HOMEWORK 7: MATH 124 (NUMBER THEORY)

1. Exercises

W let F_n denote the part of the Farey sequence of order n lying in $[0,1]$, including $0/1$ and $1/1$.

Exercise 1.1. *Determine the following.*

(1) *Write out F_6 and F_7 in increasing order. Indicate which fractions are inserted when passing from F_6 to F_7.* *(4 pts)*

(2) *Find the fractions immediately to the left and right of $3/7$ in F_{20}. Justify that they are adjacent to $3/7$ (Hint: use the property characterizing Farey neighbors shown in class).* *(3 pts)*

(3) *Find the fraction with smallest denominator lying strictly between $5/12$ and $3/7$. Justify that your answer is unique using the results in class.* *(3 pts)*

Exercise 1.2. *Let n be a positive integer and let $|F_n|$ denote the number of fractions in F_n. Show the following.*

(1)
$$|F_n| = 1 + \sum_{k=1}^{n} \phi(k),$$
where ϕ is Euler's totient function and $\phi(1) = 1$. *(3 pts)*

(2) *Show that the sum of all the fractions in F_n is $|F_n|/2$. (Hint: Pair a/b with $(b-a)/b$.)* *(3 pts)*

Exercise 1.3. *In class we parametrized the rational points on the unit circle by drawing lines through $(-1,0)$. We mentioned that one could apply the same method to*
$$x^2 + y^2 = 2.$$
We work this out in the exercise.

(1) *For $t \in \mathbb{Q}$, consider the line $y = 1 + t(x-1)$ through $(1,1)$. Find its other intersection with the circle when $t \neq -1$. What happens when $t = -1$?* *(3 pts)*

(2) *Show that your formulas, including the value $t = -1$, give every rational point on the circle except one. Identify the missing point and explain how the vertical line through $(1,1)$ accounts for it. Find a rational point whose coordinates are not integers.* *(2 pts)*

(3) *Prove that $x^2 + y^2 = 3$ has no rational solutions. (Hint: Clear denominators to obtain $a^2 + b^2 = 3c^2$ with $\gcd(a,b,c) = 1$, and consider this equation modulo 3.) Explain why the analogous construction does not work in this case.* *(3 pts)*