

HOMEWORK 8: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. EXERCISES

Exercise 1.1. Consider the infinite Galois extension $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$. We recall that we have an isomorphism $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$ by Theorem 1.10 (2) in the course notes. We now work out what infinite Galois theory is telling us in this particular case.

- (1) (4 pts) Show that we have an isomorphism $\mathbb{Z}_p^* \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ if p is odd and that $\mathbb{Z}_2^* \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$. Deduce the existence of a surjective continuous map $\hat{\mathbb{Z}}^* \rightarrow \hat{\mathbb{Z}}$ of profinite groups (Hint: use the p -adic exponential map).
- (2) (2 pts) Classify all closed subgroups of $\hat{\mathbb{Z}}^*$ using part (1) and the isomorphism $\hat{\mathbb{Z}}^* \simeq \prod_p \mathbb{Z}_p^*$. Deduce that $1 + n\hat{\mathbb{Z}}$, $\prod_p 1 + p^{e_p}\mathbb{Z}_p$ for $e_p \geq 1$, and $\langle \pm 1 \rangle$ define closed subgroups of $\hat{\mathbb{Z}}^*$.
- (3) (2 pts) Identify the (possibly infinite abelian extensions) that correspond to the closed subgroups $1 + n\hat{\mathbb{Z}}$, $\prod_p 1 + p^{e_p}\mathbb{Z}_p$ for $e_p \geq 1$, and $\langle \pm 1 \rangle$ described in (2) under infinite Galois Theory.

Exercise 1.2. (3 pts) Suppose K/\mathbb{Q}_p is a finite extension. Show that the intersection $\bigcap_{n \geq 1} (K^*)^n$ is trivial (Hint: reduce to checking it for an open finite index subgroup $V \subset \mathcal{O}_L^*$ and then argue as in the proof of Lemma 3.22 of the course notes using the exponential map).

Exercise 1.3. Using results already shown in class, explain why the following tuples (k, A, v, d) as in Definition 4.1 and 4.7 in the course notes, satisfy the conditions on the Henselian valuation and the Class Field axiom.

- (1) (3 pts) The tuple:
 - k/\mathbb{Q}_p is a p -adic field.
 - $A = \bar{k}^*$
 - $d : \text{Gal}(\bar{k}/k) \rightarrow \hat{\mathbb{Z}}$ is the map given by $\text{Gal}(\bar{k}/k) \rightarrow \text{Gal}(k^{\text{un}}/k) \simeq \hat{\mathbb{Z}}$ where k^{un}/k is the maximal unramified extension.
 - $v : A_k = k^* \rightarrow \hat{\mathbb{Z}}$ is the map given by the p -adic valuation v_K .
- (2) (3 pts) The tuple:
 - k is a finite field.
 - $d : \text{Gal}(\bar{k}/k) \rightarrow \hat{\mathbb{Z}}$ is the isomorphism of Example 2.15 of the notes.
 - A is the group \mathbb{Z} with its trivial action.
 - $v : A_k = \mathbb{Z} \rightarrow \hat{\mathbb{Z}}$ is the inclusion of \mathbb{Z} into its profinite completion.