HOMEWORK 6: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Let $M \in \mathrm{Mod}_G$ be a G-module and let e denote the identity element of G. Suppose we have a homogenous 2-cocycle $\phi : G^3 \to M$ representing a class in $H^2(G, M)$. We recall that this means that*

$$\phi(gg_0, gg_1, g_2g) = g.\phi(g_0, g_1, g_2)$$

and

$$\phi(g_1, g_2, g_3) - \phi(g_0, g_2, g_3) + \phi(g_0, g_1, g_3) - \phi(g_0, g_1, g_2) = 0$$

Moreover, this is a coboundary if and only if

$$\phi(g_0, g_1, g_2) = \rho(g_1, g_2) - \rho(g_0, g_2) + \rho(g_0, g_1) \tag{1.1}$$

for $\rho : G^2 \to M$ satisfying the condition that $\rho(gg_0, gg_1) = \rho(g_0, g_1)$. We let $M[\phi]$ be the splitting module of ϕ. As an abelian group, this is given by

$$M[\phi] := M \oplus \bigoplus_{g \in G \setminus \{e\}} \mathbb{Z}x_g$$

and we equip it with the G-action

$$g.x_h = x_{gh} - x_g + \phi(e, g, gh) \tag{1.2}$$

where in this formula $x_e := \phi(e, e, e)$ and the usual G-action of M on the first coordinate. Show the following.

1. ***(2 pts)****Check that (1.2) gives rise to a well-defined G-action.*
2. ***(2 pts)*** *Combine the above formulae to show that ϕ is a cobounadry if and only if*

   $$\phi(e, g, gh) = g\rho(e, h) - \rho(e, gh) + \rho(e, g).$$

   for some $\rho : G^2 \to M$ as above.
3. ***(2 pts)*** *Show that the natural map $M \to M[\phi]$ given by the inclusion of M induces a natural map*

   $$H^2(G, M) \to H^2(G, M[\phi]),$$

   which will send ϕ to 0, by using (2) and setting $\rho(e, g) = x_g$ for all $g \in G$.
4. ***(2 pts)*** *Show that the natural map*

   $$\alpha : M[\phi] \to \mathbb{Z}[G]$$

   sending x_g to $[g] - 1$ is a homomorphism of G-modules. Deduce the existence of a a short exact sequence

   $$0 \to M \to M[\phi] \to I_G \to 0. \tag{1.3}$$

Exercise 1.2. *Let $L = \mathbb{Q}_p(\sqrt{p})$ and $K = \mathbb{Q}_p$. Show the following.*

1. ***(2 pts)*** *The natural map*

   $$\mathrm{Nm}_{L/K} : L^* \to K^*$$

   induces an isomorphism $L^*/\mathcal{O}_L^* \simeq K^*/\mathcal{O}_K^*$.
2. ***(2 pts)*** *Compute the cohomology groups $H^i_T(\mathrm{Gal}(L/K), \mathcal{O}_L^*)$ for all $i \in \mathbb{Z}$. Show that $H^0_T(\mathrm{Gal}(L/K), \mathcal{O}_K^*) \neq 0$.*