Hodge theory for matroids

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Abstract

This expository paper provides a self-contained proof of the Heron-Rota-Welsh conjecture concerning the characteristic polynomial of a matroid. The conjecture, resolved affirmatively by Adiprasito-Huh-Katz, asserts that the absolute values of its coefficients are log-concave. A key component of the proof is a version of Hodge theory for matroids, which refers to a collection of results about the Chow ring of a matroid that are analogous to results about the cohomology of compact Kähler manifolds obtained by Hodge theory. The results are Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations, the three of which are collectively referred to as the Kähler package and are proved for matroids directly, independent of their complex-geometric analogues.

After preliminaries on matroids and the characteristic polynomial, we explain how the Kähler package implies the Heron-Rota-Welsh conjecture. The degree map of the Chow ring is then constructed using a Gröbner basis computation, and the Kähler package is proved using semi-small decompositions. This paper is the author's minor thesis, written in partial fulfillment of the mathematics PhD requirements at Harvard University.

Contents

1 Introduction

A sequence of real numbers $a_0, \ldots, a_n$ is *unimodal* if there is an index i for which

$$a_0 \leqslant \cdots \leqslant a_i \geqslant \cdots \geqslant a_n.$$

The sequence is *log-concave* if

$$a_i^2 \geqslant a_{i-1}a_{i+1} \qquad 0 < i < n.$$

A log-concave sequence of positive numbers is necessarily unimodal.[1]

Many long-standing conjectures about the unimodality of naturally occurring sequences of positive numbers in combinatorics have recently been resolved by proving the stronger conjecture that they are log-concave. Remarkably, the major breakthrough in the recent proofs of log-concavity has been a connection to Hodge theory, which is an analytic theory of compact Kähler manifolds. Initially, Hodge theory was directly used to solve problems in combinatorics [Huh12, HK12]. Later, a combinatorial theory was developed [AHK18], inspired by but logically independent from the main results of Hodge theory. This new theory, referred to as *Hodge theory for matroids*, has been applied with much success toward many combinatorial problems regarding log-concavity. The purpose of this paper is to provide a self-contained development of Hodge theory for matroids culminating in a proof of the Heron-Rota-Welsh conjecture, whose resolution [AHK18] was one of the first major successes of this new field.

Associated to a finite graph G is a polynomial with integral coefficients called its *chromatic polynomial* χ_G. By definition, the value of χ_G at a nonnegative integer q is the number of *proper q-colorings* of the graph. If G has no loops, then the polynomial $\chi_G(q)$ is divisible by q^c where c is the number of connected components of G, and the coefficients of the quotient

$$\frac{\chi_G(q)}{q^c} = a_0(G)q^n - a_1(G)q^{n-1} + \cdots + (-1)^n a_n(G) \qquad a_i > 0 \qquad i = 0, 1, \ldots, n$$

alternate in sign and are all nonzero. Read conjectured in 1968 that the sequence of positive numbers

$$a_0(G), a_1(G), \ldots, a_n(G)$$

is unimodal for any finite loopless graph G [Rea68]. Hoggar conjectured in 1974 that the sequence is log-concave [Hog74]. These conjectures were proven by Huh in [Huh12].

Both conjectures have generalizations to *matroids*, which are combinatorial objects that formalize the notion of linear independence for a collection of vectors in a vector space. A finite graph has an associated *graphic matroid* that is specified by the data of which subsets of edges are cycleless. A matroid M has a *characteristic polynomial* χ_M, which coincides with $\chi_G(q)/q^c$ when M is the graphic matroid associated to a finite graph G. If M is *loopless*, the characteristic polynomial is an integer polynomial whose coefficients are nonzero and alternate in sign. In the 1970s, Heron and Rota conjectured that the absolute values of the coefficients of the characteristic polynomial of a matroid are unimodal [Rot71, Her72], and Welsh later conjectured that they are log-concave [Wel76]. The Heron-Rota-Welsh conjecture was proven by Adiprasito, Huh, and Katz in [AHK18].

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[1] In fact, the sequence will be *trapezoidal*: there are indices $j \leqslant k$ for which $a_0 < \cdots < a_j = \cdots = a_k > \cdots > a_n$.

The *Chow ring* $\mathrm{CH}(M)$ of a matroid M is a graded algebra $\mathrm{CH}(M) = \bigoplus_{k=0}^{r} \mathrm{CH}^k(M)$ over the real numbers. The integer r is (one less than) the *rank* of the matroid M, and it turns out there is a linear isomorphism $\deg_M \colon \mathrm{CH}^r(M) \to \mathbf{R}$ called the *degree map*. Hodge theory for matroids refers to a collection of results about the Chow ring, which we now summarize.

- *Poincaré duality*: for every nonzero element $\mu \in \mathrm{CH}^k(M)$, there exists $\nu \in \mathrm{CH}^{r-k}(M)$ for which $\deg_M(\mu\nu) \neq 0$.
- *The hard Lefschetz theorem*: if $\ell \in \mathrm{CH}^1(M)$ is *ample*, the map $\mathrm{CH}^k(M) \to \mathrm{CH}^{r-k}(M)$ given by multiplication by ℓ^{r-2k} is an isomorphism for $k \leqslant r/2$.
- *The Hodge-Riemann relations*: if $\ell \in \mathrm{CH}^1(M)$ is *ample*, then the symmetric bilinear form

  $$\mathrm{CH}^k(M) \times \mathrm{CH}^k(M) \to \mathbf{R} \qquad (\mu, \nu) \mapsto (-1)^k \deg_M(\ell^{r-2k}\mu\nu)$$

  is positive-definite on the kernel of the map $\ell^{r-2k+1} \colon \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k+1}(M)$.

The set of ample classes in $\mathrm{CH}^1(M)$ is nonempty as long as $r \geqslant 1$. These three theorems are collectively referred to as the *Kähler package* and were established in [AHK18] to prove the Heron-Rota-Welsh conjecture. We note that the coefficients of the characteristic polynomial are *not* the dimensions of the graded pieces $\mathrm{CH}^k(M)$; instead, they arise as values of the Poincaré pairing. The basic inequality of log-concavity $ac - b^2 \leqslant 0$ is equivalent to the assertion that a certain symmetric 2×2 matrix has nonpositive determinant, and the Hodge-Riemann relations are what ultimately guarantee such a condition.

In section 2, we define and verify the basic properties of matroids, and show that the characteristic polynomial of a graphic matroid recovers the chromatic polynomial of the graph. In section 3, we define the Chow ring of a matroid and explain how the Kähler package implies log-concavity of the coefficients of the characteristic polynomial. In section 4, we prove the Kähler package using semi-small decompositions.

The author used [Oxl03, Oxl11] for matroid basics and [Kat16, AHK17, Bak18, Huh18] for surveys and introductions to Hodge theory for matroids. Much of this paper is drawn directly from [AHK18]. The proof of the Kähler package follows [BHM$^+$20]. Two detours were required to make the proof self-contained. The first is the construction of the degree map, where we use the Gröbner basis computation in [FY04] instead of Minkowski weights which seem to ultimately rely on the intersection theory of [FMSS95]. The author thanks Christopher Eur for suggesting this route in constructing the degree map. The second is the proof of the Hodge-Riemann relations for Boolean matroids, which is proved in [BHM$^+$20] by citing the usual Hodge theory of compact Kähler manifolds. We instead use an argument appearing in Section 5 of [ADH20] to prove this result using the coloop case of the semi-small decomposition. The author thanks June Huh for suggesting this argument for this purpose.

Acknowledgments. I would like to thank Christopher Eur and June Huh for the help which was essential for the completion of this project. I also thank Lauren Williams for advising my minor thesis, I thank Peter Kronheimer for initially sparking my interest in this subject, and I thank Siddhi Krishna for suggesting that it could be a potential minor thesis topic. This material is based upon work supported by the NSF GRFP through grant DGE-1745303.

2 Matroid preliminaries

Matroids are combinatorial objects that generalize both the notion of linear independence for a collection of vectors in a vector space and the notion of being cycleless for a collection of edges of a graph. We first define matroids and their basic properties. In section 2.1, we define the *chromatic polynomial* of graphs and the *characteristic polynomial* of matroids, and we show that the chromatic polynomial of a graph is recovered from the characteristic polynomial of its associated graphic matroid. In section 2.2, we give an alternative expression for the characteristic polynomial using the *Möbius function* and prove that its coefficients are nonzero and alternate in sign. The material from this section draws from [Zas87, Kat16].

Matroids can be axiomatized in a number of different ways. The three axiomatizations we consider are through *independent sets*, the *rank function*, and *flats*. We first explain each of these three notions for a finite set of vectors in a vector space.

Example. Let E be a finite set of vectors in a vector space over an arbitrary field. A subset $I \subseteq E$ is an *independent set* if the vectors in I are linearly independent. The *rank function* of E is the integer-valued function on the power set of E which associates to each subset $S \subseteq E$ the dimension of the span of the vectors in S. A subset $F \subseteq E$ is a *flat* if every vector in E that lies in the span of the vectors in F is already contained in F.

Each of the three pieces of data determines the other two. Suppose that the rank function of E is known. Then a set I is independent if and only if its size is equal to its rank, and a set F is a flat if and only if all sets strictly containing F are of strictly greater rank. If the independent sets are known, then the rank of a set S is the size of a maximal independent subset of S. If the flats are known, then the rank of a flat F is greatest integer r for which there are flats F_i satisfying $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r = F$, and the rank of an arbitrary subset is the rank of the smallest flat containing it.

Definition (Matroid)**.** Let E be a finite set. A *matroid* on E is defined by any of the following:

- A collection of subsets of E called *independent sets* for which
  1. The empty set is an independent set.
  2. If I is an independent set and $I' \subseteq I$, then I' is an independent set.
  3. If I_1 and I_2 are independent sets and $|I_1| < |I_2|$, then there is an element $e \in I_2 \backslash I_1$ such that $I_1 \cup e$ is an independent set.
- A function $\mathrm{rk}_M \colon \mathscr{P}(E) \to \mathbf{Z}$ for which
  1. If S is a subset of E, then $0 \leqslant \mathrm{rk}_M(S) \leqslant |S|$.
  2. If S, T are subsets of E for which $S \subseteq T$, then $\mathrm{rk}_M(S) \leqslant \mathrm{rk}_M(T)$.
  3. If S, T are subsets of E, then $r(S \cap T) \leqslant r(S) + r(T) - r(S \cup T)$.
- A collection of subsets of E called *flats* for which
  1. The set E is a flat.
  2. If F_1 and F_2 are flats, then $F_1 \cap F_2$ is a flat.
  3. If F is a flat, then any element of $E \backslash F$ is contained in exactly one flat that is minimal among flats properly containing F.

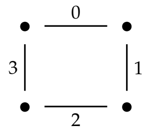
The set E is called the *ground set* of M. The *rank* of M, denoted $\mathrm{rk}(M)$, is defined to be $\mathrm{rk}_M(E)$.

It is straightforward to verify that these three axiomatizations are equivalent. If E is a finite set of vectors in a vector space, then it is clear that the independent sets, the rank function, and the flats of E satisfy these axioms, and thereby determine a matroid. This matroid is called the *linear matroid* on E. Two matroids are *isomorphic* if there is a one-to-one correspondence between their ground sets which preserves the additional structure in the obvious way. A matroid is *realizable* if there exists a field k for which it is isomorphic to the linear matroid on a set of vectors in a vector space over k, in which case it is *realizable over k*.

Example. Let G be a finite graph, potentially having loops and multiple edges, and let E be its set of edges. The *graphic matroid* on E associated to G is defined by declaring a set of edges I to be an independent set if I contains no cycles of G. A set of edges F is a flat if no edge in $E \backslash F$ has its two endpoints joined by a path in F. The rank of the matroid is the size of a maximal forest, which is just the difference of the number of vertices of G and the number of components of G.

Remark. It turns out that every graphic matroid is realizable over every field. The argument involves writing down a matrix whose entries lie in $\{-1, 0, 1\}$ and verifying that its columns provide the desired collection of vectors when interpreted over a field.

Example. We list the flats of the following graphic matroid by rank.



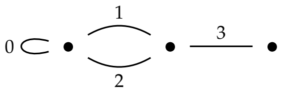
$\{0123\}$

$\{01\}$ $\{02\}$ $\{03\}$ $\{12\}$ $\{13\}$ $\{23\}$

$\{0\}$ $\{1\}$ $\{2\}$ $\{3\}$

$\varnothing$

Example. We list the flats of the following graphic matroid by rank.

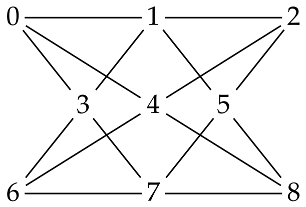


$\{0123\}$

$\{012\}$ $\{03\}$

$\{0\}$

Example. Consider the following diagram with 9 points labeled $E = \{0, 1, \ldots, 8\}$ and 8 lines where each line contains 3 points.



rank 2 flats:

$012, 037, 048, 136, 158, 246, 257, 678,$

$05, 06, 14, 17, 23, 28, 34, 35, 38, 45, 47, 56$

Let M be the matroid on E with the following flats: the only rank 0 flat is the empty set, the rank 1 flats are the 9 singletons, the rank 2 flats are the 8 triples that are contained in a line and the 12 pairs that are not contained in a line, and the only rank 3 flat is the entire set. The rank 2 flats are listed explicitly above. It is easy to verify that M is indeed a matroid. It turns out that this matroid is not realizable over any field.

We now define loops and coloops of matroids. If M is the graphic matroid associated to a graph G, then a loop of M is just a loop of the graph, and a coloop is a bridge of G. Recall that a *loop* of a graph is an edge whose two endpoints are the same vertex of G, and a *bridge* of a graph is an edge whose deletion strictly increases the number of components of the graph.

Definition (Loop)**.** An element i in the ground set E of a matroid M is a *loop* if any of the following equivalent conditions hold:

- The singleton $\{ i \}$ is not independent.
- The rank of $\{ i \}$ is zero.
- Every flat contains i.

A matroid M is *loopless* if every element of E is not a loop. Equivalently, M is loopless if any of the following equivalent conditions hold:

- Every singleton subset of E is independent.
- The only subset of E with zero rank is the empty set.
- The empty set is a flat.

Definition (Coloop)**.** An element i in the ground set E of a matroid M is a *coloop* if any of the following equivalent conditions hold:

- If I is an independent set, then $I \cup i$ is an independent set.
- $\mathrm{rk}_M(E\backslash i) = \mathrm{rk}(M) - 1$.
- $E\backslash i$ is a flat.

2.1 The chromatic and characteristic polynomials

Let G be a finite graph, and q a nonnegative integer. A *proper q-coloring of G* is a coloring of the vertices of G using q colors so that the two endpoints of each edge are colored differently. Let $\chi_G(q)$ denote the number of proper q-colorings of G.

Lemma 2.1. *The function $q \mapsto \chi_G(q)$ is a polynomial with integer coefficients.*

The polynomial $\chi_G(q)$ is called the *chromatic polynomial* of G. The proof of Lemma 2.1 is a simple consequence of the *deletion-contraction relation* that $\chi_G(q)$ satisfies. Recall that if e is an edge of G, then the *deletion* of e is the graph $G\backslash e$ with the same vertices obtained by simply deleting e, while the *contraction* of e is the graph G/e obtained by identifying the endpoints of e and then deleting e. In particular, deleting and contracting a loop are identical. Note that $\chi_G(q)$ satisfies the deletion-contraction relation

$$\chi_G(q) = \chi_{G\backslash e}(q) - \chi_{G/e}(q).$$

Indeed, suppose that the endpoints of e are v_0, v_1. We can partition the proper q-colorings of $G\backslash e$ into those for which v_0 and v_1 have the same color and those for which v_0 and v_1 have different colors. The former are in one-to-one correspondence with the proper q-colorings of G/e while the latter are in one-to-one correspondence with the proper q-colorings of G.

Proof of Lemma 2.1. We prove the result by induction on the number of edges of G. If G has no edges, then G just consists of vertices. If G has k vertices, then $\chi_G(q) = q^k$. For the inductive step, we choose an edge e and observe that both $G\backslash e$ and G/e have fewer edges than G so the deletion-contraction relation proves the result. □

We note that if G has a loop, then $\chi_G(q) = 0$. If G is loopless, then the deletion-contraction relation implies much more about $\chi_G(q)$ than it simply being an integer polynomial.

Proposition 2.2. *Let G be a loopless finite graph. Then $\chi_G(q)$ is a monic polynomial whose degree is the number of vertices of G. Furthermore, the coefficients of $\chi_G(q)$ alternate in sign*

$$\chi_G(q) = q^v - a_1(G)q^{v-1} + \cdots + (-1)^v a_v(G) \qquad a_i(G) \geqslant 0$$

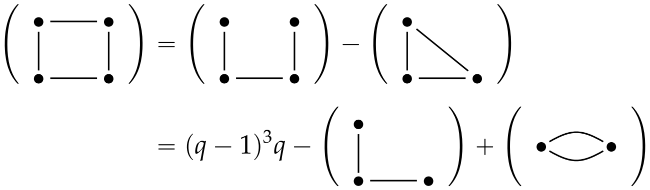
with $a_{v-i}(G) = 0$ if and only if i is less than the number of components of G.

Proof. We prove the result by induction on the number of edges of G. When G has no edges, the result is trivial. For the inductive step, we may assume that G has no multiple edges, since deleting a multiple edge does not change the chromatic polynomial. Fix an edge e of G, which we know is neither a loop nor a multiple edge. The two graphs $G\backslash e$ and G/e have fewer edges and are also loopless so their chromatic polynomials satisfy the stated properties. We show that

$$\chi_G(q) = \chi_{G\backslash e}(q) - \chi_{G/e}(q)$$

therefore also satisfies the stated properties. First, $\chi_{G\backslash e}(q)$ is monic and of degree v because G and $G\backslash e$ have the same number of vertices. Because e is not a loop, the graph G/e has one fewer vertex so $\chi_{G/e}(q)$ is monic of degree $v-1$. Thus $\chi_G(q)$ is monic of degree v. It also follows from these observations that the coefficients of $\chi_G(q)$ must alternate. Since G/e has the same number of components as G while $G\backslash e$ has at least the number of components as G, we see that the coefficient $a_{v-i}(G)$ is zero when i is less than the number of components of G. The nonvanishing the other coefficients follows from the same property for $\chi_{G\backslash e}(q)$ and $\chi_{G/e}(q)$. □

Example. Let G be the cycle with four edges. We can easily compute its chromatic polynomial using the deletion-contraction relation and quick computation that the chromatic polynomial of the path with k edges is $(q-1)^k q$.



$$\begin{aligned} &= (q-1)^3 q - (q-1)^2 q + (q-1)q \\ &= q^4 - 4q^3 + 6q^2 - 3q. \end{aligned}$$

As we see, the polynomial is monic of degree the number of vertices, and its coefficients alternate in sign and are nonzero except for its constant term. The cycle with five edges therefore has chromatic polynomial

$$(q-1)^4 q - (q^4 - 4q^3 + 6q^2 - 3q) = q^5 - 5q^4 + 10q^3 - 10q^2 + 4q.$$

We now define the characteristic polynomial of a matroid and verify its basic properties.

Definition (Characteristic polynomial)**.** Let M be a matroid on a set E. The *characteristic polynomial* of M is the polynomial

$$\chi_M(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{\mathrm{rk}(M) - \mathrm{rk}_M(S)}.$$

It is clear that $\chi_M(q)$ has integer coefficients and that if M is loopless, then $\chi_M(q)$ is monic of degree $\mathrm{rk}(M)$. Before proving various other properties of $\chi_M(q)$ analogous to those of the chromatic polynomial of a graph, we consider an example.

Example. Let G be the cycle with four edges, and let M be the associated graphic matroid. The rank of a set $S \subseteq E$ equals the rank of the smallest flat containing S, so we may explicitly compute from its list of flats that

$$\chi_M(q) = \sum_{|S|=0} q^3 - \sum_{|S|=1} q^2 + \sum_{|S|=2} q - \sum_{|S|=3} q^0 + \sum_{|S|=4} q^0 = q^3 - 4q^2 + 6q - 3$$

which we observe coincides with $\chi_G(q)/q$.

If G is a finite graph with multiple components, let G' be obtained by identifying two vertices of G lying in distinct components. The graphic matroids associated to G and G' are the same, but $\chi_G(q) = q \cdot \chi_{G'}(q)$. The chromatic polynomial of G therefore cannot be purely a function of the characteristic polynomial of the associated graphic matroid.

Proposition 2.3. *Let G be a finite graph, and let M be the associated graphic matroid. If c is the number of components of G, then*

$$\chi_G(q) = q^c\, \chi_M(q).$$

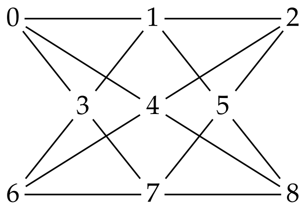
Proof. Given $S \subseteq E$, let G_S be the graph obtained from G by deleting the edges in $E \backslash S$ and contracting the edges in S. Observe that G_S has no edges, and let $|G_S|$ denote its number of vertices. The deletion-contraction relation of the chromatic polynomial implies that

$$\chi_G(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{|G_S|}.$$

It suffices to show that $|G_S| = c + \mathrm{rk}(M) - \mathrm{rk}_M(S)$ for each $S \subseteq E$. Recall that the rank of M is the size of a maximal forest. A maximal forest has size $v - c$ where v is the number of vertices of G, so we must show that $|G_S| = v - \mathrm{rk}_M(S)$ for each $S \subseteq E$.

Fix $S \subseteq E$, and choose a maximal forest I of S. Then I is an independent set of M contained in S for which $\mathrm{rk}_M(I) = \mathrm{rk}_M(S)$. The graph obtained by deleting the edges in $E \backslash S$ has v vertices, and successively contracting each edge of I lowers the number of vertices by 1. All edges of the result graph are loops by maximality of I, so their deletion does not change the number of vertices. Thus $|G_S| = v - |I| = v - \mathrm{rk}_M(I) = v - \mathrm{rk}_M(S)$ as required. □

Example. Let M be the loopless rank 3 matroid associated to the diagram



rank 2 flats:

012, 037, 048, 136, 158, 246, 257, 678,

05, 06, 14, 17, 23, 28, 34, 35, 38, 45, 47, 56 whose rank 1 flats are the 9 singletons, and whose rank 2 flats are listed explicitly. Recall that this matroid turns out not to be realizable over any field, and is in particular not a graphic matroid. By direct computation $\chi_M(q) = q^3 - 9q^2 + 28q - 20$.

Remark. Both deletion and contraction can be generalized to matroids, and the characteristic polynomial of matroids satisfies a deletion-contraction relation. Because deletion of an edge in a graph may increase the number of components of the graph, Proposition 2.3 indicates that the deletion-contraction relation for $\chi_M(q)$ must be slightly more complicated than that of $\chi_G(q)$ and should depend on whether the element $i \in E$ is a coloop.

2.2 Möbius inversion

We show that $\chi_M(q) = 0$ whenever M has a loop, and we obtain an alternate expression for $\chi_M(q)$ in terms of the *Möbius function* of the lattice of flats of M when M is loopless. We use the latter to show that the coefficients of $\chi_M(q)$ are nonzero and alternate in sign.

Definition (Möbius function)**.** Let P be a finite partially ordered set. The *Möbius function* of P is the unique function $\mu\colon P \times P \to \mathbf{Z}$ for which

- $\mu(x,x) = 1$ for all $x \in P$.
- If $x < z$, then $\sum_{x \leqslant y \leqslant z} \mu(x,y) = 0$ where the sum is over $y \in P$ satisfying $x \leqslant y \leqslant z$.
- If $x \nleqslant z$, then $\mu(x,z) = 0$.

Existence and uniqueness of the Möbius function are straightforward. Although it seems that μ is essentially a collection of independent functions $\mu(x,-)\colon P \to \mathbf{Z}$, one for each $x \in P$, there are valid formulas for μ where the second argument is fixed but the first argument varies. The following lemma is an example, from which we derive *Möbius inversion*.

Lemma 2.4. *Let P be a finite partially ordered set, and let μ be its Möbius function. If $x < z$, then*

$$\sum_{x \leqslant y \leqslant z} \mu(y,z) = 0.$$

Proof. Let $\lambda\colon P \times P \to \mathbf{Z}$ be the unique function for which

- $\lambda(x,x) = 1$ for all $x \in P$
- If $x < z$, then $\sum_{x \leqslant y \leqslant z} \lambda(y,z) = 0$ where the sum is over $y \in P$ satisfying $x \leqslant y \leqslant z$.
- If $x \nleqslant z$, then $\lambda(x,z) = 0$.

We show that $\lambda = \mu$. Consider the function $\gamma\colon P \times P \to \mathbf{Z}$ given by

$$\gamma(x,z) = \sum_{x \leqslant y \leqslant w \leqslant z} \mu(x,y)\lambda(w,z)$$

where the sum is over all pairs y, w satisfying $x \leqslant y \leqslant w \leqslant z$. It follows that

$$\mu(x,z) = \sum_{x \leqslant y \leqslant z} \mu(x,y) \sum_{y \leqslant w \leqslant z} \lambda(w,z) = \gamma(x,z) = \sum_{x \leqslant w \leqslant z} \lambda(w,z) \sum_{x \leqslant y \leqslant w} \mu(x,y) = \lambda(x,z). \qquad \square$$

Proposition 2.5 (Möbius inversion). *Let P be a finite partially ordered set. Let f and g be functions on P taking values in $\mathbf{Z}$ (or any abelian group). Then*

$$g(x) = \sum_{y \geqslant x} f(y) \quad \text{if and only if} \quad f(x) = \sum_{y \geqslant x} \mu(x,y) g(y)$$

and

$$g(y) = \sum_{x \leqslant y} f(x) \quad \text{if and only if} \quad f(y) = \sum_{x \leqslant y} g(x) \mu(x,y)$$

where μ is the Möbius function of P.

Proof. Suppose $g(x) = \sum_{y \geqslant x} f(y)$. Then

$$\sum_{y \geqslant x} \mu(x,y) g(y) = \sum_{y \geqslant x} \mu(x,y) \sum_{z \geqslant y} f(z) = \sum_{z \geqslant x} f(z) \sum_{z \geqslant y \geqslant x} \mu(x,y) = f(x).$$

The other direction is similar and uses the identity $\sum_{x \leqslant y \leqslant z} \mu(y,z) = 0$ of Lemma 2.4. The other if and only if statement is proved in the same way. ☐

In our setting, the finite partially ordered set will be the collection of flats of M ordered by inclusion, and we will only consider the Möbius function where the first argument is the empty set. Let $\mathscr{L}_M$ denote the partially ordered set consisting of the flats of M ordered by inclusion. The partially ordered set $\mathscr{L}_M$ is called the *lattice of flats* of M. If S is a subset of E, then the *closure* of S, denoted $\mathrm{cl}(S)$, is the smallest flat of M containing S. The following lemma implies that $\chi_M(q) = 0$ when M is not loopless.

Lemma 2.6. *Let F be a flat of M. Then*

$$U_F := \sum_{\substack{S \subseteq E \\ \mathrm{cl}(S) = F}} (-1)^{|S|} = \begin{cases} \mu(\varnothing, F) & \text{if } M \text{ is loopless} \\ 0 & \text{if } M \text{ is not loopless.} \end{cases}$$

Proof. The set H of loops of M is the smallest flat of M. We show that for every flat G that strictly contains H

$$\sum_{H \subseteq F \subseteq G} U_F = 0.$$

Indeed

$$\sum_{H \subseteq F \subseteq G} \sum_{\substack{S \subseteq E \\ \mathrm{cl}(S) = F}} (-1)^{|S|} = \sum_{S \subseteq G} (-1)^{|S|} = \sum_{k=0}^{|G|} (-1)^k \binom{|G|}{k} = (1-1)^{|G|} = 0.$$

If M is loopless, then $U_H = U_\varnothing = 1 = \mu(\varnothing, \varnothing)$. It follows that U_F satisfies the same defining relation as $\mu(\varnothing, F)$ so they must agree. If M has loops, then $U_H = \sum_{S \subseteq H} (-1)^{|S|} = 0$, so by induction on the rank of F, every $U_F = 0$. ☐

Corollary 2.7. *If M is not loopless, then $\chi_M(q) = 0$.*

Proof. From the definition of the characteristic polynomial and Lemma 2.6, we have

$$\chi_M(q) = \sum_{S\subseteq E}(-1)^{|S|}q^{\mathrm{rk}(M)-\mathrm{rk}_M(S)} = \sum_{F\in\mathscr{L}_M}\left(\sum_{\substack{S\subseteq E\\ \mathrm{cl}(S)=F}}(-1)^{|S|}\right)q^{\mathrm{rk}(M)-\mathrm{rk}_M(F)} = 0$$

using the fact that $\mathrm{rk}_M(S) = \mathrm{rk}_M(\mathrm{cl}(S))$. □

Corollary 2.8. *If M is loopless, then*

$$\chi_M(q) = \sum_{F\in\mathscr{L}_M}\mu(\varnothing,F)q^{\mathrm{rk}(M)-\mathrm{rk}_M(F)}.$$

We now prove a result about the Möbius function of the lattice of flats of a matroid that has the immediate corollary that the coefficients of $\chi_M(q)$ alternate in sign. This lemma is also used in the proof of the Heron-Rota-Welsh conjecture.

Lemma 2.9. *Let F be a flat of a loopless matroid M, and let μ be the Möbius function of $\mathscr{L}_M$. Then for any $i \in F$*

$$\mu(\varnothing,F) + \sum_{i\notin F'\lessdot F}\mu(\varnothing,F') = 0$$

where $F' \lessdot F$ means that F' is a flat contained in F and $\mathrm{rk}_M(F') = \mathrm{rk}_M(F) - 1$.

Proof. Fix $i \in E$, and let $\mathscr{L}_M^i$ denote the set of flats of M that contain i, ordered by inclusion. We use Möbius inversion (Proposition 2.5) for $\mathscr{L}_M^i$ to prove the result. Define $f\colon \mathscr{L}_M^i \to \mathbf{Z}$ by

$$f(F) = \mu(\varnothing,F) + \sum_{i\notin F'\lessdot F}\mu(\varnothing,F')$$

where μ is the Möbius function of $\mathscr{L}_M$. Our goal is to show that f is identically zero. By Möbius inversion, it suffices to show that the function $g\colon \mathscr{L}_M^i \to \mathbf{Z}$ defined by

$$g(F) = \sum_{\substack{F'\subseteq F\\ F'\in\mathscr{L}_M^i}} f(F') = \sum_{i\in F'\subseteq F} f(F')$$

is identically zero. But note that

$$g(F) = \sum_{i\in F'\subseteq F}\left(\mu(\varnothing,F') + \sum_{i\notin F''\lessdot F'}\mu(\varnothing,F'')\right) = \sum_{\substack{G\subseteq F\\ G\in\mathscr{L}_M}}\mu(\varnothing,G)$$

because every flat G of M that is contained in F appears exactly once in the expression for $g(F)$. Indeed, either $i \in G$ in which case G appears as F' in the sum, or $i \notin G$ in which case the unique minimal flat containing G and i appears as an F'. But now $\sum_{G\subseteq F}\mu(\varnothing,G) = 0$ so g and f are identically zero. □

Proposition 2.10. *Let M be a loopless matroid of rank $r+1$. Then the characteristic polynomial of M may be written as*

$$\chi_M(q) = w_0(M)q^{r+1} - w_1(M)q^r + \cdots + (-1)^{r+1}w_{r+1}(M) \quad \textit{with} \quad w_i(M) > 0.$$

Proof. By Corollary 2.8, we know that

$$\chi_M(q) = \sum_{F \in \mathscr{L}_M} \mu(\varnothing, F) q^{\mathrm{rk}(M) - \mathrm{rk}_M(F)}.$$

It suffices to show that $(-1)^{\mathrm{rk}_M(F)} \mu(\varnothing, F) > 0$ for each flat F. We prove the result by induction on $\mathrm{rk}_M(F)$. If $\mathrm{rk}_M(F) = 0$, then $F = \varnothing$ and $\mu(\varnothing, \varnothing) = 1 > 0$. For the inductive step, choose an element $i \in F$ so that

$$\mu(\varnothing, F) = -\sum_{i \notin F' \lessdot F} \mu(\varnothing, F')$$

by Lemma 2.9. The result immediately follows from the observation that there indeed does exist a flat F' satisfying $i \notin F' \lessdot F$ because i is not a loop. ☐

3 The Chow ring

After defining the Chow ring of a matroid, we state the main results of Hodge theory for matroids, collectively referred to as the Kähler package. In section 3.1, we prove the Heron-Rota-Welsh conjecture assuming the Kähler package. In section 3.2, we take the first step in proving the Kähler package by constructing the degree map of the Chow ring. Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations are proved in section 4. The material of this section is drawn from [FY04, AHK18, BES20].

Definition (Chow ring of a matroid)**.** Let M be a loopless matroid on the ground set E. Define the *Chow ring* of M to be the graded $\mathbf{R}$-algebra

$$\mathrm{CH}(M) = \frac{\mathbf{R}[\, x_F \mid F \text{ is a nonempty proper flat of } M \,]}{\langle x_F x_G \mid F, G \text{ incomparable} \rangle + \left\langle \sum_{i \in F} x_F - \sum_{j \in F} x_F \mid i, j \in E \right\rangle}$$

The relations of the form $x_F x_G$ are called the *incomparability relations* while the relations of the form $\sum_{i \in F} x_F - \sum_{j \in F} x_F$ are called the *linear relations*. Note that the sums appearing in the linear relations are over nonempty proper flats F that contain a fixed element i or j. The grading $\mathrm{CH}(M) = \bigoplus_{k=0}^{\infty} \mathrm{CH}^k(M)$ is inherited from the usual grading on a polynomial ring.

In the following four statements and in the rest of this paper, M is a loopless matroid of rank $r + 1 \geqslant 1$. There is an open convex subset of $\mathrm{CH}^1(M)$ that is closed under positive rescaling called the *ample cone* of M. Elements of the ample cone are called *ample classes*. The ample cone is nonempty if $r \geqslant 1$.

The degree map. *There is a linear isomorphism*

$$\deg_M \colon \mathrm{CH}^r(M) \to \mathbf{R}$$

characterized by the property that $\deg_M(x_{F_1} \cdots x_{F_r}) = 1$ *for every collection of nonempty proper flats* $F_1, \ldots, F_r$ *satisfying* $F_1 \subsetneq \cdots \subsetneq F_r$*. If* $k > r$*, then* $\mathrm{CH}^k(M) = 0$*.*

Poincaré duality. *For every nonzero element* $\mu \in \mathrm{CH}(M)$*, there exists an element* $\nu \in \mathrm{CH}(M)$ *for which* $\deg_M(\mu\nu) \neq 0$*. Equivalently, the map*

$$\mathrm{CH}^k(M) \to \mathrm{Hom}_{\mathbf{R}}(\mathrm{CH}^{r-k}(M), \mathbf{R}) \qquad \mu \mapsto (\nu \mapsto \deg_M(\mu\nu))$$

is an isomorphism for every integer k.

The hard Lefschetz theorem. *Let ℓ be an ample class of M. Then the multiplication map*

$$\ell^{r-2k}\colon \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k}(M)$$

is an isomorphism for $k \leqslant r/2$.

The Hodge-Riemann relations. *Let ℓ be an ample class of M. Then the symmetric bilinear form*

$$\mathrm{CH}^k(M) \times \mathrm{CH}^k(M) \to \mathbf{R} \qquad (\mu, \nu) \mapsto (-1)^k \deg_M(\ell^{r-2k}\mu\nu)$$

is positive-definite on the kernel of $\ell^{r-2k+1}\colon \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k+1}(M)$ for $k \leqslant r/2$.

To amplify the definition of the Chow ring and to define the ample cone, we introduce some terminology. Just as before, M is a loopless matroid with ground set E.

Definition (Linear and piecewise linear functions)**.** We call any real-valued function ℓ on the set of nonempty proper flats of M a *piecewise linear function* on M.

If f is a real-valued function on the ground set E satisfying $\sum_{i\in E} f(i) = 0$, then we define a piecewise linear function on M by the rule $F \mapsto \sum_{i\in F} f(i)$. Any piecewise linear function arising in this way is called a *linear function* on M. We note that two different functions on E may define the same linear function on M.

Two piecewise linear functions on M are *equivalent* if their difference is a linear function on M. The following lemma shows that the linear relations $\sum_{i\in F} x_F - \sum_{j\in F} x_F$ in the Chow ring capture the notion of equivalence of piecewise linear functions on M.

Lemma 3.1. *The vector space of piecewise linear functions on M modulo linear functions on M may be naturally identified with $\mathrm{CH}^1(M)$ by the map $\ell \mapsto \sum_F \ell(F)x_F$.*

Proof. Note that $\mathrm{CH}^1(M)$ is the quotient

$$\mathrm{CH}^1(M) = \frac{\mathbf{R}\langle\, x_F \mid F \text{ is a nonempty proper flat}\,\rangle}{\langle \sum_{i\in F} x_F - \sum_{j\in F} x_F\rangle}.$$

Under the rule $\ell \mapsto \sum_F \ell(F)x_F$, the space of piecewise linear functions on M is naturally identified with the vector space $\mathbf{R}\langle x_F \mid F \text{ is a nonempty proper flat}\rangle$. Given distinct elements $i, j \in E$, let $f_{ij}\colon E \to \mathbf{R}$ be the function which sends $i \mapsto 1$, $j \mapsto -1$, and all other elements to 0. The associated linear function on M corresponds to $\sum_{i\in F} x_F - \sum_{j\in F} x_F$ under the identification. It therefore suffices to prove that linear functions on M are spanned by those arising from the f_{ij}. This is true because any real-valued function f on E satisfying $\sum_{i\in E} f(i) = 0$ is a real linear combination of the f_{ij}. □

A collection $\mathscr{F} = \{F_1, \ldots, F_k\}$ of flats for which $F_1 \subsetneq \cdots \subsetneq F_k$ is called a *k-flag*. The flags we consider will always consist of nonempty proper flats of M. A *maximal* flag of nonempty proper flats is just a flag which cannot be extended to a longer flag of nonempty proper flats. If the rank of M is $r + 1$, a k-flag of nonempty proper flats is maximal if and only if $k = r$. The following lemma shows that piecewise linear functions are "linear on flags."

Lemma 3.2. *Let $F_1 \subsetneq \cdots \subsetneq F_k$ be a k-flag of nonempty proper flats of M, and let ℓ be a piecewise linear function on M. Then there is a linear function on M which agrees with ℓ on the flats $F_1, \ldots, F_k$.*

Proof. We construct a suitable function $f\colon E \to \mathbf{R}$ whose associated linear function agrees with ℓ on $F_1, \dots, F_k$. For each $j \in F_1$, let $f(j) = \ell(F_1)/|F_1|$. Assume that f has been defined on F_i. For each $j \in F_{i+1} \backslash F_i$, let $f(j) = (\ell(F_{i+1}) - \ell(F_i))/|F_{i+1} \backslash F_i|$. Once f is defined on F_k, define f on $E \backslash F_k$ in such a way to satisfy the condition $\sum_{i \in E} f(i) = 0$. $\square$

If $\mathscr{F}$ is a k-flag of nonempty proper flats, we say that a nonempty proper flat F *extends* $\mathscr{F}$ to a $(k+1)$-flag if $\mathscr{F} \cup \{F\}$ is a $(k+1)$-flag. More explicitly, F extends $\mathscr{F} = \{F_1, \dots, F_k\}$ with $F_1 \subsetneq \cdots \subsetneq F_k$ if and only if there is an index $i \in \{0, \dots, k\}$ for which $F_i \subsetneq F \subsetneq F_{i+1}$ with the convention $F_0 = \varnothing$ and $F_{k+1} = E$.

Definition (Convex and strictly convex)**.** Let $\mathscr{F}$ be a k-flag $F_1 \subsetneq \cdots \subsetneq F_k$ of nonempty proper flats. A piecewise linear function ℓ is *convex at* $\mathscr{F}$ if it is equivalent to a piecewise linear function ℓ' which is zero on each $F_i \in \mathscr{F}$ and nonnegative on each nonempty proper flat F that extends $\mathscr{F}$ to a $(k+1)$-flag. The piecewise linear function ℓ is *convex* if it is convex at every flag of nonempty proper flats.

A piecewise linear function ℓ is *strictly convex at* $\mathscr{F}$ if it is equivalent to a piecewise linear function ℓ' which is zero on each $F_i \in \mathscr{F}$ and positive on each nonempty proper flat F that extends $\mathscr{F}$ to a $(k+1)$-flag, and ℓ is *strictly convex* if it is strictly convex at every flag of nonempty proper flats.

Definition (Ample and nef)**.** The elements in $\mathrm{CH}^1(M)$ corresponding to the equivalence classes of convex piecewise linear functions are called *nef classes*. The collection of nef classes is called the *nef cone*.

The elements of $\mathrm{CH}^1(M)$ corresponding to the equivalence classes of strictly convex piecewise linear functions are called *ample classes*. The collection of ample classes is called the *ample cone*.

Lemma 3.3. *The nef cone is closed, convex, and invariant under nonnegative rescaling. The ample cone is open, convex, and invariant under positive rescaling.*

Proof. It is straightforward to verify that both cones are convex and invariant under suitable rescaling. To see that the ample cone is open, fix a k-flag $\mathscr{F}$ of proper nonempty flats, and fix a piecewise linear function ℓ that is strictly convex at $\mathscr{F}$. Up to equivalence, we may assume that ℓ vanishes on each flat of $\mathscr{F}$ and is positive on each nonempty proper flat F that extends $\mathscr{F}$ to a $(k+1)$-flag. Let ℓ' be an arbitrary piecewise linear function, which we may assume vanishes on $\mathscr{F}$. Then there is a sufficiently small $\delta > 0$ for which $\ell + \delta\ell'$ is also strictly convex at $\mathscr{F}$. By choosing a basis for the piecewise linear functions modulo linear functions, the argument extends to show that the space of piecewise linear functions that are strictly convex at $\mathscr{F}$ is open. There are finitely many flags so the ample cone, being the intersection of finitely many open sets, is open.

We show that the nef cone is closed by showing that its complement is open. Suppose ℓ is a piecewise linear function that is not convex. Then there is some k-flag $\mathscr{F}$ of nonempty proper flats with the property that for every linear function λ for which $\lambda(F_i) = \ell(F_i)$ for each $F_i \in \mathscr{F}$, there exists a nonempty proper flat F extending $\mathscr{F}$ to a $(k+1)$-flag for which $\lambda(F) > \ell(F)$. Up to equivalence, we may assume that ℓ vanishes on each flat of $\mathscr{F}$. For each linear function λ that vanishes on $\mathscr{F}$, set

$$\varepsilon_\lambda = \max_F(\lambda(F) - \ell(F))$$

where the maximum is taken over all nonempty proper flats F extending $\mathscr{F}$. Note that ε_λ is a positive continuous function on the space of linear functions λ that vanish on $\mathscr{F}$. Next, note that for any real number $c > 1$, we have

$$\varepsilon_{c\lambda} = \max_F(c\lambda(F) - \ell(F)) > \max_F(\lambda(F) - \ell(F)) = \varepsilon_\lambda.$$

It follows that the ε_λ attains a global positive minimum on the space of linear functions λ vanishing on $\mathscr{F}$. Let ε be a positive number smaller than this global minimum.

Let ℓ' be an arbitrary piecewise linear function, which up to equivalence we may assume vanishes on $\mathscr{F}$. Choose $\delta > 0$ small enough that for all nonempty proper flats F extending $\mathscr{F}$, we have

$$|\delta\ell'(F)| < \varepsilon/2.$$

Then for any linear function λ vanishing on $\mathscr{F}$, there is a nonempty proper flat F extending $\mathscr{F}$ for which $\lambda(F) - \ell(F) > \varepsilon$. It follows that $\lambda(F) - (\ell(F) + \delta\ell'(F)) > \varepsilon/2$ so $\ell + \delta\ell'$ is not nef. Again by choosing a basis for the piecewise linear functions modulo linear functions, we extend the argument to see that the complement of the nef cone is open. □

If $\operatorname{rk}(M) \leqslant 1$, then there are no nonempty proper flats so the ample cone is empty. The next lemma shows that as long as $\operatorname{rk}(M) \geqslant 2$, there exist ample classes.

Lemma 3.4. *If the rank of M is at least 2, then the ample cone is nonempty.*

Proof. Let ℓ be the piecewise linear function defined by

$$\ell(F) = |F| \cdot |E \backslash F|.$$

Fix a k-flag $\mathscr{F}$ of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$. We now define a function $f\colon E \to \mathbf{R}$ satisfying $\sum_{i\in E} f(i) = 0$ with the property that $\sum_{i\in F_j} f(i) = \ell(F_j)$ for each $F_j \in \mathscr{F}$ and for which $\sum_{i\in F} f(i) < \ell(F)$ for each nonempty proper flat F which extends $\mathscr{F}$ to a $(k+1)$-flag.

For $i \in F_1$, set $f(i) = |E\backslash F_1|$ so that $\sum_{i\in F_1} f(i) = \ell(F_1)$. Note that if F is a nonempty flat properly contained in F_1, then

$$\sum_{i\in F} f(i) = |F||E\backslash F_1| < |F||E\backslash F| = \ell(F).$$

Assume that f has been defined in F_j and satisfies the desired properties on all flats contained in F_j. For $i \in F_{j+1}\backslash F_j$ set

$$f(i) = \frac{\ell(F_{j+1}) - \ell(F_j)}{|F_{j+1}\backslash F_j|}$$

so that $\sum_{i\in F_{j+1}} f(i) = \ell(F_{j+1})$. Fix a flat F for which $F_j \subsetneq F \subsetneq F_{j+1}$. Then by direct computation

$$\ell(F) - \sum_{i\in F} f(i) = |F\backslash F_j||F_{j+1}\backslash F| > 0$$

The argument is valid when $j = k$ when we set $F_{k+1} = E$. Thus ℓ is strictly convex and therefore defines an ample class. □

Remark. A real-valued function c on the set of subsets of E satisfying $c(\varnothing) = c(E) = 0$ is called *strictly submodular* if

$$c(S) + c(T) > c(S \cap T) + c(S \cup T)$$

for every pair of incomparable subsets S, T of E. It turns out that every strictly submodular function defines an ample class of $\mathrm{CH}^1(M)$. The function $c(S) = |S||E \backslash S|$ is easily seen to be strictly submodular.

The *Boolean matroid* B on E is the matroid for which every subset of E is a flat. A piecewise linear function on B can be thought of as a function c on subsets of E satisfying $c(\varnothing) = c(E) = 0$. It turns out that a piecewise linear function on B is strictly convex if and only if it is strictly submodular.

Proposition 3.5. *If the rank of M is at least* 2*, then the closure of the ample cone is the nef cone.*

Proof. Since the nef cone contains the ample cone and is closed by Lemma 3.3, the closure of the ample cone is contained in the nef cone. By Lemma 3.4, the ample cone is nonempty, so we may choose an ample class $\ell \in \mathrm{CH}^1(M)$. Fix a nef class $\eta \in \mathrm{CH}^1(M)$ as well. It is straightforward to verify that the classes $t\ell + (1-t)\eta$ are ample for $t \in (0, 1]$, which proves the result. □

3.1 The Kähler package and log-concavity

The key observation which allows the Kähler package to establish an equality of the form $b^2 \geqslant ac$ is the following. Suppose $\ell \in \mathrm{CH}^1(M)$ is ample, and let $\eta \in \mathrm{CH}^1(M)$ be arbitrary. By the Hodge-Riemann relations in degree 1, the *Hodge-Riemann form*

$$\mathrm{CH}^1(M) \times \mathrm{CH}^1(M) \to \mathbf{R} \qquad (\mu, \nu) \mapsto -\deg_M(\ell^{r-2}\mu\nu)$$

is positive-definite on the kernel of $\ell^{r-1} \colon \mathrm{CH}^1(M) \to \mathrm{CH}^r(M)$. Since $\ell^{r-1} \cdot \ell$ is nonzero by the hard Lefschetz theorem in degree 0, we have a direct sum splitting

$$\mathrm{CH}^1(M) = \mathbf{R}\langle \ell \rangle \oplus \ker(\ell^{r-1})$$

which is easily seen to be orthogonal with respect to the Hodge-Riemann form. Furthermore, the form is negative-definite on $\mathbf{R}\langle \ell \rangle$ and positive-definite on $\ker(\ell^{r-1})$. Now consider the symmetric 2×2 matrix

$$\begin{pmatrix} -\deg_M(\ell^{r-2}(\ell\ell)) & -\deg_M(\ell^{r-2}(\ell\eta)) \\ -\deg_M(\ell^{r-2}(\eta\ell)) & -\deg_M(\ell^{r-2}(\eta\eta)) \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

If ℓ and η are linearly independent, then this 2×2 matrix represents the restriction of the Hodge-Riemann form to the subspace $\mathbf{R}\langle \ell \rangle \oplus \mathbf{R}\langle \eta \rangle$ with respect to the given basis. This matrix must have exactly one negative eigenvalue so its determinant is negative. If ℓ and η are not linearly independent, then the determinant of the matrix is 0. In any case, we find that

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 \leqslant 0$$

which is an inequality of the described form. A continuity argument extends this result to case when ℓ is only nef.

Let M be a loopless matroid of rank $r+1$, and write its characteristic polynomial as

$$\chi_M(q) = w_0(M)q^{r+1} - w_1(M)q^r + \cdots + (-1)^{r+1}w_{r+1}(M).$$

By Proposition 2.10, we have $w_k(M) > 0$ for each $k = 0, \ldots, r+1$. Expressing $\chi_M(q)$ in terms of the Möbius function of the lattice of flats of M (Corollary 2.8), we find that the evaluation of $\chi_M(q)$ at $q = 1$ yields

$$\chi_M(1) = \sum_{F \in \mathscr{L}_M} \mu(\varnothing, F)\, 1^{\operatorname{rk}(M) - \operatorname{rk}_M(F)} = \sum_{F \in \mathscr{L}_M} \mu(\varnothing, F) = 0$$

so $q-1$ divides $\chi_M(q)$. The *reduced characteristic polynomial* of M, denoted $\overline{\chi_M}(q)$, is defined to be $\chi_M(q)/(q-1)$. Write

$$\overline{\chi_M}(q) = \mu^0(M)q^r - \mu^1(M)q^{r-1} + \cdots + (-1)^r\mu^r(M)$$

and observe that $w_0(M) = \mu^0(M)$ and $w_k(M) = \mu^k(M) + \mu^{k-1}(M)$ for each $k = 1, \ldots, r$ and $w_{r+1}(M) = \mu^r(M)$. We will see that the coefficients $\mu^k(M)$ are all positive integers. It is easy to verify that log-concavity of the $\mu^k(M)$ implies that of log-concavity of the $w_k(M)$.

To establish the Heron-Rota-Welsh conjecture from the Kähler package, we define nef classes $\alpha_M, \beta_M \in \mathrm{CH}^1(M)$ and prove that

$$\mu^k(M) = \deg_M(\beta_M^k \alpha_M^{r-k}).$$

In particular, the matrix

$$\begin{pmatrix} \mu^{r-2}(M) & \mu^{r-1}(M) \\ \mu^{r-1}(M) & \mu^r(M) \end{pmatrix} = \begin{pmatrix} \deg_M(\beta_M^{r-2}(\alpha_M\alpha_M)) & \deg_M(\beta_M^{r-2}(\beta_M\alpha_M)) \\ \deg_M(\beta_M^{r-2}(\beta_M\alpha_M)) & \deg_M(\beta_M^{r-2}(\beta_M\beta_M)) \end{pmatrix}$$

has negative determinant so $\mu^{r-1}(M)\mu^{r-1}(M) \geqslant \mu^{r-2}(M)\mu^r(M)$. The inequalities for the other $\mu^k(M)$ follow from this one applied to the *truncation* $\operatorname{tr}(M)$ of M, which is a matroid with the property that $\mu^k(\operatorname{tr}(M)) = \mu^k(M)$ for $k = 0, \ldots, r-1$ but $\operatorname{rk}(\operatorname{tr}(M)) = \operatorname{rk}(M) - 1$.

We now define the elements α_M and β_M, and show that $\deg_M(\beta_M^k\alpha_M^{r-k})$ is a count of certain k-flags of nonempty proper flats of M. We then show that μ^k also equals this count using the truncation argument. For each i in the ground set E, let

$$\alpha_{M,i} = \sum_{i \in F} x_F \quad \text{and} \quad \beta_{M,i} = \sum_{i \notin F} x_F$$

The linear relations in the Chow ring assert that the class of $\alpha_{M,i}$ in $\mathrm{CH}^1(M)$ is independent of i. Let α_M be this class. It is clear that the sum $\alpha_{M,i} + \beta_{M,i}$ is independent of i so the class of $\beta_{M,i}$ in $\mathrm{CH}^1(M)$ is also independent of i. We let β_M denote this class.

Remark. Matroids with different characteristic polynomials may have isomorphic Chow rings. In this case, the classes α_M and β_M will differ between the matroids. We compute two simple examples to illustrate this.

Example. Let M be the graphic matroid associated to the path with two edges. Then the flats of M are $\varnothing, \{0\}, \{1\}$, and $E = \{0,1\}$. It follows that the Chow ring of M is

$$\mathrm{CH}(M) = \mathbf{Q}[x_0, x_1]/(x_0 - x_1, x_0x_1) = \mathbf{Q}[x_0]/(x_0^2)$$

and $\deg_M\colon \mathrm{CH}^1(M) \to \mathbf{R}$ is just the map sending $x_0 \mapsto 1$. The ample cone is $\mathbf{R}_{>0} \cdot x_0$ while the nef cone is $\mathbf{R}_{\geqslant 0} \cdot x_0$. In this example $\alpha_M = \beta_M = x_0$. The reduced characteristic polynomial of M is $\overline{\chi_M}(q) = q - 1$ so $\mu^0 = \mu^1 = 1$. Note that $\deg_M(\beta_M^0\alpha_M^1) = \deg_M(\beta_M^1\alpha_M^0) = 1$ as well.

Example. Let M be the graphic matroid associated to the cycle with three edges. The flats of M are $\varnothing, \{0\}, \{1\}, \{2\}, E$ and the Chow ring is $\mathrm{CH}(M) = \mathbf{Q}[x_0]/(x_0)^2$ where $x_0 = x_1 = x_2$. The degree map again sends $x_0 \mapsto 1$. Here $\alpha_M = x_0$ while $\beta_M = 2x_0$. We see that $\deg_M(\alpha_M) = 1$ and $\deg_M(\beta_M) = 2$ which agrees with the fact that $\overline{\chi_M}(q) = q - 2$.

Lemma 3.6. *The classes α_M and β_M in $\mathrm{CH}^1(M)$ are nef.*

Proof. Let $\mathscr{F}$ be a k-flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$. Choose $i \in F_1$ and $j \notin F_k$. Then

$$\beta_{M,i} = \sum_{i \notin F} x_F \quad \text{and} \quad \alpha_{M,j} = \sum_{j \in F} x_F$$

correspond to nonnegative piecewise linear functions that are zero on each $F_m \in \mathscr{F}$. □

Let $\mathscr{F}$ be a k-flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$ of M. We use the shorthand notation $x_{\mathscr{F}} := x_{F_1} \cdots x_{F_k}$ throughout. Fix an element $i \notin F_k$ and observe that

$$x_{\mathscr{F}}\alpha_M = x_{\mathscr{F}} \sum_{i \in F} x_F = x_{\mathscr{F}} \sum_{F_k \cup i \subseteq F} x_F = \sum_{\mathscr{F}'} x_{\mathscr{F}'}$$

where the final sum is over $(k+1)$-flags $\mathscr{F}'$ of the form $\mathscr{F} \cup F$ where F extends $\mathscr{F}$ on the right and contains i. Hence heuristically, multiplication by α_M extends flags on the right. Similarly, if we fix an element $i \in F_1$, then

$$\beta_M x_{\mathscr{F}} = \sum_{i \notin F} x_F x_{\mathscr{F}} = \sum_{i \notin F \subsetneq F_1} x_F x_{\mathscr{F}} = \sum_{\mathscr{F}'} x_{\mathscr{F}'}$$

where the final sum is over $(k+1)$-flags $\mathscr{F}'$ of the form $F \cup \mathscr{F}$ where F extends $\mathscr{F}$ on the left and does not contain i. Heuristically, multiplication by β_M extends flags on the left.

Definition (Initial)**.** Let $\mathscr{F}$ be a k-flag $F_1 \subsetneq \cdots \subsetneq F_k$ of nonempty proper flats of M. Then $\mathscr{F}$ is *initial* if $\mathrm{rk}_M(F_m) = m$ for each $m \in \{1, \ldots, k\}$.

Lemma 3.7. *Let $\mathscr{F}$ be a k-flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$.*

- *If $\mathscr{F}$ is not initial, then $x_{\mathscr{F}}\,\alpha_M^{r-k} = 0 \in \mathrm{CH}^r(M)$.*
- *If $\mathscr{F}$ is initial, then $x_{\mathscr{F}}\,\alpha_M^{r-k} = \alpha_M^r \in \mathrm{CH}^r(M)$.*

Proof. If $\mathrm{rk}_M(F_m) \neq m$ for some m, then $\mathrm{rk}_M(F_k) > k$. Thus it is not possible to extend $\mathscr{F}$ to an r-flag of nonempty proper flats by appending flats only to the right. Since multiplication by α_M extends flags to the right, we find that $x_{\mathscr{F}}\alpha_M^{r-k} = 0$. This argument is easily formalized by descending induction on k.

Now assume that $\mathrm{rk}_M(F_m) = m$ for all m. The product $x_{\mathscr{F}}\alpha_M$ is a sum of elements of the form $x_{\mathscr{F}'}$ where $\mathscr{F}'$ is a $(k+1)$-flag obtained by extending $\mathscr{F}$ on the right by a flat F that contains a fixed element $i \notin F_k$. Among such flats F, there is a unique flat F of rank $k+1$. It follows from the first assertion that $x_{\mathscr{F}}\alpha_M^{r-k} = x_{\mathscr{G}}$ where $\mathscr{G}$ is an arbitrary r-flag containing $\mathscr{F}$. In particular, $x_{\mathscr{G}} = x_{\mathscr{G}'}$ for any two r-flags $\mathscr{G}, \mathscr{G}'$ containing $\mathscr{F}$. Choosing $\mathscr{F}$ to be the 0-flag, we see that $\alpha_M^r = x_{\mathscr{G}} = x_{\mathscr{G}'}$ for any two r-flags $\mathscr{G}, \mathscr{G}'$ of nonempty proper flats. This argument is again easily formalized by ascending induction on k. □

Proposition 3.8. *The vector space* $\mathrm{CH}^k(M)$ *is spanned by elements of the form* $x_{\mathscr{F}}$ *for* $\mathscr{F}$ *a* k*-flag of nonempty proper flats. In particular,* $\mathrm{CH}^k(M) = 0$ *for* $k > r$*. Furthermore,* $\alpha_M^r = x_{\mathscr{F}}$ *for any* r*-flag* $\mathscr{F}$ *of nonempty proper flats of* M*, so* α_M^r *spans* $\mathrm{CH}^r(M)$*.*

Proof. Let

$$x_{F_1}^{k_1} \cdots x_{F_\ell}^{k_\ell}$$

be an arbitrary degree $k = k_1 + \cdots + k_\ell$ monomial, where the F_i are distinct and the k_i are positive. If any two of the F_i are incomparable, the monomial is zero. We may assume that the flats are all comparable so that they form an ℓ-flag $\mathscr{F}$. If $\ell = k$, then each $k_i = 1$ and the monomial is just $x_{\mathscr{F}}$. Otherwise, fix an index j for which $k_j > 1$, and note that there is a linear function λ for which $\lambda(F_j) = -1$ but $\lambda(F_i) = 0$ for $i \neq j$. It follows from the corresponding linear relation that

$$x_{F_j} = \sum_G \lambda(x_G) x_G$$

where the sum ranges over nonempty proper flats G for which $G \notin \mathscr{F}$. Substituting this expression in the monomial expresses the monomial as a linear combination of other monomials, each of which is one step closer to having all exponents equal to 1. □

The main result of section 3.2 is that α_M^r is nonzero. The existence and uniqueness of the degree map then follows from Proposition 3.8. The isomorphism $\deg_M \colon \mathrm{CH}^r(M) \to \mathbf{R}$ is defined by sending $\alpha_M^r \mapsto 1$ and has the property that $\deg_M(x_{\mathscr{F}}) = 1$ whenever $\mathscr{F}$ is an r-flag.

Definition (Descending)**.** Order the elements of the ground set E so that we may view E is the set of numbers $\{0, 1, \ldots, n\}$. Let $\mathscr{F}$ be a k-flag $F_1 \subsetneq \cdots \subsetneq F_k$ of nonempty proper flats of M. Then $\mathscr{F}$ is *descending* if

$$\min(F_1) > \cdots > \min(F_k) > 0$$

where $\min(F_m)$ is the smallest number in F_m thought of as a subset of $\{0, 1, \ldots, n\}$.

Whether a flag is descending depends on the ordering of E, but the following lemma shows that the sum $\sum_{\mathscr{F}} x_{\mathscr{F}} \in \mathrm{CH}^k(M)$ over all descending k-flags $\mathscr{F}$ is independent of the ordering.

Lemma 3.9. *For each positive* k*, we have*

$$\beta_M^k = \sum_{\mathscr{F}} x_{\mathscr{F}}$$

where the sum is over all descending k*-flags* $\mathscr{F}$ *of nonempty proper flats of* M*.*

Proof. We prove the result by induction on k. A descending 1-step flag of nonempty proper flats is just a single nonempty proper flat F that does not contain $0 \in E$. Note that

$$\beta_M = \beta_{M,0} = \sum_{0 \notin F} x_F$$

as required. Fix a particular descending k-step flag $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$ and let $i_{\mathscr{F}} = \min(F_1)$. Note that $i_{\mathscr{F}}$ is an element of each flag of this flat. Then $\beta_M x_{\mathscr{F}} = \sum_F x_F x_{\mathscr{F}}$ where the sum ranges over all nonempty flats F contained in F_1 that do not contain $i_{\mathscr{F}}$. For any such flat F, the flag $F \cup \mathscr{F}$ is a descending $(k+1)$-flag. It follows that $\beta_M x_{\mathscr{F}} = \sum_{\mathscr{F}'} x_{\mathscr{F}'}$ where the sum is over all descending $(k+1)$-flags whose last k terms are $\mathscr{F}$. By induction, we know that

$$\beta_M^{k+1} = \sum_{\mathscr{F}} \beta_M x_{\mathscr{F}}$$

where the sum is over all descending k-flags F. Thus $\beta_M^{k+1} = \sum_{\mathscr{F}'} x_{\mathscr{F}'}$ where the sum is over all descending $(k+1)$-flags as required. □

Let $D_k(M)$ be the set of k-flags of nonempty proper flats of M that are both initial and descending. The following is immediate from Lemmas 3.7 and 3.9.

Corollary 3.10. *For each $k \in \{0, \ldots, r\}$, we have $\beta_M^k \alpha_M^{r-k} = |D_k(M)|\, \alpha_M^r \in \mathrm{CH}^r(M)$.*

We now show that $\mu^k(M) = |D_k(M)|$ where

$$\overline{\chi_M}(q) = \mu^0(M) q^r - \mu^1(M) q^{r-1} + \cdots + (-1)^r \mu^r(M).$$

It follows from this that each $\mu^k(M)$ is a positive integer. We first show that $\mu^r(M) = |D_r(M)|$ directly, and then use a truncation argument for the other coefficients.

Lemma 3.11. *We have the equality $\mu^r(M) = |D_r(M)|$.*

Proof. Note that $(-1)^{r+1}\mu^r(M) = (-1)^{r+1} w_{r+1}(M)$ is the constant term of $\chi_M(q)$ so it suffices to show that $\chi_M(0) = (-1)^{r+1}|D_r(M)|$. By Corollary 2.8, we have

$$\chi_M(0) = \sum_{F \in \mathscr{L}_M} \mu(\varnothing, F) 0^{\mathrm{rk}(M) - \mathrm{rk}_M(F)} = \mu(\varnothing, E)$$

since the ground set $E = \{0, 1, \ldots, n\}$ is the unique flat of M of rank $r+1$.

Let F be a flat of rank $k+1 > 0$. We show by induction on k that $(-1)^{k+1}\mu(\varnothing, F)$ is just the number of initial descending k-flags of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$ for which $F_k \subsetneq F$. The result follows from the case $F = E$. If $k = 0$, then the only flat contained in F is $\varnothing$ so $-\mu(\varnothing, F) = \mu(\varnothing, \varnothing) = 1$ as required since the unique 0-flag is vacuously initial and descending. For the inductive step, we know that

$$(-1)^{k+1}\mu(\varnothing, F) = \sum_{i \notin F' \lessdot F} (-1)^k \mu(\varnothing, F')$$

by Lemma 2.9. Here i is an arbitrarily chosen element of F and the sum is over flats $F' \subsetneq F$ of rank k that do not contain i. We choose i to be $\min(F)$ so that each F' appearing in the sum satisfies $\min(F') > \min(F)$. By the induction hypothesis, we know that $(-1)^k \mu(\varnothing, F')$ is the number of initial descending $(k-1)$-flags properly contained in F', so it follows that $(-1)^{k+1}\mu(\varnothing, F)$ is the number of initial descending k-flags properly contained in F. □

Definition (Truncation)**.** Let M be a matroid of rank $r+1$ a the ground set E. The *truncation* of M is the matroid denoted $\mathrm{tr}(M)$ on E with rank function

$$\mathrm{rk}_{\mathrm{tr}(M)}(S) = \min(\mathrm{rk}_M(S), r).$$

The flats of $\mathrm{tr}(M)$ are precisely the flats F of M for which $\mathrm{rk}_M(F) \neq r$. The ground set E which is a flat of rank $r+1$ in M becomes a flat of rank r in $\mathrm{tr}(M)$. If M is loopless then $\mathrm{tr}(M)$ is loopless.

It is immediate from the definitions that $D_k(\mathrm{tr}(M)) = D_k(M)$ for $0 \leqslant k \leqslant r-1$. Together with Lemma 3.11, the following result implies that $\mu^k(M) = |D_k(M)|$ for all k.

Lemma 3.12. *There is an equality* $\mu^k(\mathrm{tr}(M)) = \mu^k(M)$ *for each* $k \in \{0, \ldots, r-1\}$.

Proof. Recall that the two sequences of numbers $w_k(M)$ and $\mu^k(M)$ are defined by

$$\begin{aligned}\chi_M(q) &= w_0(M)q^{r+1} - w_1(M)q^r + \cdots + (-1)^{r+1}w_{r+1}(M)\\ \overline{\chi_M}(q) &= \mu^0(M)q^r - \mu^1(M)q^{r-1} + \cdots + (-1)^r\mu^r(M)\end{aligned}$$

with $(q-1)\overline{\chi_M}(q) = \chi_M(q)$. Hence

$$w_0(M) = \mu^0(M) \qquad w_k(M) = \mu^k(M) + \mu^{k-1}(M) \text{ for } k \in \{1, \ldots, r\} \qquad w_{r+1}(M) = \mu^r(M).$$

It therefore suffices to show that $w_k(\mathrm{tr}(M)) = w_k(M)$ for all $k < r$. Note that

$$(-1)^k w_k(M) = \sum_{\substack{F \in \mathscr{L}_M \\ \mathrm{rk}_M(F) = k}} \mu(\varnothing, F)$$

by Corollary 2.8. Since $k < r$, we know that $(-1)^k w_k(\mathrm{tr}(M))$ is a sum over the same set, and the result now follows from the observation that the Möbius function $\mu(\varnothing, F)$ depends only on the partially ordered set of flats contained in F. □

Corollary 3.13. *There is an equality* $\mu^k(M) = |D_k(M)|$ *for each* $k \in \{0, 1, \ldots, r\}$.

Assuming the Kähler package, we now prove the Heron-Rota-Welsh conjecture.

Theorem 3.14 (Heron-Rota-Welsh conjecture)**.** *Let M be a matroid. Then the absolute values of the coefficients of $\chi_M(q)$ are log-concave.*

Proof. If M is not loopless, then $\chi_M(q) = 0$ by Corollary 2.7, so we assume M is loopless and of rank $r+1$. We claim that it suffices to prove that

$$\mu^{r-1}(M)\mu^{r-1}(M) \geqslant \mu^{r-2}(M)\mu^r(M).$$

Indeed, the inequality $\mu^k(M)\mu^k(M) \geqslant \mu^{k-1}(M)\mu^{k+1}(M)$ for $k < r-1$ follows from the given inequality for the iterated truncations of M by Lemma 3.12. Using the existence of the degree map and Corollaries 3.10 and 3.13, we have

$$\begin{pmatrix}\mu^{r-2}(M) & \mu^{r-1}(M)\\ \mu^{r-1}(M) & \mu^r(M)\end{pmatrix} = \begin{pmatrix}|D_{r-2}(M)| & |D_{r-1}(M)|\\ |D_{r-1}(M)| & |D_r(M)|\end{pmatrix}$$

$$= \begin{pmatrix} \deg_M(\beta_M^{r-2}(\alpha_M\alpha_M)) & \deg_M(\beta_M^{r-2}(\beta_M\alpha_M)) \\ \deg_M(\beta_M^{r-2}(\beta_M\alpha_M)) & \deg_M(\beta_M^{r-2}(\beta_M\beta_M)) \end{pmatrix}$$

and it suffices to show that this matrix has nonpositive determinant.

Let $\ell \in \mathrm{CH}^1(M)$ be an ample class, which exists as long as $r \geqslant 1$ by Lemma 3.4. If $r < 1$, then the result is trivially true. By the Hodge-Riemann relations in grading 1, we know that the Hodge-Riemann form

$$\mathrm{CH}^1(M) \times \mathrm{CH}^1(M) \to \mathbf{R} \qquad (\mu, \nu) \mapsto -\deg_M(\ell^{r-2}\mu\nu)$$

is positive-definite on the kernel of $\ell^{r-1}\colon \mathrm{CH}^1(M) \to \mathrm{CH}^r(M)$. The map $\ell^r\colon \mathrm{CH}^0(M) \to \mathrm{CH}^r(M)$ is an isomorphism by the hard Lefschetz theorem in grading 0 so we have a direct sum splitting

$$\mathrm{CH}^1(M) = \mathbf{R}\langle \ell \rangle \oplus \ker(\ell^{r-1}).$$

If $\nu \in \ker(\ell^{r-1})$, then $\deg_M(\ell^{r-2}\ell\nu) = 0$ so this splitting is orthogonal with respect to the Hodge-Riemann form. Furthermore, the Hodge-Riemann form is negative-definite on $\mathbf{R}\langle \ell \rangle$ because $\deg_M(\ell^r) > 0$ by the Hodge-Riemann relations in degree 0. It follows that the matrix

$$\begin{pmatrix} \deg_M(\ell^{r-2}(\alpha_M\alpha_M)) & \deg_M(\ell^{r-2}(\ell\alpha_M)) \\ \deg_M(\ell^{r-2}(\ell\alpha_M)) & \deg_M(\ell^{r-2}(\ell\ell)) \end{pmatrix}$$

has nonpositive determinant. Finally $\ell_t := t\beta_M + (1-t)\ell$ is ample for every $t \in [0, 1)$ because β_M is nef by Lemma 3.6. The given matrix with ℓ_t in place of ℓ has nonpositive determinant for each $t \in [0, 1)$ so it also has nonpositive determinant for $t = 1$ by continuity. □

Corollary 3.15 (Read's conjecture and Hoggar's conjecture)**.** *Let G be a finite graph. Then the absolute values of the coefficients of $\chi_G(q)$ are log-concave.*

Proof. The result follows from the Heron-Rota-Welsh conjecture and Proposition 2.3. □

3.2 The degree map

The purpose of this section is to prove that α_M^r is nonzero. Since $\mathrm{CH}^r(M)$ is spanned by α_M^r by Proposition 3.8, it follows that $\mathrm{CH}^r(M)$ is nonzero and the degree map $\deg_M\colon \mathrm{CH}^r(M) \to \mathbf{R}$ is well-defined. This result is proved using Minkowski weights and toric geometry in [AHK18, BHM$^+$20] and ultimately cites the main result of [FMSS95]. We instead prove this result using the Gröbner basis computation appearing in [FY04]. This route to showing that $\alpha_M^r \neq 0$ was suggested to the author by Christopher Eur. We explain the main argument here, but prove that the given generating set is in fact a Gröbner basis in the Appendix. Our exposition concerning the Chow ring partially follows [BES20] and the basic material on Gröbner bases is from [DF04].

Showing that α_M^r is nonzero in $\mathrm{CH}(M)$ is equivalent to showing that a certain polynomial does not lie in the ideal of $\mathbf{R}[\, x_F \mid F \text{ is a nonempty proper flat of } M \,]$ generated by the incomparability and linear relations. A key feature of a *Gröbner basis* of an ideal is that it provides a practical method for checking whether a given element lies in the ideal. Rather than defining a Gröbner basis for the ideal directly, we use a slightly different presentation of the Chow ring used in [FY04]. If M is a loopless matroid, then

$$\mathrm{CH}(M) = \mathbf{R}[\, z_F \mid F \text{ is a nonempty flat of } M]/\mathcal{J}$$

where $\mathcal{J}$ is the ideal

$$\mathcal{J} = \left\langle z_F z_G \mid F, G \text{ incomparable} \right\rangle + \left\langle \sum_{i \in F} z_F \mid i \in E \right\rangle.$$

We again call the $z_F z_G$ the *incomparability relations* and the $\sum_{i \in F} z_F$ the *linear relations*. Note that z_E is now a generator and we recover the original presentation by setting

$$z_F = \begin{cases} x_F & F \subsetneq E \\ -\alpha_M & F = E. \end{cases}$$

A *monomial ordering* is a well ordering on the set of monomials of a polynomial ring over a field for which $ac \geqslant bc$ whenever $a \geqslant b$ for monomials a, b, c. Fix a total ordering on the set of nonempty flats of M with the property that if $\mathrm{rk}_M(F) < \mathrm{rk}_M(G)$ then $F > G$. In this total ordering, the ground set E satisfies $F > E$ for all proper flats F. The *lexicographical ordering* on monomials in $\mathbf{R}[\, z_F \mid F \text{ is a nonempty flat of } M\,]$ is a monomial ordering. Explicitly, we may write any two monomials as

$$z_{F_1}^{n_1} \cdots z_{F_k}^{n_k} \quad \text{and} \quad z_{G_1}^{m_1} \cdots z_{G_\ell}^{n_\ell}$$

where the exponents are all positive and $F_1 > \cdots > F_k$ and $G_1 > \cdots > G_\ell$. It follows that $\mathrm{rk}_M(F_1) \leqslant \cdots \leqslant \mathrm{rk}_M(F_k)$ and $\mathrm{rk}_M(G_1) \leqslant \cdots \leqslant \mathrm{rk}_M(G_\ell)$. To compare the two monomials, we first compare F_1 and G_1. If either is larger, then the corresponding monomial is declared larger. If they agree and one n_1 and m_1 is larger, then the corresponding monomial is declared larger. If $F_1 = G_1$ and $n_1 = m_1$, then we move on to the next pair of flats and do the same comparison procedure. In this monomial ordering which is fixed for the rest of the section, the smallest monomials are $1 < z_E < z_E^2 < \cdots$.

Given a monomial ordering, any polynomial now has a *leading term* just like a single-variable polynomial. The leading term $LT(f)$ of a nonzero polynomial f is the monomial term cm where m is the largest monomial appearing in the polynomial with nonzero coefficient. In particular, the leading term is cm, the monomial with its nonzero coefficient. The leading term of the zero polynomial is defined to be zero. In our situation, note that if the leading term of a polynomial f is cz_E^k for some $k \geqslant 0$ and $c \in \mathbf{R}$, then f is a single-variable polynomial in $\mathbf{R}[z_E]$.

Definition (Gröbner basis)**.** Let I be an ideal in a polynomial ring over a field with a fixed monomial ordering. A *Gröbner basis* for I is a finite set of elements $g_1, \dots, g_m \in I$ that generate I and whose leading terms generate the *ideal of leading terms* of I, which is the ideal generated by the leading terms of all elements of I.

Remark. There is no minimality condition on $g_1, \dots, g_m$ as one might expect for a "basis". If $g_1, \dots, g_m$ is a Gröbner basis for I and $f \in I$, then $g_1, \dots, g_m, f$ is a Gröbner basis for I.

Given any collection of elements $g_1, \dots, g_m$ and an arbitrary polynomial f, we have a procedure of *polynomial division*. Set polynomials $q_1, \dots, q_m$ all equal to zero initially and repeat the following procedure:

1. If there does not exist a monomial term of f that is divisible by $LT(g_i)$ for some g_i, then terminate the procedure and let $r = f$.

2. If some monomial term μ of f is divisible by $LT(g_i)$ for some g_i, then $\mu = aLT(g_i)$ for some monomial term a. Add a to q_i, and replace f by $f - ag_i$, and repeat.

After the procedure terminates, the original polynomial f then satisfies

$$f = q_1 g_1 + \cdots + q_m g_m + r.$$

Note that the polynomials $q_1, \ldots, q_m$ may depend on the choices made during the procedure. A variation of the following proposition shows that r is independent of these choices when $g_1, \ldots, g_m$ is a Gröbner basis.

Proposition 3.16. *Let $g_1, \ldots, g_m$ be a Gröbner basis of an ideal I. Let f be a polynomial, and let $q_1, \ldots, q_m, r$ be obtained by polynomial division. Then $f \in I$ if and only if $r = 0$.*

Proof. It is clear that if $r = 0$ then $f \in I$. If $f \in I$, then we show that $r = 0$ by induction on the number of nonzero monomial terms in f. The base case $f = 0$ is trivial. In the inductive step, we know that $LT(f)$ lies in the ideal of leading terms of I, so by definition of a Gröbner basis we know that $LT(f)$ can be written as $a_1 LT(g_1) + \cdots a_m LT(g_m)$ for some polynomials a_i. However, since $LT(f)$ is just the product of a nonzero constant with a monomial, it follows that there exists an i for which $LT(f) = aLT(g_i)$. The polynomial $f - ag_i$ still lies in I and has strictly fewer nonzero monomial terms. □

Proposition 3.17. *The following collection of elements is a Gröbner basis of the ideal $\mathcal{I}$:*

$$\begin{aligned} g_{F,G} &= z_F z_G && \textit{F, G are incomparable nonempty flats} \\ g_{F,G} &= z_F \left(\textstyle\sum_{G \subseteq H} z_H\right)^{\mathrm{rk}_M(G) - \mathrm{rk}_M(F)} && F \subsetneq G \textit{ and } F, G \textit{ are nonempty flats} \\ g_{\varnothing,G} &= \left(\textstyle\sum_{G \subseteq H} z_H\right)^{\mathrm{rk}_M(G)} && G \textit{ is a nonempty flat.} \end{aligned}$$

Corollary 3.18. *The element α_M^r is nonzero in* $\mathrm{CH}^r(M)$.

Proof. Since $z_E = -\alpha_M$, it suffices to show that z_E^r does not lie in $\mathcal{I}$. We apply polynomial division to z_E^r with respect to the Gröbner basis of Proposition 3.17. The leading terms of the elements of the Gröbner basis are

$$\begin{aligned} LT(g_{F,G}) &= z_F z_G && F, G \text{ are incomparable nonempty flats} \\ LT(g_{F,G}) &= z_F z_G^{\mathrm{rk}_M(G) - \mathrm{rk}_M(F)} && F \subsetneq G \text{ and } F, G \text{ are nonempty flats} \\ LT(g_{\varnothing,G}) &= z_G^{\mathrm{rk}_M(G)} && G \text{ is a nonempty flat.} \end{aligned}$$

Note that $LT(g_{\varnothing,E}) = z_E^{r+1}$ because $\mathrm{rk}_M(E) = \mathrm{rk}(M) = r + 1$. Thus z_E^r is not divisible by any of these leading terms so polynomial division terminates with $r = z_E^r$. As z_E^r is nonzero in the polynomial ring, it follows from Proposition 3.16 that $z_E^r \notin \mathcal{I}$. □

Proof of the existence and uniqueness of the degree map. The result follows from Proposition 3.8 and Corollary 3.18. □

To prove that the elements $g_{F,G}$ defined in Proposition 3.17 form a Gröbner basis, we first prove that they generate $\mathcal{I}$, and then apply *Buchberger's criterion*.

Lemma 3.19. *The elements $g_{F,G}$ generate $\mathcal{J}$.*

Proof. It is easy to see that $\mathcal{J}$ is contained in the ideal generated by the $g_{F,G}$. It is obvious for the incomparability relations, and for any $i \in E$, if we let G be the smallest flat containing i, then $g_{\varnothing,G}$ is precisely the linear relation $\sum_{i \in H} z_H$. It suffices to show that every $g_{F,G}$ lies in $\mathcal{J}$ which is equivalent to saying that $g_{F,G} \equiv 0$ in $\mathrm{CH}(M)$. The result is trivial when F, G are incomparable. Let F, G be flats for which $F \subsetneq G$. We show that $g_{F,G} \equiv 0$ in $\mathrm{CH}(M)$ by induction on $\mathrm{rk}_M(G) - \mathrm{rk}_M(F)$. Throughout, we use the convention that $z_F = 1$ if $F = \varnothing$.

If $\mathrm{rk}_M(G) = \mathrm{rk}_M(F) + 1$, then fix an element $i \in G \backslash F$ and observe that

$$g_{F,G} = z_F \left(\sum_{G \subseteq H} z_H \right) \equiv z_F \left(\sum_{i \in H} z_H \right) \equiv 0$$

where the first equivalence follows from incomparability and the second follows from the linear relations. Now let $\mathrm{rk}_M(G) - \mathrm{rk}_M(F) = d > 1$ and again fix $i \in G \backslash F$. Then we know that

$$0 \equiv z_F \left(\sum_{i \in K} z_K \right) \left(\sum_{G \subseteq H} z_H \right)^{d-1} = g_{F,G} + z_F \left(\sum_{\substack{i \in K \\ G \nsubseteq K}} z_K \right) \left(\sum_{G \subseteq H} z_H \right)^{d-1}$$

by the linear relations. We show that $z_F z_K (\sum_{G \subseteq H} z_H)^{d-1} \equiv 0$ for each K satisfying $i \in K$ and $G \nsubseteq K$. By the incomparability relation, we may assume that $F \subsetneq K$. But then

$$z_K \left(\sum_{G \subseteq H} z_H \right)^{d-1} \equiv z_K \left(\sum_{\mathrm{cl}(G \cup K) \subseteq H} z_H \right)^{d-1}$$

again by the incomparability relations where $\mathrm{cl}(G \cup K)$ is the smallest flat containing both G and K. The expression on the right is divisible by $g_{K, \mathrm{cl}(G \cup K)}$ because

$$\mathrm{rk}_M(G \cup K) - \mathrm{rk}_M(K) \leqslant \mathrm{rk}_M(G) - \mathrm{rk}_M(G \cap K) < \mathrm{rk}_M(G) - \mathrm{rk}(F) = d$$

since $G \cap K$ strictly contains F. By induction, we know that $g_{K, \mathrm{cl}(G \cup K)} \equiv 0$. □

The *syzygy* $S(f, g)$ of two polynomials f, g is defined to be

$$S(f, g) = \frac{M}{LT(f)} f - \frac{M}{LT(g)} g$$

where M is the monic least common multiple of $LT(f)$ and $LT(g)$. Note that the syzygy of any two elements of an ideal I is also in I. Suppose $g_1, \ldots, g_m$ generate the ideal I. If they form a Gröbner basis of I, then clearly long division of $S(g_i, g_j)$ by $g_1, \ldots, g_m$ results in $r = 0$. Buchberger's criterion is the converse.

Buchberger's criterion. *Suppose $g_1, \ldots, g_m$ generate an ideal I in a polynomial ring over a field equipped with a monomial ordering. If it is possible to apply long division to each syzygy $S(g_i, g_j)$ by $g_1, \ldots, g_m$ and obtain $r = 0$, then $g_1, \ldots, g_m$ is a Gröbner basis for I.*

We prove Buchberger's criterion in the Appendix. To apply Buchberger's criterion to the elements $g_{F,G}$, we apply long division to each syzygy

$$S(g_{A,B}, g_{C,D})$$

explicitly by case work. This case work is also done in the Appendix.

4 The Kähler package

In this section, we prove the Kähler package which consists of Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations. For all three results, the argument is by induction on the size of the ground set, and the inductive step is proven using the *semi-small decomposition*. Given a matroid M and an element i in the ground set E, there is a matroid $M\backslash i$ on $E\backslash i$ called the *deletion* of i from M which generalizes deletion of an edge from a graph.

Definition (Deletion)**.** Let M be a matroid and let i be an element of the ground set E. The *deletion* of i from M is the matroid denoted $M\backslash i$ on the ground set $E\backslash i$ defined by any of the following equivalent conditions:

- A set $I \subseteq E\backslash i$ is independent in $M\backslash i$ if and only if I is independent in M.
- If $S \subseteq E\backslash i$, then $\mathrm{rk}_{M\backslash i}(S) = \mathrm{rk}_M(S)$.
- A set $F \subseteq E\backslash i$ is a flat of $M\backslash i$ if and only if $F = G\backslash i$ for some flat G of M.

Note that if M is loopless, then $M\backslash i$ is loopless.

There is a graded algebra homomorphism from the Chow ring of $M\backslash i$ to the Chow ring of M denoted $\vartheta_i\colon \mathrm{CH}(M\backslash i) \to \mathrm{CH}(M)$ which is defined by

$$\vartheta_i(x_G) = x_G + x_{G\cup i}$$

where a variable on the right is set to zero if its label is not a flat of M. It is easy to verify that ϑ_i sends the incomparability and linear relations to zero and is therefore well-defined. The algebra map ϑ_i allows us to view $\mathrm{CH}(M)$ as a module over $\mathrm{CH}(M\backslash i)$. It turns out that $\mathrm{CH}(M)$ decomposes as a direct sum of indecomposable $\mathrm{CH}(M\backslash i)$-modules. The *semi-small decomposition* of $\mathrm{CH}(M)$ refers to this decomposition, and further characterizes the direct summands.

Let $\mathcal{S}_i$ be the collection of nonempty proper subsets F of $E\backslash i$ for which F and $F \cup i$ are both flats of M. Let $\mathrm{CH}_{(i)} \subseteq \mathrm{CH}(M)$ denote the image of ϑ_i. Assume that $|E| \geqslant 2$.

The semi-small decomposition. *Let M be a loopless matroid, and let $i \in E$. If i is not a coloop, then there is a direct sum decomposition*

$$\mathrm{CH}(M) = \mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\,\mathrm{CH}_{(i)}$$

into indecomposable graded $\mathrm{CH}(M\backslash i)$*-modules, where all pairs of distinct summands are orthogonal under the Poincaré pairing of* $\mathrm{CH}(M)$*. If i is a coloop, then there is a direct sum decomposition*

$$\mathrm{CH}(M) = \mathrm{CH}_{(i)} \oplus x_{E\backslash i}\,\mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\,\mathrm{CH}_{(i)}$$

into indecomposable graded $\mathrm{CH}(M\backslash i)$*-modules, where all pairs of distinct summands except the first two are orthogonal under the Poincaré pairing of* $\mathrm{CH}(M)$*.*

Remark. The *Poincaré pairing* is the bilinear map

$$\mathrm{CH}^k(M) \times \mathrm{CH}^{r-k}(M) \to \mathbf{R} \qquad (\mu,\nu) \mapsto \deg_M(\mu\nu).$$

Poincaré duality is equivalent to the assertion that this pairing is nondegenerate. Subspaces $V \subseteq \mathrm{CH}^k(M)$ and $W \subseteq \mathrm{CH}^{r-k}(M)$ are *orthogonal* under the Poincaré pairing if $\deg_M(\mu\nu) = 0$ whenever $\mu \in V$ and $\nu \in W$.

In section 4.2, we prove the semi-small decomposition and Poincaré duality simultaneously by induction on the size of the ground set. In section 4.3, we prove the hard Lefschetz theorem and the Hodge-Riemann relations simultaneously by induction on the size of the ground set using both the semi-small decomposition and Poincaré duality. In both arguments, the inductive step assumes the corresponding results for all loopless matroids on strictly smaller ground sets. The summands appearing in the semi-small decomposition are not Chow rings of matroids on smaller sets. In order to apply the induction hypothesis on these summands, we identify them with tensor products of Chow rings of certain matroids on strict subsets of the ground set in section 4.1. The material in this section is from [BHM$^+$20, ADH20]. Step 6 of the proof of the hard Lefschetz theorem and the Hodge-Riemann relations was suggested to the author by June Huh.

4.1 The pullback and pushforward maps

In this section, we identify the summands appearing in the semi-small decomposition with tensor products of Chow rings of certain matroids on strict subsets of the ground set. The identification will be as $\mathrm{CH}(M\backslash i)$-modules and in each case there will be some version of compatibility with the degree maps. The following proposition provides an example of compatibility with the degree maps.

Proposition 4.1. *Let M be a loopless matroid, let $i \in E$, and let $\vartheta_i \colon \mathrm{CH}(M\backslash i) \to \mathrm{CH}(M)$ be the graded algebra map associated with the deletion. If i is not a coloop, then*

$$\deg_{M\backslash i} = \deg_M \circ \vartheta_i.$$

If i is a coloop, then

$$\deg_{M\backslash i} = \deg_M \circ x_{E\backslash i} \circ \vartheta_i = \deg_M \circ \alpha_M \circ \vartheta_i$$

where the middle maps in the composites denote multiplication.

Proof. Assume that i is not a coloop. Then

$$\vartheta_i(\alpha_{M\backslash i}) = \sum_{j \in G\backslash i} x_{G\backslash i} + x_{G\cup i}$$

where the sum is over proper flats of $M\backslash i$ that contain a fixed element $j \in M\backslash i$. It is easy to verify that this sum is just $\sum_{j\in H} x_H = \alpha_M$ using the fact that $E\backslash i$ is not a flat of M. It follows that $\vartheta_i(\alpha^r_{M\backslash i}) = \alpha^r_M$ because ϑ_i is an algebra map. Both $M\backslash i$ and M are of rank $r+1$ because i is not a coloop so the formula follows from the definition of the degree map.

Now assume that i is a coloop. A similar computation as in the previous case shows that $\vartheta_i(\alpha_{M\backslash i}) = \alpha_M - x_{E\backslash i}$. Thus

$$\alpha_M \vartheta_i(\alpha^{r-1}_{M\backslash i}) = \alpha_M(\alpha_M - x_{E\backslash i})^{r-1} = \alpha^r_M$$

where we use the identity $x_{E\backslash i}\alpha_M = x_{E\backslash i}\sum_{i\in G} x_G = 0$. Thus $\deg_{M\backslash i} = \deg_M \circ \alpha_M \circ \vartheta_i$. Next observe that

$$\deg_M \circ \alpha_M \circ \vartheta_i = \deg_M \circ \vartheta_i \circ \alpha_{M\backslash i} + \deg_M \circ x_{E\backslash i} \circ \vartheta_i = \deg_M \circ x_{E\backslash i} \circ \vartheta_i$$

using the fact that $\alpha_M = \vartheta_i(\alpha_{M\backslash i}) + x_{E\backslash i}$ and that $\mathrm{CH}^r(M\backslash i) = 0$. □

The degree formula of Proposition 4.1 has the following consequence. Suppose $\mathrm{CH}(M\backslash i)$ satisfies Poincaré duality. If i is not a coloop, then the map

$$\vartheta_i\colon\ \mathrm{CH}(M\backslash i) \to \mathrm{CH}_{(i)}$$

is an isomorphism of $\mathrm{CH}(M\backslash i)$-modules. Indeed, if $\mu \in \mathrm{CH}(M\backslash i)$ is nonzero, there exists an element $\nu \in \mathrm{CH}(M\backslash i)$ for which $\deg_{M\backslash i}(\mu\nu) \neq 0$ by Poincaré duality. Then by the degree formula $\deg_M(\vartheta_i(\mu)\vartheta_i(\nu)) \neq 0$ so in particular $\vartheta_i(\mu) \neq 0$. Thus ϑ_i is injective and is therefore an isomorphism onto its image. By the same reasoning, if i is a coloop, then the maps

$$\vartheta_i\colon\ \mathrm{CH}(M\backslash i) \to \mathrm{CH}_{(i)} \qquad x_{E\backslash i} \circ \vartheta_i\colon\ \mathrm{CH}(M\backslash i) \to x_{E\backslash i}\,\mathrm{CH}_{(i)}$$

are isomorphisms of $\mathrm{CH}(M\backslash i)$-modules.

To provide similar descriptions for the other summands $x_{F\cup i}\,\mathrm{CH}_{(i)}$ in the semi-small decomposition, we first define the relevant auxiliary matroids, and then we then define *pullback* and *pushforward* maps which relate the Chow ring of M to those of the auxiliary matroids.

Definition (Localization and contraction)**.** Let M be a matroid on the ground set E, and let F be a nonempty proper flat of M. The *localization* of M at F is the matroid denoted M^F on the ground set F defined by any of the following equivalent conditions:

- A subset $I \subseteq F$ is independent in M^F if and only if I is independent in M.
- If $S \subseteq F$, then $\mathrm{rk}_{M^F}(S) = \mathrm{rk}_M(S)$.
- A subset $G \subseteq F$ is a flat of M^F if and only if G is a flat of M.

Note that if $i \in E$ is a coloop, then $E\backslash i$ is a flat and $M^{E\backslash i} = M\backslash i$. The *contraction* of M by F is the matroid denoted M_F on the ground set $E\backslash F$ defined by any of the following equivalent conditions:

- A subset $I \subseteq E\backslash F$ is independent in M_F if and only if for every maximal independent subset I_F of F, the set $I \cup I_F$ is independent in M.
- If $S \subseteq E\backslash F$, then $\mathrm{rk}_{M_F}(S) = \mathrm{rk}_M(S \cup F) - \mathrm{rk}_M(F)$.
- A subset $G \subseteq E\backslash F$ is a flat of M_F if and only if $G \cup F$ is a flat of M.

Note that $\mathscr{L}_{M^F}$ is just the lattice of flats of M that are contained in F while $\mathscr{L}_{M_F}$ is the lattice of flats of M that contain F. If M is loopless, then M^F and M_F are loopless.

Lemma 4.2 (Pullback map)**.** *Let M be a loopless matroid, and let F be a nonempty proper flat of M. There is a unique graded algebra homomorphism*

$$\varphi^F_M\colon\ \mathrm{CH}(M) \to \mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F)$$

called the pullback map *for which*

$$\varphi_M^F(x_G) = \begin{cases} 0 & F \text{ and } G \text{ are incomparable} \\ 1 \otimes x_G & G \subsetneq F \\ x_{G \setminus F} \otimes 1 & F \subsetneq G \end{cases}$$

The map is surjective and additionally satisfies

$$\begin{aligned} \varphi_M^F(x_F) &= -(1 \otimes \alpha_{M^F} + \beta_{M_F} \otimes 1) \\ \varphi_M^F(\alpha_M) &= \alpha_{M_F} \otimes 1 \\ \varphi_M^F(\beta_M) &= 1 \otimes \beta_{M_F} \end{aligned}$$

Proof. For uniqueness, it suffices to show that φ_M^F must send x_F to $-(1 \otimes \alpha_{M^F} + \beta_{M_F} \otimes 1)$ if it has the described behavior on x_G for $G \neq F$. First, observe that for $j \notin F$

$$\varphi_M^F(\alpha_M) = \sum_{j \in G} \varphi_M^F(x_G) = \sum_{F \cup j \subseteq G} x_{G \setminus F} \otimes 1 = \alpha_{M_F} \otimes 1$$

whereas for $i \in F$

$$\begin{aligned} \varphi_M^F(\alpha_M) &= \sum_{i \in G \subsetneq F} 1 \otimes x_G + \varphi_M^F(x_F) + \sum_{F \subsetneq G} x_{G \setminus F} \otimes 1 \\ &= 1 \otimes \alpha_{M^F} + \varphi_M^F(x_F) + (\alpha_{M_F} + \beta_{M_F}) \otimes 1 \end{aligned}$$

Thus $\varphi_M^F(x_F) = -(1 \otimes \alpha_{M^F} + \beta_{M_F} \otimes 1)$ as required. The computation that $\varphi_M^F(\beta_M) = 1 \otimes \beta_{M_F}$ is straightforward.

To see existence, we must check that if φ_M^F has the described behavior on each x_G for G a nonempty proper flat of M, then it respects the incomparability relation and that $\varphi_M^F(\sum_{i \in G} x_G)$ is independent of i. The former is straightforward to verify and the latter follows from our computations above. Surjectivity is easy to verify as well. □

Lemma 4.3 (Pushforward map)**.** *Let M be a loopless matroid, and let F be a nonempty proper flat of M. There is a unique* $\mathrm{CH}(M)$*-module homomorphism*

$$\psi_M^F \colon \mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F) \to \mathrm{CH}(M)$$

called the pushforward map *for which $\psi_M^F(1) = x_F$. The tensor product* $\mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F)$ *is viewed as a* $\mathrm{CH}(M)$*-module via the pullback map φ_M^F. The pushforward map satisfies the identity*

$$\deg_{M_F} \otimes \deg_{M^F} = \deg_M \circ \psi_M^F.$$

Proof. Uniqueness follows from the fact that the pullback φ_M^F is surjective. Existence is also straightforward and involves checking the incompatibility and linear relations for M_F and M^F. To verify the degree formula, we simply observe that if $F_1 \subsetneq \cdots \subsetneq F_r$ is a maximal flag of M for which $F_i = F$ for some index i, then ψ_M^F sends

$$(x_{F_{i+1} \setminus G} \cdots x_{F_r \setminus G}) \otimes (x_{F_1} \cdots x_{F_{i-1}}) \mapsto x_{F_1} \cdots x_{F_r}. \qquad \square$$

Remark. The composite $\psi^F_M \circ \varphi^F_M$ is just multiplication by x_F, while the composite $\varphi^F_M \circ \psi^F_M$ is multiplication by $\varphi^F_M(x_F)$.

Using the pullback and pushforward maps, we identify the summand $x_{F\cup i}\,\mathrm{CH}_{(i)}$ in the semi-small decomposition as a $\mathrm{CH}(M\backslash i)$-module and provide a degree formula.

Lemma 4.4. *Let M be a loopless matroid, let $i \in E$, and let $F \in \mathcal{S}_i$. Note that i is a coloop of $M^{F\cup i}$ and let*

$$\vartheta_i^{F\cup i}\colon\ \mathrm{CH}(M^F) \to \mathrm{CH}(M^{F\cup i})$$

be the graded algebra map associated with the deletion of i from $M^{F\cup i}$. There is a surjective algebra map q which makes the following diagram commute

$$\begin{tikzcd}
\mathrm{CH}(M\backslash i) \arrow[r, "\vartheta_i"] \arrow[d, two heads, "\varphi^F_{M\backslash i}"] & \mathrm{CH}(M) \arrow[dd, two heads, "\varphi^{F\cup i}_M"] \\
\mathrm{CH}((M\backslash i)_F) \otimes \mathrm{CH}((M\backslash i)^F) \arrow[d, two heads, "q"] & \\
\mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F) \arrow[r, "\mathrm{Id}\otimes\vartheta_i^{F\cup i}"] & \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^{F\cup i})
\end{tikzcd}$$

Proof. Note that $(M\backslash i)^F$ and M^F are matroids on F with

$$\begin{aligned}
\text{flats of } (M\backslash i)^F &= \{\, G\backslash i \mid G\backslash i \subseteq F \text{ and } G \text{ is a flat of } M \,\} \\
\text{flats of } M^F &= \{\, G \mid G \subseteq F \text{ and } G \text{ is a flat of } M \,\}
\end{aligned}$$

These two sets are the same so $(M\backslash i)^F = M^F$. Next, observe that $(M\backslash i)_F$ and $M_{F\cup i}$ are both matroids on $M\backslash(F\cup i)$ and that

$$\begin{aligned}
\text{flats of } (M\backslash i)_F &= \{\, G\backslash(F\cup i) \mid G \text{ is a flat of } M \text{ containing } F \,\} \\
\text{flats of } M_{F\cup i} &= \{\, G\backslash(F\cup i) \mid G \text{ is a flat of } M \text{ containing } F\cup i \,\}
\end{aligned}$$

Thus every flat of $M_{F\cup i}$ is a flat of $(M\backslash i)_F$ so there is a surjective algebra map

$$\mathrm{CH}((M\backslash i)_F) \to \mathrm{CH}(M_{F\cup i}) \qquad x_H \mapsto \begin{cases} x_H & H \text{ is a flat of } M_{F\cup i} \\ 0 & H \text{ is not a flat of } M_{F\cup i}. \end{cases}$$

The algebra map q is defined to be the tensor product of this surjective map with the identity $\mathrm{CH}((M\backslash i)^F) \to \mathrm{CH}(M^F)$. To see that the diagram is commutative, it is straightforward to verify that $(\mathrm{Id}\otimes\vartheta_i^{F\cup i}) \circ q \circ \varphi^F_{M\backslash i}$ and $\varphi^{F\cup i}_M \circ \vartheta_i$ are both given by

$$x_{G\backslash i} \mapsto \begin{cases} x_{G\backslash(F\cup i)} \otimes 1 & F\cup i \subsetneq G\cup i \text{ and } G\cup i \text{ is a flat of } M \\ 1 \otimes (x_{G\backslash i} + x_{G\cup i}) & G\backslash i \subsetneq F \\ 1\otimes x_F - 1\otimes \alpha_{M^{F\cup i}} - \beta_{M_{F\cup i}} \otimes 1 & F = G\backslash i \\ 0 & \text{otherwise} \end{cases}$$

for $G\backslash i$ a nonempty proper flat of $M\backslash i$. □

Proposition 4.5. *Let M be a loopless matroid, let $i \in E$, and let $F \in \mathscr{S}_i$. Then there is a surjective* $\mathrm{CH}(M\backslash i)$*-module map*

$$\Psi_i^F \colon \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F) \to x_{F\cup i}\,\mathrm{CH}_{(i)}$$

which increases grading by one with the property that for any $\mu, \nu \in \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$

$$\deg_M(\Psi_i^F(\mu)\Psi_i^F(\nu)) = -(\deg_{M_{F\cup i}} \otimes \deg_{M^F})(\mu\nu).$$

The map $q \circ \varphi^F_{M\backslash i} \colon \mathrm{CH}(M\backslash i) \to \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$ of Lemma 4.4 defines the module structure on the tensor product.

Remark. If $\mathrm{CH}(M_{F\cup i})$ and $\mathrm{CH}(M^F)$ satisfy Poincaré duality, then Ψ_i^F is an isomorphism of $\mathrm{CH}(M\backslash i)$-modules. Indeed, if $\mu \in \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$ is nonzero, then by Poincaré duality for the two Chow rings, it follows that there exists a $\nu \in \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$ for which $(\deg_{M_{F\cup i}} \otimes \deg_{M^F})(\mu\nu) \neq 0$. The degree formula then implies that $\Psi_i^F(\mu) \neq 0$.

Proof. Let Ψ_i^F be the composite $\psi_M^{F\cup i} \circ (\mathrm{Id} \otimes \vartheta_i^{F\cup i})$. The fact that Ψ_i^F is a surjective $\mathrm{CH}(M\backslash i)$-module map follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}(M) \arrow[rr, "x_{F\cup i}"] \arrow[dr, "\varphi_M^{F\cup i}"] & & \mathrm{CH}(M) \\
& \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^{F\cup i}) \arrow[ur, "\psi_M^{F\cup i}"] &
\end{tikzcd}$$

and Lemma 4.4. It suffices to prove the degree formula.

For ease of notation, we temporarily let $\psi = \psi_M^{F\cup i}$, $\vartheta = \vartheta_i^{F\cup i}$, and $\varphi = \varphi_M^{F\cup i}$. Observe that

$$\deg_M(\Psi_i^F(\mu)\Psi_i^F(\nu)) = \deg_M(\psi((\mathrm{Id} \otimes \vartheta)\mu) \cdot \psi((\mathrm{Id} \otimes \vartheta)\nu)).$$

Because ψ is a $\mathrm{CH}(M)$-module map, the expression equals

$$\deg_M \psi \left(\varphi\psi((\mathrm{Id} \otimes \vartheta)\mu) \cdot (\mathrm{Id} \otimes \vartheta)\nu\right).$$

Since $\deg_M \circ \psi = \deg_{M_{F\cup i}} \otimes \deg_{M^{F\cup i}}$ and $\varphi \circ \psi$ is multiplication by $\varphi(x_{F\cup i})$, we obtain

$$-(\deg_{M_{F\cup i}} \otimes \deg_{M^{F\cup i}})((1 \otimes \alpha_{M^{F\cup i}} + \beta_{M_{F\cup i}} \otimes 1) \cdot (\mathrm{Id} \otimes \vartheta)(\mu\nu))$$

Note that $(\beta_{M_{F\cup i}} \otimes 1) \cdot (\mathrm{Id} \otimes \vartheta_i)\mu \cdot (\mathrm{Id} \otimes \vartheta)\nu$ lies in $\mathrm{CH}(M_{F\cup i}) \otimes \vartheta(\mathrm{CH}(M^F))$. The rank of M^F is less than the rank of $M^{F\cup i}$ so $\deg_{M^{F\cup i}}$ vanishes on $\vartheta(\mathrm{CH}(M^F))$. Thus, our expression is equal to

$$\begin{aligned}
&- (\deg_{M_{F\cup i}} \otimes \deg_{M^{F\cup i}})((1 \otimes \alpha_{M^{F\cup i}}) \cdot (\mathrm{Id} \otimes \vartheta)(\mu\nu)) \\
= &- (\deg_{M_{F\cup i}} \otimes (\deg_{M^{F\cup i}} \circ \alpha_{M^{F\cup i}} \circ \vartheta))(\mu\nu).
\end{aligned}$$

The result now follows from the formula $\deg_{M^F} = \deg_{M^{F\cup i}} \circ \alpha_{M^{F\cup i}} \circ \vartheta$ of Proposition 4.1. $\square$

4.2 The semi-small decomposition and Poincaré duality

We prove the semi-small decomposition and Poincaré duality for M when $|E| \geqslant 2$ from the assumption that Poincaré duality holds for all loopless matroids on nonempty proper subsets of E. The base case of our induction is the trivial fact that Poincaré duality holds when $|E| = 1$.

Proof of the semi-small decomposition and Poincaré duality. Let M be a loopless matroid with $|E| \geqslant 2$, and assume that Poincaré duality holds for all matroids on nonempty proper subsets of E, and fix an element $i \in E$. We prove the result in four steps.

Step 1: *The subspace $x_{F \cup i} \mathrm{CH}_{(i)}$ is zero in grading r for each $F \in \mathcal{S}_i$.* The surjective map

$$\Psi_i^F \colon \mathrm{CH}(M_{F \cup i}) \otimes \mathrm{CH}(M^F) \to x_{F \cup i} \mathrm{CH}_{(i)}$$

of Proposition 4.5 increases grading by one, so it suffices to show that $\mathrm{CH}(M_{F \cup i}) \otimes \mathrm{CH}(M^F)$ is zero in grading $r - 1$. Note that $\mathrm{rk}(M_{F \cup i}) = r - \mathrm{rk}_M(F)$ and $\mathrm{rk}(M^F) = \mathrm{rk}_M(F)$, so the top grading in which $\mathrm{CH}(M_{F \cup i}) \otimes \mathrm{CH}(M^F)$ is nonzero is $(r - \mathrm{rk}_M(F) - 1) + (\mathrm{rk}_M(F) - 1) = r - 2$.

Step 2: *Nondegeneracy of the Poincaré pairing on the subspaces.*

- The Poincaré pairing is nondegenerate on each $x_{F \cup i} \mathrm{CH}_{(i)}$. By surjectivity of Ψ_i^F, an arbitrary nonzero element of $x_{F \cup i} \mathrm{CH}_{(i)}$ is of the form $\Psi_i^F(\mu)$ for some nonzero element $\mu \in \mathrm{CH}(M_{F \cup i}) \otimes \mathrm{CH}(M^F)$. Poincaré duality holds for both $\mathrm{CH}(M_{F \cup i})$ and $\mathrm{CH}(M^F)$, so there is an element $\nu \in \mathrm{CH}(M_{F \cup i}) \otimes \mathrm{CH}(M^F)$ for which $(\deg_{M_{F \cup i}} \otimes \deg_{M^F})(\mu\nu) \neq 0$. It follows that
$$\deg_M(\Psi_i^F(\mu)\Psi_i^F(\nu)) = -(\deg_{M_{F \cup i}} \otimes \deg_{M^F})(\mu\nu) \neq 0$$
as required.
- If i is not a coloop, then the Poincaré pairing is nondegenerate on $\mathrm{CH}_{(i)}$. This result follows from the formula $\deg_M \circ \vartheta_i = \deg_{M \setminus i}$ of Proposition 4.1. Indeed, if $\vartheta_i(\mu) \in \mathrm{CH}_{(i)}$ is nonzero, by Poincaré duality for $M \setminus i$ there is an element $\nu \in \mathrm{CH}(M \setminus i)$ for which
$$0 \neq \deg_{M \setminus i}(\mu\nu) = \deg_M(\vartheta_i(\mu\nu)) = \deg_M(\vartheta_i(\mu)\vartheta_i(\nu)).$$
- If i is a coloop, then the Poincaré pairing is nondegenerate on $\mathrm{CH}_{(i)} + x_{E \setminus i} \mathrm{CH}_{(i)}$. We have the formula $\deg_{M \setminus i} = \deg_M \circ x_{E \setminus i} \circ \vartheta_i$ from Proposition 4.1. Given a nonzero element $\vartheta_i(\mu) \in \mathrm{CH}_{(i)}$, there exists $\nu \in \mathrm{CH}(M \setminus i)$ for which $\deg_{M \setminus i}(\mu\nu) \neq 0$ by Poincaré duality for $M \setminus i$. The element $x_{E \setminus i}\vartheta_i(\nu) \in x_{E \setminus i} \mathrm{CH}_{(i)}$ satisfies
$$\deg_M(x_{E \setminus i}\vartheta_i(\nu)\vartheta_i(\mu)) = \deg_{M \setminus i}(\nu\mu) \neq 0.$$

  The same argument shows that for any nonzero element $x_{E \setminus i}\vartheta_i(\mu) \in x_{E \setminus i} \mathrm{CH}_{(i)}$, there exists $\vartheta_i(\nu) \in \mathrm{CH}_{(i)}$ for which $\deg_M(x_{E \setminus i}\vartheta_i(\mu)\vartheta_i(\nu)) \neq 0$.

Step 3: *Orthogonality of the subspaces under the pairing and trivial pairwise intersection.* We first show that the relevant pairs are orthogonal with respect to the Poincaré pairing. It follows that each such pair intersects trivially by Step 2.

- Assume F, G are distinct elements of $\mathcal{S}_i$. If F, G are incomparable, then $x_{F\cup i}\operatorname{CH}_{(i)}$ and $x_{G\cup i}\operatorname{CH}_{(i)}$ are clearly orthogonal. Assume that $F \subsetneq G$ and note that
$$x_{F\cup i}x_{G\cup i} = x_{F\cup i}(x_G + x_{G\cup i}) = x_{F\cup i}\vartheta_i(x_G)$$
so that $x_{F\cup i}\operatorname{CH}_{(i)} \cdot x_{G\cup i}\operatorname{CH}_{(i)} \subseteq x_{F\cup i}\operatorname{CH}_{(i)}$. Since $x_{F\cup i}\operatorname{CH}_{(i)}$ is zero in degree r, the subspaces are orthogonal.
- If i is not a coloop, then $\operatorname{CH}_{(i)} \cdot x_{F\cup i}\operatorname{CH}_{(i)} \subseteq x_{F\cup i}\operatorname{CH}_{(i)}$ which is zero in degree r so $\operatorname{CH}_{(i)}$ is orthogonal to each $x_{F\cup i}\operatorname{CH}_{(i)}$.
- If i is a coloop, then $\operatorname{CH}_{(i)} \cdot x_{F\cup i}\operatorname{CH}_{(i)} \subseteq x_{F\cup i}\operatorname{CH}_{(i)}$ which is zero in degree r, and $x_{E\setminus i}x_{F\cup i} = 0$ because $E\setminus i$ and $F \cup i$ are incomparable.

It remains to show that $\operatorname{CH}_{(i)}$ and $x_{E\setminus i}\operatorname{CH}_{(i)}$ intersect trivially when i is a coloop. If μ is a nontrivial element in their intersection, then there exists $\nu \in \operatorname{CH}_{(i)}$ for which $\deg_M(\mu\nu) \neq 0$ by the third part of Step 5. However, $\mu\nu$ is an element of $\operatorname{CH}_{(i)}$ which is zero in grading r. Thus $\operatorname{CH}_{(i)} \cap x_{E\setminus i}\operatorname{CH}_{(i)} = 0$.

Step 4: *The direct sum spans* $\operatorname{CH}(M)$. The result is clear in grading 0. To show the result in grading 1, it suffices to show that x_G lies in the direct sum for each nonempty proper flat G. If $G\setminus i$ is not a flat, then $x_G = \vartheta_i(x_{G\setminus i}) \in \operatorname{CH}_{(i)}$. If $G\setminus i$ and G are distinct nonempty flats, then $x_{G\setminus i} \in \mathcal{S}_i$ and $x_G \in x_{(G\setminus i)\cup i}\operatorname{CH}_{(i)}$. If $G\setminus i = G$, then $x_G = \vartheta_i(x_G) - x_{G\cup i}$ lies in $\operatorname{CH}_{(i)}$ or $\operatorname{CH}_{(i)} \oplus x_{G\cup i}\operatorname{CH}_{(i)}$ depending on whether $G \cup i$ is a flat. The last case is $G = \{i\}$, which is handled by the observation that x_G can be written as a linear combination of the variables x_H for $H \neq G$.

Note that the direct sum of the semi-small decomposition is just the $\operatorname{CH}(M\setminus i)$-submodule of $\operatorname{CH}(M)$ generated by $\operatorname{CH}^1(M)$. To show that the direct sum is all of $\operatorname{CH}(M)$, it suffices to prove that
$$\operatorname{CH}^1_{(i)} \cdot \operatorname{CH}^k(M) = \operatorname{CH}^{k+1}(M) \quad \text{for each } k \geqslant 1.$$
In fact, it suffices to prove the case $k = 1$. Indeed
$$\operatorname{CH}^1_{(i)} \cdot \operatorname{CH}^k(M) = \operatorname{CH}^1_{(i)} \cdot \operatorname{CH}^1 \cdot \operatorname{CH}^{k-1}(M) = \operatorname{CH}^2(M) \cdot \operatorname{CH}^{k-1}(M) = \operatorname{CH}^{k+1}(M)$$
for each $k \geqslant 1$.

Assume that i is not a coloop. Since
$$\operatorname{CH}^2(M) = \operatorname{CH}^1(M) \cdot \operatorname{CH}^1(M) = \left(\operatorname{CH}^1_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\operatorname{CH}^0_{(i)}\right) \cdot \operatorname{CH}^1(M)$$
it suffices to show that $x_{F\cup i} \cdot \operatorname{CH}^1(M) \subseteq \operatorname{CH}^1_{(i)} \cdot \operatorname{CH}^1(M)$ for each $F \in \mathcal{S}_i$. Next, since
$$x_{F\cup i} \cdot \operatorname{CH}^1(M) = x_{F\cup i}\left(\operatorname{CH}^1_{(i)} \oplus \bigoplus_{G\in\mathcal{S}_i} x_{G\cup i}\operatorname{CH}^0_{(i)}\right)$$
it suffices to show that $x_{F\cup i}x_{G\cup i}$ is contained in $\operatorname{CH}^1_{(i)} \cdot \operatorname{CH}^1(M)$ for each $G \in \mathcal{S}_i$. If F and G are distinct, then
$$x_{F\cup i}x_{G\cup i} = \begin{cases} 0 & F \text{ and } G \text{ are incomparable} \\ x_{F\cup i}\vartheta_i(x_G) & F \subsetneq G \\ \vartheta_i(x_F)x_{G\cup i} & G \subsetneq F \end{cases}$$

which lies in $\mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(M)$. What remains is to show that $x_{F\cup i}^2 \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(M)$.

We first observe that

$$x_{F\cup i}\alpha_M = x_{F\cup i}\sum_{i\in G} x_G = x_{F\cup i}\left(\sum_{i\in G\subsetneq F\cup i} x_G\right) + x_{F\cup i}^2 + x_{F\cup i}\left(\sum_{F\cup i\subsetneq G} x_G\right)$$

while at the same time for any $j \notin F \cup i$

$$x_{F\cup i}\alpha_M = x_{F\cup i}\sum_{j\in G} x_G = x_{F\cup i}\sum_{F\cup\{i,j\}\subseteq G} x_G.$$

Thus

$$-x_{F\cup i}^2 = x_{F\cup i}\left(\sum_{i\in G\subsetneq F\cup i} x_G + \sum_{F\cup i\subsetneq G} x_G - \sum_{F\cup\{i,j\}\subseteq G} x_G\right)$$

If $i \in G \subsetneq F \cup i$, then $x_{F\cup i}x_G = \vartheta_i(x_F)x_G$. Similarly, if $F \cup i \subsetneq G$, then $x_{F\cup i}x_G = x_{F\cup i}\vartheta_i(x_{G\setminus i})$ so $x_{F\cup i}^2 \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(M)$ as required.

If i is a coloop, then

$$\mathrm{CH}^2(M) = \left(\mathrm{CH}^1_{(i)} \oplus x_{E\setminus i}\,\mathrm{CH}^0_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\,\mathrm{CH}^0_{(i)}\right) \cdot \mathrm{CH}^1(M).$$

The previous arguments reduce the problem to showing that $x_{E\setminus i}^2 \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(M)$. Using a similar observation as before, we have

$$0 = x_{E\setminus i}\sum_{i\in G} x_G = x_{E\setminus i}\alpha_M = x_{E\setminus i}\sum_{j\in G} x_G = x_{E\setminus i}^2 + x_{E\setminus i}\sum_{j\in G\subsetneq E\setminus i} x_G$$

for some $j \neq i$. Since $x_{E\setminus i}x_G = x_{E\setminus i}\vartheta_i(x_G)$, it follows that $x_{E\setminus i}^2 \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(M)$ as required. □

4.3 The hard Lefschetz theorem and the Hodge-Riemann relations

Remark. Let M be a loopless matroid. If ℓ is an arbitrary element of $\mathrm{CH}^1(M)$, we say that $\mathrm{CH}(M)$ *satisfies the Hodge-Riemann relations for* ℓ if the conclusion of the Hodge-Riemann relations hold for ℓ, which is to say that the *Hodge-Riemann form* associated to ℓ

$$\mathrm{CH}^k(M) \times \mathrm{CH}^k(M) \to \mathbf{R} \qquad (\mu, \nu) \mapsto (-1)^k \deg_M(\ell^{r-2k}\mu\nu)$$

is positive-definite on $\ker(\ell^{r-2k+1})$ for each $k \leqslant r/2$.

Since M satisfies Poincaré duality, the Hodge-Riemann relations for $\ell \in \mathrm{CH}^1(M)$ imply the hard Lefschetz theorem for ℓ. If $\eta \in \mathrm{CH}^k(M)$ satisfies $\ell^{r-2k}\eta = 0$, then η lies in the kernel of ℓ^{r-2k+1}. But $(-1)^k \deg_M(\ell^{r-2k}\eta\eta) = (-1)^k \deg_M(0) = 0$, and since the Hodge-Riemann form is positive-definite on the kernel of ℓ^{r-2k+1}, it follows that $\eta = 0$. By Poincaré duality, the injective map $\ell^{r-2k}\colon\ \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k}(M)$ is an isomorphism.

We also note that the hard Lefschetz theorem for ℓ is equivalent to the nondegeneracy of the Hodge-Riemann form of ℓ, which is easily seen by induction.

Proposition 4.6. *If M and N are loopless matroids for which* $\mathrm{CH}(M)$ *and* $\mathrm{CH}(N)$ *satisfy the Hodge-Riemann relations for $\ell \in \mathrm{CH}^1(M)$ and $h \in \mathrm{CH}^1(N)$, respectively, then* $\mathrm{CH}(M) \otimes \mathrm{CH}(N)$ *satisfies the Hodge-Riemann relations for $\ell \otimes 1 + 1 \otimes h$.*

Proof. Let $\mathrm{rk}(M) = r + 1$ and $\mathrm{rk}(N) = s + 1$, and set

$$P^k := \ker\left(\ell^{r-2k+1} \colon \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k+1}(M)\right)$$
$$R^j := \ker\left(h^{s-2j+1} \colon \mathrm{CH}^j(N) \to \mathrm{CH}^{s-j+1}(N)\right)$$

for $k \leqslant r/2$ and $j \leqslant s/2$. The Hodge-Riemann relations yield the direct sum decompositions

$$\mathrm{CH}^k(M) = \ell^k P^0 \oplus \ell^{k-1} P^1 \oplus \cdots \oplus \ell P^{k-1} \oplus P^k$$
$$\mathrm{CH}^j(N) = h^j R^0 \oplus h^{j-1} R^1 \oplus \cdots \oplus h R^{j-1} \oplus R^j$$

which are orthogonal with respect to their Hodge-Riemann forms.

To prove that $\mathrm{CH}(M) \otimes \mathrm{CH}(N)$ satisfies the Hodge-Riemann relations for $\ell \otimes 1 + 1 \otimes h$ in each degree $k \leqslant (r+s)/2$, first note that the k-graded part of $\mathrm{CH}(M) \otimes \mathrm{CH}(N)$ decomposes as a direct sum

$$\bigoplus_{a+i+b+j=k} \ell^a P^i \otimes h^b R^j$$

where a, i, b, j are nonnegative integers that sum to k. For each pair of nonnegative integers i, j for which $i \leqslant r/2$, $j \leqslant s/2$, and $i + j \leqslant k$, let

$$Q_k^{ij} = \bigoplus_{a+b=k-(i+j)} \ell^a P^i \otimes h^b R^j.$$

It is straightforward to verify that Q_k^{ij} and $Q_k^{i'j'}$ are orthogonal with respect to the Hodge-Riemann form of $\ell \otimes 1 + 1 \otimes h$ when $(i, j) \neq (i', j')$. It therefore suffices to prove the result on each such summand.

Fix nonnegative integers i, j for which $i \leqslant r/2$ and $j \leqslant s/2$. We prove the Hodge-Riemann relations on

$$\bigoplus_{k=i+j}^{r+s-(i+j)} Q_k^{ij}$$

which is sufficient because multiplication by $\ell \otimes 1 + 1 \otimes h$ preserves i, j. By choosing a basis for $P^i \otimes R^j$ and using the Hodge-Riemann relations for ℓ and h, the argument reduces to showing that $\mathbf{R}[\ell, h]/(\ell^{c+1}, h^{d+1})$ satisfies the Hodge-Riemann relations for $\ell + h$ where $c = r - 2i$ and $d = s - 2j$. The result then follows from the usual Hodge-Riemann relations for the compact Kähler manifold $\mathbf{CP}^c \times \mathbf{CP}^d$ or by a more direct combinatorial argument using the *Lindström-Gessel-Viennot* lemma, explained in the proof of [AHK18, Lemma 7.8]. We only briefly outline the argument.[2]

Without loss of generality, we may assume that $c \leqslant d$. It suffices to consider gradings $k \leqslant c$, since the map $(\ell + h)^{r-2k+1}$ is injective when $c < k \leqslant (c+d)/2$. For $k \leqslant c$, a basis for the

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[2] The author may return to this later to provide the details.

k-graded part of $\mathbf{R}[\ell, h]/(\ell^{c+1}, h^{d+1})$ is $\ell^k, \ell^{k-1}h, \dots, h^k$. The entries of the matrix representing the Hodge-Riemann form with respect to this basis are binomial coefficients

$$\deg((\ell + h)^{c+d-2k}\ell^{i+j}h^{k-i+k-j}) = \binom{c+d-2k}{c-i-j}$$

A combinatorial argument using the Lindström-Gessel-Viennot lemma computes the sign of the determinant of the Hodge-Riemann form from this observation. The result then follows using the fact that the kernel of $(\ell + h)^{r-2k+1}$ is one-dimensional for $k \leqslant c$. □

Lemma 4.7. *Let M be a loopless matroid, and suppose M satisfies the hard Lefschetz theorem with respect to $\ell \in \mathrm{CH}^1(M)$. Then M satisfies the Hodge-Riemann relations with respect to ℓ if and only if the signature of its Hodge-Riemann form on $\mathrm{CH}^k(M)$ is*

$$\sum_{j=0}^{k}(-1)^{k-j}(\dim \mathrm{CH}^j(M) - \dim \mathrm{CH}^{j-1}(M))$$

for each $k \leqslant r/2$.

Proof. Since M satisfies the hard Lefschetz theorem with respect to ℓ, we have the splitting

$$\mathrm{CH}^k(M) = \ker(\ell^{r-2k+1}) \oplus \ell(\mathrm{CH}^{k-1}(M))$$

which is orthogonal with respect to the Hodge-Riemann form of ℓ. If the Hodge-Riemann relations are valid for ℓ, then its Hodge-Riemann form has the stated signature by induction on k using this decomposition. The converse is similarly verified. □

Proof of the hard Lefschetz theorem and the Hodge-Riemann relations. We prove both results by simultaneous induction on the size of $|E|$. If $|E| = 1$, then $\mathrm{rk}(M) = r + 1 = 1$ and there are no ample classes. Hence both results are vacuous. We assume that $|E| \geqslant 2$. If $r = 0$, then again there are no ample classes and the results are vacuous. If $r = 1$, then $\mathrm{CH}^0(M) = \mathbf{R} = \mathrm{CH}^1(M)$. A class $\ell \in \mathrm{CH}^1(M)$ is ample if and only if $\deg_M(\ell) > 0$ so both the hard Lefschetz theorem and the Hodge-Riemann relations are easy to see. Thus, we assume that $r \geqslant 2$. By induction, we assume that both the hard Lefschetz theorem and the Hodge-Riemann relations hold for all matroids on nonempty proper subsets of E.

Our proof is in six steps which we now outline. Step 1 is to prove the hard Lefschetz theorem for M. The remaining steps are to prove the Hodge-Riemann relations for M. Step 2 reduces the Hodge-Riemann relations for all ample classes to the Hodge-Riemann relations for a single nef class. The proof of Step 2 uses the result of Step 1. For Steps 3 - 6, we fix an element $i \in E$ and use the semi-small decomposition to prove the Hodge-Riemann relations for a nef class. If i is not a coloop, then the nef class we use is $\vartheta_i(\ell)$ where $\ell \in \mathrm{CH}^1(M \backslash i)$ is ample. If i is a coloop, then the nef class we use is $\vartheta_i(\ell) + \varepsilon x_{E\backslash i}$ for a sufficiently small $\varepsilon > 0$. The proof of the Hodge-Riemann relations is done summand-by-summand in the semi-small decomposition. Step 3 handles the summands $x_{F\cup i}\mathrm{CH}_{(i)}$ for $F \in \mathcal{S}_i$. Step 4 finishes the case where i is not a coloop. Step 5 verifies that $\vartheta_i(\ell) + \varepsilon x_{E\backslash i}$ for small $\varepsilon > 0$ is indeed nef when i is a coloop. Step 6 finishes the case where i is a coloop.

Step 1: *The hard Lefschetz theorem for M.* Let $\ell \in \mathrm{CH}^1(M)$ be ample, and suppose $\eta \in \mathrm{CH}^k(M)$ satisfies $\ell^{r-2k}\eta = 0$ where $k \leqslant r/2$. We show that $x_F\eta = 0$ for each nonempty proper flat F of

M using the Hodge-Riemann relations for $\mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F)$. It follows that $\mu\eta = 0$ for all $\mu \in \mathrm{CH}^{r-k}(M)$ so $\eta = 0$ by Poincaré duality for M. Thus $\ell^{r-2k}\colon \mathrm{CH}^k(M) \to \mathrm{CH}^{r-k}(M)$ is injective and is therefore an isomorphism again by Poincaré duality for M.

Fix a nonempty proper flat F. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}(M) \arrow[rr, "x_F"] \arrow[dr, "\varphi_M^F"'] & & \mathrm{CH}(M) \\
& \mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F) \arrow[ur, "\psi_M^F"'] &
\end{tikzcd}$$

where φ_M^F and ψ_M^F are the pushforward and pullback maps defined in Lemmas 4.2 and 4.3, respectively. Let $\eta_F = \varphi_M^F(\eta)$ and $\ell_F = \varphi_M^F(\ell)$. Write $\ell = \sum_G c_G x_G$ with $c_F = 0$ and $c_G > 0$ whenever G, F are distinct and comparable. Then

$$\ell_F = \left(\sum_{F \subsetneq G} c_G x_{G \setminus F}\right) \otimes 1 + 1 \otimes \left(\sum_{G \subsetneq F} c_G x_G\right)$$

It is straightforward to verify that $\sum_{F \subsetneq G} c_G x_{G \setminus F} \in \mathrm{CH}^1(M_F)$ is ample as along as the sum is nonempty, and similarly that $\sum_{G \subsetneq F} c_G x_G \in \mathrm{CH}^1(M^F)$ is ample as long as the sum is nonempty. At least one of the sums is nonempty by assumption that $r \geqslant 2$. If one of the sums is empty, then the corresponding Chow ring is simply a copy of $\mathbf{R}$ in grading 0. Hence, in all cases, $\mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F)$ satisfies the Hodge-Riemann relations for ℓ_F by Proposition 4.6. From the identity $\deg_F \coloneqq \deg_{M_F} \otimes \deg_{M^F} = \deg_M \circ \psi_M^F$, we see that

$$\deg_F(\ell_F^{r-2k-1}\eta_F\eta_F) = \deg_M(x_F\ell^{r-2k-1}\eta\eta).$$

Write $\ell = \sum_F c_F x_F$ where each c_F is positive, and observe that

$$0 = \deg_M(\ell^{r-2k}\eta_F\eta_F) = \sum_F c_F \deg_M(x_F\ell^{r-2k-1}\eta\eta) = \sum_F c_F \deg_F(\ell_F^{r-2k-1}\eta_F\eta_F).$$

Since c_F is positive for every F, we find that $\deg_F(\ell_F^{r-2k-1}\eta_F\eta_F) = 0$ for every F. Since η_F lies in the kernel of ℓ_F^{r-2k}, the Hodge-Riemann relations for $\mathrm{CH}(M_F) \otimes \mathrm{CH}(M^F)$ imply that $\eta_F = 0$. Since $\psi_M^F(\eta_F) = x_F\eta$, we have shown that $x_F\eta = 0$ for every F as required.

Step 2: *Reduction of the Hodge-Riemann relations from the ample cone to a single nef class.* We show that if the Hodge-Riemann relations are true with respect to a nef class of M, then they are true with respect to every ample class of M.

Let $\lambda \in \mathrm{CH}^1(M)$ be a nef class for which the Hodge-Riemann form

$$(\mu, \nu) \mapsto (-1)^k \deg_M(\lambda^{r-2k}\mu\nu)$$

on $\mathrm{CH}^k(M)$ is positive-definite on the kernel of λ^{r-2k+1} for each $k \leqslant r/2$. The hard Lefschetz theorem is therefore valid for λ. Thus, there is a decomposition

$$\mathrm{CH}^k(M) = \lambda(\mathrm{CH}^{k-1}(M)) \oplus \ker(\lambda^{r-2k+1})$$

for each $k \leqslant r/2$ which is orthogonal with respect to the Hodge-Riemann form. By Lemma 4.7, the signature of the Hodge-Riemann form associated to λ on $\mathrm{CH}^k(M)$ is

$$\sigma_k \coloneqq \sum_{j=0}^{k} (-1)^{k-j} (\dim \mathrm{CH}^j(M) - \dim \mathrm{CH}^{j-1}(M)).$$

If $\ell \in \mathrm{CH}^1(M)$ is ample, then $\ell_t \coloneqq t\ell + (1-t)\lambda$ is ample for each $t \in (0,1]$. Thus, the Hodge-Riemann form on $\mathrm{CH}^k(M)$ associated with ℓ_t is nondegenerate for each $t \in [0,1]$ by Step 1. It follows that the Hodge-Riemann forms on $\mathrm{CH}^k(M)$ associated with ℓ and λ have the same signature. Indeed, signature is the difference between the number of positive and negative eigenvalues, so it locally constant on the space of nondegenerate forms. The Hodge-Riemann relations for ℓ then follow from Lemma 4.7.

Step 3: *For any $i \in E$ and $F \in \mathcal{S}_i$, the Hodge-Riemann relations for $x_{F \cup i} \mathrm{CH}_{(i)}$ are valid for $\vartheta_i(\ell)$ if $\ell \in \mathrm{CH}^1(M\backslash i)$ is ample.* More precisely, the Hodge-Riemann form of $\vartheta_i(\ell)$ restricted to $x_{F\cup i} \mathrm{CH}_{(i)}^{k-1}$ is positive-definite on the kernel of

$$\vartheta_i(\ell)^{r-2k} \colon x_{F\cup i} \mathrm{CH}_{(i)}^{k-1} \to x_{F\cup i} \mathrm{CH}_{(i)}^{r-k-1}$$

for each $k \leqslant r/2$. Let ℓ_F be the image of ℓ under the map

$$q \circ \varphi_{M\backslash i}^{F} \colon \mathrm{CH}(M\backslash i) \to \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$$

appearing in Lemma 4.4 which defines the $\mathrm{CH}(M\backslash i)$-module structure on the tensor product. By the argument in Step 1, we know that

$$\varphi_{M\backslash i}^{F}(\ell) = \ell' \otimes 1 + 1 \otimes \ell'$$

where at least one of ℓ', ℓ'' is ample, and both are ample unless one of the two Chow rings is just a copy of $\mathbf{R}$ in grading 0. From the proof of Lemma 4.4, every flat of $M_{F\cup i}$ is a flat of $(M\backslash i)_F$ and the map q is the tensor product of the map

$$q' \colon \mathrm{CH}((M\backslash i)_F) \to \mathrm{CH}(M_{F\cup i}) \qquad x_G \mapsto \begin{cases} x_G & G \text{ is a flat of } M_{F\cup i} \\ 0 & \text{otherwise.} \end{cases}$$

with the identity $\mathrm{CH}((M\backslash i)^F) \to \mathrm{CH}(M^F)$. It is easy to see that q' sends ample classes to ample classes. Since ℓ_F is of the form $q'(\ell') \otimes 1 + 1 \otimes \ell''$, it follows from Proposition 4.6 that $\mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$ satisfies the Hodge-Riemann relations for ℓ_F.

By Proposition 4.5, an arbitrary nonzero element in the kernel of

$$\vartheta_i(\ell)^{r-2k} \colon x_{F\cup i} \mathrm{CH}_{(i)}^{k-1} \to x_{F\cup i} \mathrm{CH}_{(i)}^{r-k-1}$$

is of the form $\Psi_i^F(\mu)$ for some $\mu \in \mathrm{CH}(M_{F\cup i}) \otimes \mathrm{CH}(M^F)$ in grading $k-1$. Next

$$0 = \vartheta_i(\ell)^{r-2k+1} \Psi_i^F(\mu) = \Psi_i^F(\ell_F^{r-2k+1} \mu)$$

because Ψ_i^F is a $\mathrm{CH}(M\backslash i)$-module map. Poincaré duality for $M_{F\cup i}$ and M^F imply that Ψ_i^F is injective so

$$\ell_F^{(r-2)-2(k-1)+1} \mu = 0.$$

By the Hodge-Riemann relations for ℓ_F, it follows that

$$0 < (-1)^{k-1}(\deg_{M_{F\cup i}} \otimes \deg_{M^F})(\ell_F^{r-2k+1}\mu\mu) = (-1)^k \deg_M(\vartheta_i(\ell)^{r-2k+1}\Psi_i^F(\mu)\Psi_i^F(\mu))$$

as required.

Step 4: *If $i \in E$ is not a coloop and $\ell \in \mathrm{CH}^1(M\backslash i)$ is ample, then $\mathrm{CH}(M)$ satisfies the Hodge-Riemann relations with respect to the nef class $\vartheta_i(\ell)$.* It is straightforward to verify that $\vartheta_i(\ell)$ is nef. We have the semi-small decomposition

$$\mathrm{CH}(M) = \mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\,\mathrm{CH}_{(i)}$$

which is orthogonal with respect to the Poincaré pairing. Because $\vartheta_i(\ell) \in \mathrm{CH}_{(i)}$, the induced decomposition of $\mathrm{CH}^k(M)$ is also orthogonal with respect to the Hodge-Riemann form. It therefore suffices to show that the Hodge-Riemann form is positive definite on the kernel of $\vartheta_i(\ell)^{r-2k+1}$ on each summand. The summands of the form $x_{F\cup i}\,\mathrm{CH}_{(i)}$ for $F \in \mathcal{S}_i$ are handled by Step 3.

Fix a nonzero element $\vartheta_i(\mu)$ in the kernel of

$$\vartheta_i(\ell)^{r-2k+1}\colon\ \mathrm{CH}^k_{(i)} \to \mathrm{CH}^{r-k+1}_{(i)}\,.$$

Then $0 = \vartheta_i(\ell)^{r-2k+1}\vartheta_i(\mu) = \vartheta_i(\ell^{r-2k+1}\mu)$. Because $M\backslash i$ satisfies Poincaré duality, the map ϑ_i is injective so $\mu \in \mathrm{CH}^k(M\backslash i)$ lies in the kernel of ℓ^{r-2k+1}. By the Hodge-Riemann relations for $M\backslash i$, we have that

$$0 < (-1)^k \deg_{M\backslash i}(\ell^{r-2k}\mu\mu) = (-1)^k \deg_M(\vartheta_i(\ell)^{r-2k}\vartheta_i(\mu)\vartheta_i(\mu)).$$

Step 5: *If $i \in E$ is a coloop and $\ell \in \mathrm{CH}^1(M\backslash i)$ is ample, then $\vartheta_i(\ell) + \varepsilon x_{E\backslash i} \in \mathrm{CH}^1(M)$ is nef for all sufficiently small $\varepsilon > 0$.* Let $\ell_\varepsilon = \vartheta_i(\ell) + \varepsilon x_{E\backslash i}$ for each $\varepsilon > 0$. We show that ℓ_ε is nef when ε is sufficiently small by showing that for each flag $\mathscr{F}$ of nonempty proper flats of M, the class ℓ_ε is convex at $\mathscr{F}$ for sufficiently small ε. Since there are finitely many flags, the result follows.

- Suppose $\mathscr{F}$ is a k-flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$ of M for which $F_k = E\backslash i$. Then $F_1 \subsetneq \cdots \subsetneq F_{k-1}$ is a $(k-1)$-flag of nonempty proper flats of $M\backslash i$. Thus, we may write
$$\ell_\varepsilon = \sum_G c_G(x_G + x_{G\cup i}) + \varepsilon x_{E\backslash i}$$
where the sum ranges over nonempty proper flats G of $M\backslash i$ with the property that $c_{F_j} = 0$ for $j = 1, \ldots, k-1$ and $c_G > 0$ for all nonempty proper flats G of M that extend $\mathscr{F}$ to a $(k+1)$-flag. Indeed, any such flat G is also a nonempty proper flat of $M\backslash i$ which extends $F_1 \subsetneq \cdots \subsetneq F_{k-1}$ to a k-flag of $M\backslash i$. Now let λ be an arbitrary linear function on M for which $\lambda(F_j) = 0$ for $j = 1, \ldots, k-1$ and $\lambda(E\backslash i) = 1$. If $\varepsilon > 0$ is sufficiently small, then the linear function $\varepsilon\lambda$ agrees with the piecewise linear function ℓ_ε on $\mathscr{F}$ and $\varepsilon\lambda(G) < c_G$ for any nonempty proper flat G of M extending $\mathscr{F}$ to a $(k+1)$-flag. It follows that ℓ_ε is strictly convex at $\mathscr{F}$ for sufficiently small $\varepsilon > 0$.

- Assume $\mathscr{F}$ is a k-flag of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_k$ of M for which $F_k \neq E\backslash i$. Then $F_1\backslash i \subseteq \cdots \subseteq F_k\backslash i$ is a flag of proper flats of $M\backslash i$. We may write
$$\ell_\varepsilon = \sum_G c_G(x_G + x_{G\cup i}) + \varepsilon x_{E\backslash i}$$
where the sum ranges over nonempty proper flats G of $M\backslash i$, so that the coefficient of x_{F_j} is zero in this expression for each $j = 1, \ldots, k$ and so that $c_G > 0$ for each nonempty proper flat G of $M\backslash i$ that extends $F_1\backslash i \subseteq \cdots \subseteq F_k\backslash i$. If H is a nonempty proper flat of M extending $\mathscr{F}$ to a $(k+1)$-flag, then $H\backslash i$ is comparable to each of $F_1\backslash i \subseteq \cdots \subseteq F_k\backslash i$. It follows the coefficient of x_H is nonnegative. Thus ℓ_ε is convex at $\mathscr{F}$ independent of the choice of $\varepsilon > 0$.

Step 6: *If $i \in E$ is a coloop and $\ell \in \mathrm{CH}^1(M\backslash i)$ is ample, then $\mathrm{CH}(M)$ satisfies the Hodge-Riemann relations with respect to the nef class $\vartheta_i(\ell) + \varepsilon x_{E\backslash i}$ for all sufficiently small $\varepsilon > 0$.* We have the semi-small decomposition
$$\mathrm{CH}(M) = S \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\,\mathrm{CH}_{(i)} \qquad \text{where} \qquad S = \mathrm{CH}_{(i)} \oplus x_{E\backslash i}\,\mathrm{CH}_{(i)}$$
whose induced decomposition of $\mathrm{CH}^k(M)$ is easily seen to be orthogonal with respect to the Hodge-Riemann form of ℓ_ε. It therefore suffices to prove the Hodge-Riemann relations for ℓ_ε on each summand. The summands $x_{F\cup i}\,\mathrm{CH}_{(i)}$ are handled by Step 3 because multiplication by ℓ_ε is the same as multiplication by $\vartheta_i(\ell)$ on $x_{F\cup i}\,\mathrm{CH}_{(i)}$. If $0 \leqslant k \leqslant (r-1)/2$, then let $P^k \subseteq \mathrm{CH}^k(M\backslash i)$ denote the kernel of the map ℓ^{r-2k} so that $\mathrm{CH}^k(M\backslash i) = P^k \oplus \ell(\mathrm{CH}^{k-1}(M\backslash i))$ by the Hodge-Riemann relations for ℓ. Thus
$$S^k = \vartheta_i(P^k) \oplus \vartheta_i(\ell)\,\mathrm{CH}^{k-1}_{(i)} \oplus x_{E\backslash i}\,\mathrm{CH}^{k-1}_{(i)}\,.$$
If r is even and $k = r/2$, then $\mathrm{CH}^{r/2}(M\backslash i) = \ell(\mathrm{CH}^{r/2-1}(M\backslash i))$ so the same decomposition is valid after defining $P^{r/2} = 0$.

We will first show that the Hodge-Riemann form of ℓ_ε is nondegenerate on S^k for $k \leqslant r/2$ for sufficiently small $\varepsilon > 0$. To then prove the Hodge-Riemann relations for ℓ_ε on S^k, it suffices to show that the form is positive-definite on S^0 and that its signature σ_k on S^k equals $\dim S^k - \dim S^{k-1} - \sigma_{k-1}$ for $1 \leqslant k \leqslant r/2$. Note that this is equivalent to showing that $\sigma_k = \dim P^k$. Indeed $\dim P^0 = \dim S^0$ and it is straightforward to verify that
$$\dim S^k - \dim S^{k-1} - \dim P^{k-1} = \dim P^k.$$
Our goal then is to show that the Hodge-Riemann form of ℓ_ε restricted to S^k is nondegenerate and has signature P^k for all small $\varepsilon > 0$.

With respect to the given splitting of S^k, let the symmetric matrix
$$\begin{pmatrix} H_{11}(\varepsilon) & H_{12}(\varepsilon) & H_{13}(\varepsilon) \\ H_{21}(\varepsilon) & H_{22}(\varepsilon) & H_{23}(\varepsilon) \\ H_{31}(\varepsilon) & H_{23}(\varepsilon) & H_{33}(\varepsilon) \end{pmatrix}$$
represent the Hodge-Riemann form of ℓ_ε restricted to S^k. This matrix is congruent to
$$H(\varepsilon) = \begin{pmatrix} \varepsilon^{-1}H_{11}(\varepsilon) & \varepsilon^{-1}H_{12}(\varepsilon) & H_{13}(\varepsilon) \\ \varepsilon^{-1}H_{21}(\varepsilon) & \varepsilon^{-1}H_{22}(\varepsilon) & H_{23}(\varepsilon) \\ H_{31}(\varepsilon) & H_{23}(\varepsilon) & \varepsilon H_{33}(\varepsilon) \end{pmatrix}.$$

Each entry is a polynomial in ε, so we may define $H(0)$ to be the limit of the matrices $H(\varepsilon)$ as $\varepsilon \to 0$. We show that $H(0)$ is nondegenerate and has signature $\dim P^k$, which implies the same for $H(\varepsilon)$ for all sufficiently small ε. First, we explicitly calculate $H(0)$. If $\mu, \nu \in P^k$, then

$$\begin{aligned}\frac{(-1)^k}{\varepsilon} \deg_M(\ell_\varepsilon^{r-2k} \vartheta_i(\mu)\vartheta_i(\nu)) &= (-1)^k \deg_M((r-2k)x_{E\setminus i}\vartheta_i(\ell^{r-1-2k}\mu\nu)) + O(\varepsilon) \\ &= (r-2k)(-1)^k \deg_{M\setminus i}(\ell^{r-1-2k}\mu\nu) + O(\varepsilon)\end{aligned}$$

so the limit of $\varepsilon^{-1}H_{11}(\varepsilon)$ as $\varepsilon \to 0$ is just a positive multiple of the Hodge-Riemann form of ℓ restricted to P^k when $r > 2k$. If $r = 2k$, then $\varepsilon^{-1}H_{11}(\varepsilon)$ is just the empty matrix. By similar calculations, we find that $\varepsilon^{-1}H_{12}(\varepsilon)$, $H_{13}(\varepsilon)$, and $\varepsilon H_{33}(\varepsilon)$ go to zero as $\varepsilon \to 0$. We also find that $\varepsilon^{-1}H_{22}(\varepsilon)$ and $H_{23}(\varepsilon)$ both limit to negative multiples of the Hodge-Riemann form of ℓ on $\mathrm{CH}^{k-1}(M\setminus i)$. In particular

$$H(0) = \begin{pmatrix} (r-2k)Q_\ell^k|_{P^k} & 0 & 0 \\ 0 & -(r-2k)Q_\ell^{k-1} & -Q_\ell^{k-1} \\ 0 & -Q_\ell^{k-1} & 0 \end{pmatrix}$$

where $Q_\ell^k|_{P^k}$ is the Hodge-Riemann form of ℓ restricted to $P^k \subseteq \mathrm{CH}^k(M\setminus i)$ and where Q_ℓ^{k-1} is the Hodge-Riemann form of ℓ on $\mathrm{CH}^{k-1}(M\setminus i)$. It follows from the nondegeneracy of $Q_\ell^k|_{P^k}$ and Q_ℓ^{k-1} that $H(0)$ is nondegenerate.

The signature of $H(0)$ is just the sum of the dimension of P^k with the signature of

$$\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} := \begin{pmatrix} -(r-2k)Q_\ell^{k-1} & -Q_\ell^{k-1} \\ -Q_\ell^{k-1} & 0 \end{pmatrix}$$

because $Q_\ell^k|_{P^k}$ is positive-definite. We claim that the signature of this 2×2 matrix is zero. Note that both A and B are symmetric matrices, and B is invertible. If $r > 2k$, then A is also invertible. In this case

$$\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ BA^{-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -BA^{-1}B \end{pmatrix} \begin{pmatrix} \mathrm{Id} & A^{-1}B \\ 0 & \mathrm{Id} \end{pmatrix}$$

from which the result follows because A and $BA^{-1}B$ have the same signature. If $r = 2k$, then $A = 0$. Let $A(\delta)$ be an invertible symmetric matrix for each $\delta > 0$ for which $A(\delta) \to A(0) := 0$ as $\delta \to 0$. Then the signature of

$$\begin{pmatrix} A(\delta) & B \\ B & 0 \end{pmatrix}$$

is zero for all $\delta > 0$ and so the signature is also zero when $\delta = 0$ by nondegeneracy of the matrix. Thus $H(0)$ is nondegenerate and has signature $\dim P^k$ from which it follows that the same is true for the Hodge-Riemann form of ℓ_ε on S^k for sufficiently small $\varepsilon > 0$. □

Appendix

We prove Buchberger's criterion and use it to verify Proposition 3.17 which asserts that the elements

$g_{F,G} = z_F z_G$ — F, G are incomparable nonempty flats

$$
\begin{aligned}
g_{F,G} &= z_F \left(\textstyle\sum_{G\subseteq H} z_H\right)^{\mathrm{rk}_M(G)-\mathrm{rk}_M(F)} && F \subsetneq G \text{ and } F, G \text{ are nonempty flats}\\
g_{\varnothing,G} &= \left(\textstyle\sum_{G\subseteq H} z_H\right)^{\mathrm{rk}_M(G)} && G \text{ is a nonempty flat.}
\end{aligned}
$$

of $\mathbf{R}[\,x_F \mid F \text{ is a nonempty flat}\,]$ form a Gröbner basis of

$$\mathcal{J} = \left\langle z_F z_G \mid F, G \text{ incomparable}\right\rangle + \left\langle \sum_{i\in F} z_F \mid i \in E\right\rangle.$$

Proof of Buchberger's criterion. If f is a nonzero polynomial, then its *degree* is the monomial of $LT(f)$. We compare the degrees of two polynomials by comparing the monomials using the monomial ordering.

Let f be a nonzero polynomial in I and write $f = \sum_i h_i g_i$ for some polynomials $h_1, \ldots, h_m$. Let M be the maximal degree among $h_1 g_1, \ldots, h_m g_m$. If M equals the degree of f, then it follows that the leading term of f lies in the ideal generated by the leading terms of $g_1, \ldots, g_m$. Otherwise, M is greater than the degree of f. We will find a new collection of polynomials $h'_1, \ldots, h'_m$ for which $f = \sum_i h'_i g_i$ and for which the maximal degree of the $h'_i g_i$ is less than M. The result then follows from the assumption that a monomial ordering is a well ordering.

Let $J \subseteq \{1, \ldots, m\}$ be the set of indices j for which the degree of $h_j g_j$ is M. Then

$$f = \sum_{j\in J} LT(h_j) g_j + \sum_{j\in J} (h_j - LT(h_j)) g_j + \sum_{i\notin J} h_i g_i$$

where the degree of each term in the latter two sums is less than M. Note that the first sum

$$s = \sum_{j\in J} LT(h_j) g_j$$

therefore also has strictly smaller degree than M. We claim that s may be written as

$$s = \sum_{i,j\in J} b_{i,j} S(LT(h_i)g_i, LT(h_j)g_j)$$

where $b_{i,j}$ are constants in the ground field. To see this, first note that

$$S(LT(h_i)g_i, LT(h_j)g_j) = \lambda_i LT(h_i) g_i - \lambda_j LT(h_j) g_j$$

where λ_i, λ_j are nonzero constants because $LT(h_i)g_i$ and $LT(h_j)g_j$ for $i, j \in J$ have the same degree M. It follows that the polynomials $LT(h_j)g_j$ for $j \in J$ have the span linear span as the syzygies $S(LT(h_i)g_i, LT(h_j)g_j)$ for $i, j \in J$ together with a fixed $LT(h_k)g_k$ for $k \in J$. Hence s is a linear combination of the syzygies along with $LT(h_k)g_k$, but because the degrees of s and the syzygies are strictly smaller than that of $LT(h_k)g_k$, the coefficient of $LT(h_k)g_k$ must be zero.

Hence

$$f = \sum_{i,j\in J} b_{i,j} S(LT(h_i)g_i, LT(h_j)g_h) + \sum_{j\in J} (h_j - LT(h_j)) g_j + \sum_{i\notin J} h_i g_i.$$

The degree of each $S(LT(h_i)g_i, LT(h_j)g_j)$ is less than M so if we can express each such syzygy as a sum $\sum_k \ell_k g_k$ where each $\ell_k g_k$ is of degree at most the degree of the syzygy, then we are done. Note that $S(LT(h_i)g_i, LT(h_j)g_j)$ is the product of $S(g_i, g_j)$ and a monomial term. Because long division of $S(g_i, g_j)$ by $g_1, \ldots, g_m$ may be done to obtain $r = 0$, it follows that $S(LT(h_i)g_i, LT(h_j)g_j)$ can indeed be expressed as a sum $\sum_k \ell_k g_k$ where each $\ell_k g_k$ has degree at most the degree of the syzygy. □

Proof of Proposition 3.17. Because the elements $g_{F,G}$ generate $\mathcal{J}$ by Lemma 3.19, it suffices by Buchberger's criterion to apply long division to each syzygy $S(g_{A,B}, g_{C,D})$ by the elements $g_{F,G}$ and obtain $r = 0$. Throughout, we use the convention that $z_F = 1$ if $F = \varnothing$. Note that if a polynomial f is divisible by $g_{F,G}$, then $LT(f)$ is divisible by $LT(g_{F,G})$. Thus if at any stage of long division, the polynomial f is divisible by some $g_{F,G}$, we are done.

Case: *Both pairs $\{A, B\}$ and $\{C, D\}$ are incomparable.* Then $S(z_A z_B, z_C z_D) = 0$ since the syzygy of any two monomials is zero.

Case: *One pair is incomparable while the other pair is comparable.* We may assume that A, B are incomparable and $C \subsetneq D$. Let $d = \mathrm{rk}_M(D) - \mathrm{rk}_M(C)$ so that $LT(g_{C,D}) = z_C z_D^d$. If the sets $\{A, B\}$ and $\{C, D\}$ are disjoint, then the syzygy is just

$$S(g_{A,B}, g_{C,D}) = LT(g_{C,D})g_{A,B} + LT(g_{A,B})g_{C,D} = LT(g_{C,D})g_{A,B} + g_{A,B}g_{C,D}$$

and is therefore divisible by $g_{A,B}$. The two sets $\{A, B\}$ and $\{C, D\}$ cannot be equal, so we may assume that $A \in \{C, D\}$ and $B \notin \{C, D\}$. If $A = C$, then the syzygy is

$$S(g_{A,B}, g_{A,D}) = z_D^d z_A z_B - z_B z_A (\textstyle\sum_{D \subseteq H} z_H)^d$$

which is again divisible by $g_{A,B} = z_A z_B$. Finally assume that $A = D$. Then

$$-S(g_{A,B}, g_{C,A}) = z_B z_C (\textstyle\sum_{A \subseteq H} z_H)^d - z_B z_C z_A^d$$

One term in the expansion of the sum $z_B z_C (\sum_{A \subseteq H} z_H)^d$ is $z_B z_C z_A^d$ which is canceled. Among the rest of the terms, we may subtract the monomials that are divisible by $z_A z_B = g_{A,B}$. The result is

$$z_B z_C (\textstyle\sum_{A \subsetneq H} z_H)^d$$

If B and C are incomparable, then the polynomial is divisible by $g_{B,C}$, so assume that they are comparable. It follows that $C \subsetneq B$ because B and A are incomparable. Now again by subtracting monomials in which two incomparable flats appear, we obtain

$$z_B z_C (\textstyle\sum_{\mathrm{cl}(A \cup B) \subseteq H} z_H)^d$$

We claim that this polynomial is divisible by $g_{B,\mathrm{cl}(A \cup B)}$. Indeed

$$\mathrm{rk}_M(A \cup B) - \mathrm{rk}_M(B) \leqslant \mathrm{rk}(A) - \mathrm{rk}(A \cap B) \leqslant \mathrm{rk}(A) - \mathrm{rk}(C) = d.$$

Case: *Both pairs are comparable and $B = D$.* We have $A \subsetneq B$ and $C \subsetneq B$. Let $d = \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ and $e = \mathrm{rk}_M(B) - \mathrm{rk}_M(C)$ and assume without loss of generality that $e \geqslant d$. Then the syzygy $S(g_{A,B}, g_{C,B})$ is

$$z_C z_B^{e-d} z_A (\textstyle\sum_{B \subseteq H} z_H)^d - z_A z_C (\sum_{B \subseteq H} z_H)^e = z_A z_C (\sum_{B \subseteq H} z_H)^d (z_B^{e-d} - (\sum_{B \subseteq H} z_H)^{e-d})$$

which is divisible by $g_{A,B}$.

Case: *Both pairs are comparable and $A = C$.* Here $A \subsetneq B$ and $A \subsetneq D$. Let $d = \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ and $e = \mathrm{rk}_M(D) - \mathrm{rk}_M(A)$. Then the syzygy $S(g_{A,B}, g_{A,D})$ is

$$z_D^e z_A (\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d z_A (\sum_{D \subseteq H} z_H)^e.$$

If B, D are incomparable, then by dropping incomparable terms, we obtain

$$z_D^e z_A(\textstyle\sum_{\mathrm{cl}(B\cup D)\subseteq H} z_H)^d - z_B^d z_A(\sum_{\mathrm{cl}(B\cup D)\subseteq H} z_H)^e.$$

There is no cancellation between these sums. The first sum is divisible by $g_{D,\mathrm{cl}(B\cup D)}$ while the second is divisible by $g_{B,\mathrm{cl}(B\cup D)}$. If B, D are comparable, then assume that $B \subsetneq D$ without loss of generality. In the syzygy, the lead terms of the two polynomials cancel, so the syzygy is difference of the two polynomials

$$z_A z_D^e((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d) - z_A z_B^d((\sum_{D\subseteq H} z_H)^e - z_D^e).$$

Notice that every monomial term in the second polynomial is divisible by z_B^d while no term of the first polynomial is. Thus, there is no cancellation between the two polynomials. Every term of the second polynomial is divisible by the leading term of $g_{A,B}$, so we may add

$$g_{A,B}((\textstyle\sum_{D\subseteq H} z_H)^e - z_D^e)$$

to the syzygy. We thereby obtain

$$\begin{aligned} &z_A z_D^e((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d) + z_A((\sum_{B\subseteq H} z_H)^d - z_B^d)((\sum_{D\subseteq H} z_H)^e - z_D^e) \\ &= z_A((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d)(\sum_{D\subseteq H} z_H)^e \end{aligned}$$

which is divisible by $g_{A,D}$.

Case: *Both pairs are comparable and* $B = C$. Here $A \subsetneq B \subsetneq D$, and we let $d = \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ and $e = \mathrm{rk}_M(D) - \mathrm{rk}_M(B)$. The syzygy $S(g_{A,B}, g_{B,D})$ is

$$z_A z_D^e((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d) - z_A z_B^d((\sum_{D\subseteq H} z_H)^e - z_D^e).$$

Expressed as the difference of the two given polynomials, there is no cancellation between the two. The second polynomial is divisible by $LT(g_{A,B}) = z_A z_B^d$ so we add $g_{A,B}((\sum_{D\subset H} z_H)^e - z_D^e)$ to syzygy to obtain

$$z_A((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d)(\sum_{D\subseteq H} z_H)^e.$$

The sum of the terms that are divisible by $LT(g_{B,D}) = z_B z_D^e$ is

$$z_A((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d - (\sum_{B\subsetneq H} z_H)^d) z_D^e.$$

We subtract the corresponding multiple of $g_{B,D}$

$$z_A((\textstyle\sum_{B\subseteq H} z_H)^d - z_B^d - (\sum_{B\subsetneq H} z_H)^d)(\sum_{D\subseteq H} z_H)^e$$

from the expression to obtain

$$z_A(\textstyle\sum_{B\subsetneq H} z_H)^d(\sum_{D\subseteq H} z_H)^e.$$

Suppose some monomial μ of $(\sum_{B\subsetneq H} z_H)^d$ is divisible by z_Y where $Y \subsetneq D$. Then $z_A\mu(\sum_{D\subseteq H} z_H)^e$ is divisible by $g_{Y,D}$ so we delete these terms. Of the remaining monomials of $(\sum_{B\subsetneq H} z_H)^d$, consider the ones from $(\sum_{D\subseteq H} z_H)^d$. The sum of their corresponding terms is $z_A(\sum_{D\subseteq H} z_H)^{d+e}$

which is just $g_{A,D}$ so we delete these as well. Every remaining term is divisible by a polynomial of the form $z_A z_W (\sum_{D \subseteq H} z_H)^e$ where $B \subsetneq W$ and W, D are incomparable. Note that by incomparability, this polynomial reduces to

$$z_A z_W (\textstyle\sum_{\mathrm{cl}(D \cup W) \subseteq H} z_H)^e$$

which is divisible by $g_{W,\mathrm{cl}(D \cup W)}$.

Last case: *Both pairs are comparable and* $\{A, B\}$ *and* $\{C, D\}$ *are disjoint*. Here $A \subsetneq B$ and $C \subsetneq D$, and let $d = \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$ and $e = \mathrm{rk}_M(D) - \mathrm{rk}_M(C)$. The syzygy $S(g_{A,B}, g_{C,D})$ is

$$z_C z_D^e z_A ((\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d) - z_A z_B^d z_C ((\sum_{D \subseteq H} z_H)^e - z_D^e)$$

If A, C are incomparable, this polynomial is divisible by $g_{A,C}$. Assume they are comparable and that $A \subsetneq C$ without loss of generality.

- **Subcase**: B, C are incomparable. The second sum is divisible by $g_{B,C}$ so we drop it. If B, D are incomparable, then by further dropping incomparable terms, we obtain

  $$z_A z_C z_D^e (\textstyle\sum_{\mathrm{cl}(B \cup D) \subseteq H} z_H)^d$$

  which is divisible by $g_{D,\mathrm{cl}(B \cup D)}$. If B, D are comparable, then $B \subsetneq D$ and by dropping incomparable terms, we have obtain

  $$z_A z_C z_D^e (\textstyle\sum_{\mathrm{cl}(B \cup C) \subseteq H} z_H)^d$$

  which is divisible by $g_{C,\mathrm{cl}(B \cup C)}$.
- **Subcase**: $C \subsetneq B$. If B and D are incomparable, then by dropping incomparable terms, we obtain

  $$z_C z_D^e z_A (\textstyle\sum_{\mathrm{cl}(B \cup D) \subseteq H} z_H)^d - z_A z_B^d z_C (\sum_{\mathrm{cl}(B \cup D) \subseteq H} z_H)^e$$

  with no cancellation. The first polynomial is divisible by $g_{D,\mathrm{cl}(B \cup D)}$ and the second is divisible by $g_{B,\mathrm{cl}(B \cup D)}$. If B and D are comparable, then the syzygy is the difference

  $$z_A z_C z_D^e ((\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d) - z_A z_B^d z_C ((\sum_{D \subseteq H} z_H)^e - z_D^e)$$

  with no cancellation. If $B \subsetneq D$, then add a multiple of $g_{A,B}$ to obtain

  $$z_A z_C ((\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d)(\sum_{D \subseteq H} z_H)^e$$

  which is a multiple of $g_{C,D}$. If $D \subsetneq B$, then subtract a multiple of $g_{C,D}$ to obtain

  $$z_A z_C (z_D^e - (\textstyle\sum_{D \subseteq H} z_H)^e)(\sum_{B \subseteq H} z_H)^d$$

  which is a multiple of $g_{A,B}$.
- **Subcase**: $B \subsetneq C$. We write the syzygy as the difference

  $$z_A z_C z_D^e ((\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d) - z_A z_B^d z_C ((\sum_{D \subseteq H} z_H)^e - z_D^e)$$

  which has no cancellation. Adding the appropriate multiple of $g_{A,B}$ corresponding to the terms divisible by $z_A z_B^d$, we obtain

  $$z_A z_C ((\textstyle\sum_{B \subseteq H} z_H)^d - z_B^d)(\sum_{D \subseteq H} z_H)^e$$

  which is divisible by $g_{C,D}$. □

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