HOMEWORK 2: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Let $G = C_p$ be the cyclic group of order p, and let $k = \mathbf{F}_p$. We consider the G-modules*

$$M = k[G] \qquad N = k,$$

where $k[G]$ is the group ring of G introduced above. We consider the natural map.

$$\varepsilon : k[G] \to k, \qquad \varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g.$$

which is known as the augmentation morphism

(1) ***(1 pt)*** *Show that ϵ is sujrective.*
(2) ***(1 pt)*** *Show that M^G is one-dimensional and is spanned by*

$$s = \sum_{g\in G} g.$$

(3) ***(1 pt)*** *Compute the induced map on G-invariants*

$$\varepsilon^G : M^G \longrightarrow N^G$$

and show that it is the zero map.

Exercise 1.2. ***(3 pts)*** *Show that, for every $A \in \mathrm{Mod}_G$, there exists an injection $A \hookrightarrow M$ in Mod_G such that M is injective (Hint: first think about the case of usual abelian groups (e.g when G is trivial). We already discussed examples of injective objects in this category in Exercise 1.4 of HW1.)*

Exercise 1.3. ***(3 pts)*** *Show that if we are given a map $f : M \to N$ in Mod_G such that $f = 0$ then, for any lifts f^i filling in a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & \cdots \\ & & \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \\ 0 & \longrightarrow & N & \longrightarrow & J^0 & \xrightarrow{d^0} & J^1 & \xrightarrow{d^1} & \cdots \end{array}$$

between injective resolutions $M \to I^*$ and $N \to I^*$, we may construct morphisms $g^i : I^{i+1} \to J^i$ such that

$$f^i = g^i \circ d^i + d^{i-1} \circ g^{i-1}.$$

(Hint: Proceed by induction on i and use the lifting property for injective objects).

Exercise 1.4. *Show the following.*

(1) ***(2 pts)*** *Show that if $M \in \mathrm{Mod}_G$ is injective then it is acyclic. I.e that*

$$H^i(G, M) = 0.$$

(Hint: We discussed how an injective map from an injective module must split, so apply this to the injective resolution.).

(2) ***(2 pts)*** *Let $M \to M_*$ be an acyclic resolution of $M \in \mathrm{Mod}_G$. We apply G-invariants to the terms of the resolution and consider the resulting complex*

$$M_0^G \xrightarrow{(d^0)^G} M_1^G \to \cdots M_i^G \xrightarrow{(d^i)^G} M_{i+1}^G \to \cdots$$

and consider, for all $i \geq 0$, the resulting cohomology

$$\mathrm{Ker}((d^i)^G)/\mathrm{Im}((d^{i-1})^G).$$

where we set $d_{-1}^G : 0 \to M_0^G$. Show that this is isomorphic to $H^i(G, M)$ (Hint: inductively apply the long exact cohomology sequence).

Exercise 1.5. *Show that the following is true.*

(1) ***(2 pts)*** *Show that the G-module $C^n(G, M)$ introduced in class is expressible as $\mathrm{Ind}_e^G(C^n(G, M)_0)$, where $C^n(G, M)_0$ is the subset of $C^n(G, M)$ of functions for which $\phi(g_0, \ldots, g_n) = 0$ when $g_0 \neq e$. In particular, we have that $C^0(G, M) = \mathrm{Ind}_e^G(M)$ which using Frobenius reciprocity is equipped with a natural G-equivariant embedding $M \to \mathrm{Ind}_e^G(M)$.*

(2) ***(1 pt)*** *Check that the map d^n introduced in class is indeed G-equivariant for the G-action on the $C^n(G, M)$ described in class.*

(3) ***(1 pt)*** *Show that for all $n \geq 0$, we have that*

$$d^{n+1} \circ d^n = 0.$$

(4) ***(2 pts)*** *Check that we have an exact complex of G-modules*

$$0 \to M \to C^0(G, M) \xrightarrow{d^0} C^1(G, M) \xrightarrow{d^1} \cdots,$$

and deduce that we have an isomorphism

$$H^n(G, M) \simeq \mathrm{Ker}((d^n)^G)/\mathrm{Im}((d^{n-1})^G)).$$

for all $n \geq 0$, where we set $(d^{-1})^G : 0 \to C^0(G, M)^G$.

(5) ***(3 pts)*** *Show, for all $n \geq 0$, that we have an isomorphism*

$$C^n(G, M)^G \simeq C(G^n, M),$$

where $C(G^n, M)$ denotes the space of all functions $\phi : G^n \to M$. Show that, under this isomorphism, we have an identification of

$$(d^n)^G : C(G^n, M) \to C(G^{n+1}, M)$$

with

(1.1)

$$(d^n)^G(\phi)(g_1, \ldots, g_{n+1}) = g_1.\phi(g_2, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \ldots, g_n).$$