

HOMEWORK 3: MATH 124 (NUMBER THEORY)

1. EXERCISES

Exercise 1.1. For each of the following congruences, determine whether a solution exists. If a solution exists, find all solutions in $\mathbb{Z}/m\mathbb{Z}$, where m is the given modulus. If no solution exists, explain why (2 pts each).

- (a) $7x \equiv 5 \pmod{12}$.
- (b) $6x \equiv 9 \pmod{15}$.
- (c) $8x \equiv 6 \pmod{14}$.
- (d) $12x \equiv 8 \pmod{18}$.
- (e) $15x \equiv 10 \pmod{35}$.

Exercise 1.2. Determine whether each of the following system of congruences have a solution. If they do find all $x \in \mathbb{Z}$ which solve the following equations (4 pts each).

(1)

$$\begin{aligned}x &\equiv 2 \pmod{3} \\x &\equiv 3 \pmod{5} \\x &\equiv 1 \pmod{2}.\end{aligned}$$

(2)

$$\begin{aligned}x &\equiv 1 \pmod{4} \\x &\equiv 0 \pmod{3} \\x &\equiv 5 \pmod{7}\end{aligned}$$

(3)

$$\begin{aligned}x &\equiv 5 \pmod{7} \\x &\equiv 7 \pmod{10} \\x &\equiv 2 \pmod{15}\end{aligned}$$

(4)

$$\begin{aligned}x &\equiv 17 \pmod{24} \\x &\equiv 29 \pmod{10} \\x &\equiv 49 \pmod{60}\end{aligned}$$

Exercise 1.3. Suppose I am trying to send a message to my friend. I encode the message in a three digit security code N . In order to decode the code, my friend has been given a cipher. Namely, he knows that my secret codes always satisfy

$$N \equiv 3 \pmod{7}, \quad N \equiv 1 \pmod{11}, \quad N \equiv 10 \pmod{13}.$$

What was the number N encoding my message? How can my friend figure out what it is? Why is the assumption that N is a three digit number important? (4 pts)

Exercise 1.4. Consider the polynomial

$$f(x) = 6x^4 - 30x^2 + 24 \in \mathbb{Z}[x]$$

In class we explained that this admits a unique factorization in terms of irreducible elements of $\mathbb{Z}[x]$. Write down the unique factorization explicitly and explain why each of the terms are irreducible (Hint: set $s = x^2$ and factor the resulting polynomial in s), (5 pts).