HOMEWORK 7: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Let L/K be a finite extension of p-adic fields. Show that the following is true.*

1. ***(3 pts)*** *Show that* $\mathrm{Nm}_{L/K}(L^*) \subset K^*$ *is an open subgroup of finite index.*
2. ***(3 pts)*** *Show that, for all* $n \geq 1$*, the subgroup* $(K^*)^n \subset K^*$ *is an open subgroup of finite index.*

Exercise 1.2. ***(Kummer Theory)*** *Let K be a field of characteristic 0. Let $\overline{K}/K$ denote the algebraic closure. We write $\mu_n \subset \overline{K}^*$ for the subgroup defined by the elements $x \in \overline{K}^*$ such that $x^n = 1$. We note that this has the natural structure of an object in $\mathrm{Mod}_{\mathrm{Gal}(\overline{K}/K),\mathrm{cont}}$. We consider the short exact sequence*

$$0 \to \mu_n \to \overline{K}^* \to \overline{K}^* \to 0, \tag{1.1}$$

in $\mathrm{Mod}_{\mathrm{Gal}(\overline{K}/K),\mathrm{cont}}$, where the last map is given by the multiplication by n-map. We assume that $\mu_n \subset K^*$.

1. ***(1 pt)*** *Show that we have an isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ as $\mathrm{Gal}(\overline{K}/K)$-modules, where $\mathbb{Z}/n\mathbb{Z}$ has trivial Galois action.*
2. ***(2 pts)*** *Use the long exact cohomology sequence attached to (1.1) to deduce the existence of an injective map*
$$K^*/(K^*)^n \hookrightarrow \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), \mu_n).$$
$$a \mapsto \{\sigma \mapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}\},$$
3. ***(1 pt)*** *If $a \in K^\times$ and $\alpha \in \overline{K}^*$, show that $\alpha^n = a$, show that for every $\sigma \in \mathrm{Gal}(K(\alpha)/K)$*
$$\sigma(\alpha) = \zeta\alpha \quad \text{for some } \zeta \in \mu_n.$$
Deduce that $K(\alpha)/K$ is abelian of exponent dividing n.
4. ***(3 pts)*** *If $\Delta \subset K^\times$ contains $(K^\times)^n$, define*
$$L_\Delta = K\big(\sqrt[n]{a} \mid a \in \Delta\big).$$
Show that L_Δ/K is abelian of exponent dividing n. Assume $\Delta/(K^\times)^n$ is finite. Prove that
$$\mathrm{Gal}(L_\Delta/K) \simeq \Delta/(K^\times)^n$$
5. ***(2 pts)*** *Conclude that finite abelian extensions of K of exponent dividing n correspond to finite-index subgroups of $K^\times$ containing $(K^\times)^n$.*
6. ***(2 pts)*** *Let $K/\mathbb{Q}_p$ be a p-adic field. Describe the extension corresponding to the open finite index subgroup $(K^*)^n \subset K^*$ in 1.1 (2) under local class field theory.*
7. ***(2 pts)*** *Let $K/\mathbb{Q}_p$ be a p-adic field (not necessarily containing μ_ℓ) and let ℓ be a prime. Show that if $x \in \mathrm{Nm}_{L/K}(L^*)$ for all finite extensions L/K then x is an ℓth power.*