

1 Simplicial Sets

In class, we talked about semisimplicial sets and the category Δ_{inj} . As a reminder:

Definition 1. The category Δ_{inj} has as its objects finite non-empty linearly-ordered sets, and its morphisms are the order-preserving injections.

Remark 1. All objects in Δ_{inj} are isomorphic to $[n] = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$, and all morphisms factor through $\delta_i : [n] \rightarrow [n+1]$, where $\delta_i(j) = j$ for $j < i$ and $\delta_i(j) = j+1$ for $j \geq i$.

Definition 2. A semi-simplicial set (Δ_{inj} -complex) is a functor from Δ_{inj}^{op} to Sets

A natural extension of the idea of taking order-preserving injections between the objects of Δ_{inj} is to take all order-preserving functions.

Definition 3. Define a category Δ with finite non-empty linearly-ordered sets as its objects and order-preserving functions as its morphisms.

Similarly as for Δ_{inj} , the objects in Δ are all isomorphic to $[n]$ for some $n \in \mathbb{N}$.

Definition 4. Define morphisms $\delta_i : \Delta_{n-1} \rightarrow \Delta_n$ and $\sigma_i : [n+1] \rightarrow [n]$ by

$$\begin{aligned} \delta_i(j) &= j \text{ if } j < i; & \delta_i(j) &= j+1 \text{ if } j \geq i \\ \sigma_i(j) &= j \text{ if } j \leq i; & \sigma_i(j) &= j-1 \text{ if } j > i. \end{aligned}$$

Basically, δ_i is the injection that skips i , and σ_i is the morphism that doubles up on i .

Proposition 1. All non-identity morphisms $\mu : \Delta_n \rightarrow \Delta_m$ in Δ can be written as

$$\mu = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t},$$

where $0 \leq i_s < \cdots < i_1 \leq m$ and $0 \leq j_1 < \cdots < j_t < n$, and $n - t + s = m$.

Proof. Exercise. □

Just as we had the identity $\delta_j \delta_i = \delta - i \delta_{j-1}$ if $i < j$ in Δ_{inj} , we have analogous identities in Δ :

Proposition 2.

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1} \text{ if } i < j, \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \text{ if } i \leq j \\ \sigma_j \delta_i &= \delta_i \sigma_{j-1} \text{ if } i < j \\ \sigma_j \delta_j &= \delta_{j+1} \sigma_j = \text{id} \\ \sigma_j \delta_i &= \delta_{j-1} \sigma_i \text{ if } i > j+1. \end{aligned}$$

Proof. Exercise. □

With the notion of the category Δ , we can define a simplicial set:

Definition 5. A simplicial set is a functor from Δ^{op} to sets.

Thus, to the morphisms $\delta_j : \Delta_{n-1} \rightarrow \Delta_n$, there corresponds a morphism $\partial_j : X_n \rightarrow X_{n-1}$, and to the morphisms $\sigma_j : \Delta_{n+1} \rightarrow \Delta_n$, there corresponds a morphism $s_j : X_n \rightarrow X_{n+1}$.

Example 1. The simplest non-empty semi-simplicial set is $X_0 = \{*\}$ and $X_i = \emptyset$, with $\partial_j : X_n \rightarrow X_{n-1}$ being the unique function from the empty set. This, however, does not extend to a simplicial set, since there are no maps to an empty set, except from an empty set!

Thus, the simplest non-empty simplicial set is actually $X_n = \{*\}$ for all $n \geq 0$. From this example, we see that by adding the non-injective functions, simplicial sets are actually much more complicated than semisimplicial sets.

Example 2. Via the Yoneda embedding, we have a functor $\Delta \rightarrow \text{Fun}(\Delta^{op}, \text{Sets})$. Thus, we can regard each $[n]$ as a simplicial set X , where $X_m = \text{Hom}_\Delta([m], [n])$. This is also known as the standard n -simplex, and we'll denote it by $\Delta[n]$.

What is this simplex more concretely? We note that X_0 is just the morphisms from $\{0\}$ to $\{0, \dots, n\}$; thus, if we draw $n + 1$ points, we can think of X_0 as the collection of these points. What is X_1 ? It is the collection of order-preserving morphisms from $\{0, 1\}$ to $\{0, \dots, n\}$, which can be identified with 2-element subsets, or alternatively, edges between the vertices. We see that in general, X_m can be thought of as the m -dimensional faces connecting the vertices. However, in addition to these faces, which correspond to the injective mappings in Δ , there are other degenerate faces.

2 Geometric realization of simplicial set

Recall the definition of the standard geometrical n -simplex:

Definition 6. The standard (geometrical) n -simplex is the set $\{(x_0, x_1, \dots, x_n) | x_i \in [0, 1], \sum x_i = 1\}$.

Proposition 3. *There exists a functor from Δ to Spaces sending $[n]$ to Δ^n .*

Remark 2. We note a strong connection between the standard combinatorial n -simplex and the standard geometric one. In particular, one-element subsets of $\{0, 1, \dots, n\}$ correspond to vertices of Δ^n , two element subsets with edges, and so forth.

We now define the geometric realization of a simplicial set.

Definition 7. The geometric realization of a simplicial set $X_\bullet$ is the space

$$|X_\bullet| = \frac{\bigsqcup_n X_n \times \Delta^n}{(\partial_i x, u) \sim (x, \delta_i u), (s_i x, u) \sim (x, \sigma_i u)}$$

Example 3. Let us consider the geometric realization of the simplest simplicial set, $X_n = \{*\}$. Each “level” of $|X_\bullet|$ consists of elements $(*, u)$, where $u \in \Delta^n$ is a point in the standard n -simplex. We now identify the points $(\partial_i *, u) \in |X|_{n-1}$ with the point $(*, \delta_i u) \in |X|_n$, as well as the points $(s_i *, u) \in |X|_{n+1}$ with $(*, \sigma_i u) \in |X|_n$. We thus see that every point in level n of $|X_\bullet|$ is identified with a point in level $n - 1$ via the degeneracy map. Thus, inducting downwards, we see that every point in $|X_\bullet|$ is identified with a point in level 0, which is just a point. Thus, the geometric realization of the infinite simplicial set X_n is just a point.

Exercise 1. Show that $|\Delta[n]| = \Delta^n$. This formalizes our intuition that the standard n -simplex $\Delta[n]$ behaves like Δ^n .

In addition to the geometric realization of a simplicial set, we also can construct a simplicial set from a space X .

Definition 8. Let X be a space. Then, $\text{Sing}(X)$ is the simplicial set with $\text{Sing}(X)_n = \text{Hom}_{\text{Top}}(\Delta^n, X)$.

Remark 3. So far, simplicial sets seem to just complicate matters; why then do we want to study simplicial sets when semisimplicial sets are so much easier?

One important reason that we study simplicial sets in addition to semisimplicial sets is that simplicial sets are much better behaved under products. Intuitively, if $X_\bullet$ and $Y_\bullet$ are (semi)simplicial sets, we expect that $X_\bullet \times Y_\bullet$ is the (semi)simplicial set with levels $X_n \times Y_n$. Moreover, we would expect that the geometric realization of $X_\bullet \times Y_\bullet$ would simply be $|X_\bullet| \times |Y_\bullet|$, the product of topological spaces. The first statement is true for both semisimplicial and simplicial sets, but the latter statement is true for simplicial sets and false for semisimplicial sets!

2.1 Adjunctions of Simplicial Sets

Definition 9. Let $\mathcal{D}$ and $\mathcal{C}$ be categories. An adjunction between the two categories is a pair of functors $F : \mathcal{D} \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$ along with a family of bijections $\text{Hom}_{\mathcal{C}}(FY, X) \cong \text{Hom}_{\mathcal{D}}(Y, GX)$ which is natural in $X \in \mathcal{C}_o$ and $Y \in \mathcal{D}_o$. We say that F is a left adjoint functor and G is right adjoint functor.

What does this definition mean? So suppose there is an adjunction between category $\mathcal{C}$ and $\mathcal{D}$. Now, if $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we can certainly consider a morphism $F(Y) \rightarrow X$ in $\mathcal{C}$. It would be nice to “invert” F and get a morphism in $\mathcal{D}$; adjunction provides a way to associate the above morphism with a morphism $Y \rightarrow G(X)$. What the naturality condition gives is that if $\phi : F(Y) \rightarrow X$ is any morphism and $\Phi_{Y,X}(\phi)$ is the bijection associating ϕ with a morphism in $\text{Hom}_{\mathcal{D}}(Y, GX)$, then $G(f) \circ \Phi_{Y,X}(\phi) = \Phi_{Y,X'}(f \circ \phi)$ for any $f : X \rightarrow X'$ (a similar condition holds for a morphism $G \rightarrow G'$).

Example 4. Let $F : \text{Sets} \rightarrow \text{Groups}$ be the functor assigning to each set the free group generated by the set, and $G : \text{Groups} \rightarrow \text{Sets}$ be the functor that treats a group as a set. This forms an adjunction between Sets and Groups, as we saw in the last problem set.

Proposition 4. *The geometric realization $|\cdot|$ and $\text{Sing}(\cdot)$ are adjoint.*

Proof. Let $\phi : X \rightarrow \text{Sing}(Y)$; that is, ϕ is a collection of morphisms $\phi_n : X_n \rightarrow \text{Hom}(\Delta^n, Y)$ so that $\phi_n(\text{SSet}_X(f)(x_n)) = \text{SSet}_{\text{Sing}(Y)}(f)(\phi_n(x_n))$, where f is a composition of degeneracies and boundary maps (and SSet_X is the functor turning X into a simplicial set). By the definition of singular chain complexes, $\text{SSet}_{\text{Sing}(Y)}(f)(\pi) = \pi \circ \text{Space}_\Delta(f)$, where $\text{Space}_\Delta : \Delta \rightarrow \text{Top}$ is the functor sending $[n] \mapsto \Delta^n$ and $\pi : \Delta^n \rightarrow Y$ is an element of $\text{Sing}(Y)_n = \text{Hom}(\Delta^n, Y)$.

We define $\Phi_{XY}(\phi)$ as the homeomorphism sending the point (x_n, u) with $x_n \in X_n$, $u \in \Delta^n$ to the point $\phi(x_n)(u)$. To see that this map is well defined, we note that two points (x_n, u_n) and (x_m, u_m) are related if and only if we can find a composite of degeneracies and boundaries f such that $\text{SSet}_X(f)(x_n) = x_m$ and $\text{Space}_\Delta(f)(u_m) = u_n$; then

$$\begin{aligned} \phi(x_m)(u_m) &= \phi(\text{SSet}_X(f)(x_n))(u_m) = \text{SSet}_{\text{Sing}(Y)}(f)(\phi(x_n))(u_m) \\ &= \phi(x_n)(\text{Space}_\Delta(f)(u_m)) = \phi(x_n)(u_n). \end{aligned}$$

Now, we show naturality with respect to Y . Let $g : Y \rightarrow Y'$ be any morphism of spaces and $\phi : X \rightarrow \text{Sing}(Y)$. Then, $\text{Sing}(g)(\phi)$ is the morphism from X to $\text{Sing}(Y')$ defined by $(\text{Sing}(g) \circ \phi)(x)(u) = g((\phi(x))(u))$ for $x \in X$ and $u \in \Delta^n$. Thus, $\Phi_{XY'}(\text{Sing}(g) \circ \phi)$ is the map sending $(x_n, u_n) \in |X|$ to $g(\phi_n(x_n)(u_n))$. This is precisely $g \circ \Phi_{XY}(\phi)$.

Now suppose we have a map $g : X' \rightarrow X$ of semisimplicial sets. Under geometric realization, this gets sent to a continuous map $|g|$ between $|X'|$ and $|X|$ defined by $|g|(x'_n, u) = (g_n(x'_n), u)$; this map is well defined since g , as morphisms between simplicial sets, respects the relations we are quotienting by. The map $\Phi_{X'Y}(\phi \circ g)$ sends (x'_n, u_n) to $\phi(g(x'_n))(u_n)$. On the other hand, $\Phi_{XY}(\phi) \circ |g|$ is the map that sends (x'_n, u) first to $(g_n(x'_n), u_n)$ and subsequently to $\phi(g_n(x'_n))(u_n)$. This is the same value as before.

Finally, if we have a map $\pi : |X| \rightarrow Y$, then, for each $x_n \in X$, $\pi(x_n, \cdot)$ is a continuous function from $\Delta^n \rightarrow Y$. Thus, we can define $\Phi^{-1}(\pi)$ to be simplicial set morphism that, on each level, sends x_n to $\pi(x_n, \cdot) \in \text{Hom}(\Delta^n \rightarrow Y)$. A similar proof as above will show all the other naturality conditions hold. $\square$

It turns out that for any cocomplete category E and a functor $F : \Delta \rightarrow E$, we can define a pair of adjoints, with the right adjoint given by $R : E \rightarrow \text{SSet}$ sending an object e to the set $\text{Hom}_E(F([n]), e)$ (similar to the definition of singular complexes) and the left adjoint being somewhat more

complicated (it is the left Kan extension of F along the Yoneda embedding). When E is the category of spaces, then the right adjoint is the singular complex functor and the left adjoint is geometric realization. When E is the category of categories, the right adjoint is the nerve functor discussed later and the left adjoint takes simplicial sets to its “fundamental category”.

3 Kan Complexes

Which simplicial sets arise from $\text{Sing}(Y)$ for some space Y . To partially answer this question, we introduce Kan complexes.

Definition 10. The horn Λ_k^n is the simplicial set obtained from the the boundary of the standard n -simplex by removing the k -th face. Here, k ranges from 0 to n .

Definition 11. Let K be a simplicial set. We say K is a Kan condition if for any map $\phi : \Lambda_k^n \rightarrow K$, there exists a map $\phi' : \Delta[n] \rightarrow K$ such that $\phi = \phi' \circ i$, where i is the inclusion $\Lambda_k^n \subset \Delta[n]$.

Proposition 5. *Let Y be a topological space. Then, $\text{Sing}(Y)$ is a Kan complex.*

Proof. Suppose we have a map $\phi : \Lambda_k^n \rightarrow \text{Sing}(Y)$. Since geometric realization and Sing are adjoint, this corresponds to a map $\phi' : |\Lambda_k^n| \rightarrow Y$. We know that there is a deformation retract d from $\Delta^n = |\Delta[n]|$ to $|\Lambda_i^n|$. Thus, we can find a map $\pi : |\Delta[n]| \rightarrow Y$ given by $\pi = \phi' \circ d$. If $i : \Lambda_k^n \rightarrow \Delta^n$ is the inclusion map, then $\pi \circ |i| = \phi' \circ d \circ |i|$, and since d is a deformation retract, $d \circ |i|$ is just the identity map on Λ_i^n .

So now we have a map $\pi : |\Delta[n]| \rightarrow Y$ staisfying $\pi \circ |i| = \phi'$. By the adjointness of geometric realization and Sing , we get a map $\pi' : \Delta[n] \rightarrow \text{Sing}(Y)$, and since the correspondance is natural, we have $\pi' \circ i = \phi$. This shows that $\text{Sing}(Y)$ is a Kan complex. $\square$

From this proposition, we see evidence for the correspondance between Kan complexes and spaces. In Higher Topos Theory, Laurie points out that there is a “simple combinatorial recipe” for extracting homotopy groups from K (which are isomorphic to the homotopy groups for $|K|$).