HOMEWORK 3: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Suppose we are given a two cocycle $\phi(g,h) : G^2 \to M$ for $M \in \mathrm{Mod}_G$ then we can produce an extension $0 \to M \to E \to G \to 0$, as follows. As a set, we define $E := M \times G$. However, we endow E with the binary operation*

$$(m,g).(m',h) := (m + g.m' + \phi(g,h), gh).$$

Check the following, by using the two cocycle condition on M. Explicitly, for $g_1, g_2, g_3 \in G$ we recall that this says that

$$g_1\phi(g_2,g_3) - \phi(g_1g_2,g_3) + \phi(g_1,g_2g_3) - \phi(g_1,g_2) = 0.$$

(1) ***(2 pts)****Show that the element $(-\phi(e,e),e)$ is a two side identity element for the group operation described above.*
(2) ***(2 pts)*** *Show that the binary operation is associative.*
(3) ***(2 pts)*** *Check that $(-g^{-1}.m - \phi(g^{-1},g) - \phi(e,e), g^{-1})$ is a 2-sided inverse to (m,g).*

In particular, by (1)-(3) E is a group. We note that there are natural maps

$$E \to G$$
$$(m,g) \mapsto g$$

and natural maps

$$M \to E$$
$$m \mapsto (m - \phi(e,e), e)$$

which will sit in a short exact sequence

$$0 \to M \to E \to G \to 0$$

of groups.

(4) ***(3 pts)*** *Suppose that $\phi'(g,h) = \phi(g,h) + g.b(h) - b(gh) + b(g)$ for some function $b : G \to M$ and let E' be the group attached to ϕ' as above. Show that the map*

$$\alpha : E \to E'$$

defined by $(m,g) \mapsto (m - b(g), g)$ is an isomorphism and that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}_M} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_G} & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Exercise 1.2. *Let L/K be a finite Galois extension. Consider the additive group $(L,+) \in \mathrm{Mod}_{\mathrm{Gal}(L/K)}$ show the following is true.*

(1) ***(2 pts)*** *Show, using the explicit description of H^1 provided in class that one has $H^1(\mathrm{Gal}(L/K), L) = 0$ (Hint: You need to use the normal basis theorem which says that there exists $\alpha \in L$ such that its conjugates under $\mathrm{Gal}(L/K)$ form a basis for L as a K-vector space).*

(2) ***(2 pts)*** *Prove that L is actually an induced* $\mathrm{Gal}(L/K)$*-module and conclude that*

$$H^i(\mathrm{Gal}(L/K), L) = 0$$

for all $i \geq 1$.

(3) *(Extra Credit: **2** pts) Can you show the analogue of the above results if L/K is an infinite Galois Extension?*

Exercise 1.3. *Show the following.*

(1) ***(2 pts)*** *Show that Proposition 2.55 in the notes is true in the case of $n = 0$.*

(2) ***Extra Credit: (4 pts)*** *Can you explain how to prove this in the case of higher n using the dimension shifting principle sketched in class?*

Exercise 1.4. *[Ser94, Section 2.4] Let $f : G \to G'$ be any continuous morphism of profinite groups and p be a prime number.*

(1) ***(3 pts)*** *Show the equivalence of the following two properties.*
- *The index of $f(G)$ in G' is prime to p*
- *For any G'-module M equal to its p-primary part, the homomorphism*

$$H^1(G', M) \to H^1(G, M)$$

  is injective.

(2) ***(4 pts)*** *Show the equivalence of the following properties.*
- *f is surjective.*
- *For any G'-module M, the homomorphism*

$$H^1(G', M) \to H^1(G, M)$$

  is injective.
- *For any finite G'-module M, the homomorphism*

$$H^1(G', M) \to H^1(G, M)$$

  is injective.

References

[Ser94] Jean-Pierre Serre. *Cohomologie galoisienne*. Fifth. Vol. 5. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1994, pp. x+181. ISBN: 3-540-58002-6. DOI: `10.1007/BFb0108758`. URL: `https://doi.org/10.1007/BFb0108758`.