HOMEWORK 8: MATH 124 (NUMBER THEORY)

1. Exercises

We write $\langle a_0, a_1, \ldots, a_m\rangle$ for a simple continued fraction. Recall that, for an infinite continued fraction $\langle a_0, a_1, a_2, \ldots,\rangle$, the *nth convergent* of $\langle a_0, a_1, a_2, \ldots\rangle$ is the rational number obtained by truncating after a_n:

$$r_n = \langle a_0, a_1, \ldots, a_n\rangle, \qquad n \geq 0.$$

Thus $r_0 = \langle a_0\rangle = a_0$ and $r_1 = \langle a_0, a_1\rangle = a_0 + 1/a_1$ and so on.

Exercise 1.1. *Compute the following continued fractions and rational numbers.*

(1) *Use the Euclidean algorithm to express* $415/93$ *and* $-43/19$ *as finite simple continued fractions whose last partial quotient is greater than* 1*. Display the divisions you use.* *(6 pts)*

(2) *Express* $\langle 0, 2, 3, 4\rangle$ *and* $\langle -2, 1, 3\rangle$ *as fractions in lowest terms.* *(4 pts)*

Exercise 1.2. *Use the results in class about convergence and tails of infinite continued fractions.*

(1) *Determine the value of* $\langle 2, 2, 2, \ldots\rangle$*. Explain which root of your quadratic equation gives the value.* *(3 pts)*

(2) *Let*

$$x = \langle 1, 2, 1, 2, \ldots\rangle, \qquad y = \langle 2, 1, 2, 1, \ldots\rangle.$$

Find equations relating x *and* y*, and determine both values.* *(4 pts)*

(3) *Compute the first six convergents of* x*, beginning with* $\langle 1\rangle$*. Use the results in class to give a rational interval containing* x *whose length is less than* $1/100$*.* *(3 pts)*

Exercise 1.3. *Let* $F_0 = 0$*,* $F_1 = 1$*, and* $F_{n+1} = F_n + F_{n-1}$ *for* $n \geq 1$ *be the Fibonacci numbers. Set* $\phi = (1 + \sqrt{5})/2$*.*

(1) *Prove that, for every* $n \geq 2$*,*

$$\underbrace{\langle 1, 1, \ldots, 1\rangle}_{n-1 \text{ entries}} = \frac{F_n}{F_{n-1}}.$$

(5 pts)

(2) *Using the convergence results in class, deduce that*

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi.$$

Explain how you identify the limit; you do not need to re-prove convergence. *(5 pts)*

Exercise 1.4. *We investigate how well the Fibonacci ratios approximate ϕ. Write*

$$r_n = \frac{F_n}{F_{n-1}}, \qquad E_n = F_{n-1}^2 |r_n - \phi|.$$

A calculator or a short program may be used in part (1). Keep full precision during the computations and round only the displayed answers.

(1) *For $n = 3, 4, \ldots, 12$, make a table of r_n, $|r_n - \phi|$, and E_n, rounded to six decimal places (You may use a calculator). Compare E_n with $1/\sqrt{5}$. Based on your table, conjecture which indices satisfy*

$$\left| \frac{F_n}{F_{n-1}} - \phi \right| < \frac{1}{\sqrt{5}\, F_{n-1}^2}.$$

(4 pts)

(2) *Use the error formula from class to show that*

$$E_n = \frac{1}{\phi + F_{n-2}/F_{n-1}} = \frac{1}{r_n + \phi - 1}.$$

Use this identity and the alternating behavior of convergents to prove your conjecture from part (1). *(4 pts)*

(3) *Determine $\lim_{n\to\infty} E_n$. Explain why these Fibonacci ratios give infinitely many distinct rational approximations satisfying Hurwitz's bound.* *(2 pts)*

Exercise 1.5. *We show that Hurwitz's theorem would be false if $\sqrt{5}$ were replaced by $\sqrt{7}$, using ϕ as a counterexample. You may use the result in class that, for an irrational x, any reduced fraction p/q with $q > 0$ satisfying*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

must be a convergent to x.

(1) *Use the limit computed in Exercise 1.4 to show that only finitely many Fibonacci ratios satisfy*

$$\left| \phi - \frac{F_n}{F_{n-1}} \right| < \frac{1}{\sqrt{7}\, F_{n-1}^2}.$$

(3 pts)

(2) *Deduce that only finitely many reduced fractions p/q with $q > 0$ satisfy*

$$\left| \phi - \frac{p}{q} \right| < \frac{1}{\sqrt{7}\, q^2}.$$

Explain why this disproves the proposed version of Hurwitz's theorem. Why is it necessary to consider all reduced fractions, rather than just the Fibonacci ratios? *(4 pts)*

(3) *Explain why the same argument works with any constant $C > \sqrt{5}$ in place of $\sqrt{7}$. Conclude that $\sqrt{5}$ is the optimal constant in Hurwitz's theorem.* *(3 pts)*