

HOMEWORK 4: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. EXERCISES

Exercise 1.1. We recall in class we introduced the following examples of central simple algebras attached to a finite cyclic extension L/K . We write σ for the generator of the Galois group $\text{Gal}(L/K)$. For $a \in K^*$, we define the cyclic algebra

$$A = (L/K, \sigma, a)$$

as follows. We consider the K -algebra

$$A := \bigoplus_{i=0}^{n-1} Lu^i$$

generated by L and the symbol u subject to the relationship that

$$u^n = a,$$

and, for all $x \in L$, we have that

$$ux = \sigma(x)u.$$

Show the following claims.

(1) **(Basic properties)**

- (a) (2 pts) Show that $A = (L/K, \sigma, a)$ is a central simple K -algebra of dimension n^2 over K .
- (b) (2 pts) Show that the natural inclusion $L \hookrightarrow A$ defines a maximal commutative subfield of A .
- (c) (4 pts) Prove that

$$A \otimes_K L \cong M_n(L).$$

of central simple L -algebras. (Hint: Recall that we have an isomorphism $L \otimes_K L \simeq \prod_{\tau \in \text{Gal}(L/K)} L$ for any Galois extension L/K . In the case of a cyclic extension, this takes the form of sending $x \otimes y \mapsto (\sigma^i(x)y)_{i=0}^{n-1}$ in the case of a cyclic extension. Use this map to define the isomorphism $A \otimes_K L \simeq M_n(L)$.)

- (2) **(Relationship to the Hamilton Quaternions)** Let $K = \mathbb{R}$ and $L = \mathbb{C}$, and let σ be complex conjugation. Consider the cyclic algebra

$$A = (\mathbb{C}/\mathbb{R}, \sigma, -1)$$

- (a) (1 pt) Let $u \in A$ be the adjoined generator, so that $u^2 = -1$ and $uz = \bar{z}u$ for $z \in \mathbb{C}$. Define elements

$$i := \sqrt{-1} \in \mathbb{C} \subset A, \quad j := u, \quad k := ij.$$

Show that $i^2 = j^2 = k^2 = -1$ and that

$$ij = k, \quad ji = -k,$$

- (b) (1 pt) Show that every element of A can be written uniquely as

$$a + bi + cj + dk \quad (a, b, c, d \in \mathbb{R}),$$

and conclude that A is isomorphic to the classical Hamilton quaternion division algebra \mathbb{H} described in class.

- (3) (**The Splitting Criterion.**) We will now be interested in showing the following Theorem. We let $\text{Nm}_{L/K} : L^* \rightarrow K^*$ denote the norm map.

Theorem 1.2. *The central simple algebra $A = (L/K, \sigma, a)$ is isomorphic to $M_n(K)$ if and only if $a = \text{Nm}_{L/K}(b)$ for some $b \in L^*$.*

Assume that $a = \text{Nm}_{L/K}(b)$, for some $b \in L^\times$.

- (a) (1 pt) Set $v := b^{-1}u \in A$. Compute v^i for $i = 0, \dots, n-1$ and show that $v^n = 1$.
 (b) (2 pts) Consider the element $e = \sum_{i=0}^{n-1} v^i \in A$. Show that Ae is a nonzero left ideal of A of K -dimension n . Deduce that $A \cong M_n(K)$, by arguing similarly to the proof of Wedderburn's theorem. Conclude the converse direction of Theorem 1.2.

We now establish the forward direction. Suppose that $A \simeq M_n(K)$.

- (c) (1 pt) Let V be a simple left A -module. Show that $\dim_K V = n$ and that, via the embedding $L \hookrightarrow A$, the space V becomes a 1-dimensional vector space over L .
 (d) (1 pt) Choose $0 \neq v \in V$. Since $u^n = a \in K \subset L$, show that

$$u^n v = av.$$

- (e) (1 pt) Because V is 1-dimensional over L , there exists $\lambda \in L^\times$ with $uv = \lambda v$. Using the relation $ux = \sigma(x)u$, prove that

$$u^n v = \lambda \sigma(\lambda) \cdots \sigma^{n-1}(\lambda) v = \text{Nm}_{L/K}(\lambda) v.$$

- (f) (1 pt) Combine the two previous steps to deduce that $a = \text{Nm}_{L/K}(\lambda)$.
 (4) (2 pts) Combine Theorem 1.2 with Wedderburn's theorem to conclude that $A = (L/K, \sigma, a)$ is isomorphic to $M_r(D)$ for $D \not\cong K$ if and only if $a \neq \text{Nm}_{L/K}(b)$ for some $b \in L^*$.

Exercise 1.3. We recall the definition of the Brauer group from class.

Definition 1.4. Fix a field K , we define the following.

- (1) We say two finite dimensional central simple algebras A, B over K are equivalent $A \sim B$ if there exists a finite-dimensional division algebra D over K and positive integers $n, m \geq 1$ such that

$$A \simeq M_n(D)$$

and

$$B \simeq M_m(D),$$

where we note that such a D always exists by Wedderburn's Theorem (Theorem ??). We denote the equivalence class of such a finite dimensional central simple algebra A over k by $[A]$.

- (2) We write $\text{Br}(k)$ for the set of equivalence classes of finite central simple algebras over k .
 (3) For a finite extension L/K , we write $\text{Br}(L/K) \subset \text{Br}(K)$ for the subset of $[A]$ such that $A \otimes_K L \simeq M_n(L)$ for some $n \geq 1$. This is referred to as the Brauer group of the finite extension L/K .

We want to show that this does indeed deserve the name of being called a group. Show that the following is true.

- (1) (2 pts) If A, B are two finite dimensional central simple algebras over K . Show that $A \otimes_K B$ is again a finite dimensional central simple algebra over K .
 (2) (1 pt) Check that the map

$$\text{Br}(K) \times \text{Br}(K) \rightarrow \text{Br}(K)$$

$$([A], [B]) \mapsto [A \otimes_K B]$$

gives rise to a well-defined binary, commutative, and associative operation on the set of equivalence classes of finite-dimensional central simple algebras over K with identity element K .

- (3) (2 pts) *Given a finite-dimensional central simple algebra A of dimension n over K , show that*

$$A \otimes A^{\text{op}} \simeq M_n(K).$$

Conclude that $\text{Br}(K)$ is a commutative group. For a finite extension L/K , prove that $\text{Br}(L/K) \subset \text{Br}(K)$ is a subgroup.