HOMEWORK 12: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Prove the following.*

(1) ***(2 pts)****Recall that the class number of* $\mathbb{Q}[\sqrt{-5}]$ *is* 2*. Show that* $\mathbb{Q}[\sqrt{-1}, \sqrt{-5}]$ *is the Hilbert class field of* $\mathbb{Q}[\sqrt{-5}]$*. Choose a non-principal ideal generating the class group. Show it becomes principal inside* $\mathbb{Q}[\sqrt{-1}, \sqrt{-5}]$.

(2) *Let* $p(x) \in \mathbb{Z}[x]$ *be a polynomial with squarefree discriminant* D*. Let* $K = \mathbb{Q}[x]/(p(x))$ *and let* $L/\mathbb{Q}$ *be the Galois closure of* K*.*

 (a) ***(2 pts)*** *Prove that* $\mathrm{Gal}(L/\mathbb{Q})$ *has Galois group* S_n *over* $\mathbb{Q}$ *(Hint: there is a natural embedding* $\mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow S_n$ *given by permuting the roots of* $p(x)$*. Look at the projection to* $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$ *and relate that to the sign of the permutation).*

 (b) ***(3 pts)*** *Prove that* L *is unramified over* $\mathbb{Q}[\sqrt{D}]$ *by analyzing decomposition groups, as in Exercise 1.7 of the course notes.*

(3) ***(2 pts)*** *Show that the class field* $\mathbb{Q}(\sqrt{-23})$ *has a* 3*-torsion element. By Corollary 4.84 of the course notes, it in turn admits an everywhere abelian unramified extension of degree* 3*. Describe this extension explicitly using Part (2) (Hint: take* $p(x) = x^3 - x - 1$*).*

Exercise 1.2. *Let* L/K *be a Galois extension of number fields. Consider the natural map* $\mathrm{Cl}(K) \to \mathrm{Cl}(L)^{\mathrm{Gal}(L/K)}$ *described in class.*

(1) ***(2 pts)*** *Given an example for which the map* $\mathrm{Cl}(K) \to \mathrm{Cl}(L)^{\mathrm{Gal}(L/K)}$ *is not injective (Hint: use the principal ideal theorem).*

(2) ***(2 pts)*** *Show that the map* $\mathrm{Cl}(K) \to \mathrm{Cl}(L)^{\mathrm{Gal}(L/K)}$ *is not surjective if we take* $K = \mathbb{Q}$ *and* $L = \mathbb{Q}(\sqrt{-21})$.