HOMEWORK 5: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Consider the short exact sequence in* Mod_G

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{1.1}$$

defined by the augmentation map. Here $\mathbb{Z}$ *has the trivial action. Show the following.*

(1) ***(2 pts)*** *By considering the long exact cohomology sequence of homology groups attached to (1.1) show that we have an isomorphism*

$$H_1(G, \mathbb{Z}) \simeq I_G/I_G^2,$$

where I_G^2 *is the ideal generated by products of elements in* I_G *with elements in* I_G.

(2) ***(2 pts)*** *Show that* I_G *is generated as an ideal by the elements* $(g-1)$.

(3) ***(2 pts)*** *Consider the map*

$$\phi : G \to I_G/I_G^2$$

sending $\phi(g)$ *to* $g-1$ *mod* I_G^2. *Show that this is a well-defined homomorphism where* I_G/I_G^2 *is equipped with its additive group structure.*

(4) ***(2 pts)*** *Show that* ϕ *factors through* $G^{\mathrm{ab}} := G/[G,G]$, *where* $[G,G]$ *is the subgroup of commutators and that the resulting map*

$$\phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to I_G/I_G^2$$

is an isomorphism. Conclude that $H_1(G,\mathbb{Z}) \simeq G^{\mathrm{ab}}$ *as abelain groups.*

Exercise 1.2. *Suppose we have a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. *Show the following.*

(1) ***(2 pts)*** *Check that we have a commutative diagram of the form*

$$\begin{array}{ccccccccccc}
 & & & & H_T^{-1}(G,A) & \longrightarrow & H_T^{-1}(G,B) & \longrightarrow & H_T^{-1}(G,C) & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{-1}(G,C) & \longrightarrow & A_G & \longrightarrow & B_G & \longrightarrow & C_G & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \overline{\mathrm{Nm}}_G} & & \downarrow{\scriptstyle \overline{\mathrm{Nm}}_G} & & \downarrow{\scriptstyle \overline{\mathrm{Nm}}_G} & & \\
 & & 0 & \longrightarrow & A^G & \longrightarrow & B^G & \longrightarrow & C^G & \longrightarrow & H^1(G,A) \longrightarrow \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & H_T^0(G,A) & \longrightarrow & H_T^0(G,B) & \longrightarrow & H_T^0(G,C) & & ,
\end{array} \tag{1.2}$$

where the second and third row are given by the long exact sequence for homology and cohomology, respectively, and the remaining arrows are the obvious ones.

(2) ***(2 pts)*** *Check that by applying the snake lemma to the middle part of (1.2) that we obtain a long exact sequence*

$$\cdots \to H_T^{i-1}(G,C) \to H_T^i(G,A) \xrightarrow{H_T^i(f)} H_T^i(G,B) \xrightarrow{H_T^i(g)} H_T^i(G,C) \to H_T^{i+1}(G,A) \to \cdots \tag{1.3}$$

for all $i \in \mathbb{Z}$.

Exercise 1.3 (Division Algebras over Finite and p-adic Fields). *Fix a prime p and let $q = p^f$ for some integer $f \geq 1$. Let $g \geq 1$ and consider the finite cyclic Galois extension $\mathbb{F}_{q^g}/\mathbb{F}_q$. We write $\mathrm{Frob}_q : \mathbb{F}_{q^g} \to \mathbb{F}_{q^g}$ for the qth power Frobenius $x \mapsto x^q$.*

Similarly, we consider $\mathbb{Q}_q/\mathbb{Q}_p$ to be an unramified extension of degree f (i.e the natural map $\mathrm{Gal}(\mathbb{Q}_{p^g}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p)$) given by reduction mod p is an isomorphism. We let $\mathbb{Q}_{q^g}/\mathbb{Q}_q$ be the unramified extension of $\mathbb{Q}_q$ of degree g, and write $\sigma \in \mathrm{Gal}(\mathbb{Q}_{q^g}/\mathbb{Q}_q)$ to be the natural lift of Frob_q under the isomorphism $\mathrm{Gal}(\mathbb{Q}_{q^g}/\mathbb{Q}_q) \to \mathrm{Gal}(\mathbb{F}_{q^g}/\mathbb{F}_q)$ given by mod p-reduction.

Recall that, for any finite extension $K/\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and uniformizing element $\pi \in \mathcal{O}_K$, we have an isomorphism

$$K^* \simeq \mathcal{O}_K^* \times \langle \pi^{\mathbb{Z}} \rangle, \tag{1.4}$$

of abelian groups. Moreover, we have a short exact sequence

$$0 \to U_K^1 \to \mathcal{O}_K^* \to (\mathcal{O}_K/\pi)^* \to 0 \tag{1.5}$$

of abelian groups, where the last map is given by mod π-reduction. Show the following.

(1) ***(2 pts)*** *Consider the norm map*

$$\mathrm{Nm}_{\mathbb{F}_{q^g}/\mathbb{F}_q} : \mathbb{F}_{q^g}^* \to \mathbb{F}_q^*.$$
$$x \mapsto \prod_{i=0}^{g-1} (\mathrm{Frob}_q)^i(x).$$

Show that $\mathrm{Nm}_{\mathbb{F}_{q^g}/\mathbb{F}_q}$ is surjective and compute the kernel. Describe the Tate cohomology groups $H_T^i(\mathrm{Gal}(\mathbb{F}_{q^g}/\mathbb{F}_q), \mathbb{F}_{q^g}^*)$ for all $i \in \mathbb{Z}$.

(2) ***(1 pt)*** *Using part (1), deduce the following.*

Theorem 1.4. ***(The Artin-Wedderburn Theorem)*** *Let $\mathbb{F}_q$ be a finite field then every finite-dimensional division algebra over $\mathbb{F}_q$ splits over a finite extension of $\mathbb{F}_q$.*

(3) ***(1 pt)*** *Show that the short exact sequence (1.5) when specialized to $K = \mathbb{Q}_{q^g}$ and $K = \mathbb{Q}_q$ gives rise to a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^1_{\mathbb{Q}_{q^g}} & \longrightarrow & \mathcal{O}_{\mathbb{Q}_{q^g}^*} & \longrightarrow & \mathbb{F}_{q^g}^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Nm}_{\mathbb{Q}_{q^g}/\mathbb{Q}_q}} & & \downarrow{\scriptstyle \mathrm{Nm}_{\mathbb{Q}_{q^g}/\mathbb{Q}_q}} & & \downarrow{\scriptstyle \mathrm{Nm}_{\mathbb{F}_{q^g}/\mathbb{F}_q}} & & \\ 0 & \longrightarrow & U^1_{\mathbb{Q}_q} & \longrightarrow & \mathcal{O}_{\mathbb{Q}_q^*} & \longrightarrow & \mathbb{F}_{q^g}^* & \longrightarrow & 0 \end{array} \tag{1.6}$$

of short exact sequences.

(4) ***(3 pts)*** *Show that $\mathrm{Nm}_{\mathbb{Q}_{q^g}/\mathbb{Q}_q} : U^1_{\mathbb{Q}_{q^g}} \to U^1_{\mathbb{Q}_q}$ is surjective (Hint: Use Hensel's Lemma).*

(5) ***(2 pts)*** *Combine (1), (3), and (4) with the product decomposition 1.4 to deduce that*

$$\mathbb{Q}_q^*/\mathrm{Nm}_{\mathbb{Q}_{q^g}/\mathbb{Q}_q}(\mathbb{Q}_{q^g}^*) \simeq \mathbb{Z}/g\mathbb{Z}.$$

What can you conclude about division algebras over $\mathbb{Q}_q$?