrm{ad}}}$ the morphism induced by ϕ.*

For $\tau\in I^{\mathrm{ad}}(\mathbb{A}_f)$, the subgroup $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I\subset\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ is the stabilizer of $S_\tau(\phi)\subset S_\tau(G,\phi_0)$. If $g\in X(\phi)$, there exists a $\langle\Phi\rangle\times Z_G(\mathbb{Q}_p)\times\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant bijection

$$S_\tau(\phi)\xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g,\tau))\quad g'\mapsto g^{-1}y_\phi g'.$$

Proof. That $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$ is the stabilizer of $S_{\mathrm{\tau}}(\phi) \subset S_{\mathrm{\tau}}(G, \phi_0)$ follows from the injectivity of $\beta \mapsto \phi^\beta$ in (4.5.6). The $(I(\mathbb{A}_f), \langle\Phi\rangle \times G(\mathbb{A}_f^p))$-bivariant bijection

$$X(\phi) \xrightarrow{\sim} X_v(\delta) \times G(\mathbb{A}_f)$$

induces a $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$-equivariant bijection $S_{\mathrm{\tau}}(\phi) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(\phi, \mathrm{\tau}))$. We have to check that this bijection is $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$-equivariant.

Let $h \in G(\bar{\mathbb{Z}}_{(p)})_+^{\natural,I}$ and $i_h \in I_\phi(\bar{\mathbb{Q}})$ as in (4.3.4). For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, write $z_\tau = h^{-1}\tau(h)$. By definition the action of h on $S_{\mathrm{\tau}}(G, \phi_0)$ is given by right multiplication, and sends $X(\phi)$ into $X(z \cdot \phi)$. On the other hand, left multiplication by i_h^{-1} (acting via the canonical action of I) sends $X(z \cdot \phi)$ isomorphically into $X(\mathrm{Int}(i_h^{-1}) \circ z \cdot \phi) = X(\phi)$. Thus, if $g = (g_0, g^p) \in X(\phi)$, then $i_h^{-1} gh \in X(\phi)$. One checks immediately that $\mathrm{\tau} i_h^{-1} \mathrm{\tau}^{-1} i_h \in I(\mathbb{A}_f)$. Hence

$$\mathrm{\tau} i_h^{-1} \mathrm{\tau}^{-1} gh = (\mathrm{\tau} i_h^{-1} \mathrm{\tau}^{-1} i_h)(i_h^{-1} gh) \in X(\phi),$$

and $\mathrm{\tau} i_h^{-1} \mathrm{\tau}^{-1} gh = gh$ as elements of $S_{\mathrm{\tau}}(G, \phi_0)$.

Comparing this with the definition of the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I}$ on $S(\tilde{\mathfrak{k}}(g, \mathrm{\tau}))$ proves the lemma. □

(4.5.10) Let $h \in G^{\mathrm{ad}}(\mathbb{Q})^+$, and $\beta \in H^1(\mathbb{Q}, Z_G)$ the corresponding cocycle. Let $\tilde{h} \in G(\bar{\mathbb{Q}})$ be a lift of h, and take $\phi^\beta = z \cdot \phi$ where $z_\tau = \tilde{h}^{-1}\tau(\tilde{h})$.

Proposition (4.5.11). *Let $g \in X(\phi)$, and $\mathrm{\tau} \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$, there exists a $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant bijection*

$$S_{\mathrm{\tau}}(\phi^\beta) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g, \mathrm{\tau})^\beta) \quad g' \mapsto \tilde{h} g^{-1} y_\phi g'.$$

Moreover, under the projection

$$c_G : S_{\mathrm{\tau}}(\phi^\beta) \to \pi(G, \phi_0)$$

we have $c_G(g') = \tilde{h} g^{-1} g' c_G(g)$.

Proof. Note that $I = I_\phi = I_{\phi^\beta}$. If $g' \in X(\phi^\beta)$, then by (4.5.9), we have $\langle\Phi\rangle \times Z_G(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant bijection

$$S_{\mathrm{\tau}}(\phi^\beta) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g', \mathrm{\tau})).$$

Keeping in mind the definition of a triple attached to an element of $X(\phi^\beta)$, in (4.5.1), one sees as in (4.3.2) that multiplication by $\tilde{h} g^{-1} g'$ induces an isomorphism

$$S(\tilde{\mathfrak{k}}(g', \mathrm{\tau})) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g, \mathrm{\tau})^\beta).$$

We take the map in the Proposition to be the composite of these two maps. From the definitions, one checks that this map is independent of the choice of g', and sends g' to $\tilde{h} g^{-1} y_\phi g'$.

To check the final claim note that in (3.6.2) the identification of $X(\phi_{\widetilde{\mathrm{ab}}}) \xrightarrow{\sim} X(\phi_{\widetilde{\mathrm{ab}}}^\beta)$ is given by multiplication by $\tilde{h}^{-1}$. Hence the image of g under

$$X(\phi) \to X(\phi_{\widetilde{\mathrm{ab}}}) \xrightarrow{\sim} X(\phi_{\widetilde{\mathrm{ab}}}^\beta)$$

is $[\tilde{h}^{-1}g]$. The final claim follows. $\square$

(4.6) Proof of the main theorem: In this subsection we finish the proof of the main theorem. We continue to use the notation and assumptions introduced above, but we now assume that (G,X) is of Hodge type as in §1,2. We also assume that Z_G is a torus.

(4.6.1) Let (T,h_T,i) be as in (3.5.8). We say that an isogeny class $\mathscr{I}\subset \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$ is *attached* to (T,h_T,i) if the reduction mod p of the points in the image of $i:\mathrm{Sh}(T,h_T)\to \mathrm{Sh}_{\mathsf{K}_p}(G,X)$ lie in $\mathscr{I}$. By (2.2.3) any $\mathscr{I}$ is attached to some (T,h_T,i). The second description of $\mathfrak{k}(T,h_T)$ given in (4.3.9), together with the Shimura-Taniyama reciprocity law, and [Ko 4, Lem. 12.1] (cf. also [Wi 3, 4.2.5]), implies that $\mathfrak{k}(\mathscr{I})\sim i_*(\mathfrak{k}(T,h_T))$.

Similarly, we say that an admissible morphism $\phi:\mathfrak{Q}\to G$ is attached to (T,h_T,i) if ϕ is conjugate to $i\circ\psi_{\mu_{h_T}}$. By (3.5.11) any ϕ is attached to some (T,h_T,i). An exercise using the definitions shows that $\mathfrak{k}(\phi)\sim i_*(\mathfrak{k}(T,h_T))$.

Now fix $\mathscr{I}$ and let

$$\mathscr{I}_0=\cup_{h\in G^{\mathrm{ad}}(\mathbb{Q})^+}\mathscr{I}^{[h]}\subset \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p)$$

where $[h]$ denotes the image of h in $\text{Ш}^\infty_G(\mathbb{Q},I)$, and we use the notation introduced in §4.4.

Proposition (4.6.2). *Suppose that $\mathscr{I}$ is attached to (T,h_T,i), and let $\phi:\mathfrak{Q}\to G$ be conjugate to $i\circ\psi_{\mu_{h_T}}$. There exists a $\tau\in I^{\mathrm{ad}}_{\phi_0}(\mathbb{A}_f)$ and a $\langle\Phi\rangle\times Z(\mathbb{Q}_p)\times\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant bijection $\xi:\mathscr{I}_0\xrightarrow{\sim}S_\tau(G,\phi_0)$ such that the diagram*

(4.6.3)
$$\begin{array}{ccc}\mathscr{I}_0 & \xrightarrow[\xi]{\sim} & S_\tau(G,\phi_0)\\ \downarrow{\scriptstyle c_G} & & \downarrow{\scriptstyle c_G}\\ \pi(G,X) & \xrightarrow[\vartheta_G]{\sim} & \pi(G,\phi_0)\end{array}$$

commutes. Moreover, for each $h\in G^{\mathrm{ad}}(\mathbb{Q})^+$, ξ takes $\mathscr{I}^{[h]}$ isomorphically onto $S_\tau(\phi^{[h]})$.

Proof. We may assume that $\phi=i\circ\psi_{\mu_{h_T}}$. We use the notation introduced in (3.6.6). Let $t=(t_p,t^p)\in X_p(\psi_{\mu_{h_T}})\times X^p(\psi_{\mu_{h_T}})$. We again denote by $(t_p,t^p)\in T(\mathbb{A}_f)$ a point lifting the image of t under the map of (3.6.7). Let $x\in\mathscr{I}$ be the reduction of the image of t under

$$T(\mathbb{A}_f)\to \mathrm{Sh}(T,h_T)\xrightarrow{i}\mathrm{Sh}_{\mathsf{K}_p}(G,X)$$

By construction the image of t in $\pi(G,\phi_0)$ maps to $c_G(x)$ under ϑ_G. By the reciprocity law we have $\mathfrak{k}(x)=(\gamma_0,(\gamma_l)_{l\neq p},\delta)$ with $\gamma_l=\gamma_0\in G(\bar{\mathbb{Q}})$ for all $l\neq p$, and $\mathfrak{k}(x)\sim i_*(\mathfrak{k}(T,h_T))$.

Now let $g_p\in X_p(\phi)$, and set $g^p=i(t^p)\in X^p(\phi)$, and $g=(g_p,g^p)$. An exercise using the definitions shows that $\mathfrak{k}(g)=(\gamma_0,(\gamma_l)_{l\neq p},\delta')$, for some $\delta'\in G(\widehat{\mathbb{Q}}^{\mathrm{ur}}_p)$. Since $\mathfrak{k}(\phi)\sim i_*(\mathfrak{k}(T,h_T))$, we may choose g_p so that $\mathfrak{k}(g)=\mathfrak{k}(x)$. We claim that we can choose g_p so that we also have $\vartheta_G\circ c_G(g)=c_G(x)$. To do this it suffices to arrange that g_p and t_p have the same image in $X_p(\phi_{\widetilde{\mathrm{ab}}})$. By (3.6.2)(2), these two images differ by an element of $G(\mathbb{Q}_p)/\tilde{G}(\mathbb{Q}_p)G(\mathbb{Z}_p) = \pi_1(G)^\Gamma$ (cf. (1.2.3)). Hence the claim follows from (4.6.4) below which shows that the map $J_\delta(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$ is surjective.

Choose $\tau \in I^{\mathrm{ad}}_{\phi_0}(\mathbb{A}_f)$ such that $\tilde{\mathfrak{k}}(x) = \tilde{\mathfrak{k}}(g,\tau)$, and write $I = I_{\phi_0}$. By (4.5.9) and (4.4.14) there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{I} & \xrightarrow{\sim} & S(\tilde{\mathfrak{k}}(g,\tau)) & \xrightarrow{\sim} & S_\tau(\phi) \\ \downarrow & & & & \downarrow \\ \pi(G,X) & & \xrightarrow[\vartheta_G^{-1}]{\sim} & & \pi(G,\phi_0) \end{array}$$

where the maps in the top line are $\langle\Phi\rangle \times Z(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant, and the diagram commutes because the top line sends x to g. We take $\xi_{\mathscr{I}} : \mathscr{I} \xrightarrow{\sim} S_\tau(\phi)$ to be the composite of the maps in the top line.

By (4.4.14) and (4.5.9) $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^I \subset \mathscr{A}(G_{\mathbb{Z}_{(p)}})$ is the stabilizer of both $\mathscr{I} \subset \mathscr{I}_0$ and $S_\tau(\phi) \subset S_\tau(G,\phi_0)$. Hence $\xi_{\mathscr{I}}$ extends uniquely to a $\langle\Phi\rangle \times Z(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant bijection

$$\cup_{h \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+} \mathscr{I}^{[h]} \xrightarrow{\sim} \cup_{h \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+} S_\tau(\phi^{[h]}),$$

which commutes with the projections to $\pi(G,X)$ and $\pi(G,\phi_0)$ as both of these are $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant.

If $h \in G^{\mathrm{ad}}(\mathbb{Q})^+$, then by (4.4.17) and (4.5.11) we have $\langle\Phi\rangle \times Z(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})^I$-equivariant isomorphisms

$$\mathscr{I}^\beta \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(x)^\beta) \xrightarrow{\sim} S(\tilde{\mathfrak{k}}(g,\tau)^\beta) \xrightarrow{\sim} S_\tau(\phi^\beta).$$

Moreover if $x' \in \mathscr{I}^\beta$ maps to g' under the composite of these maps, then using the notation of (4.4.17), $\tilde{h}g_{x'}^{-1} = \tilde{h}g^{-1}y_\phi g'$. Hence, using (4.4.17) and (4.5.11), we have

$$c_G(x') = \tilde{h}g_{x'}^{-1}c_G(x) = \tilde{h}g^{-1}y_\phi g' c_G(x) = \vartheta_G(\tilde{h}g^{-1}y_\phi g' c_G(g)) = \vartheta_G(c_G(g')).$$

Here we have used that y_ϕ acts trivially on $\pi(G,\phi_0)$, since $g_0^{-1}y_\phi g_0 \in G(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$. Hence $\mathscr{I}^\beta \xrightarrow{\sim} S_\tau(\phi^\beta)$ is compatible with the maps c_G.

Applying this for a set of representatives of the orbits of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on the image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $\text{Ш}^\infty_G(\mathbb{Q}, I)$, and keeping in mind (4.4.8) and (4.5.6), gives a diagram as in (4.6.3). The final claim in the proposition follows from the construction. □

Lemma (4.6.4). *Let $b \in G(L)$. Then the map $J_b(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$ is surjective.*

Proof. As usual, let $M = M_b$ be the centralizer of ν_b, so that J_b is an inner form of M. We consider the maps

$$J_b(\mathbb{Q}_p) \to H^0(\mathbb{Q}_p, \tilde{J}_b \to J_b) \to \pi_1(M)^\Gamma \to \pi_1(G)^\Gamma.$$

The first map is surjective as $H^1(\mathbb{Q}_p, \tilde{J}_b(\mathbb{Q}_p))$ is trivial. The second term can be identified with $H^0(\mathbb{Q}_p, \tilde{M}_b \to M_b)$. As $M_b(\mathbb{Q}_p)$ surjects onto $\pi_1(M)^\Gamma$ by (1.2.3), the second map is surjective. Finally, again applying (1.2.3), the third map is surjective. □

Corollary (4.6.5). *Suppose that (G,X) is of Hodge type, and let $\mathsf{K}_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ be a hyperspecial subgroup. Then there is a $\langle\Phi\rangle \times Z(\mathbb{Q}_p) \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant bijection $\mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi_0]} S_{\tau(\phi_0)}(G,\phi_0)$ such that the diagram*

$$\begin{array}{ccc} \mathscr{S}_{\mathsf{K}_p}(G,X)(\bar{\mathbb{F}}_p) & \xrightarrow{\sim} & \coprod_{[\phi_0]} S_{\tau(\phi_0)}(G,\phi_0) \\ \downarrow & & \downarrow \\ \pi(G,X) & \xrightarrow[\vartheta_G]{} & \coprod_{[\phi_0]} \pi(G,\phi_0) \end{array}$$

commutes. Here $[\phi_0]$ runs over conjugacy classes of admissible morphisms $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$, and $\tau(\phi_0)$ is some element of $I^{\mathrm{ad}}_{\phi_0}(\mathbb{A}_f)$.

Proof. Let $\mathscr{I}$ and ϕ each be attached to (T,h_T,i) as in (4.6.2), and let $\beta \in Ш^{\infty}_G(\mathbb{Q},I)$ be a class arising from a cocycle $z \in Z^1(\mathbb{Q},Z_I)$. We claim that there is a triple (T,h_T,i^β) such that $\mathscr{I}^\beta$ and ϕ^β are each attached to (T,h_T,i^β). In particular, this applies to β arising from an element of $G^{\mathrm{ad}}(\mathbb{Q})^+$, as well as to $\beta \in Ш_G(\mathbb{Q},I)$, by the Hasse principle for I^{ad}. Assuming this, the corollary follows from (4.6.2), together with (4.4.13) and (4.5.7).

To see the claim, let $g \in G(\bar{\mathbb{Q}})$ be such that $g^{-1}\tau(g) = z_\tau$ for $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We take $i^\beta = gig^{-1} : T \to G$, which is defined over $\mathbb{Q}$ as $z_\tau \in T(\bar{\mathbb{Q}})$. As in (4.4.7), $i^\beta \circ h_T \in X$. By (4.2.6), $\mathscr{I}^\beta$ is attached to (T,h_T,i^β).

We have to show that ϕ^β is attached to (T,h_T,i^β). Now ϕ^β is conjugate to $g(z\cdot i\circ\psi_{\mu_{h_T}})g^{-1}$. Let $q \in \mathfrak{Q}$, and write $\psi_{\mu_{h_T}}(q) = t \rtimes \tau$. Then

$$g(z\cdot i\circ\psi_{\mu_{h_T}})g^{-1}(q) = gz_\tau i(t) \rtimes \tau g^{-1} = gi(t)g^{-1} \rtimes \tau = i^\beta \circ \psi_{\mu_{h_T}}(q).$$

So ϕ^β is attached to (T,h_T,i^β). □

Lemma (4.6.6). *Let (H,Y) be a Shimura datum of abelian type with H an adjoint group. Then there exists a central isogeny $H^\sharp \to H$ such that*

1. *If (G,X) is a Shimura datum of Hodge type such that $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is isomorphic to (H,Y) then G^{der} is a quotient of $H^\sharp$.*
2. *There exists a Shimura datum of Hodge type (G,X) such that $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is isomorphic to (H,Y) and $G^{\mathrm{der}} = H^\sharp$, with Z_G a torus.*
3. *Suppose that H is quasi-split and unramified at a prime p. Then (G,X) in (2) can be chosen so that G is quasi-split and unramified at p, and such that $E(G,X)_p = E(G^{\mathrm{ad}},X^{\mathrm{ad}})_p$. (The notation being as in (3.8.12).)*

Proof. This is deduced in [Ki 2, 3.4.13] from Deligne's results [De 2], except for the claim in (2) that (G,X) can be chosen so that Z_G is a torus, and the final claim in (3) that (G,X) may be chosen so that $E(G,X)_p = E(G^{\mathrm{ad}},X^{\mathrm{ad}})_p$. That (G,X) may be chosen to satisfy the former condition is shown in [KP, 4.6.22]. The construction of (G,X) in (2) uses [De 2, 2.3.10]. We briefly explain how to make the choices in *loc. cit* so that the conditions of (3) hold.

Let $(H,Y) = (H_1,Y_1)\times\cdots\times(H_r,Y_r)$ with H_i adjoint and $\mathbb{Q}$-simple. We have $H^\sharp = H_1^\sharp\times\cdots\times H_r^\sharp$, and it suffices to construct (G,X) when H is $\mathbb{Q}$-simple. Then H has the form $R_{F/\mathbb{Q}}H^s$ with F a totally real field, and H^s absolutely simple [De 2, 2.3.4]. If H is quasi-split and unramified at p, then p is unramified in F. Let I_c

and I_{nc} respectively denote the set of embeddings $F \hookrightarrow \mathbb{R}$ for which H^s is compact (resp. non-compact).

Let K/F be a quadratic, totally imaginary extension, such that every prime of F over p splits completely in K. Let J denote the set of embeddings $K \hookrightarrow \mathbb{C}$. Then, keeping in mind the field embeddings fixed in §1.3, J may be identified with the set of embeddings $K \hookrightarrow \bar{\mathbb{Q}}_p$, and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on J.

Since $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is the reflex field of (G, X), the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/E(G^{\mathrm{ad}}, X^{\mathrm{ad}}))$ on J preserves the set of elements which map to I_c. If $w \in J$, and $\bar{w}$ is the conjugate of w, then the orbits of w and $\bar{w}$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_p)$ are disjoint, since the primes of F over p split completely in K. Thus, there exists a subset $T \subset J$ which maps bijectively onto I_c, and which is stable by $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_p)$.

Now let $h_T : \mathbb{S} \to (K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ be the cocharacter which induces the identity on $\mathbb{C}^\times$ when composed with $w \in T$, and is trivial when composed with $w \in J$ such that $w, \bar{w} \notin T$. Then $E(R_{K/\mathbb{Q}}\mathbb{G}_m, h_T)_p$ is contained in $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_p$ by what we saw above, and [De 2, 2.3.10] implies that (G, X) can be chosen to satisfy (3). □

Theorem (4.6.7). *Let (G_2, X_2) be a Shimura datum of abelian type, and $\mathsf{K}_{2,p} \subset G_2(\mathbb{Q}_p)$ a hyperspecial subgroup. Let $r_2 = [E(G_2, X_2)_p : \mathbb{Q}_p]$.*

Then $\mathrm{Sh}_{\mathsf{K}_{2,p}}(G_2, X_2)$ admits an integral canonical model $\mathscr{S}_{\mathsf{K}_{2,p}}(G_2, X_2)$ over $\mathcal{O}_{E(G_2,X_2)_p}$, and there is a $\langle\Phi_{r_2}\rangle \times Z_{G_2}(\mathbb{Q}_p) \times \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})$-equivariant isomorphism

$$\mathscr{S}_{\mathsf{K}_{2,p}}(G_2, X_2)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \textstyle\coprod_{[\phi]} S_{\tau(\phi_0)}(\phi) \tag{4.6.8}$$

where ϕ runs over conjugacy classes of admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_{G_2}$, and $\tau(\phi_0) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ depends only on ϕ_0. Here $G_{2,\mathbb{Z}_{(p)}}$ is a reductive model of G_2 over $\mathbb{Z}_{(p)}$, with $\mathsf{K}_{2,p} = G_{2,\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$.

Proof. Let (G, X) be a Shimura datum of Hodge type which satisfies the conclusion of (4.6.6) applied to $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$. In particular, we have that G is quasi-split and unramified, there is a central isogeny $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$, inducing an isomorphism $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\sim} (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$, and r_2 is divisible by

$$r_E = [E(G, X)_p : \mathbb{Q}_p] = [E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_p : \mathbb{Q}_p].$$

An argument as in [Ki 2, 3.4.14] shows that there exists a reductive model $G_{\mathbb{Z}_{(p)}}$ of G over $\mathbb{Z}_{(p)}$, and a central isogeny $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \to G^{\mathrm{der}}_{2,\mathbb{Z}_{(p)}}$, extending the isogeny $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$.

Fix a connected component $X^+ \subset X_2$. After replacing X by a $G^{\mathrm{ad}}(\mathbb{R})$ conjugate, we may assume $X^+ \subset X$. Taking the fibre over $X^+ \in \pi(G, X)$ in the commutative diagram in (4.6.5), we obtain an $\mathscr{E}_p^{r_2}(G^{\mathrm{der}}_{\mathbb{Z}_{(p)}})$-equivariant isomorphism

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p)^+ \xrightarrow{\sim} \textstyle\coprod_{[\phi_0]} S_{\tau(\phi_0)}(G^{\mathrm{der}}, \phi_0)^+ \ . \tag{4.6.11}$$

Now apply $[\mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \times \_]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ$ to both sides of (4.6.11). Using [Ki 2, 3.4.11] and (3.8.13) we obtain an integral canonical model $\mathscr{S}_{\mathsf{K}_{2,p}}(G_2, X_2)$ of $\mathrm{Sh}_{\mathsf{K}_p}(G_2, X_2)$ and a $\langle\Phi_{r_2}\rangle \times \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})$-equivariant isomorphism

$$\mathscr{S}_{\mathsf{K}_{2,p}}(G_2, X_2)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \textstyle\coprod_{[\phi_0]} S_{\tau(\phi_0)}(G_2, \phi_0). \tag{4.6.12}$$

The right hand sides of (4.6.12) and (4.6.8) are canonically isomorphic by (3.4.16). We have constructed a $\langle\Phi_{r_2}\rangle \times \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})$-equivariant isomorphism as in (4.6.8), and the same remark as in (3.7.10) shows that this isomorphism is also $Z_{G_2}(\mathbb{Q}_p)$-equivariant. □

Errata for [Ki 2]

We are grateful to Yihang Zhu and Tom Lovering from bringing the following to our attention. All references are to [Ki 2].

E.1 The statement of (1.5.2) is missing the condition that $\mathrm{Lie}\, H$ is stable under the action of Frobenius $h \mapsto \varphi \circ h \circ \varphi^{-1}$ on $\mathrm{End}(M_0)$. This is needed to insure that the series, obtained by substituting the formula for β into itself, has coefficients in $\mathrm{Lie}\, H$, as claimed, because the formula for $\varphi(\beta)$ involves conjugation by φ.

As a result, (1.5.3) is incorrect and should be deleted. The construction in (1.5.4) is correct, but the connection ∇_{R_G} has coefficients in $\mathrm{Lie}\, G$ not $\mathrm{Lie}\, U_G^\circ$ as claimed.

The only result from §1.5 which is used in the rest of the paper is (1.5.8) and its Corollary (1.5.11). The statement of (1.5.8) is fortunately still correct, and we now explain how to modify its proof taking into account the corrected version of (1.5.2).

E.2 Using the notation of §1.5, let A (resp. A_G) be the completion of $R[1/p]$ (resp. $R_G[1/p]$) at $(t_1, \dots, t_n)$. We write $M_A = M \otimes_R A$, and similarly for M_{A_G}. Let $\sigma_\alpha \in M_A^\otimes$ denote the parallel transport of s_α. That is σ_α is the unique section such that $\nabla(\sigma_\alpha) = 0$, and σ_α modulo $(t_1, \dots, t_n)$ is s_α.

Lemma E.3. *Let* $\mathrm{Spf}\, A'_G \subset \mathrm{Spf}\, A$ *denote the largest closed subspace such that* $\sigma_\alpha \in \mathrm{Fil}^0 M^\otimes_{A'_G}$. *Then* $A'_G = A_G$.

Proof. Since ∇_{R_G} has coefficients in $\mathrm{Lie}\, G$, $\sigma_\alpha|_{A_G} = s_\alpha \otimes 1$, so $\mathrm{Spf}\, A_G \subset \mathrm{Spf}\, A'_G$. Since A_G is a regular ring, it suffices to show that A_G and A'_G have the same tangent space, or that any map of K_0-algebras $A'_G \to K_0[u]/u^2$ factors through A_G. This can be proved by exactly the same argument as in (1.5.6), keeping in mind that the induced map $A \to K_0[u]/u^2$ is compatible with the Frobenii $t_i \mapsto t_i^p$ on A and $u \mapsto u^p = 0$ on $K_0[u]/u^2$. □

E.4 In the proof of (1.5.8), replace the fourth and fifth paragraphs by the following:

To show that $\tilde{\varpi}$ factors through R_G it suffices to show that the induced map $\tilde{\varpi} : A \to K_0[\![u]\!]$ factors through A_G. Since the $\tilde{s}_\alpha \in M^\otimes_{\widehat{S}} \subset M^\otimes_{K_0[\![u]\!]}$ are Frobenius invariant, they are parallel for the connection on $M^\otimes_{K_0[\![u]\!]}$. Hence $\tilde{\varpi}^*(\sigma_\alpha) = \tilde{s}_\alpha$. By the construction $\mathscr{G}_{\tilde{\varpi}}$, $\tilde{s}_\alpha \in \mathrm{Fil}^0 M^\otimes_{K_0[\![u]\!]}$. Hence $\tilde{\varpi}$ factors through A_G by (E.3).

E.5 The assertion that $\ker(\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G_{2,\mathbb{Z}_{(p)}}))$ is finite, in (3.3.9) is not correct. (This assertion also appears in [Mo, 3.21.1]). This is used in the argument that the quotient

$$\mathscr{S}_{K_{2,p}}(G_2, X_2) = [\mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p}(G, X)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \tag{E.5.1}$$

in (3.4.11) is well defined and satisfies the extension property. To see that this is the case without using the finiteness, denote the kernel above by $\Delta(G, G_2)^\circ$ and write

$$\Delta(G, G_2) = \ker(\mathscr{A}(G_{\mathbb{Z}(p)}) \to \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})).$$

We have the following Lemma.

Lemma E.6. *The subgroup* $Z_G(\mathbb{A}_f^p)/Z_{G_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)})^- \cap \Delta(G, G_2)$ *has finite index in* $\Delta(G, G_2)$. *In particular,* $Z_{G^{\mathrm{der}}}(\mathbb{A}_f^p)/Z_{G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)})^- \cap \Delta(G, G_2)^\circ$ *has finite index in* $\Delta(G, G_2)^\circ$.

Proof. It suffices to prove the lemma when $G_2 = G^{\mathrm{ad}}$, which we now assume. If $(h, g) \in \Delta(G, G^{\mathrm{ad}})$, then $g \in G^{\mathrm{ad}}(\mathbb{Q})$ lifts to $G(\mathbb{Q}_l)$ for $l \neq p$. Since, the group

$$\ker\left(H^1(\mathbb{Z}_{(p)}, Z_G) \to \prod_{l \neq p} H^1(\mathbb{Q}_l, Z_G)\right)$$

is finite (cf. [De 2, 2.0.4]), after passing to a finite subgroup of $\Delta(G, G^{\mathrm{ad}})$, we may assume that g lifts to a point in $G(\mathbb{Z}_{(p)})_+$, so $(h, g) \sim (hg^{-1}, 1)$. Since hg^{-1} maps to 1 in $G^{\mathrm{ad}}(\mathbb{A}_f^p)$, $hg^{-1} \in Z_G(\mathbb{A}_f^p)/Z_{G_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)})^-$.

The second claim follows since $\Delta(G, G_2)^\circ \subset \Delta(G^{\mathrm{der}}, G_2)$. □

E.7 Now for any *closed* subgroup $K^p \subset G(\mathbb{A}_f^p)$, write $K = K_pK^p$, and

$$\mathscr{S}_{K_p}(G, X) = \varprojlim \mathscr{S}_{K_pK^{p\prime}}(G, X)$$

where $K^{p\prime} \subset G(\mathbb{A}_f^p)$ runs over compact open subgroups containing K^p. We define (over $\mathcal{O}_{(p)}^p$)) $\mathscr{S}_{K_p}(G, X)^+$ analogously.

By (E.6), if K^p is compact open and sufficiently small, then $\Delta(G, G_2)^\circ$ acts on $\mathscr{S}_K(G, X)^+$ through a finite quotient, and we may construct the scheme in (E.5.1) as a union of copies of $\varprojlim \mathscr{S}_K(G, X)^+/\Delta(G, G_2)^\circ$ where K^p runs through compact open subgroups of $G(\mathbb{A}_f^p)$.

To check the extension property, let $l \neq p$ be a prime, and let $K^p \subset G(\mathbb{A}_f^{p.l})$ be compact open, where $\mathbb{A}_f^{p,l}$ denotes the adeles with trivial components at p and l. Then $\mathscr{S}_K(G, X)^+$ satisfies the extension property, and the second claim of (E.6) implies that $\Delta(G, G_2)^\circ$ acts on $\mathscr{S}_K(G, X)^+$ through a finite quotient. Hence $\mathscr{S}_K(G, X)^+/\Delta(G, G_2)^\circ$ satisfies the extension property, which implies that $\mathscr{S}_{K_p}(G, X)^+/\Delta(G, G_2)^\circ$ does also.

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