HOMEWORK 9: MATH 223B (GALOIS COHOMOLOGY AND CLASS FIELD THEORY)

1. Exercises

Exercise 1.1. *Let $\mathbb{A}_K^* \subset \mathbb{A}_K$ denote the set of multiplicative units inside $\mathbb{A}_K$ and let $\mathbb{I}_K$ denote the set of ideles, as defined in Definition 4.4.2 of class. Show that the following is true.*

(1) ***(2 pts)*** *Show that, as subgroups of $\prod_v K_v$, we have an equality $\mathbb{A}_K^* = \mathbb{I}_K$.*
(2) ***(1 pt)*** *Show that the subspace topology on $\mathbb{A}_K^* \subset \mathbb{A}_K$ does not agree with the topology on $\mathbb{I}_K$ as a restricted product under the identification of (1), and that the map $\mathbb{I}_K \to \mathbb{A}_K$ is continuous.*
(3) ***(4 pts)*** *Consider the natural embedding*

$$\mathbb{I}_K \to \mathbb{A}_K \times \mathbb{A}_K$$
$$x \mapsto (x, x^{-1}).$$

Show that the restricted product topology on $\mathbb{I}_K$ is given by the subspace topology for $\mathbb{A}_K \times \mathbb{A}_K$ equipped with the product topologyunder this embedding. Moreover, show that this embedding has closed image.

Exercise 1.2. *(**3 pts**) Let K be a number field, and let $\mathbb{A}_K$ denote its ring of adèles. Show that the diagonal embedding*

$$\iota_K : K \hookrightarrow \mathbb{A}_K$$

has discrete image in the natural topology on $\mathbb{A}_K$. (Hint: Use the product formula. Try to find an open neighborhood U of $0 \in A_K$ such that $U \cap K = \{0\}$)