

COMPATIBILITY OF THE FARGUES-SCHOLZE AND GAN-TAKEDA LOCAL LANGLANDS

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ABSTRACT. Given a prime p , a finite extension L/\mathbb{Q}_p , a connected p -adic reductive group G/L , and a smooth irreducible representation π of $G(L)$, Fargues-Scholze [FS21] recently attached a semisimple L -parameter to such π , giving a general candidate for the local Langlands correspondence. It is natural to ask whether this construction is compatible with known instances of the correspondence after semisimplification. For $G = \mathrm{GL}_n$ and its inner forms, Fargues-Scholze and Hansen-Kaletha-Weinstein [HKW21] showed that the correspondence is compatible with the correspondence of Harris-Taylor/Henniart [Hen00; HT01]. We verify a similar compatibility for $G = \mathrm{GSp}_4$ and its unique non-split inner form $G = \mathrm{GU}_2(D)$, where D is the quaternion division algebra over L , assuming that L/\mathbb{Q}_p is unramified and $p > 2$. In this case, the local Langlands correspondence has been constructed by Gan-Takeda and Gan-Tantono [GT11; GT14]. Analogous to the case of GL_n and its inner forms, this compatibility is proven by describing the Weil group action on the cohomology of a local Shimura variety associated to GSp_4 , using basic uniformization of abelian type Shimura varieties due to Shen [She20], combined with various global results of Kret-Shin [KS23] and Sorensen [Sor10] on Galois representations in the cohomology of global Shimura varieties associated to inner forms of GSp_4 over a totally real field. After showing the parameters are the same, we apply some ideas from the geometry of the Fargues-Scholze construction explored recently by Hansen [Han20]. This allows us to give a more precise description of the cohomology of this local Shimura variety, verifying a strong form of the Kottwitz conjecture in the process.

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1. INTRODUCTION

1.1. Background and Main Theorems. Fix distinct primes $\ell \neq p$, let \mathbb{Q}_p denote the p -adic numbers, and let G/\mathbb{Q}_p be a connected reductive group. Set $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ to be the completion of the algebraic closure of \mathbb{Q}_p . We fix an isomorphism $i : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$. Let $W_{\mathbb{Q}_p}$ be the Weil group of \mathbb{Q}_p and set \hat{G} to be the reductive group over $\overline{\mathbb{Q}_\ell}$ with root datum dual to G . Let Q be the finite quotient through which $W_{\mathbb{Q}_p}$ acts on \hat{G} . We define the L -group ${}^L G := Q \ltimes \hat{G}$. We let $\Pi(G)$ denote the set of isomorphism classes of smooth irreducible representations of the p -adic group $G(\mathbb{Q}_p)$, and let $\Phi(G)$ denote the set of L -parameters, i.e the set of conjugacy classes of homomorphisms

$$\phi : W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G(\mathbb{C})$$

where $\mathrm{SL}_2(\mathbb{C})$ acts via an algebraic representation and $W_{\mathbb{Q}_p}$ acts via a continuous semisimple homomorphism in a way that commutes with the natural projection ${}^L G(\mathbb{C}) \rightarrow Q$, where ${}^L G(\mathbb{C})$ is endowed with the discrete topology. The local Langlands correspondence is a conjectural map

$$\mathrm{LLC}_G : \Pi(G) \rightarrow \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

that builds a bridge between L -parameters and the smooth irreducible representations of $G(\mathbb{Q}_p)$. Conjecturally (under some additional constraints on $\Phi(G)$ if G is not split), these maps should be surjective with finite fibers called L -packets and satisfy various properties such as compatibility with products, maps of L -groups, character twists, as well as L , ϵ , and γ -factors. Moreover, one expects that the correspondence is uniquely characterized by some such finite list of properties.

In general, the existence and uniqueness of such a correspondence is completely unknown. However, very recently, Fargues and Scholze [FS21], using the action of the excursion algebra on the moduli space of G -bundles on the Fargues-Fontaine curve, were able to construct a completely general candidate, analogous to the work of V. Lafforgue in the function field setting [Laf18]. Namely, they construct a map

$$\mathrm{LLC}_G^{\mathrm{FS}} : \Pi(G) \rightarrow \Phi^{\mathrm{ss}}(G)$$

$$\pi \mapsto \phi_\pi^{\mathrm{FS}},$$

where $\Phi^{\text{ss}}(G)$ denotes the set of conjugacy classes of continuous semisimple maps

$$\phi : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

that commute with the projection ${}^L G(\overline{\mathbb{Q}}_\ell) \rightarrow Q$ as above. Fargues and Scholze showed that their map has several good properties such as compatibility with parabolic induction; however, one would also like to check that this correspondence agrees with known instances of the local Langlands correspondence. Precisely, given a candidate for the local Langlands correspondence

$$\begin{aligned} \text{LLC}_G : \Pi(G) &\rightarrow \Phi(G) \\ \pi &\mapsto \phi_\pi \end{aligned}$$

we expect a commutative diagram of the form

$$\begin{array}{ccc} \Pi(G) & \xrightarrow{\text{LLC}_G} & \Phi(G) \\ & \searrow \text{LLC}_G^{\text{FS}} & \downarrow (-)^{\text{ss}} \\ & & \Phi^{\text{ss}}(G) \end{array}$$

where the semisimplification map $(-)^{\text{ss}}$ precomposes an L -parameter $\phi \in \Phi(G)$ with the map

$$g \in W_{\mathbb{Q}_p} \mapsto \left(g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{-\frac{1}{2}} \end{pmatrix} \right) \in W_{\mathbb{Q}_p} \times \text{SL}_2(\mathbb{C})$$

and then applies the fixed isomorphism $i^{-1} : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell$, where $|\cdot| : W_{\mathbb{Q}_p} \rightarrow W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^* \rightarrow \mathbb{C}^*$ is the norm character. We make the following definition.

Definition 1.1. For $\pi \in \Pi(G)$, we say that a local Langlands correspondence LLC_G is compatible with the Fargues-Scholze local Langlands correspondence if we have an equality: $\phi_\pi^{\text{FS}} = \phi_\pi^{\text{ss}}$, as conjugacy classes of semi-simple L -parameters.

For GL_n , the local Langlands correspondence was constructed by Harris-Taylor/Henniart [Hen00; HT01] and is uniquely characterized by the preservation of L , ϵ , and γ -factors. In this case, compatibility with the Fargues-Scholze local Langlands correspondence follows from the description of the cohomology of the Lubin-Tate and Drinfeld towers proven in [HT01] and was verified by Fargues and Scholze [FS21, Theorem I.9.6]. The main goal of this note is to extend compatibility of the correspondence to GSp_4 and its inner form. To this end, we now fix a finite extension L/\mathbb{Q}_p and set G to be $\text{Res}_{L/\mathbb{Q}_p} \text{GSp}_4$ and J to be its unique non-split inner form $\text{Res}_{L/\mathbb{Q}_p} \text{GU}_2(D)$, where D/L is the quaternion division algebra. In this case, the local Langlands correspondence has been constructed by Gan-Takeda and Gan-Tantono, respectively [GT11; GT14]. It is constructed from the local Langlands correspondence for GL_n and theta lifting and admits a similar unique characterization in terms of the preservation of L , ϵ , and γ factors. We note that we can and do identify $\Phi(G)$ and $\Phi(J)$ with a subset of homomorphisms:

$$\phi : W_L \times \text{SL}_2(\mathbb{C}) \rightarrow \hat{G}(\mathbb{C}) = \text{GSpin}_5(\mathbb{C}) \simeq \text{GSp}_4(\mathbb{C})$$

This allows us to introduce a bit of terminology. Namely, we say that a parameter ϕ in $\Phi(G)$ or $\Phi(J)$ is supercuspidal if the $\text{SL}_2(\mathbb{C})$ -factor acts trivially and ϕ does not factor through any proper

Levi subgroup of GSp_4 . This terminology is justified by the fact that this is precisely the case when the L -packets over ϕ contain only supercuspidal representations. In what follows, we will often abuse notation and drop the superscript $(-)^{\mathrm{ss}}$ when speaking about such parameters, as in this case it merely corresponds to forgetting the trivially acting $\mathrm{SL}_2(\mathbb{C})$ -factor and applying the isomorphism i^{-1} . We now come to our main theorem.

Theorem 1.1. *The following is true.*

- (1) *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*
- (2) *If L/\mathbb{Q}_p is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Remark 1.1. As will be explained more below, the restrictions in the case where the parameter is supercuspidal are necessary to apply basic uniformization of the generic fiber of abelian type Shimura varieties due to Shen [She20]. If one were not to impose this assumption, the relevant Shimura varieties would have bad reduction at p , which, to the best of our knowledge, prevents the methods of Shen from working. In particular, if one could establish the expected description of basic locus in the sense of the isomorphism (2) of Definition 4.1, for Shimura varieties associated to the group $\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}$, where \mathbf{G} is an inner form of GSp_4 over a totally real field F with an inert prime p such that $F_p \simeq L$ for L any extension then our result would hold in complete generality.

As mentioned above, the proof of compatibility for GL_n uses the results of Harris-Taylor [HT01] on the cohomology of the Lubin-Tate/Drinfeld Towers. In particular, if one looks at the rigid generic fiber of the Lubin-Tate tower

$$\lim_{m \rightarrow \infty} \mathrm{LT}_{n,m,\check{\mathbb{Q}}_p}$$

a tower of $n - 1$ -dimensional rigid spaces over $\check{\mathbb{Q}}_p$, for fixed $n \geq 1$ and varying $m \geq 1$, where $\check{\mathbb{Q}}_p$ denotes the completion of the maximal unramified extension of \mathbb{Q}_p . The cohomology of this tower

$$R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell) := \mathrm{colim}_{m \rightarrow \infty} R\Gamma_c(\mathrm{LT}_{n,m,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

based changed to \mathbb{C}_p carries commuting actions of $\mathrm{GL}_n(\mathbb{Q}_p)$ and $D_{\frac{1}{n}}^*$, the units in the division algebra over \mathbb{Q}_p of invariant $\frac{1}{n}$, as well as an action of the Weil group $W_{\mathbb{Q}_p}$. In particular, given $\pi \in \Pi(\mathrm{GL}_n)$ (resp. $\rho \in \Pi(D_{\frac{1}{n}}^*)$), we can consider the complexes

$$R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi] := R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell) \otimes_{\mathcal{H}(\mathrm{GL}_n)}^{\mathbb{L}} \pi$$

and

$$R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\rho] := R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell) \otimes_{\mathcal{H}(D_{\frac{1}{n}}^*)}^{\mathbb{L}} \rho$$

where $\mathcal{H}(\mathrm{GL}_n) := C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ (resp. $\mathcal{H}(D_{\frac{1}{n}}^*)$) is the usual smooth Hecke algebra of G (resp. $D_{\frac{1}{n}}^*$). Then the key result of Harris-Taylor and later refined by Boyer is as follows.

Theorem 1.2. [HT01; Boy09] Fix $\pi \in \Pi(\mathrm{GL}_n)$, a supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, let

$$\mathrm{JL} : \Pi(D_{\frac{1}{n}}^*) \rightarrow \Pi(\mathrm{GL}_n(\mathbb{Q}_p))$$

be the map defined by the Jacquet-Langlands correspondence and $\rho := \mathrm{JL}^{-1}(\pi) \in \Pi(D_{\frac{1}{n}}^*)$ a Jacquet-Langlands lift of π . Then the complexes $R\Gamma_c(G, b, \mu)[\pi]$ and $R\Gamma_c(G, b, \mu)[\rho]$ are concentrated in middle degree $n - 1$. The middle degree cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ is isomorphic to

$$\rho \boxtimes \phi_\pi^\vee \otimes |\cdot|^{(1-n)/2}$$

as a $D_{\frac{1}{n}}^* \times W_{\mathbb{Q}_p}$ representation. Similarly, the middle degree cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ is isomorphic to

$$\pi \boxtimes \phi_\pi \otimes |\cdot|^{(1-n)/2}$$

where $\phi_\pi \in \Phi^{\mathrm{ss}}(G)$ is the (semisimplified) L -parameter associated to π by Harris-Taylor.

To see why this result is relevant for compatibility, we invoke the observation, due to Scholze-Weinstein [SW13; SW20], that, using Grothendieck-Messing theory, the Lubin-Tate tower at infinite level

$$\mathrm{LT}_{n,\infty} := \lim_{m \rightarrow \infty} \mathrm{LT}_{n,m,\check{\mathbb{Q}}_p}$$

is representable by a space admitting a moduli interpretation as a space of shtukas over the Fargues-Fontaine curve; namely, the space denoted $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)_\infty$ in the notation of [SW20], where $b \in B(\mathrm{GL}_n)$ is an element in the Kottwitz set of GL_n corresponding to a rank n isocrystal of slope $\frac{1}{n}$ and $\mu = (1, 0, \dots, 0, 0)$ is a dominant cocharacter of GL_n . If X denotes the Fargues-Fontaine curve, then this parametrizes modifications

$$\mathcal{O}_X(-\frac{1}{n}) \rightarrow \mathcal{O}_X^n$$

of type $(1, 0, \dots, 0, 0)$ (i.e this map is an embedding with cokernel a length 1 torsion sheaf on X), where $\mathcal{O}_X(-\frac{1}{n})$ is the unique rank n vector bundle on X of slope $-\frac{1}{n}$. This interpretation allows one to relate the complex $R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi]$ to the action of a Hecke operator $T_{\mu^{-1}}$ on Bun_G , acting on a sheaf \mathcal{F}_π constructed from the supercuspidal representation π , where $\mu^{-1} = (0, 0, \dots, 0, -1)$ is a dominant inverse of μ . The Fargues-Scholze parameter of π is built from the action of the excursion algebra on the sheaf \mathcal{F}_π , which in turn is built from Hecke operators equipped with a factorization structure coming from geometric Satake. It thus is reasonable to expect that the cohomology group $R\Gamma_c(\mathrm{LT}_{n,\infty}, \overline{\mathbb{Q}}_\ell)[\pi]$ should have $W_{\mathbb{Q}_p}$ -action given by the Fargues-Scholze parameter $\phi_\pi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \rightarrow {}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \simeq \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ composed with the highest weight representation of ${}^L\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ corresponding to the dominant cocharacter μ^{-1} . However, this is just the dual of the standard representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Thus, using Theorem 1.2, we can see that

$$\phi_\pi^\vee = (\phi_\pi^{\mathrm{FS}})^\vee$$

where the twist by the norm-character $|\cdot|^{(1-n)/2}$ is cancelled out by a perverse normalization (also related to the middle degree being the relevant one) in the definition of the Hecke operator $T_{\mu^{-1}}$. This implies compatibility for supercuspidal π , which, by using compatibility of the Fargues-Scholze

correspondence with parabolic induction, is enough to conclude the general case. In a similar fashion, using the description of the ρ -isotypic part one can prove compatibility for the inner form $D_{\frac{1}{n}}^*$.

One may expect, given the above sketch of compatibility for GL_n , that, to prove Theorem 1.1, one must similarly provide a description of the cohomology of the $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$)-isotypic part of a local Shimura variety/shtuka space at infinite level associated to G . In the case where the associated L -parameters are supercuspidal, this is the content of the Kottwitz conjecture [RV14, Conjecture 7.3]. To this end, we consider the cohomology of a shtuka space associated to the group $G = \mathrm{Res}_{L/\mathbb{Q}_p} \mathrm{GSp}_4$. Namely, the space denoted $\mathrm{Sht}(G, b, \mu)_\infty$, where μ is the Siegel cocharacter and $b \in B(G)$ is a basic element in the Kottwitz set of G , corresponding to a rank 4 isocrystal with a polarization and automorphism group equal to $J = \mathrm{Res}_{L/\mathbb{Q}_p}(\mathrm{GU}_2(D))$. This space carries a commuting $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$ action. The quotients

$$\mathrm{Sht}(G, b, \mu)_K := \mathrm{Sht}(G, b, \mu)_\infty / \underline{K}$$

for varying compact open $K \subset G(\mathbb{Q}_p)$ are, as before, representable by rigid analytic varieties over $\mathrm{Spa}(\check{L})$ of dimension 3, where $\check{L} := L\check{\mathbb{Q}}_p$. They define a tower of local Shimura varieties in the sense of Rapoport-Viehmann [RV14], which uniformize the basic locus of certain global Shimura varieties analogous to the Lubin-Tate case described above. Letting $\mathrm{Sht}(G, b, \mu)_{K, \mathbb{C}_p}$ be the base-change of these spaces to \mathbb{C}_p , we can then consider the analog of the complexes described above

$$R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \rightarrow 1} R\Gamma_c(\mathrm{Sht}(G, b, \mu)_{K, \mathbb{C}_p}, \overline{\mathbb{Q}}_\ell)$$

This complex is concentrated in degrees $0 \leq i \leq 6 = 2\dim(\mathrm{Sht}(G, b, \mu)_K)$ and admits an action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_L$. This allows one to consider the ρ and π -isotypic parts, i.e we set

$$R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J)}^{\mathbb{L}} \rho$$

and

$$R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(G)}^{\mathbb{L}} \pi$$

where $\mathcal{H}(G)$ (resp. $\mathcal{H}(J)$) are the usual smooth Hecke algebra of G (resp. J). To deduce compatibility, one needs to realize the (semi-simplified) L -parameter ϕ_π (resp. ϕ_ρ) of Gan-Takeda (resp. Gan-Tantono) in these two cohomology groups. We will sketch how to do this in the next section using uniformization and global methods. Interestingly, after knowing compatibility, one can use ideas from the geometry of the Fargues-Scholze construction to provide a more precise of the complexes $R\Gamma_c(G, b, \mu)[\rho]$ and $R\Gamma_c(G, b, \mu)[\pi]$. Namely, recent work of Hansen [Han20] allows us to deduce that if ϕ_ρ (resp. ϕ_π) is supercuspidal the complexes $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) are concentrated in middle degree 3. It then follows from work of Hansen-Kaletha-Weinstein [HKW21] on a weakening of the Kottwitz conjecture and work of Fargues-Scholze [FS21, Section X.2] describing the Hecke action on objects with supercuspidal Fargues-Scholze parameter that one can actually deduce a strong form of the Kottwitz conjecture for these representations. To state this result, we first note that, if we are given a supercuspidal parameter $\phi : W_L \rightarrow \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$, even though the parameter ϕ is irreducible, its composition with the standard embedding $\mathrm{std} : \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ may not be. In particular, the size of the L -packets $\Pi_\phi(G) := \mathrm{LLC}_G^{-1}(\phi)$ and $\Pi_\phi(J) := \mathrm{LLC}_J^{-1}(\phi)$ over ϕ are governed by this.

- (1) (stable) $\text{std} \circ \phi$ is irreducible. In this case, the L -packets each contain one supercuspidal member.
- (2) (endoscopic) $\text{std} \circ \phi \simeq \phi_1 \oplus \phi_2$, where $\phi_i : W_L \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ are distinct irreducible 2-dimensional representations with $\det(\phi_1) = \det(\phi_2)$. In this case, the L -packets over ϕ each contain two supercuspidal members.

This allows us to state our main consequence of Theorem 1.1, which (almost) verifies the strong form of the Kottwitz conjecture for GSp_4/L and $\text{GU}_2(D)/L$.

Theorem 1.3. *Let L/\mathbb{Q}_p be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L -packet over a supercuspidal parameter $\phi : W_L \rightarrow \text{GSp}_4(\overline{\mathbb{Q}}_\ell)$ as above. Then the complexes*

$$R\Gamma_c(G, b, \mu)[\pi]$$

and

$$R\Gamma_c(G, b, \mu)[\rho]$$

are concentrated in middle degree 3.

- (1) *If ϕ is stable supercuspidal, with singleton L -packets $\{\pi\} = \Pi_\phi(G)$ and $\{\rho\} = \Pi_\phi(J)$, then the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho \boxtimes (\text{std} \circ \phi)^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$ -module, and the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi \boxtimes \text{std} \circ \phi \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$ -module.

- (2) *If ϕ is an endoscopic parameter, with L -packets $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$ ¹, the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

or

$$\rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$ -module. Similarly, the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

or

$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$ -module. Here we write $\text{std} \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct irreducible 2-dimensional representations of W_L and $\det(\phi_1) = \det(\phi_2)$.

Moreover, both possibilities for the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_\phi(J)$ (resp. $\pi \in \Pi_\phi(G)$). In particular, knowing the precise form of either $R\Gamma_c(G, b, \mu)[\rho]$ or $R\Gamma_c(G, b, \mu)[\pi]$ for some $\rho \in \Pi_\phi(J)$ or $\pi \in \Pi_\phi(G)$ determines the precise form of the cohomology in all other cases.

¹For an explanation of the notation, see the discussion at the end of section 2.2.

- Remark 1.2.* (1) Results of this form when $L = \mathbb{Q}_p$ have also been announced by Ito-Meida [IM21].
- (2) If one knew Arthur's multiplicity formula for inner forms of GSp_4 over totally real fields, one should be able to determine the cohomology in the endoscopic case more precisely, using basic uniformization and the more precise description of the cohomology of the global Shimura variety this multiplicity formula would provide (See for example [Ngu19, Section 3.2] for this kind of analysis in the case of unitary groups.). However, to our knowledge the multiplicity formula is unknown in this case. In the case that $L = \mathbb{Q}_p$, one can apply what is known about the multiplicity formula for $\mathrm{GSp}_4/\mathbb{Q}$ [Art04; GT19]. This is carried out by Ito-Meida [IM21]. The correct answer, for the ρ -isotypic part, should be that, if $\rho = \rho_1$, we are in the first case, and if $\rho = \rho_2$, we are in the second case. Similarly, for the π -isotypic part, if $\pi = \pi^+$ is the unique generic member of the L -packet for a fixed choice of Whittaker datum, we should be in the first case and, if $\pi = \pi^-$, we should be in the second case. It might also be possible to show this using a weaker argument. Our analysis reduces us to checking that $R\Gamma_c(G, b, \mu)[\rho_1]$ admits a sub-quotient isomorphic to $\pi_+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$, which may be possible to show through basic uniformization and a small global argument.
- (3) We hope that the perspective we take on the Kottwitz conjecture in this paper will help provide further advancements in our knowledge of the cohomology of local Shimura varieties. In particular, by invoking the use of these very general geometric tools from the Fargues-Scholze construction, we require global input only to show compatibility, which, as we will see in the next section, requires substantially less than the input needed to determine the precise form of the cohomology such as a multiplicity formula for the automorphic spectrum.

We will now conclude the introduction by providing a sketch of the proof of Theorem 1.1.

1.2. Proof Sketch of the Main Theorems. As before, we set $G = \mathrm{Res}_{L/\mathbb{Q}_p} \mathrm{GSp}_4$ and $J = \mathrm{Res}_{L/\mathbb{Q}_p} \mathrm{GU}_2(D)$. Similar to the case of GL_n , the idea behind proving compatibility for G and J is to use the compatibility of the Fargues-Scholze local Langlands correspondence with parabolic induction to reduce to the case where π is a supercuspidal representation. However, this is a little bit more subtle than the case of GL_n . Unlike GL_n , the local Langlands correspondence for these groups is not a bijection. As seen before, the L -packets can be either of size 1 or 2. Given an L -parameter $\phi : W_L \times \mathrm{SL}_2(\overline{\mathbb{Q}_\ell}) \rightarrow \mathrm{GSp}_4(\overline{\mathbb{Q}_\ell})$, there are three distinct possibilities.

- (1) The L -packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ do not contain any supercuspidal representations.
- (2) The L -packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain a mix of supercuspidal and non-supercuspidal representations.
- (3) The L -packets $\Pi(G)_\phi$ and $\Pi(J)_\phi$ contain only supercuspidals.

Case (1) is straight forward. Since compatibility is known for GL_n and its inner forms and any proper Levi subgroup of G (resp. J) is a product of such groups, it follows from compatibility of the Fargues-Scholze correspondence with parabolic induction and products that the correspondences are compatible for any representation lying in such an L -packet.

Case (2) is a bit more subtle, here ϕ^{ss} factors through a Levi subgroup of $\mathrm{GSp}_4(\overline{\mathbb{Q}_\ell})$, but ϕ itself does not. In particular, the restriction to the SL_2 factor of ϕ is non-trivial. In this case, we

can write $\Pi_\phi(G) = \{\pi_{disc}, \pi_{sc}\}$ (resp. $\Pi_\phi(J) := \{\rho_{disc}, \rho_{sc}\}$ or $\Pi_\phi(J) = \{\rho_{disc}^1, \rho_{disc}^2\}$, depending on whether the parameter is of Saito-Kurokawa or Howe-Piatetski-Schapiro type), where π_{disc} (resp. $\rho_{disc}^1, \rho_{disc}^2$, and ρ_{disc}) are non-supercuspidal (essentially) discrete series representation of G (resp. J), and π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of $G(\mathbb{Q}_p)$ (resp. $J(\mathbb{Q}_p)$). The key observation is that π_{disc} (resp. ρ_{disc}) is an irreducible sub-quotient of a parabolic induction, so, in this case, we can apply the same argument as in case (1) to deduce compatibility of the two correspondences. It remains to see that the same is true for π_{sc} . To do this, we use a description of the ρ -isotypic part of the shtuka space $\text{Sht}(G, b, \mu)_\infty$ introduced in section 1.1. Namely, we consider the complex

$$R\Gamma_c(G, b, \mu)[\rho_{disc}] \simeq R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(G)}^{\mathbb{L}} \rho_{disc}$$

of $G(\mathbb{Q}_p) \times W_L$ -modules. Recent work of Hansen-Kaletha-Weinstein [HKW21] then tells us the form of this cohomology group (or rather a small variant thereof) as a $J(\mathbb{Q}_p)$ -representation. In particular, if we let $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ denote the summand of $R\Gamma_c(G, b, \mu)[\rho_{disc}]$ where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation, then in the Grothendieck group of admissible $J(\mathbb{Q}_p)$ -representations of finite length $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ is equal to $-2\pi_{sc}$.

Similar to the case of $G = \text{GL}_n$, this complex describes the action of the Hecke operator T_μ acting on a sheaf $\mathcal{F}_{\rho_{disc}}$ constructed from ρ_{disc} on Bun_G . Moreover, the complex $R\Gamma_c(G, b, \mu)[\rho_{disc}]_{sc}$ can be interpreted as a complex of sheaves on the open Harder-Narasimhan(=HN)-strata $\text{Bun}_G^1 \subset \text{Bun}_G$ corresponding to the trivial G -bundle on the Fargues-Fontaine curve X . It follows from the above description in the Grothendieck group that the excursion algebra will act on this complex via eigenvalues valued in the parameter $\phi_{\pi_{sc}}^{\text{FS}}$. However, since the excursion algebra is built from Hecke operators, it will also commute with the action of Hecke operators on $\mathcal{F}_{\rho_{disc}}$. This allows us to conclude that it also must act via eigenvalues valued in $\phi_{\rho_{disc}}^{\text{FS}}$ giving a chain of equalities

$$\phi_{\pi_{sc}}^{\text{FS}} = \phi_{\rho_{disc}}^{\text{FS}} = \phi_{\rho_{disc}}^{\text{ss}} = \phi_{\pi_{sc}}^{\text{ss}}$$

where the first equality follows from the previous argument and the second equality follows from the above analysis of induced representations. Similarly, one deduces compatibility for ρ_{sc} by applying a similar argument to the π -isotypic part.

Case (3) is by far the most involved and takes up the majority of the paper. This is the case in which the L -parameter ϕ is supercuspidal. First, one can make a reduction to showing compatibility for just $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ , by using the commutation of Hecke operators and the excursion algebra similar to what was done in case (2). Now, the key point again is that the complex $R\Gamma_c(G, b, \mu)[\rho]$ describes the action of the Hecke operator T_μ on a sheaf \mathcal{F}_ρ on Bun_G . To make further progress towards compatibility, we use that the Hecke operators can be in turn described using the spectral action of the derived category of perfect complexes on the stack of L -parameters, as constructed in [FS21, Chapter X]. In particular, a Hecke operator defines a vector bundle on the stack of L -parameters, whose action on the sheaf \mathcal{F}_ρ via the spectral action is precisely T_μ . Using this, we argue using the support of the spectral action of certain averaging operators, considered by [AL21] in the case of GL_n , to show that, if $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$ occurs as a

W_L -stable sub-quotient of the complex

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

we have an equality:

$$\text{std} \circ \phi_\rho^{\text{FS}} = \text{std} \circ \phi$$

for all $\rho \in \Pi_\phi(J)$. Now a GSp_4 -valued parameter is in turn determined by its composition with std and its similitude character, which is precisely the central character of ρ . Therefore, since the Fargues-Scholze correspondence is compatible with central characters, this is enough to conclude that $\phi = \phi_\rho^{\text{FS}}$. This reduces the question of showing compatibility for $\rho \in \Pi(\text{GU}_2(D))$ with supercuspidal Gan-Tantono parameter ϕ to the following.

Proposition 1.4. *Let ϕ be a supercuspidal parameter with associated L -packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

is concentrated in middle degree 3 and admits a non-zero W_L -stable sub-quotient with W_L -action given by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Just as one does in proving Theorem 1.2, the key idea is to directly relate the complex

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

to the cohomology of a global Shimura variety using basic uniformization of the generic fiber as proven by Shen [She20] and an analogue of Boyer's trick [Boy99]. This allows us to in turn prove Proposition 1.4 using global results on Galois representations in the cohomology of these Shimura varieties due to Kret-Shin [KS23] and Sorensen [Sor10]. More specifically, in this case the relevant Shimura datum is given by (\mathbf{G}, X) , where \mathbf{G} is a \mathbb{Q} -inner form of $\mathbf{G}^* := \text{Res}_{F/\mathbb{Q}}(\text{GSp}_4)$ for F/\mathbb{Q} a totally real extension with p inert and $F_p \simeq L$. The relevant uniformization result is then applicable if L/\mathbb{Q}_p is an unramified extension and $p > 2$. To state the key consequence of this uniformization result, we introduce some notation. We let \mathbb{A} and \mathbb{A}_f denote the adèles and finite adèles of \mathbb{Q} , respectively. If $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ denotes the level away from p and $K_p \subset G(\mathbb{Q}_p)$ denotes the level at p , we let $\mathcal{S}(\mathbf{G}, X)_{K_p K^p}$ be the rigid analytic Shimura variety over \mathbb{C}_p of level $K_p K^p$. We set ξ be a regular weight of an algebraic representation \mathcal{V}_ξ of \mathbf{G} over \mathbb{Q} and let \mathcal{L}_ξ denote the associated $\overline{\mathbb{Q}}_\ell$ local system on $\mathcal{S}(\mathbf{G}, X)_{K_p K^p}$. We then define

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi) := \text{colim}_{K_p \rightarrow \{1\}} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K_p K^p}, \mathcal{L}_\xi)$$

The basic uniformization result of Shen then furnishes a \mathbb{Q} -inner form \mathbf{G}' of \mathbf{G} satisfying that $\mathbf{G}'_{\mathbb{Q}_p} \simeq J$ together with a $G(\mathbb{Q}_p) \times W_L$ -invariant map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

functorial in the level K^p . Here $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi)$ denotes the space of algebraic automorphic forms of level K^p valued in the algebraic representation \mathcal{V}_ξ in the sense of [Gro99]. We want to use this uniformization map to apply global results on the cohomology of $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$ to study the action of W_L on $R\Gamma_c(G, b, \mu)$. To do this, we show an analogue of Boyer's trick, which says that the non-basic Newton strata of the adic flag variety $\mathcal{F}\ell_{G, \mu^{-1}} := (G/P_{\mu^{-1}})^{ad}$ are parabolically induced as spaces with $G(\mathbb{Q}_p)$ -action. Using the Hodge-Tate period map from the Shimura variety $\mathcal{S}(\mathbf{G}, X)_{K^p}$ to $\mathcal{F}\ell_{G, \mu^{-1}}$, this implies that, if we pass to the part of the cohomology on both sides where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation, we get a $W_L \times G(\mathbb{Q}_p)$ -equivariant isomorphism:

$$\Theta_{sc} : R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

After showing this, we fix a ρ having supercuspidal Gan-Tantono parameter ϕ , and, via an argument using the simple trace formula, choose a globalization of ρ to a cuspidal automorphic representation Π' of \mathbf{G}' , which occurs as a $J(\mathbb{Q}_p)$ -stable direct summand of $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi)$ and is an unramified twist of Steinberg at some non-empty set of places S_{st} , for some sufficiently large regular weight ξ and sufficiently small level K^p . We set S to be a finite set of places outside of which Π' is unramified. The Hecke eigenvalues of Π' then define a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the abstract commutative Hecke algebra of \mathbf{G}' away from the finite places S . Regarding both sides of Θ as \mathbb{T}^S -modules, we can localize at \mathfrak{m} to get a map

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

We write $K^p = K^{S_{st} \cup \{p\}} K_{\{p\} \cup S_{st}}$ for $K^{S_{st} \cup \{p\}} \subset \mathbf{G}(\mathbb{A}_f^{S_{st} \cup \{p\}})$. Taking colimits on both sides as $K_{\{p\} \cup S_{st}} \rightarrow \{1\}$, we see that $\Theta_{\mathfrak{m}}$ induces a map:

$$(R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}} \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

Since we know that Θ_{sc} is an isomorphism, we have an isomorphism

$$(R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}}^{st} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}}^{st}$$

Noting that $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi)$ is semi-simple, we can project to the summand where $\mathbf{G}(F_v) \simeq \mathbf{G}'(F_v)$ acts via an unramified twist of the Steinberg representation for all $v \in S_{st}$. This implies that we have an isomorphism

$$\Theta_{\mathfrak{m}, sc}^{st} : (R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}}^{st} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}, sc}^{st}$$

The key point is now, by analyzing the simple twisted trace formula of Kottwitz-Shelstad [KS99] and stable trace formulas of Arthur [Art02], we can prove a strong multiplicity one type result (Proposition 5.4), for cuspidal automorphic representations that are unramified twists of Steinberg at some sufficiently large non-empty set of places and regular of weight ξ at infinity. This implies that the representations of \mathbf{G}' occurring on LHS of $\Theta_{\mathfrak{m}}^{st}$ must have local constituent at p with Langlands parameter ϕ , since we localized at the Hecke eigensystem defined by Π' at the unramified places.

and, since the local constituents of the automorphic representations of \mathbf{G}' at p occurring in the LHS are all in the L -packet $\Pi_\phi(J)$ by the strong multiplicity one result, we can reduce Proposition 1.4 to showing that $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m}, sc}^{st}$ is concentrated in degree 3 and has W_L -action given (up to multiplicity) by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$. This follows from the analysis carried out in Kret-Shin [KS23]. In particular, it follows from their results that this complex will be concentrated in degree 3 and that the traces of Frobenius in $\Gamma_F := \text{Gal}(\overline{F}/F)$ on the étale cohomology of the associated global Shimura variety over \overline{F} are given by $\text{std} \circ \phi_{\tau_v}$, where τ_v are the local constituents of some weak transfer τ of Π' to an automorphic representation of $\text{Res}_{F/\mathbb{Q}}\text{GSp}_4 =: \mathbf{G}^*$ and ϕ_{τ_v} is the associated Gan-Takeda parameter. This allows one, up to multiplicities, to describe the Galois action on the global Shimura variety in terms of the composition $\text{std} \circ \rho_\tau$, where ρ_τ is a global $\text{GSp}_4(\overline{\mathbb{Q}}_\ell)$ -valued representation of the absolute Galois group of F constructed by Sorensen [Sor10] from τ characterized by the property that $i\text{WD}(\text{std} \circ \rho_\tau)|_{W_{F_v}}^{F\text{-s.s.}} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$ for all but finitely many places v of F . This would give one precisely the desired description of the W_L -action on $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ if one knew that $\phi_{\tau_p} = \phi_\rho$. Since Π' is globalization of ρ , one needs to choose τ to be a strong transfer of Π' at the prime p . This latter goal is accomplished using analysis of the simple trace formula as done in Kret-Shin [KS23, Section 6] combined with the character identities proven by Chan-Gan [CG15]. These results on strong transfers also aid us in deducing the strong multiplicity one type result mentioned above.

In section 2, we give an overview of the Gan-Takeda and Gan-Tantono local Langlands correspondence, putting it in the framework of the refined local Langlands correspondence of Kaletha in preparation for applications to the Kottwitz conjecture. In section 3, we describe the Fargues-Scholze local Langlands correspondence and related ideas, giving the proof of compatibility in cases (1) and (2) and reducing case (3) to Proposition 1.4, via some properties of the spectral action discussed in section 3.2. In section 4, we discuss basic uniformization of the relevant Shimura varieties and prove the aforementioned analogue of Boyer's trick, showing that the uniformization map Θ_{sc} is an isomorphism. In section 5, we analyze the simple trace formula with fixed central character in a fashion similar to Kret-Shin [KS23] to deduce the existence of the required strong transfers, as well as combine this with analysis of the simple twisted trace formula to deduce the required strong multiplicity one result. In section 6, we apply the results of section 5 combined with results of Kret-Shin [KS23] and Sorensen [Sor10] to compute the relevant Galois action on the global Shimura variety. Finally, in section 7, we put the results of the previous sections together to prove Proposition 1.4. We then conclude with the application to the proofs of Theorem 1.1 and 1.3, as well as formally deduce compatibility for the local Langlands correspondence for Sp_4 and its non quasi-split inner form $\text{SU}_2(D)$, as constructed by Gan-Takeda [GT10] and Choï [Cho17], respectively. We finish the section with a brief discussion of an application to the cohomology of the related (non-minuscule) local shtuka spaces.

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CONVENTIONS AND NOTATIONS

For a diamond or v -stack, we freely use the formalism in [Sch18; FS21] of ℓ -adic cohomology of diamonds and v -stacks. We will fix isomorphisms $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \xrightarrow{\cong} \mathbb{C}$ and use the (geometric) normalization of local class field theory that sends the Frobenius to the inverse of the uniformizer. For a supercuspidal L -parameter, we will often abuse notation and use ϕ to denote both the L -parameter and the semisimplified parameter ϕ^{ss} , as in this case this merely corresponds to forgetting the trivially acting $\text{SL}_2(\mathbb{C})$ -factor and applying the isomorphism i . For a reductive group H/\mathbb{Q}_p , we will write $R\mathcal{H}om_{H(\mathbb{Q}_p)}(-, \overline{\mathbb{Q}}_\ell) : \text{D}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{\text{op}} \rightarrow \text{D}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ for the derived smooth duality functor, where $\text{D}(H(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ is the unbounded derived category of smooth representations. Namely, it is derived functor induced by the left exact smooth duality functor $V \mapsto (V^*)^{\text{sm}}$, where $(V^*)^{\text{sm}}$ is the set of all functions $f : V \rightarrow \overline{\mathbb{Q}}_\ell$ such that there exists $K \subset H(\mathbb{Q}_p)$ a compact open such that for all $v \in V$ and $k \in K$ we have that $f(kv) = f(v)$. Normally, in the literature the space $\text{Sht}(G, b, \mu)_\infty$ parametrizes modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}_b$ with meromorphy μ . For us, it will denote the space parametrizing modifications of type μ^{-1} . This convention limits the appearances of duals (cf. Remark 3.8).

2. LOCAL LANGLANDS FOR GSp_4 AND $\text{GU}_2(D)$

2.1. Local Langlands for GSp_4 . In this section, we will describe the local Langlands correspondence of Gan-Takeda for the group $G := \text{GSp}_4/L$, where L/\mathbb{Q}_p is a finite extension. We fix a choice of Whittaker datum $\mathfrak{m} := (B, \psi)$ throughout section 2, where B is the Borel and ψ is a generic character of L .

As before, we consider the set $\Phi(G)$ of admissible homomorphisms

$$\phi : W_L \times \text{SL}_2(\mathbb{C}) \rightarrow \hat{G}(\mathbb{C}) = \text{GSpin}_5(\mathbb{C}) \simeq \text{GSp}_4(\mathbb{C})$$

taken up to \hat{G} -conjugacy, where W_L acts via a continuous semisimple homomorphism with respect to the discrete topology and $\text{SL}_2(\mathbb{C})$ acts via an algebraic representation. Similarly, let $\Pi(G)$ denote the isomorphism classes of smooth irreducible representations of the group $G(L)$. We can now state the main theorem of Gan-Takeda.

Theorem 2.1. [GT11] *There is a surjective finite to one map*

$$\text{LLC}_G : \Pi(\text{GSp}_4) \rightarrow \Phi(\text{GSp}_4)$$

$$\pi \mapsto \phi_\pi$$

with the following properties:

- (1) π is an (essentially) discrete series representation of $\mathrm{GSp}_4(L)$ if and only if its L -parameter does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.
- (2) Given an L -parameter ϕ , we set $S_\phi := Z_{\hat{G}}(\mathrm{Im}(\phi))$ to be the centralizer of ϕ . The fiber $\Pi_\phi(G)$ can be naturally parametrized by the set of irreducible characters of the component group

$$A_\phi := \pi_0(S_\phi) \simeq \pi_0(S_\phi/Z(\mathrm{GSp}_4))$$

which is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in $\Pi_\phi(G)$ is generic for the fixed choice of Whittaker datum, and is indexed by the trivial character of A_ϕ .

- (3) The similitude character $\mathrm{sim}(\phi_\pi)$ of $\mathrm{GSp}_4(L)$ is equal to the central character ω_π via the isomorphism given by local class field theory.
- (4) Given a character χ of L^* and letting $\lambda : \mathrm{GSp}_4 \rightarrow L^*$ be the similitude character of $\mathrm{GSp}_4(L)$, we have, via local class field theory, that the L -parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$.
- (5) If $\pi \in \Pi(\mathrm{GSp}_4)$ is a generic supercuspidal representation or non-supercuspidal representation then, for any $r \geq 1$ and any smooth irreducible representation σ of $\mathrm{GL}_r(L)$, we have that

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$L(s, \pi \times \sigma, \psi) = L(s, \phi_\pi \otimes \phi_\sigma, \psi)$$

$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\sigma, \psi),$$

where the RHS are the Artin local factors associated to the representations of $W_L \times \mathrm{SL}_2(\mathbb{C})$, and the LHS are the local factors of Shahidi [Sha90] with respect to the morphisms of L -groups defined in [GT11, Section 4]. In the case that $r \leq 2$ and π is a non-generic supercuspidal representation, we have the same relationship where the LHS is given by the local factors defined by Townsend [Tow13].

The map LLC_G is uniquely determined by the properties (1), (3), and (5), where one can take $r \leq 2$ in (5).

Remark 2.1. When the paper of Gan-Takeda was released there was no good theory of L, ϵ , and γ factors for nongeneric supercuspidal representations satisfying the usual properties. (See the 10-Commandments in [LR05]) Instead, to uniquely characterize the correspondence for these representations, they use an equality between the Plancharel measure on the family of inductions from $\mathrm{GSpin}_5(L) \times \mathrm{GL}_r(L) \simeq \mathrm{GSp}_4(L) \times \mathrm{GL}_r(L)$ to $\mathrm{GSpin}_{2r+5}(L)$ for $r \leq 2$. However, this theory of L, ϵ , and γ factors was later constructed by Nelson Townsend in his PhD thesis [Tow13].

We now make the following definition.

Definition 2.1. Write $\mathrm{std} : \mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$ for the standard embedding. We say a discrete L -parameter is stable if the L -packet $\Pi_\phi(G)$ has size 1 and is endoscopic if it has size 2. Equivalently, by Theorem 2.1 (2), this is equivalent to saying that the character group A_ϕ^\vee of the component group A_ϕ has cardinality 1 or 2, respectively. By [GT11, Lemma 6.2], this can be characterized as follows.

- (stable) $\mathrm{std} \circ \phi$ is an irreducible representation of $W_L \times \mathrm{SL}_2(\mathbb{C})$. In this case, $S_\phi = Z(\hat{G}) = \mathbb{G}_m$, so A_ϕ is trivial.

- (endoscopic) $\text{std} \circ \phi \simeq \phi_1 \oplus \phi_2$, where the $\phi_i : W_L \times \text{SL}_2(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{C})$ for $i = 1, 2$ are distinct irreducible 2-dimensional representations of $W_L \times \text{SL}_2(\mathbb{C})$ with $\det(\phi_1) = \det(\phi_2)$. In this case, $A_\phi \simeq \mathbb{Z}/2\mathbb{Z}$. We recall that $\text{GSp}_4(\mathbb{C})$ has a unique endoscopic group $\text{GSO}_{2,2}$, and the endoscopic parameters lie in the image of the map $\Phi(\text{GSO}_{2,2}) \rightarrow \Phi(\text{GSp}_4)$. More specifically, the dual group of $\text{GSO}_{2,2}$ is GSpin_4 and one has an identification $\text{GSpin}_4 \simeq (\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))^0 := \{(g_1, g_2) \in \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \mid \det(g_1) = \det(g_2)\}$ and the map $\Phi(\text{GSO}_{2,2}) \rightarrow \Phi(\text{GSp}_4) \xrightarrow{\text{std} \circ} \Phi(\text{GL}_4)$ comes from the inclusion $(\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))^0 \subset \text{GL}_4(\mathbb{C})$. Using this, we can compute that one has an identification

$$S_\phi \simeq \{(a, b) \in \mathbb{C}^* \times \mathbb{C}^* \mid a^2 = b^2\} \subset (\text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}))^0$$

where the center $Z(\text{GSp}_4)(\mathbb{C}) \simeq \mathbb{C}^*$ embeds diagonally.

For an L -parameter ϕ we see, by Theorem 2.1 (2), that the size of the L -packet $\Pi_\phi(G)$ is at most 2, this allows us to subdivide into three cases:

- (1) The L -packet $\Pi_\phi(G)$ does not contain any supercuspidal representations.
- (2) The L -packet $\Pi_\phi(G)$ contains one supercuspidal and one non-supercuspidal.
- (3) The L -packet contains only supercuspidals.

In Case (1) the parameter will not be discrete. Case (2) is where the parameter ϕ does not factor through a Levi-subgroup so it is discrete, but its semisimplification ϕ^{ss} as defined in section 1 does. Case (2) does not occur when the parameter is a stable discrete parameter, by definition. The relevant case is when the parameter is discrete endoscopic. To understand this, we let $\nu(n)$ denote the unique n -dimension irreducible representation of $\text{SL}_2(\mathbb{C})$ then there are two cases:

- (1) (Saito-Kurokawa Type) We have $\text{std} \circ \phi = \phi_0 \oplus \chi \boxtimes \nu(2)$, where ϕ_0 is a 2-dimensional irreducible representation of W_L and χ is a character, with $\chi^2 = \det(\phi_0)$. Therefore, the semisimplification ϕ^{ss} satisfies: $\text{std} \circ \phi^{\text{ss}} = \phi_0 \oplus \chi \otimes |\cdot|^{\frac{1}{2}} \oplus \chi \otimes |\cdot|^{-\frac{1}{2}}$.
- (2) (Howe-Piatetski-Shapiro Type) We have $\text{std} \circ \phi = \chi_1 \boxtimes \nu(2) \oplus \chi_2 \boxtimes \nu(2)$, where χ_1 and χ_2 are distinct characters of W_L satisfying $\chi_1^2 = \chi_2^2$. Therefore, the semisimplification ϕ^{ss} satisfies: $\text{std} \circ \phi^{\text{ss}} = \chi_1 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_1 \otimes |\cdot|^{-\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{-\frac{1}{2}}$.

Remark 2.2. (1) The terminology here is explained by Arthur's classification [Art04] of the global automorphic representations of GSp_4 appearing in the papers [Kur78] and [HP79], respectively.

- (2) We will mention in the next section how to distinguish these two cases via the number of supercuspidals in the L -packet $\Pi_\phi(\text{GU}_2(D))$ defined by the Gan-Tantono local Langlands correspondence.

Case (3) is the situation where the parameter ϕ is supercuspidal as defined in the introduction. In particular, in the supercuspidal case the restriction of the parameter ϕ to the $\text{SL}_2(\mathbb{C})$ factor is trivial, so the irreducible representations occurring in the decomposition of $\text{std} \circ \phi$ are just representations of W_L .

For the purposes of applying the weak form of the Kottwitz Conjecture proven in Hansen-Kaletha-Weinstein [HKW21], we formulate this correspondence in terms of the refined local Langlands of

Kaletha [Kal16] with respect to the fixed choice of Whittaker datum \mathfrak{m} . Now, given a parameter ϕ that is either mixed supercuspidal or supercuspidal, we have by Theorem 2.1 (2) a correspondence between the L -packet $\Pi_\phi(G)$ and the set of irreducible characters A_ϕ^\vee . This in turn gives rise to an irreducible character of the group S_ϕ via the composition:

$$S_\phi \rightarrow \pi_0(S_\phi) = A_\phi$$

This allows us to make the following definition.

Definition 2.2. For ϕ a supercuspidal or mixed-supercuspidal parameter ϕ as above and $\pi \in \Pi_\phi(G)$, we denote the character of S_ϕ described above by τ_π .

2.2. Local Langlands for $\mathrm{GU}_2(D)$. In this section, we describe the local Langlands correspondence for the unique non-split inner form $J = \mathrm{GU}_2(D)$, the group of similitudes of the unique 2-dimensional Hermitian vector space over the quaternion division algebra D/L . As in the previous section, we let $\Pi(J)$ denote the set of irreducible admissible representations of J , and $\Phi(J)$ be the set of L -parameters of J . This is a subset of the previous set $\Phi(\mathrm{GSp}_4)$ as we will now explain. J has a unique up to conjugacy minimal parabolic whose Levi factor is

$$D^* \times \mathrm{GL}_1.$$

This defines a form of the Siegel parabolic of GSp_4 and it determines a dual parabolic subgroup $P^\vee(\mathbb{C})$ in the dual group $\mathrm{GSp}_4(\mathbb{C})$ of $\mathrm{GU}_2(D)$. This is the Heisenberg parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$, its conjugacy class is said to be relevant for J while all other conjugacy classes of proper parabolics are said to be irrelevant. We say $\phi \in \Phi(\mathrm{GSp}_4)$ is relevant if it does not factor through any irrelevant parabolic subgroups of $\mathrm{GSp}_4(\mathbb{C})$. We define $\Phi(J)$ to be the subset of relevant ϕ in $\Phi(\mathrm{GSp}_4)$. We set $B_\phi := \pi_0(Z_{\mathrm{Sp}_4}(\mathrm{Im}(\phi)))$. One has an exact sequence:

$$\langle \pm 1 \rangle \rightarrow B_\phi \rightarrow A_\phi \rightarrow 0$$

Implying that one has an injection on the group of irreducible characters $\hat{A}_\phi \hookrightarrow \hat{B}_\phi$, which identifies \hat{A}_ϕ as the subgroup of (index at most 2) of characters trivial on the image of the center $Z(\mathrm{Sp}_4)(\mathbb{C})$. One can check that $\hat{B}_\phi \neq \hat{A}_\phi$ if and only if ϕ is relevant for $\mathrm{GU}_2(D)$. Now we can state the main theorem of Gan and Tanton.

Theorem 2.2. [GT14] *There is a natural surjective finite-to-one map*

$$\mathrm{LLC}_J : \Pi(\mathrm{GU}_2(D)) \rightarrow \Phi(\mathrm{GU}_2(D))$$

$$\rho \mapsto \phi_\rho$$

with the following properties:

- (1) ρ is an (essentially) discrete series representation of $\mathrm{GU}_2(D)$ if and only if its parameter ϕ_ρ does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.
- (2) For an L -parameter ϕ , the fiber $\Pi_\phi(J)$ can be naturally parametrized by the set $\hat{B}_\phi \setminus \hat{A}_\phi$. This set has size either 1 or 2.
- (3) The similitude character $\mathrm{sim}(\phi_\rho)$ of ϕ_ρ is equal to the central character ω_ρ of ρ , via the isomorphism given by local class field theory.

- (4) Given a character χ of L^* and letting $\lambda : \mathrm{GU}_2(D) \rightarrow L^*$ be the similitude character of $\mathrm{GU}_2(D)$, we have, via local class field theory, that the L -parameter of $\rho \otimes (\chi \circ \lambda)$ is equal to $\phi_\rho \otimes \chi$.
- (5) If $\rho \in \Pi(\mathrm{GU}_2(D))$ is a non-supercuspidal representation then, for any smooth irreducible representation σ of $\mathrm{GL}_r(L)$, we have that

$$\begin{aligned}\gamma(s, \rho \times \sigma, \psi) &= \gamma(s, \phi_\rho \otimes \phi_\sigma, \psi) \\ L(s, \rho \times \sigma, \psi) &= L(s, \phi_\rho \otimes \phi_\sigma, \psi) \\ \epsilon(s, \rho \times \sigma, \psi) &= \epsilon(s, \phi_\rho \otimes \phi_\sigma, \psi)\end{aligned}$$

where the RHS are the Artin local factors associated to the representations of $W_L \times \mathrm{SL}_2(\mathbb{C})$ and the LHS are the local factors of Shahidi, as defined in [GT14, Section 8].

- (6) Suppose that ρ is a supercuspidal representation. For any irreducible supercuspidal representation σ of $\mathrm{GL}_r(L)$ with L -parameter ϕ_σ , if $\mu(s, \rho \boxtimes \sigma, \psi)$ denotes the Plancherel measure associated to the family of induced representations $I_P(\pi \boxtimes \sigma, s)$ on $\mathrm{GSpin}_{r+4, r+1}$, where we have regarded $\rho \boxtimes \sigma$ as a representation of the Levi subgroup $\mathrm{GSpin}_{4,1} \times \mathrm{GL}_r \simeq \mathrm{GU}_2(D) \times \mathrm{GL}_r$, then $\mu(s, \rho \boxtimes \sigma)$ is equal to

$$\gamma(s, \phi_\rho^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\rho \otimes \phi_\sigma^\vee, \bar{\psi}) \cdot \gamma(2s, \mathrm{Sym}^2 \phi_\sigma \otimes \mathrm{sim}(\phi_\rho)^{-1}, \psi) \cdot \gamma(-2s, \mathrm{Sym}^2 \phi_\sigma^\vee \otimes \mathrm{sim}(\phi_\rho), \bar{\psi})$$

The map LLC_J is uniquely determined by the properties (1), (3), (5), and (6), with $r \leq 4$ in (5) and (6).

We now further elaborate on the structure of the L -packets $\Pi_\phi(J) := \mathrm{LLC}_J^{-1}(\phi)$ in the case where the parameter ϕ is mixed supercuspidal. If the parameter ϕ is of this form, then, it follows from [GT14, Proposition 5.4] and the description of LLC_J provided in [GT14, Section 7], that the L -packet $\Pi_\phi(J)$ has following structure, as alluded to in Remark 2.2 (2).

- (1) (Saito-Kurokawa Type) The L -packet $\Pi_\phi(J) = \{\rho_{disc}, \rho_{sc}\}$ contains one supercuspidal representation ρ_{sc} and one non-supercuspidal representation ρ_{disc} .
- (2) (Howe-Piatetski-Shapiro Type) The L -packet $\Pi_\phi(J) = \{\rho_{disc}^1, \rho_{disc}^2\}$ contains no supercuspidal representations.

We now would also like to briefly comment on the structure of the set $\hat{B}_\phi \setminus \hat{A}_\phi$, confirming the expectation that the size of the L -packets $\Pi_\phi(G)$ and $\Pi_\phi(J)$ is always the same.

- (stable) In the case that the L -parameter ϕ is stable, we have that $B_\phi = \mathbb{Z}/2\mathbb{Z}$ and, as noted in section 2.1, $A_\phi = 1$. This means the set $\hat{B}_\phi \setminus \hat{A}_\phi$ consists of one element corresponding to the non-trivial character.
- (endoscopic) In the case that the parameter ϕ is endoscopic, we have that the decomposition $\mathrm{std} \circ \phi \simeq \phi_1 \oplus \phi_2$ induces an exact sequence

$$A_\phi = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Delta} B_\phi = Z(\mathrm{SL}_2) \times Z(\mathrm{SL}_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and so the set $\hat{B}_\phi \setminus \hat{A}_\phi$ has size 2 and is indexed by two characters η_{+-} and η_{-+} each non-trivial on one of the two \mathbb{C}^* -factors under the isomorphism

$$S_\phi \simeq \{(a, b) \in \mathbb{C}^* \times \mathbb{C}^* \mid a^2 = b^2\} \subset (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0$$

from section 2.1.

We now wish to put this local Langlands correspondence for the inner form in the context of the refined local Langlands correspondence of Kaletha [Kal16]. We consider the Kottwitz set $B(G)$ [Kot85; RR96] and let $b \in B(G)$ be the basic element whose associated σ -centralizer $J_b = J$. We take this to be the basic element whose slope homomorphism is the dominant rational cocharacter of G given by $(1/2, 1/2, 1/2, 1/2)$. Let $Z(\mathrm{GSp}_4) \simeq \mathbb{G}_m$ be the center. We recall that we have an isomorphism $\pi_1(G) \simeq X_*(Z(\hat{G})) \simeq \mathbb{Z}$ and that the κ -invariant of b is sent to the element $1 \in \mathbb{Z}$ under this isomorphism. This indexes the identity representation of \mathbb{G}_m , denoted $id_{\mathbb{G}_m}$. Thus, given a discrete parameter (= supercuspidal or mixed supercuspidal) $\phi : W_L \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GSp}_4(\mathbb{C})$, the refined local Langlands correspondence asserts bijections

$$\Pi_\phi(G) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t. } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

$$\pi \mapsto \tau_\pi$$

$$\Pi_\phi(J) \longleftrightarrow \{\text{irreducible algebraic representations } \tau \text{ of } S_\phi \text{ s.t. } \tau|_{Z(\hat{G})} = id_{\mathbb{G}_m}\}$$

$$\rho \mapsto \tau_\rho$$

where $\mathbf{1}$ is the trivial representation. In section 2.1, we saw how for $\pi \in \Pi_\phi(G)$ to construct the desired τ_π . Here it is uniquely pinned down by the property that the trivial representation corresponds to the unique \mathfrak{m} -generic representation. In the case of the inner form, the situation is a bit more tricky. Consider $\rho \in \Pi(J)$ with associated L -parameter ϕ_ρ . If ϕ_ρ is stable then $S_\phi = \mathbb{G}_m$ and τ_ρ is simply $id_{\mathbb{C}^*}$. If ϕ is endoscopic, then, as noted in section 2.1, we have an inclusion:

$$Z(\mathrm{GSp}_4)(\mathbb{C}) = \mathbb{C}^* \xrightarrow{\Delta} S_\phi \simeq \{(a, b) \in \mathbb{C}^* \times \mathbb{C}^* \mid a^2 = b^2\}$$

We consider the characters $\tau_i : S_\phi \rightarrow \mathbb{C}^*$ for $i = 1, 2$ given by projecting to the first and second coordinate. These satisfy the property that $\tau_i|_{\mathbb{C}^*} = id_{\mathbb{C}^*}$ on the diagonally embedded center as desired. Similarly, under the parametrization of Gan-Tantono $\Pi_\phi(J) = \{\rho_{+-}, \rho_{-+}\}$, where ρ_{+-} and ρ_{-+} correspond to the characters η_{+-} and η_{-+} described above. Specifically, if π_1 and π_2 are the unique discrete series representations of $\mathrm{GL}_2(L)$ in the L -packet over ϕ_1 and ϕ_2 then

$$\rho_{+-} := \theta(\mathrm{JL}^{-1}(\pi_2) \boxtimes \pi_1) \text{ and } \rho_{-+} := \theta(\mathrm{JL}^{-1}(\pi_1) \boxtimes \pi_2)$$

where θ denotes the non-zero local theta lift from $D^* \times \mathrm{GL}_2(F)$ to $\mathrm{GU}_2(D)$, as in [GT14, Proposition 5.4], and $\mathrm{JL} : \Pi(D^*) \rightarrow \Pi(\mathrm{GL}_2)$ is the Jacquet-Langlands correspondence. Now we would like to match these two representations with τ_1 and τ_2 under the refined local Langlands of Kaletha. Suppose we fix such a matching. We consider a refined endoscopic datum \mathfrak{c} for the quasi-split reductive group G , which we recall is a tuple $(H, s, \mathcal{H}, \eta)$ which consists of

- a quasi-split group H over F ,
- an extension \mathcal{H} of W_F by \hat{H} such that the map $W_F \rightarrow \mathrm{Out}(\hat{H})$ coincides with the map $\rho_H : W_F \rightarrow \mathrm{Out}(\hat{H})$ induced by the action of W_F on $\hat{H} \subset {}^L H$,
- an element $s \in Z(\hat{H})^\Gamma$,
- an L -homomorphism $\eta : \mathcal{H} \rightarrow {}^L G$,

satisfying the condition:

- we have $\eta(\hat{H}) = Z_{\hat{G}}(s)^\circ$.

Considering J as the σ -centralizer of the unique basic element $b_1 \in B(G)$ of the Kottwitz invariant 1, this defines for us an extended pure inner twisting of $(\xi, b) : G \rightarrow J_b$ in the sense of [Kot97, Section 5.2], and we can attach a canonical transfer factor $\Delta[\mathfrak{m}, \mathfrak{c}, b_1]$ to this datum, as defined in [Kal16, Section 4.1]. Given a test function $f \in C_c^\infty(J(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})$, we can use these transfer factors to say what it means for $f^\mathfrak{c} \in C_c^\infty(H(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})$ to be matching in the sense that their stable orbital integrals normalized with respect to these transfer factors match up.

Now, suppose we have a discrete parameter $\phi \in \Phi(J)$ and a refined endoscopic datum \mathfrak{c} , such that $\phi = \eta \circ \phi^\mathfrak{c}$ as conjugacy classes of parameters for an L -parameter $\phi^\mathfrak{c} : W_L \times \mathrm{SL}(2, \overline{\mathbb{Q}_\ell}) \rightarrow \mathcal{H}$. Then the matching is uniquely described using the endoscopic character identities. This asserts an equality

$$\Theta_{\phi^\mathfrak{c}}^1(f^\mathfrak{c}) = \sum_{\pi \in \Pi_{\phi^\mathfrak{c}}(H)} \mathrm{tr}(1|\tau_\pi)\theta_\pi(f^\mathfrak{c}) = e(J) \sum_{\rho \in \Pi_\phi(J)} \mathrm{tr}(s|\tau_\rho)\theta_\rho(f) = \Theta_\phi^s(f),$$

where $e(J)$ is the Kottwitz sign of J , as defined in [Kot83] and θ_π denotes the Harish-Chandra character of π . Using the linear independence of the distributions θ_π and the fact that the packets $\Pi_\phi(J)$ are disjoint, we can see that the matching between $\rho \mapsto \tau_\rho$ is uniquely characterized by these relations. To show that there exists a matching between the representations τ_1 and τ_2 and ρ_{+-} and ρ_{-+} , we need to show that these identities are satisfied under the parametrization of Gan-Takeda-Tantono. This will follow from the endoscopic character identities verified by Chan-Gan [CG15]. Namely, in the case that the parameter ϕ is stable, one only needs to consider the trivial endoscopic datum $\mathfrak{c}_{\mathrm{triv}} = (G, 1, {}^L G, \mathrm{id})$, and these identities follow from [CG15, Proposition 11.1 (1)]. In the case that the parameter ϕ is endoscopic, the case $\mathfrak{c}_{\mathrm{triv}} = (G, 1, {}^L G, \mathrm{id})$ follows from [CG15, Proposition 11.1 (1)], but one also needs to consider the refined endoscopic datum given by $\mathfrak{c} = (\mathrm{GSO}_{2,2}, (1, -1), \mathrm{GSpin}_4, i : \mathrm{GSpin}_4 \rightarrow \mathrm{GSpin}_5 \simeq \mathrm{GSp}_4)$, where i is the map from before and $(1, -1) \in S_\phi \subset (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0$. In this case, the identities (up to a sign) follow from combining [CG15, Proposition 11.1 (1)] as before and [CG15, Proposition 11.1 (2)], where we note that since τ_1 and τ_2 are by definition two projections their traces against $(1, -1)$ have the opposite sign, which is consistent with [CG15, Proposition 11.1 (2)].

In order for the above argument for deducing the refined local Langlands from the results of Chan-Gan to work, one needs to compare the transfer factors used in [CG15, Section 4] with those of Kaletha, so that the notion of matching functions between f and $f^\mathfrak{c}$ coincide. We comment briefly on this now. When discussing the transfer factors $\Delta[\mathfrak{m}, \mathfrak{c}]$ used for transferring between two quasi-split groups both Kaletha and Chan-Gan work with the Whittaker normalized transfer factors considered in [KS12]. Kaletha on the one hand works with the renormalized factors Δ'_λ in the notation of [KS12, Section 5.5] (See [Kal16, Page 221]) while Chan-Gan work with the original factors denoted Δ_λ considered in [KS99, Section 5.3] (See [CG15, Section 4.1]); however, as noted at the end of [KS12, Section 5.5] when the element s in the refined endoscopic datum satisfies $s = s^{-1}$, the two transfer factors agree, and this will always be true for the endoscopic datum mentioned above. We also note that Chan-Gan unlike Kaletha do not include the Weyl discriminant term Δ_{IV} in their definition of the Whittaker normalized transfer factors, as noted in [CG15, Section 4.1]. Instead they include it in their definition of the orbital integrals used when defining the notion of matching functions. Moreover, both authors use the arithmetic normalization of local class field theory. These observations allow one to see that that the transfer factors $\Delta[\mathfrak{m}, \mathfrak{c}]$ considered in

[Kal16, Page 221] and the transfer factors considered [CG15] give rise to the same definition of matching functions when transferring between two quasi-split groups.

Therefore, the main difficulty when comparing the two conventions of transfer factors lies when considering transfer between a quasi-split group and a non quasi-split inner form. In particular, one needs to compare the transfer factors considered in [CG15, Section 4.3] with the transfer factor $\Delta[\mathfrak{m}, \mathfrak{c}, b_1]$ mentioned above. This is determined up to a norm one element in \mathbb{C} , as mentioned in [CG15, Section 4]. For Chan-Gan, fixing this normalization is accomplished by matching the Levi subgroups $D^* \times \mathrm{GL}_1 \subset \mathrm{GU}_2(D)$ and $\mathrm{GL}_2 \times \mathrm{GL}_1 \subset \mathrm{GSp}_4$ under the inner twisting, and then fixing the normalization by setting the transfer factor to be equal to the Kottwitz sign $e(D^* \times \mathrm{GL}_1) = -1$ on the pairs (δ, δ^*) of strongly regular elements, where $\delta \in (D^* \times \mathrm{GL}_1)(L)$ is the norm of an element $\delta^* \in (\mathrm{GL}_2 \times \mathrm{GL}_1)(L)$, where we recall that Chan-Gan incorporate the Weyl determinant into their orbital integrals. Then one chooses normalization of the transfer factors between GSp_4 and $\mathrm{GU}_2(D)$ to be compatible with the one on the Levi, as defined in [CG15, Section 4.3]. Similarly, when contemplating the transfer factors between $\mathrm{GU}_2(D)$ and $\mathrm{GSO}_{2,2}$ for the endoscopic case, Chan-Gan perform a similar procedure where one matches up the Levi $D^* \times \mathrm{GL}_1 \subset \mathrm{GU}_2(D)$ with either $\mathrm{GL}_2 \times \mathrm{GL}_1 \subset \mathrm{GSO}_{2,2}$ and $\mathrm{GL}_1 \times \mathrm{GL}_2 \subset \mathrm{GSO}_{2,2}$, which determine transfer factors that differ up to a sign, as remarked in [CG15, Section 4.4]. Then for the endoscopic character identities in [CG15, Proposition 11.1 (2)] an ad hoc choice is made in the beginning of [CG15, Section 11]. On the other hand, as mentioned above Kaletha normalizes his choice by realizing $\mathrm{GU}_2(D)$ as an extended pure inner twist GSp_4 , where one obtains an identity

$$\Delta[\mathfrak{m}, \mathfrak{c}, b_1](\gamma, \delta) = \Delta[\mathfrak{m}, \mathfrak{c}](\mathrm{inv}[z](\delta^*, \delta), s).$$

Here γ and δ are strongly regular elements of $H(L)$ and $\mathrm{GU}_2(L)$, respectively, and the terms $\delta^* \in \mathrm{GSp}_4(F)$ and $\langle \mathrm{inv}[z](\delta^*, \delta), s \rangle$ are as constructed in the discussion around [Kal16, Equation 6] (Here Kaletha is discussing pure inner twists as opposed to extended pure inner twists, but the constructions are almost identical). The element $\langle \mathrm{inv}[z](\delta^*, \delta), s \rangle$ is a certain cocycle evaluated on the element s , and, by virtue of s being an element of order 2 in the cases of interest, this is given by ± 1 . This sign can in general depend on (δ^*, δ) , but if $H = G$ in the endoscopic datum \mathfrak{c} then its value is determined by evaluating $\kappa(b_1) \in X^*(Z(\hat{G})^\Gamma)$ on the element $s \in Z(\hat{G})^\Gamma$ (see [HKW21, Fact 3.2.3]). By studying the restriction of the construction of $\langle \mathrm{inv}[z](\delta^*, \delta), s \rangle$ to Levi subgroups (where one can check that if (γ, δ) lie in Levis which transfer to each other then δ^* can also be chosen to lie in the associated Levi of GSp_4) in the case of the endoscopic datum $(\mathrm{GSO}_{2,2}, (1, -1), \mathrm{GSpin}_4, i : \mathrm{GSpin}_4 \rightarrow \mathrm{GSpin}_5 \simeq \mathrm{GSp}_4)$ and using the previous observations, one can conclude that the transfer factors used in [Kal16] and [CG15] differ by a sign, which allows us to conclude the desired endoscopic character identities for the refined local Langlands (up to a sign in the endoscopic case), but one needs to carefully trace through all the constructions to determine which one. Namely, one needs to match up the normalization obtained from Kaletha's construction with the ad-hoc choice Chan-Gan make for the statement of [CG15, Proposition 11.1 (2)].

The above argument shows the refined local Langlands correspondence of Kaletha holds for the group G ; however, in order to describe the exact matching between ρ_{+-} and ρ_{-+} with τ_1 and τ_2 one needs to exactly compare the transfer factors. The above argument only shows that there is some matching; nonetheless, for our purposes the choice ends up being irrelevant, so we denote the representations in the L -packet $\Pi_\phi(J)$ corresponding to the projections τ_1 and τ_2 by ρ_1 and

ρ_2 , respectively. Similarly, for the representations obtained by pre-composing a character with the composition

$$S_\phi \rightarrow A_\phi$$

we denote the elements of the L -packet $\Pi_\phi(G)$ corresponding to the trivial (non-trivial) character of A_ϕ by π^+ (resp. π^-). We note that, by Theorem 2.1 (2), π^+ can be characterized by the unique \mathfrak{m} -generic representation of this L -packet.

Definition 2.3. Given a supercuspidal or mixed supercuspidal L -parameter ϕ as above and $\rho \in \Pi_\phi(J)$, we let τ_ρ be the irreducible representation of S_ϕ associated to it via the matching described above. Given $\pi \in \Pi_\phi(G)$ and $\rho \in \Pi_\phi(J)$, we set

$$\delta_{\pi,\rho} := \tau_\pi^\vee \otimes \tau_\rho$$

where τ_π^\vee denotes the contragredient.

Remark 2.3. Changing the choice of Whittaker datum scales the representations by a 1-dimensional character of S_ϕ that is trivial when restricted to the center, so in particular this pairing is independent of the choice of Whittaker datum (See [HKW21, Lemma 2.3.3]).

3. THE FARGUES-SCHOLZE LOCAL LANGLANDS CORRESPONDENCE

We will now discuss the Fargues-Scholze local Langlands correspondence and deduce compatibility in the cases where the Gan-Takeda/Gan-Tantono parameter is not supercuspidal. We will then conclude by reducing the question of compatibility in the supercuspidal case to Proposition 1.4.

3.1. Overview of the Fargues-Scholze Local Langlands Correspondence. For now, let G be any connected reductive group over \mathbb{Q}_p . Since we are going to be using geometric Satake, we fix a choice of the square root of p in $\overline{\mathbb{Q}_\ell}$, so that half Tate-twists are well-defined. For us, we will always take $i^{-1}(\sqrt{p})$, where $i : \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$ is the fixed isomorphism. Fargues-Scholze [FS21] consider the moduli space of G -bundles on the Fargues-Fontaine curve X , denoted Bun_G . This moduli space is an Artin v -stack (in the sense of [FS21, Section IV.I]) and has the structure that the underlying points of its topological space $|\text{Bun}_G|$ are in natural bijection with elements of the Kottwitz set $B(G)$, where the slopes of the G -isocrystal associated to $b \in B(G)$ are the negatives of the slopes of the associated vector bundle \mathcal{E}_b and the specializations between points of $|\text{Bun}_G|$ is dictated by the partial ordering on $B(G)$ induced by the kappa invariant and the slope homomorphism [Vie21]. In particular, the connected components of Bun_G are in bijection with $B(G)_{\text{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma$. Specifically, for any $b \in B(G)_{\text{basic}}$, there is a unique open Harder-Narasimhan strata $\text{Bun}_G^b \subset \text{Bun}_G$ dense inside the associated connected component. We recall that the elements of $B(G)_{\text{basic}}$ parametrize extended pure inner forms of G , via sending an element $b \in B(G)_{\text{basic}}$ to its σ -centralizer J_b/\mathbb{Q}_p . For such a basic b , we have an identification $\text{Bun}_G^b \simeq [*/J_b(\mathbb{Q}_p)] =: B_{J_b}(\mathbb{Q}_p)$ of the HN-strata defined by b and the classifying stack of $J_b(\mathbb{Q}_p)$. For any Artin v -stack Z , Fargues-Scholze define a triangulated category $\mathbf{D}_\bullet(Z, \overline{\mathbb{Q}_\ell})$ of solid $\overline{\mathbb{Q}_\ell}$ -sheaves [FS21, Section VII.1] and isolate a nice full subcategory $\mathbf{D}_{\text{lis}}(Z, \overline{\mathbb{Q}_\ell}) \subset \mathbf{D}_\bullet(Z, \overline{\mathbb{Q}_\ell})$ of lisse-étale $\overline{\mathbb{Q}_\ell}$ -sheaves [FS21, Section VII.6.], which may be roughly thought of as the unbounded derived category of étale $\overline{\mathbb{Q}_\ell}$ sheaves on Z , where one has made an enlargement to capture information about the topology of p -adic groups. In any case, the key point for us is that we have the following basic result.

Lemma 3.1. [FS21, Proposition VII.7.1] *There is an equivalence of categories*

$$D_{\text{lis}}(B\mathcal{J}_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq D(\mathcal{J}_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS denotes unbounded derived category of smooth $\mathcal{J}_b(\mathbb{Q}_p)$ -representations with coefficients in $\overline{\mathbb{Q}}_\ell$. Under this equivalence, Verdier duality corresponds to smooth duality.

Remark 3.1. The main reason for constructing this category D_{lis} is that, if one were to take the usual definition for the category of étale $\overline{\mathbb{Q}}_\ell$ -sheaves on $B\mathcal{J}_b(\mathbb{Q}_p)$, this equivalence would no longer be true. In particular, one would obtain the bounded derived category of representations of $\mathcal{J}_b(\mathbb{Q}_p)$ admitting a $\mathcal{J}_b(\mathbb{Q}_p)$ -stable $\overline{\mathbb{Z}}_\ell$ -lattice, where the representation is continuous with respect to the ℓ -adic topology on the target. This would limit the scope of the Fargues-Scholze LLC as, in general, one wants to consider smooth $\overline{\mathbb{Q}}_\ell$ -representations of $\mathcal{J}_b(\mathbb{Q}_p)$, and hence the need for the enlargement of the derived category to D_{lis} .

Lemma 3.1 tells us that, given an irreducible smooth representation π of $G(\mathbb{Q}_p)$, we can consider the associated sheaf, denoted \mathcal{F}_π , on Bun_G^1 the open HN-strata corresponding to the trivial element $\mathbf{1} \in B(G)$, and take the extension by zero along the open inclusion $j_!(\mathcal{F}_\pi)^2$. This realizes the representation π in terms of a sheaf on the moduli space Bun_G in an analogous way to how the function-sheaf dictionary realizes cuspidal automorphic forms as functions associated to sheaves in the context of curves over finite fields. Following V. Lafforgue [Laf18], Fargues and Scholze construct a semisimple L-parameter associated to this sheaf by looking at the action of the excursion algebra on this category $D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$. This relies on a form of the geometric Satake correspondence for the B_{dR}^+ -affine Grassmannians. For any finite set I , let X^I be the product of I -copies of the diamond $X = \text{Spd}(\check{\mathbb{Q}}_p)/\text{Frob}^{\mathbb{Z}}$. We then have the Hecke stack

$$\begin{array}{ccc} & \text{Hck} & \\ & \swarrow \scriptstyle h^\leftarrow & \searrow \scriptstyle h^\rightarrow \times \text{supp} \\ \text{Bun}_G & & \text{Bun}_G \times X^I \end{array}$$

defined as the functor that parametrizes, for S a perfectoid space in characteristic p together with a map $S \rightarrow X^I$ defining a tuple of Cartier divisors in the relative Fargues-Fontaine X_S over S , corresponding to characteristic 0 untilts S_i^\sharp for $i \in I$ of S , a pair of G -torsors $\mathcal{E}_1, \mathcal{E}_2$ together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \bigcup_{i \in I} S_i^\sharp} \xrightarrow{\cong} \mathcal{E}_2|_{X_S \setminus \bigcup_{i \in I} S_i^\sharp}$$

where $h^\leftarrow((\mathcal{E}_1, \mathcal{E}_2, i, (S_i^\sharp)_{i \in I})) = \mathcal{E}_1$ and $h^\rightarrow \times \text{supp}((\mathcal{E}_1, \mathcal{E}_2, \beta, (S_i^\sharp)_{i \in I})) = (\mathcal{E}_2, (S_i^\sharp)_{i \in I})$. We set ${}^L G^I$ to be I -copies of the Langlands dual group of G , i.e. ${}^L G = Q \times \hat{G}(\overline{\mathbb{Q}}_\ell)$, where \hat{G} is the reductive group having dual root datum to G and is viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$. The Weil group acts on \hat{G} via the induced action on root datum through some finite quotient Q , which we now fix. Let $\text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$ denote the category of algebraic $\overline{\mathbb{Q}}_\ell$ -representations of I -copies of ${}^L G$. For

²The shriek push-forward is not in general well-defined in the context of solid $\overline{\mathbb{Q}}_\ell$ -sheaves. However, for the inclusion of HN-strata into Bun_G , its existence follows from [FS21, Proposition VII.7.3], using [FS21, Proposition VII.6.7].

each element $W \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, the geometric Satake correspondence of Fargues-Scholze [FS21, Chapter VI] furnishes a solid $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{S}_W on Hck . This allows us to define Hecke operators.

Definition 3.1. For each $W \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, we define the Hecke operator

$$\begin{aligned} T_W &: \text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{D}_{\blacksquare}(\text{Bun}_G \times X^I) \\ A &\mapsto R(h^\rightarrow \times \text{supp})_{\natural}(h^{\leftarrow*}(A) \otimes^{\mathbb{L}} \mathcal{S}_W) \end{aligned}$$

where \mathcal{S}_W is a solid $\overline{\mathbb{Q}}_\ell$ -sheaf and the functor $R(h^\rightarrow \times \text{supp})_{\natural}$ is the natural push-forward. I.e the left adjoint to the restriction functor in the category of solid $\overline{\mathbb{Q}}_\ell$ -sheaves [FS21, Proposition VII.3.1].

Remark 3.2. These satisfy various compatibilities with respect to composition and restriction to the diagonal. In particular, given two representations $V, W \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$, we have that

$$(T_V \times \text{id})(T_W)(\cdot)|_{\Delta} \simeq T_{V \otimes W}(\cdot)$$

where $\Delta : X \rightarrow X^2$ is the diagonal map.

We then consider $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}^I}$, the category of objects in $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ with continuous action by $W_{\mathbb{Q}_p}^I$. Examples of objects in this category are objects of $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ tensored by a continuous representation of $W_{\mathbb{Q}_p}^I$, for a more precise description see [FS21, Section IX.1]. With this in hand, we then have the following theorem of Fargues-Scholze.

Theorem 3.2. [FS21, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3] *The Hecke operator T_W for $W \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$*

$$T_W : \text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{D}_{\blacksquare}(\text{Bun}_G \times X^I)$$

induces a functor

$$\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}^I}$$

and the induced endofunctors of $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ given by forgetting the Weil group action preserve compact and ULA objects.

Remark 3.3. This should be thought of as a manifestation of Drinfeld's Lemma, where (roughly) the étale fundamental group of $\text{Spd}(\check{\mathbb{Q}}_p)/\text{Frob}^{\mathbb{Z}} = X$ should be the same as $W_{\mathbb{Q}_p}$.

From now on, when talking about Hecke operators we shall always refer to this induced functor, which we will also abusively denote by T_W . Theorem 3.2 has direct implications for the cohomology of local Shimura varieties. To study this, consider a minuscule cocharacter μ with field of definition E , and let $b \in B(G, \mu)$ be the unique basic element in the μ -admissible locus (See [RV14, Definition 2.3]). We say that the triple (G, b, μ) defines a local Shimura datum in the sense of Rapoport-Viehmann [RV14]. Attached to such a data, Scholze-Weinstein [SW20] construct a tower of diamonds

$$p_K : (\text{Sht}(G, b, \mu)_K)_{K \subset G(\mathbb{Q}_p)} \rightarrow \text{Spd}(\check{E})$$

for varying open compact $K \subset G(\mathbb{Q}_p)$. This is obtained by considering the space $\text{Sht}(G, b, \mu)_{\infty}$ which parametrizes modifications $\mathcal{E}_b \rightarrow \mathcal{E}_0$ with meromorphy bounded by μ , where \mathcal{E}_b (resp. \mathcal{E}_0) is the bundle corresponding to $b \in B(G)$ (resp. the trivial bundle) on the Fargues-Fontaine curve.

It has commuting actions by $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ given by acting via automorphisms on \mathcal{E}_0 and \mathcal{E}_b , respectively. The tower is then given by considering the quotients of this space for varying open compact $K \subset G(\mathbb{Q}_p)$ under the action of $G(\mathbb{Q}_p)$.

Definition 3.2. Let $\text{Sht}(G, b, \mu)_{K, \mathbb{C}_p}$ be the base-change of the above tower to \mathbb{C}_p . We define the complex

$$R\Gamma_c(G, b, \mu) := \text{colim}_{K \rightarrow \{1\}} R\Gamma_c(\text{Sht}(G, b, \mu)_{K, \mathbb{C}_p}, \mathbb{Z}_\ell)$$

a colimit of $W_E \times J_b(\mathbb{Q}_p)$ -modules with a $G(\mathbb{Q}_p)$ -action, where W_E is the Weil group of E . The complex $R\Gamma_c(\text{Sht}(G, b, \mu)_{K, \mathbb{C}_p}, \mathbb{Z}_\ell)$ has the natural structure of a continuous W_E -representation and a smooth $J_b(\mathbb{Q}_p)$ -representation by [FS21, Theorem IX.3.1] (cf. [Han20, Proposition 2.3]). A priori it only has an action by the inertia group, but this space admits a non-effective Frobenius descent datum. We then define, for ρ a smooth admissible $J_b(\mathbb{Q}_p)$ -representation, the complex

$$R\Gamma_c(G, b, \mu)[\rho] := \text{colim}_{K \rightarrow \{1\}} R\Gamma_c(G, b, \mu)_K \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \rho,$$

where $\mathcal{H}(J_b)$ is the usual smooth Hecke algebra. We also define

$$R\Gamma_c^b(G, b, \mu)[\rho] := \text{colim}_{K \rightarrow \{1\}} R\mathcal{H}om_{J_b(\mathbb{Q}_p)}(R\Gamma_c(G, b, \mu)_K, \rho)[-2d](-d).$$

Similarly, for π a smooth admissible $G(\mathbb{Q}_p)$ -representation, we define $R\Gamma_c(G, b, \mu)[\pi]$ and $R\Gamma_c^b(G, b, \mu)[\pi]$.

Remark 3.4. We note that, by Hom-Tensor duality, $R\mathcal{H}om(R\Gamma_c(G, b, \mu)[\rho], \overline{\mathbb{Q}}_\ell)[-2d](-d)$ is isomorphic to $R\Gamma_c^b(G, b, \mu)[\rho^*]$, where ρ^* is the contragredient. We will end up using both of these cohomology groups throughout this manuscript. The former is more natural from the point of view of basic uniformization, while the latter is disposable to the results of Hansen-Kaletha-Weinstein [HKW21] on the Kottwitz conjecture.

To study these complexes, we specialize the above discussion of Hecke operators to the case where $W = V_{\mu^{-1}}$ is specified by the highest weight representation of highest weight μ^{-1} a dominant inverse of μ and $I = \{*\}$ is a singleton. The sheaf \mathcal{S}_W will then be supported on the closed subspace $\text{Hck}_{\leq \mu^{-1}} = \text{Hck}_{\mu^{-1}}$ of Hck , parametrizing modifications with meromorphy bounded by or equal to μ^{-1} , where the equality follows by the minuscule assumption. The space $\text{Hck}_{\mu^{-1}}$ is cohomologically smooth of dimension $d := \langle 2\rho_G, \mu \rangle$, and the sheaf \mathcal{S}_W , as in the geometric Satake correspondence of [MV07], behaves like the intersection cohomology of this space, so we have $\mathcal{S}_W \simeq \overline{\mathbb{Q}}_\ell[d](\frac{d}{2})$. This implies that, to study the action of the Hecke operator T_W on Bun_G , we can look at the restriction of the diagram defining the Hecke correspondence to this subspace

$$\begin{array}{ccc} & \text{Hck}_{\mu^{-1}} & \\ \swarrow h_{\mu^{-1}}^{\leftarrow} & & \searrow h_{\mu^{-1}}^{\rightarrow} \times \text{supp} \\ \text{Bun}_G & & \text{Bun}_G \times \text{Spd}(\check{E})/\text{Frob}^{\mathbb{Z}} \end{array}$$

In particular, we have an isomorphism:

$$T_{\mu^{-1}}(A) := T_W(A) \simeq R(h_{\mu^{-1}}^{\rightarrow} \times \text{supp})_{\natural}(h_{\mu^{-1}}^{\leftarrow*}(A))[d](\frac{d}{2})$$

Now consider a smooth admissible representation π of $G(\mathbb{Q}_p)$ and apply the Hecke operator to the sheaf:

$$j_!(\mathcal{F}_\pi)$$

Then the fiber of $\text{Hck}_{\mu^{-1}}$ of $h_{\mu^{-1}}^{\leftarrow}$ over Bun_G^1 is identified with

$$[Gr_{G,\mu^{-1}}/G(\mathbb{Q}_p)]$$

the Schubert cell/variety associated to μ^{-1} in the B_{dR}^+ -affine Grassmannian, quotiented out by $G(\mathbb{Q}_p)$ acting on the trivial bundle via automorphisms. The sheaf

$$T_{\mu^{-1}}j_!(\mathcal{F}_\pi)$$

is then supported on the HN-strata given by the Kottwitz elements in $B(G, \mu)$ since, by [Rap18, Proposition A.9], any G -bundle occurring as a modification of type μ^{-1} of the trivial bundle has associated Kottwitz element lying in this set. We then consider the restriction

$$j_b^*T_{\mu^{-1}}j_!(\mathcal{F}_\pi) \in D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

where $j_b : \text{Bun}_G^b \hookrightarrow \text{Bun}_G$ is the inclusion of the open HN-strata defined by b . The complex $j_b^*T_{\mu^{-1}}j_!(\mathcal{F}_\pi)$ will be computed in terms of the cohomology of sheaves supported on the Newton strata

$$[Gr_{G,\mu^{-1}}^b/G(\mathbb{Q}_p)]$$

parametrizing modifications of type μ^{-1} of the trivial bundle such that the resulting bundle has associated Kottwitz element of type b after pulling back to each geometric point, modulo automorphisms of the trivial bundle. The space $\text{Sht}(G, b, \mu)_\infty$ defined above is a pro-étale $J_b(\mathbb{Q}_p)$ -torsor with respect to the $J_b(\mathbb{Q}_p)$ -action by automorphisms of \mathcal{E}_b

$$\text{Sht}(G, b, \mu)_\infty \rightarrow Gr_{G,\mu^{-1}}^b$$

over this Newton strata. Using this description of the infinite level Shimura variety, it then follows from base change (See [FS21, Chapter IX.3] for details) that we have an isomorphism

$$R\Gamma_c(G, b, \mu)[\pi][d]\left(\frac{d}{2}\right) \simeq j_b^*T_{\mu^{-1}}j_!(\mathcal{F}_\pi) \in D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_E}$$

of $J_b(\mathbb{Q}_p) \times W_E$ -modules. For our purposes, it will be also useful to have a description of the ρ -isotypic part of this cohomology in terms of Hecke operators, for ρ a smooth irreducible representation of $J_b(\mathbb{Q}_p)$. In particular, analysis similar to the above gives us an isomorphism

$$R\Gamma_c(G, b, \mu)[\rho][d]\left(\frac{d}{2}\right) \simeq j_{\mathbf{1}}^*T_{\mu}j_{b\mathbf{1}}(\mathcal{F}_\rho)$$

as $G(\mathbb{Q}_p) \times W_E$ -modules. We record these two isomorphisms as a corollary of the above discussion.

Corollary 3.3. *Given a local Shimura datum (G, b, μ) as above and π (resp. ρ) a smooth irreducible representation of $G(\mathbb{Q}_p)$ (resp. $J_b(\mathbb{Q}_p)$). There exists an isomorphism*

$$R\Gamma_c(G, b, \mu)[\rho][d]\left(\frac{d}{2}\right) \simeq j_{\mathbf{1}}^*T_{\mu}j_{b\mathbf{1}}(\mathcal{F}_\rho)$$

of complexes of $G(\mathbb{Q}_p) \times W_E$ -modules and an isomorphism

$$R\Gamma_c(G, b, \mu)[\pi][d]\left(\frac{d}{2}\right) \simeq j_b^* T_{\mu^{-1}} j_{1!}(\mathcal{F}_\pi)$$

of complexes of $J_b(\mathbb{Q}_p) \times W_E$ -modules.

We have the following basic structural result which, in more generality, follows from the analysis in Fargues-Scholze, but, in the case of a local Shimura datum, also partially follows from standard finiteness results for rigid spaces (See [RV14, Section 6]). In particular, one can show the following.

Theorem 3.4. [FS21, Corollary I.7.3, Page 317] *For a local Shimura datum (G, b, μ) as above, the cohomology groups of $R\Gamma_c^b(G, b, \mu)[\rho]$ and $R\Gamma_c(G, b, \mu)[\rho]$ are valued in smooth admissible $G(\mathbb{Q}_p)$ -representations of finite length with an action of W_E . Moreover, they are concentrated in degrees $0 \leq i \leq 2d$.*

Remark 3.5. A sheaf $\mathcal{F} \in \mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ being ULA is equivalent to its stalks at different HN-strata being valued in complexes of smooth admissible representations [FS21, Theorem V.7.1, Proposition VII.7.9], so indeed the admissibility of the above complex is a consequence of Theorem 3.2 and Corollary 3.3.

Fargues-Scholze use the endofunctors defined by the Hecke algebra on $\mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ to define the excursion algebra.

Definition 3.3. For a finite set I , a representation $W \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, maps $\alpha : \overline{\mathbb{Q}}_\ell \rightarrow \Delta^* W$ and $\beta : \Delta^* W \rightarrow \overline{\mathbb{Q}}_\ell$, and elements $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$, one defines the excursion operator on $\mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ to be the composition:

$$\text{id} = T_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} T_{\Delta^* W} = T_W \xrightarrow{(\gamma_i)_{i \in I}} T_W = T_{\Delta^* W} \xrightarrow{\beta} T_{\overline{\mathbb{Q}}_\ell} = \text{id}$$

where $\Delta^* W$ is the precomposition of W with the diagonal embedding ${}^L G \rightarrow {}^L G^I$.

This defines a natural endomorphism of the identity functor on $\mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$. If one looks at the induced endofunctor given by the inclusion $\mathbf{D}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \subset \mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ induced by the open immersion $j : \text{Bun}_G^1 \hookrightarrow \text{Bun}_G$ then one obtains a natural endomorphism of the identity functor on $\mathbf{D}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$. In other words, we get a family of compatible endomorphisms for all complexes of smooth representations of $G(\mathbb{Q}_p)$; namely, an element of the Bernstein center. One can verify that this excursion algebra satisfies similar properties to that considered by V. Lafforgue, so, using Lafforgue's reconstruction theorem [Laf18, Proposition 11.7], one can show the following.

Theorem 3.5. *To an irreducible smooth $\overline{\mathbb{Q}}_\ell$ -representation π of $G(\mathbb{Q}_p)$ (or more generally $A \in \mathbf{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ any Schur-irreducible object (i.e. $\text{End}(A) = \overline{\mathbb{Q}}_\ell$)), there is a unique continuous semisimple map*

$$\phi_\pi^{\text{FS}} : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

characterized by the property that for all I, W, α, β , and $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$, the corresponding endomorphism of π defined above is given by multiplication by the scalar that results from the composite

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi_\pi(\gamma_i))_{i \in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

By further studying the geometry of Bun_G and the Hecke stacks, one can deduce various good properties of this correspondence.

Theorem 3.6. [FS21, Theorem I.9.6] *The mapping defined above*

$$\pi \mapsto \phi_\pi^{\text{FS}}$$

enjoys the following properties:

- (1) *(Compatibility with Local Class Field Theory) If $G = T$ is a torus, then $\pi \mapsto \phi_\pi$ is the usual local Langlands correspondence*
- (2) *The correspondence is compatible with character twists, passage to contragredients, and central characters.*
- (3) *(Compatibility with products) Given two irreducible representations π_1 and π_2 of two connected reductive groups G_1 and G_2 over \mathbb{Q}_p , respectively. We have*

$$\pi_1 \boxtimes \pi_2 \mapsto \phi_{\pi_1}^{\text{FS}} \times \phi_{\pi_2}^{\text{FS}}$$

under the Fargues-Scholze local Langlands correspondence for $G_1 \times G_2$.

- (4) *(Compatibility with parabolic induction) Given a parabolic subgroup $P \subset G$ with Levi factor M and a representation π_M of M , then the semisimple L -parameter corresponding to any sub-quotient of $\text{ind}_P^G(\pi_M)$ the (normalized) parabolic induction is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_{\pi_M}^{\text{FS}}} {}^L M(\overline{\mathbb{Q}}_\ell) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

where the map ${}^L M(\overline{\mathbb{Q}}_\ell) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is the natural embedding.

- (5) *(Compatibility with Harris-Taylor/Henniart LLC) For $G = \text{GL}_n$ or an inner form of G the semisimple L -parameter associated to π is the (semi-simplified) parameter ϕ_π^{ss} associated to π by Harris-Taylor/Henniart.*
- (6) *(Compatibility with Restriction of Scalars) The above story works the same for G' a connected reductive group over any finite extension E'/\mathbb{Q}_p , where one then gets a semisimple L -parameter valued on $W_{E'}$. If $G = \text{Res}_{E'/\mathbb{Q}_p} G'$ is the Weil restriction of some G'/E' then L -parameters for G/\mathbb{Q}_p agree with L -parameters for G'/E' in the usual sense.*
- (7) *(Compatibility with Isogenies) If $G' \rightarrow G$ is a map of reductive groups inducing an isomorphism of adjoint groups, π is an irreducible smooth representation of $G(E)$ and π' is an irreducible constituent of $\pi|_{G'(E)}$ then $\phi_{\pi'}$ is the image of ϕ_π under the induced map $\hat{G} \rightarrow \hat{G}'$.*

Remark 3.6. In (5), the compatibility of the Fargues-Scholze local Langlands correspondence with the Harris-Taylor/Henniart local Langlands correspondence for an arbitrary inner form of GL_n is not included in the paper of Fargues-Scholze [FS21]. However, it follows from the work of Hansen-Kaletha-Weinstein on the Kottwitz conjecture [HKW21, Theorem 1.0.3].

3.2. The Spectral Action. With these basic structural properties out of the way, we turn our attention to the "spectral action" on $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$, which will be very important to proving compatibility of the two correspondences in the case where the parameter is supercuspidal, as well as deducing applications to the Kottwitz conjecture. We recall that an L -parameter over $\overline{\mathbb{Q}}_\ell$ can be thought of as a continuous (not necessarily semisimple) homomorphism

$$\phi : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

commuting with the natural projection to Q . Assuming the Frobenius semi-simplicity of ϕ , one can use the classical construction of Grothendieck-Deligne to see that this coincides with the definition given in section 2 for GSp_4 after applying the isomorphism i , where the monodromy operation is recovered through the exponential of the action of $W_{\mathbb{Q}_p}$ on the ℓ -power roots of unity. Such a continuous map can be thought of as a continuous 1-cocycle $W_{\mathbb{Q}_p} \rightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$, with respect to the action of $W_{\mathbb{Q}_p}$ on $\hat{G}(\overline{\mathbb{Q}}_\ell)$. If we let A/\mathbb{Z}_ℓ be any \mathbb{Z}_ℓ -algebra endowed with a topology given by writing $A = \mathrm{colim}_{A' \subset A} A'$, where A' is a finitely generated \mathbb{Z}_ℓ -module with its ℓ -adic topology, then we can define a moduli space, denoted $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})$, over \mathbb{Z}_ℓ , whose A -points are the continuous 1-cocycles $W_{\mathbb{Q}_p} \rightarrow \hat{G}(A)$ with respect to the natural action of $W_{\mathbb{Q}_p}$ on $\hat{G}(A)$. This defines a scheme considered in [Dat+20] and [Zhu20] which, by [FS21, Theorem I.9.1], can be written as a union of open and closed affine subschemes $\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ as P runs through subgroups of wild inertia of W_E , where each $\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})$ is a flat local complete intersection over \mathbb{Z}_ℓ of dimension $\dim(G)$. This allows us to consider the Artin stack quotient $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}]$, where \hat{G} acts via conjugation. We then consider the base change to $\overline{\mathbb{Q}}_\ell$, denoted $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ and referred to as the stack of Langlands parameters, as well as the category $\mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ of perfect complexes of coherent sheaves on this space. We let $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ denote the triangulated subcategory of compact objects in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ (which are precisely the objects with quasi-compact support on Bun_G and which restrict to compact objects in $\mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ for all $b \in B(G)$ by [FS21, Theorem V.4.1, Proposition VII.7.4]). We then have the key theorem of Fargues-Scholze.

Theorem 3.7. [FS21, Corollary X.I.3] *There exists a natural compactly supported $\overline{\mathbb{Q}}_\ell$ -linear action of $\mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ satisfying the property that the restriction along the map*

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G) \rightarrow \mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

induces the action of Hecke operators

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I) \rightarrow \mathrm{End}(\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW_{\mathbb{Q}_p}^I}$$

for a varying finite index set I .

Remark 3.7. (1) Here the map

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G) \rightarrow \mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

associates to a representation V , with associated map $r_V : {}^L G(\overline{\mathbb{Q}}_\ell) \rightarrow GL(V)(\overline{\mathbb{Q}}_\ell)$ a vector bundle on $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ of rank equal to $\dim(V)$ with $W_{\mathbb{Q}_p}$ -action. This bundle, denoted C_V , has the property that its evaluation at a $\overline{\mathbb{Q}}_\ell$ -point corresponding to a parameter $\phi : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is precisely $r_V \circ \phi$.

- (2) The compactly supported condition means that, for all $\mathcal{F} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$, the functor $\mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]) \rightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ induced by acting on \mathcal{F} factors through an action of $\mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}/P, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$, where P is a subgroup of wild inertia. Fargues and Scholze state this action in terms of a $(\infty, 1)$ -category acting on an $(\infty, 1)$ -category, we

suppress this technicality for simplicity. However, we note that this enhanced action has the concrete implication that if one has a morphism, cone, or homotopy limit/colimit in the derived category $\text{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ that acting on an object $A \in \text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ will produce a corresponding morphism, cone, or homotopy limit/colimit.

- (3) In fact, Fargues-Scholze show that giving such a compactly supported $\overline{\mathbb{Q}}_\ell$ -linear action is equivalent (when properly formulated) to giving a $\text{Rep}_{\overline{\mathbb{Q}}_\ell}(Q^I)$ -linear monoidal functor

$$\text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G^I) \rightarrow \text{End}_{\overline{\mathbb{Q}}_\ell}(\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega)^{BW_{\mathbb{Q}_p}^I}$$

In the case that $I = \{*\}$, the fact that the Hecke action satisfies this monoidal property is precisely Remark 3.2.

To study this spectral action, we consider, as in [FS21, Section VIII.3.], the coarse quotient in the category of schemes

$$\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} // \hat{G}$$

of $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}$ by the action of \hat{G} via conjugation. Given an L -parameter $\phi : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$, it follows by [Dat+20, Proposition 4.13] or [FS21, Proposition VIII.3.2], that the \hat{G} -orbit of ϕ defines a closed $\overline{\mathbb{Q}}_\ell$ -point of $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} // \hat{G}$ if and only if ϕ is a semisimple parameter. Moreover, the natural map

$$\pi : [\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}] \rightarrow \mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} // \hat{G}$$

evaluated on a $\overline{\mathbb{Q}}_\ell$ -point in the stack quotient defined by an L -parameter ϕ defines a closed $\overline{\mathbb{Q}}_\ell$ -point in the coarse moduli space given by its semisimplification ϕ^{ss} . We can fit excursion operators into this picture as follows. We let $\mathcal{Z}^{\text{spec}}(G, \overline{\mathbb{Q}}_\ell) := \mathcal{O}(\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell})^{\hat{G}}$ be the ring of functions on the stack of L -parameters/the coarse moduli space, which we refer to as the spectral Bernstein center. As noted above, the excursion operators define a family of commuting endomorphisms of the identity functor on $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$. We let $\mathcal{Z}^{\text{geom}}(G, \overline{\mathbb{Q}}_\ell)$ be the ring of such endomorphisms as in [FS21, Definition IX.0.2], which we refer to as the geometric Bernstein center. In [FS21, Corollary IX.0.3], Fargues and Scholze construct a canonical map of rings

$$\mathcal{Z}^{\text{spec}}(G, \overline{\mathbb{Q}}_\ell) \rightarrow \mathcal{Z}^{\text{geom}}(G, \overline{\mathbb{Q}}_\ell)$$

which is given by excursion operators in the following sense. By [FS21, Theorem VIII.3.6], there is an identification between $\mathcal{Z}^{\text{spec}}(G, \overline{\mathbb{Q}}_\ell)$ and the algebra of excursion operators. In particular, an excursion operator, as in Definition 3.3, associated to the datum I, W, α, β , and $\gamma_i \in W_{\mathbb{Q}_p}$ for $i \in I$ defines a function $f_{I, W, \alpha, \beta, (\gamma_i)_{i \in I}} \in \mathcal{O}_{[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}]} = \mathcal{Z}^{\text{spec}}(G, \overline{\mathbb{Q}}_\ell)$ on $\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} // \hat{G}$, whose evaluation on the closed point of the coarse moduli space associated to a semisimple parameter $\phi : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is precisely the scalar that results from the endomorphism:

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi(\gamma_i))_{i \in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

We note that multiplication by $f_{I, W, \alpha, \beta, (\gamma_i)_{i \in I}}$ defines an endomorphism

$$\mathcal{O}_{[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}]} \rightarrow \mathcal{O}_{[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}]}$$

of the structure sheaf on the Artin stack $[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}]$. If we act on a Schur-irreducible object $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$ then we obtain an endomorphism

$$\{\mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}]} \star A = A \rightarrow \mathcal{O}_{[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}]} \star A = A\} \in \mathrm{End}(A) = \overline{\mathbb{Q}_\ell}$$

which will be precisely the scalar given by evaluating ϕ_A^{FS} on the excursion datum (See also [Zou22, Theorem 5.2]). In this way, we see that the action of excursion operators can be obtained through the spectral action. We will leverage this interpretation of excursion operators to prove the following key lemma.

Lemma 3.8. *Let $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$ be any Schur-irreducible object with Fargues-Scholze parameter ϕ_A^{FS} . Set x to be the closed point defined by the parameter ϕ_A^{FS} in the coarse moduli space $Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}$ and let $\pi^{-1}(x)$ denote the closed subset defined by the preimage in $[Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}]$. Suppose we have $C \in \mathrm{Perf}([Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}])$ with support disjoint from $\pi^{-1}(x)$ then C acts by zero on A via the spectral action.*

Proof. If we look at the action on $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$ via the map

$$\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Q}_\ell}) \rightarrow \mathcal{Z}^{\mathrm{geom}}(G, \overline{\mathbb{Q}_\ell})$$

given by excursion operators this factors through the maximal ideal $\mathfrak{m}_A \subset \mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Q}_\ell}) = \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G})$ defined by the closed point ϕ_A^{FS} in the coarse moduli space. By the conditions on the support of C , this implies that we can write the identity element as $1 = 1_C + 1_A \in \mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Q}_\ell})$, where 1_C is a function that annihilates C and 1_A is in the annihilator of $\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Q}_\ell})/\mathfrak{m}_A$. We consider the spectral action of C on A

$$C \star A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$$

and look at the endomorphism induced by multiplication by 1 on C

$$C \star A \rightarrow C \star A$$

which is just the identity. However, since 1_C annihilates C , this is the same as the action of 1_A on $C \star A$, but, it follows by the above discussion that acting via multiplication by 1_A is the same as acting via the map

$$\mathcal{Z}^{\mathrm{spec}}(G, \overline{\mathbb{Q}_\ell}) \rightarrow \mathcal{Z}^{\mathrm{geom}}(G, \overline{\mathbb{Q}_\ell})$$

given by excursion operators, and the action of 1_A after applying this map is zero. This would lead to a contradiction unless $C \star A$ is also zero. \square

To take advantage of this lemma, we now introduce the following endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$, which are analogues of the averaging operators considered by [AL21] in the Fargues-Scholze geometric Langlands correspondence for GL_n and by [FGV02; Gai04] in the classical geometric Langlands correspondence over function fields.

Definition 3.4. Let ϕ be a representation of $W_{\mathbb{Q}_p}$ and V a representation of ${}^L G$ with T_V the associated Hecke operator. We consider the endofunctor of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$

$$A \mapsto R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$$

where $R\Gamma(W_{\mathbb{Q}_p}, -) : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}} \rightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ is the derived functor given by continuous group cohomology with respect to $W_{\mathbb{Q}_p}$. We denote this endofunctor by $\mathcal{A}v_{V,\phi} : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$.

We now would like to realize the functor $\mathcal{A}v_{V,\phi}$ as the spectral action of an object in $\mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])$ similar to [AL21, Section 5.5]. An obvious guess would be that one should take the vector bundle C_V corresponding to the Hecke operator T_V , as in Remark 3.7 (1), and then twist this by the constant sheaf defined by ϕ^\vee , which we denote by

$$C_V \otimes \phi^\vee \in \mathrm{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}])^{BW_{\mathbb{Q}_p}}$$

More precisely, this is the vector bundle with $W_{\mathbb{Q}_p}$ -action whose evaluation at a $\overline{\mathbb{Q}}_\ell$ -point corresponding to a L -parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is the vector space with $W_{\mathbb{Q}_p}$ -action given by tensoring the representation

$$\tilde{\phi} : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell) \xrightarrow{r_V} GL(V)$$

with ϕ^\vee . To obtain the desired perfect complex, it is natural to apply $R\Gamma(W_{\mathbb{Q}_p}, -)$ to the vector bundle $C_V \otimes \phi^\vee$, which we denote by $\mathcal{A}v_{V,\phi}$. We note that $\mathcal{A}v_{V,\phi}$ is a perfect complex. Indeed, as p is invertible in $\overline{\mathbb{Q}}_\ell$, the wild inertia $P \subset W_{\mathbb{Q}_p}$ will always act through a finite quotient on $C_V \otimes \phi^\vee$ and has no higher cohomology which implies that the invariants

$$(C_V \otimes \phi^\vee)^P$$

are a direct summand of the vector bundle $C_V \otimes \phi^\vee$. If we choose a generator $\tau \in I/P$ in the tame quotient of the inertia subgroup $I \subset W_{\mathbb{Q}_p}$ together with a Frobenius lift $\sigma \in W_{\mathbb{Q}_p}/P$ then the complex $\mathcal{A}v_{V,\phi}$ is computed as the homotopy limit of the diagram:

$$\begin{array}{ccc} (C_V \otimes \phi^\vee)^P & \xrightarrow{\tau^{-1}} & (C_V \otimes \phi^\vee)^P \\ \downarrow \sigma^{-1} & & \downarrow \sigma^{(1+\tau+\dots+\tau^{p-1})-1} \\ (C_V \otimes \phi^\vee)^P & \xrightarrow{\tau^{-1}} & (C_V \otimes \phi^\vee)^P \end{array} \quad (1)$$

This gives a presentation of $\mathcal{A}v_{\phi,V}$ as a perfect complex, as in [AL21, Page 21]. To see this, note that σ and τ generate a dense discrete subgroup of $W_{\mathbb{Q}_p}/P$ subject to the relationship that $\sigma^{-1}\tau\sigma = \tau^p$, and it follows that the limit of the diagram is just $((C_V \otimes \phi^\vee)^P)^{W_{\mathbb{Q}_p}/P}$ and that the homotopy limit is in turn the derived functor $R\Gamma(W_{\mathbb{Q}_p}, C_V \otimes \phi^\vee) \simeq R\Gamma(W_{\mathbb{Q}_p}/P, (C_V \otimes \phi^\vee)^P)$, where we have used the vanishing of the higher Galois cohomology with respect to P for the last isomorphism.

Then we have the following Lemma which is a verbatim generalization of [AL21, Lemma 5.7].

Lemma 3.9. *There exists a canonical identification*

$$\mathcal{A}v_{V,\phi}(-) \simeq \mathcal{A}v_{V,\phi} \star (-)$$

of endofunctors of $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$

Proof. Equation (1) gives a diagram of perfect complexes on $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}]$. Acting via the spectral action on an object $\mathcal{F} \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ then gives a diagram

$$\begin{array}{ccc} (T_V(\mathcal{F}) \otimes \phi^\vee)^P & \xrightarrow{\tau-1} & (T_V(\mathcal{F}) \otimes \phi^\vee)^P \\ \downarrow \sigma-1 & & \downarrow \sigma(1+\tau+\dots+\tau^{p-1})-1 \\ (T_V(\mathcal{F}) \otimes \phi^\vee)^P & \xrightarrow{\tau-1} & (T_V \otimes \phi^\vee)^P \end{array}$$

However, if we take the homotopy limit of the diagram in (1), the claim follows from the fact that the spectral action commutes with homotopy limits, as noted in Remark 3.7 (3). \square

With this identification in hand, we can apply Lemma 3.8 to prove the following key consequence.

Lemma 3.10. *Let $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ be an ULA Schur-irreducible object with Fargues-Scholze parameter ϕ_A^{FS} , V a representation of ${}^L G$, and ϕ an irreducible representation of $W_{\mathbb{Q}_p}$. If the cohomology sheaves of $T_V(A) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ with respect to the standard t -structure on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$ have a non-zero sub-quotient as $W_{\mathbb{Q}_p}$ -modules with $W_{\mathbb{Q}_p}$ -action given by ϕ or $\phi(1)$ then $r_V \circ \phi_A^{\mathrm{FS}}$ also has such a sub-quotient.*

Remark 3.8. During the creation of this manuscript, a similar result was obtained by Koshikawa through a similar but simpler proof [Kos21, Theorem 1.3]. As we will similarly conclude in Section 3.3, he shows that if the cohomology $R\Gamma_c(G, b, \mu)[\rho](\frac{d}{2})$ admits a sub-quotient with W_E -action given by an irreducible representation ϕ then the Fargues-Scholze parameter ϕ_ρ^{FS} admits a sub-quotient given by ϕ^\vee . However, in his paper, the shtuka space $\mathrm{Sht}(G, b, \mu)_\infty$ parametrizes modifications of the form $\mathcal{E}_0 \dashrightarrow \mathcal{E}_b$ of type μ , whereas for us it parametrizes modifications of type μ^{-1} . This explains why, under our conventions, there is no dual appearing.

Proof. We first show that the assumption on the cohomology sheaves of $T_V(A)$ implies that averaging operator

$$Av_{\phi, V}(A) = R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$$

is non-trivial. Since A is compact we know by Theorem 3.2 that $T_V(A)$ is also compact and therefore supported on a finite number of HN-strata. By applying excision with respect to the HN-strata of Bun_G [FS21, Proposition VII.7.3], we can assume, without loss of generality, that $T_V(A)$ is supported on a single stalk, where it is given by a complex of smooth irreducible representations of $J_b(\mathbb{Q}_p)$ for $b \in B(G)$. Moreover, since A is ULA, by Theorem 3.2 the sheaf $T_V(A)$ is also ULA and therefore it follows that, for all open compact $K \subset J_b(\mathbb{Q}_p)$, $T_V(A)^K$ is a perfect complex of $\overline{\mathbb{Q}}_\ell$ -vector spaces. Writing $T_V(A) := \mathrm{colim}_{K \rightarrow \{1\}} T_V(A)^K$ and using that cohomology commutes with colimits, this allows us to apply results from the Galois cohomology of $W_{\mathbb{Q}_p}$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$ vector spaces to $T_V(A)$. To do this, we consider the spectral sequence

$$E_2^{p,q} = H^p(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee)) \implies H^{p+q}(R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee))$$

where cohomology is being taken with respect to the standard t -structure on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. Now recall that the cohomological dimension of $W_{\mathbb{Q}_p}$ acting on finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces is 2.

Therefore, this sequence degenerates at the E_3 page. Moreover, the only non-zero degeneracy maps are given by

$$E_2^{0,q+1} = H^0(W_{\mathbb{Q}_p}, H^{q+1}(T_V(A) \otimes \phi^\vee)) \rightarrow E_2^{2,q} = H^2(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee))$$

However, using local Tate-duality on the RHS, we can rewrite this differential as a map:

$$(H^{q+1}(T_V(A)) \otimes \phi^\vee)^{W_{\mathbb{Q}_p}} \rightarrow H^0(W_{\mathbb{Q}_p}, H^q(T_V(A) \otimes \phi^\vee)^\vee(1))^\vee \simeq ((H^q(T_V(A)))^\vee \otimes \phi(1))^{W_{\mathbb{Q}_p}^\vee}$$

Now the term on the RHS (resp. LHS) will only be non-zero if $\phi(1)$ (resp. ϕ) occurs as a sub-quotient of $H^q(T_V(A))$ (resp. $H^{q+1}(T_V(A))$). By assumption, this will be true for some value of q , but now, since the Euler-Poincaré characteristic of $W_{\mathbb{Q}_p}$ acting on $\overline{\mathbb{Q}_\ell}$ -vector spaces is 0, one of these values being non-zero implies the $H^1(W_{\mathbb{Q}_p}, -)$ of some cohomology sheaf of T_V must also be non-zero. This will then give rise to a non-zero contribution to the cohomology of $R\Gamma(W_{\mathbb{Q}_p}, T_V(A) \otimes \phi^\vee)$ so the averaging operator $Av_{V,\phi}(A)$ applied to A is non-zero. Lemma 3.9 therefore tells us that the spectral action of the perfect complex $\mathcal{A}_{V,\phi}$ on A is non-trivial. If x denotes the closed $\overline{\mathbb{Q}_\ell}$ -point in the coarse moduli space of Langlands parameters defined by ϕ_A^{FS} with preimage $\pi^{-1}(x)$ in the stack of Langlands parameters, Lemma 3.8 tells us that $\mathcal{A}_{V,\phi}$ must have non-zero support on $\pi^{-1}(x)$. The $\overline{\mathbb{Q}_\ell}$ -points of $\pi^{-1}(x)$ correspond to the set of Langlands parameters whose semisimplification is precisely ϕ_A^{FS} . The previous analysis tells us that the evaluation of the perfect complex $R\Gamma(W_{\mathbb{Q}_p}, C_V \otimes \phi^\vee)$ at some such point, corresponding to an L -parameter $\tilde{\phi} : W_{\mathbb{Q}_p} \rightarrow {}^L G(\overline{\mathbb{Q}_\ell})$, must be non-zero. This evaluation is precisely the complex

$$R\Gamma(W_{\mathbb{Q}_p}, r_V \circ \tilde{\phi} \otimes \phi^\vee)$$

However, by again applying local Tate-duality, this can only be the case if $r_V \circ \tilde{\phi}$ has a sub-quotient isomorphic to ϕ or $\phi(1)$. Since ϕ is irreducible, this can only happen if $r_V \circ \tilde{\phi}^{\text{ss}} = r_V \circ \phi_A^{\text{FS}}$ has this property. \square

We conclude this section by reviewing how the spectral action behaves on objects with supercuspidal Fargues-Scholze parameter along the lines of [FS21, Section X.2]. These results will be used to deduce a strong form of the Kottwitz conjecture from compatibility. We recall that a supercuspidal parameter, viewed as a continuous 1-cocycle with respect to the $W_{\mathbb{Q}_p}$ -action on $\hat{G}(\overline{\mathbb{Q}_\ell})$, denoted $\phi : W_{\mathbb{Q}_p} \rightarrow \hat{G}(\overline{\mathbb{Q}_\ell})$, satisfies the property that it doesn't factor through any $\hat{P}(\overline{\mathbb{Q}_\ell})$ for P a parabolic subgroup of G . Equivalently, this is the same as insisting that, if $S_\phi := Z_{\hat{G}}(\text{Im}(\phi))$ as before, then the quotient $S_\phi/Z(\hat{G})^\Gamma$ is finite, where $\Gamma := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. From this, it follows by deformation theory (See the proof of [Dat+20, Theorem 1.6] for more details) that the unramified twists of the parameter ϕ define a connected component

$$C_\phi \hookrightarrow [\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}]$$

giving rise to a direct summand

$$\text{Perf}(C_\phi) \hookrightarrow \text{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}])$$

Therefore, the spectral action gives rise to a corresponding direct summand

$$D_{\text{lis}}^{C_\phi}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega \subset D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$$

such that the Schur-irreducible objects in this subcategory all have Fargues-Scholze parameter given by an unramified twist of ϕ . Now, it follows by Proposition 3.14 in the next section and Theorem 3.6 (4), that, since ϕ is supercuspidal, the restriction of any object in $D_{\text{lis}}^{C_\phi}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$ to any non-basic HN-strata of Bun_G must be zero. Therefore, one obtains a decomposition

$$D_{\text{lis}}^{C_\phi}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega \simeq \bigoplus_{b \in B(G)_{\text{basic}}} D^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$$

where J_b is the σ -centralizer of b and $D^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega \subset D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$ is a full subcategory of the derived category of compact objects in smooth representations of $J_b(\mathbb{Q}_p)$. It also follows again by Theorem 3.6 (4) that the Schur-irreducible objects of any $D^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$ must lie only in the supercuspidal components of the Bernstein center. Now to further analyze this we fix a character χ of $Z(G)(\mathbb{Q}_p)$ and consider the subcategory

$$D_{\text{lis}, \chi}^{C_\phi}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega \simeq \bigoplus_{b \in B(G)_{\text{basic}}} D_\chi^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$$

where $D_\chi^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$ is the derived subcategory of $D^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$ generated by compact objects with fixed central character χ , via the natural isomorphism $Z(J_b)(\mathbb{Q}_p) \simeq Z(G)(\mathbb{Q}_p)$ (where we recall that J_b is an extended pure inner form of G). One can see that the spectral action of $\text{Perf}(C_\phi)$ preserves this subcategory. Indeed, it follows by [FS21, Theorem I.8.2] that $\text{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}])$ and in turn $\text{Perf}(C_\phi)$ is generated under cones and retracts by the image of $\text{Rep}_{\overline{\mathbb{Q}_\ell}}({}^L G)$ in $\text{Perf}([\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}_\ell}}/\hat{G}])$. This reduces us to checking that Hecke operators preserve this subcategory, which, in turn reduces to the observation that, if one looks at the simultaneous action of $J_b(\mathbb{Q}_p) \times J_{b'}(\mathbb{Q}_p)$ on the space parametrizing modifications $\mathcal{E}_b \rightarrow \mathcal{E}_{b'}$, for b and b' in $B(G)_{\text{basic}}$ that, under the canonical identification $Z(J_{b'})(\mathbb{Q}_p) \simeq Z(J_b)(\mathbb{Q}_p)$, the diagonally embedded center acts trivially. This follows since an element in the center of $J_b(\mathbb{Q}_p)$ acts on the modification by the inverse of an element in the corresponding center of $J_{b'}(\mathbb{Q}_p)$, where the inverse appears from the fact that $J_b(\mathbb{Q}_p)$ is acting on the left and $J_{b'}(\mathbb{Q}_p)$ is acting on the right.

Now, via local class field theory, we take χ to be the central character determined by ϕ and local class field theory (as in [Bor79, Section 10.1]). Since all Schur-irreducible objects in $D_\chi^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega$ lie in the supercuspidal component of the Bernstein-center and have fixed central character χ and, by [Cas95, Theorem 5.4.1], supercuspidal representations are injective/projective in the category of smooth representations with fixed central character, we can write $D_\chi^{C_\phi}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})^\omega = \bigoplus_\pi \text{Perf}(\overline{\mathbb{Q}_\ell}) \otimes \pi$, where π runs over all supercuspidal representations of $J_b(\mathbb{Q}_p)$ with central character χ which, a priori, have Fargues-Scholze parameter given by an unramified twist of ϕ , but, by Theorem 3.6 (2), must indeed be equal to ϕ . All in all, we conclude that we have a decomposition

$$D_{\text{lis}, \chi}^{C_\phi}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega = \bigoplus_{b \in B(G)_{\text{basic}}} \bigoplus_{\pi_b} \text{Perf}(\overline{\mathbb{Q}_\ell}) \otimes \pi_b$$

where the π_b runs over all supercuspidal representations of $J_b(\mathbb{Q}_p)$ with Fargues-Scholze parameter $\phi_{\pi_b}^{\text{FS}} = \phi$. Moreover, the RHS carries an action of $\text{Rep}_{\overline{\mathbb{Q}_\ell}}(S_\phi)$, the category of finite-dimensional

$\overline{\mathbb{Q}}_\ell$ -representations of S_ϕ .

Now, C_ϕ is the quotient of a torus by S_ϕ , and vector bundles on $[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/S_\phi]$ can be described in terms of representations of S_ϕ . Therefore, given $W \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)$, we get a vector bundle on C_ϕ attached to W by pulling back along the map $C_\phi \rightarrow [\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/S_\phi]$, and we can spectrally act to get an object:

$$\mathrm{Act}_W(\pi_b) \in \bigoplus_{b \in B(G)_{\mathrm{basic}}} \bigoplus_{\pi_b} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi_b.$$

Assume that $W|_{Z(\hat{G})^\Gamma}$ is isotypic, given by some character $\eta : Z(\hat{G})^\Gamma \rightarrow \overline{\mathbb{Q}}_\ell^*$. As $Z(\hat{G})^\Gamma$ is diagonalizable with characters given by $B(G)_{\mathrm{basic}} \xrightarrow{\simeq} \pi_1(G)_\Gamma$ via the κ map, we obtain an element $b_\eta \in B(G)_{\mathrm{basic}}$. Then $\mathrm{Act}_W(\pi_b)$ is concentrated on the basic HN-strata given by $b' = b + b_\eta$. Therefore, we get an isomorphism

$$\mathrm{Act}_W(\pi_b) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$$

where $V_{\pi_{b'}} \in \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)$ and $\pi_{b'}$ runs over all supercuspidals of $J_{b'}$ with Fargues-Scholze parameter $\phi_{\pi_{b'}}^{\mathrm{FS}} = \phi$. With this in hand, we can elucidate the $W_{\mathbb{Q}_p}$ -action on the Hecke operator applied to a smooth irreducible object with supercuspidal Fargues-Scholze parameter, similar to what was done in Lemma 3.10 in the general case. Namely, given $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$, we obtain a vector bundle on $[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/S_\phi]$ with $W_{\mathbb{Q}_p}$ -action given by ϕ via pulling back to the closed substack defined by the parameter ϕ . In other words, we have a functor:

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G) \rightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}.$$

Theorem 3.7 and the above discussion imply that the action of the image of V in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)^{BW_{\mathbb{Q}_p}}$ acting via the spectral action on $D_{\mathrm{lis}, \chi}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ described above is precisely the Hecke operator T_V . In particular, we note that, since every object $D_{\mathrm{lis}, \chi}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ has Fargues-Scholze parameter equal to ϕ , the spectral action of the Hecke operator on objects in $D_{\mathrm{lis}, \chi}^{C_\phi}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ factors over the base-change to the localization around the closed point defined by ϕ .

This tells us that, if we decompose $r_V \circ \phi$ viewed as a representation of S_ϕ as a direct sum $\bigoplus_{i \in I} W_i \otimes \sigma_i$ where $W_i \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then we have an isomorphism

$$T_V(\pi) \simeq \bigoplus_{i \in I} \mathrm{Act}_{W_i}(\pi) \otimes \sigma_i$$

as $J_{b'}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ -modules. We now summarize the above discussion as a corollary for future use.

Corollary 3.11. *Let ϕ be a supercuspidal parameter of G , $b \in B(G)_{\mathrm{basic}}$ a basic element, $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$ an irreducible representation of some highest weight μ with dominant inverse μ^{-1} , and π_b a representation of $J_b(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal to ϕ . We set $b_\mu \in B(G, \mu^{-1})$ to be the unique basic element and $b' = b + b_\mu$. If we decompose $r_V \circ \phi$ viewed as representation of*

S_ϕ as a direct sum $\bigoplus_{i \in I} W_i \otimes \sigma_i$, where $W_i \in \text{Rep}(S_\phi)$ is irreducible and σ_i is a continuous finite-dimensional representation of $W_{\mathbb{Q}_p}$, then there exists an isomorphism of $W_{\mathbb{Q}_p} \times J_{b'}(\mathbb{Q}_p)$ -modules

$$T_\mu(\pi_b) \simeq \bigoplus_{i \in I} \text{Act}_{W_i}(\pi) \otimes \sigma_i$$

where $\text{Act}_{W_i}(\pi) \simeq \bigoplus_{\pi_{b'}} V_{\pi_{b'}} \otimes \pi_{b'}$ with $V_{\pi_{b'}} \in \text{Perf}(\overline{\mathbb{Q}_\ell})$ and $\pi_{b'}$ ranging over supercuspidal representation of $J_{b'}(\mathbb{Q}_p)$ with Fargues-Scholze parameter equal to ϕ .

Remark 3.9. As we will start to see in the next section, the work of Hansen [Han20], Hansen-Kaletha-Weinstein [HKW21], and compatibility of the Fargues-Scholze and Gan-Takeda/Gan-Tantono local Langlands correspondence will allow us to use this Corollary to prove Theorem 1.3. This is suggested already by Corollary 3.3, which shows us that $T_\mu(\pi_b)$ can be computed explicitly using the cohomology of local Shimura varieties.

3.3. Compatibility with the Local Langlands for GSp_4 and $\text{GU}_2(D)$. With the results of the previous section in place, we can now start making progress towards our goal of proving compatibility. So we again let $G = \text{Res}_{L/\mathbb{Q}_p} \text{GSp}_4$ and $J = \text{Res}_{L/\mathbb{Q}_p} \text{GU}_2(D)$, for L/\mathbb{Q}_p a finite extension. As mentioned in section 1, the case where a representation $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$), is a sub-quotient of a parabolic induction easily follows from Theorem 3.6 (3), (4), (5), and compatibility of the (semi-simplified) Gan-Takeda (resp. Gan-Tantono) parameter with parabolic induction. We record this as a corollary now.

Corollary 3.12. *Let $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) be representations occurring as a sub-quotient of a parabolic induction. For such π (resp. ρ), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

To tackle the remaining cases where the L -parameter ϕ is mixed supercuspidal or supercuspidal, we note that these are the cases where the L -parameter is discrete (i.e the L -parameter does not factor through a Levi subgroup). This case is disposable to the results of Hansen-Kaletha-Weinstein [HKW21]. We will now let μ be the Siegel cocharacter of G so that the σ -centralizer of the unique basic element $b \in B(G, \mu)$ is J . We now state the main result of [HKW21] specialized to the case of the Shimura datum (G, b, μ) .

Theorem 3.13. *[HKW21, Theorem 1.0.2] Let ϕ be a mixed supercuspidal or supercuspidal parameter and $S_\phi := Z_{\hat{G}}(\text{Im}(\phi))$ as before. Let $\Pi_\phi(G)$ and $\Pi_\phi(J)$ denote the L -packets over ϕ . Set $\pi \in \Pi_\phi(G)$ (resp. $\rho \in \Pi_\phi(J)$) to be smooth irreducible representations of $G(\mathbb{Q}_p)$ (resp. $J(\mathbb{Q}_p)$). If ϕ is supercuspidal or mixed supercuspidal, we have the following equality in the Grothendieck group $K_0(G(\mathbb{Q}_p))^{\text{ell}}$ of elliptic admissible representations of $G(\mathbb{Q}_p)$ of finite length*

$$[R\Gamma_c^b(G, b, \mu)[\rho]] = - \sum_{\pi \in \Pi_\phi(G)} \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, \text{std} \circ \phi)\pi$$

and the following equality

$$[R\Gamma_c^b(G, b, \mu)[\pi]] = - \sum_{\rho \in \Pi_\phi(J)} \text{Hom}_{S_\phi}(\delta_{\pi, \rho}^\vee, (\text{std} \circ \phi)^\vee)\rho$$

in the Grothendieck group of elliptic admissible $J(\mathbb{Q}_p)$ -representations of finite length, where $\delta_{\pi,\rho}$ is the algebraic representation of S_ϕ in Definition 2.3. Moreover, if the Fargues-Scholze parameter of π (resp. ρ) is supercuspidal this is true in the Grothendieck group $K_0(G(\mathbb{Q}_p))$ (resp. $K_0(J_b(\mathbb{Q}_p))$) of all admissible representations of finite length.

Remark 3.10. (1) To deduce the result for the π -isotypic part, we have implicitly used the two towers isomorphism $\text{Sht}(G, b, \mu)_\infty \simeq \text{Sht}(J, \hat{b}, \mu^{-1})_\infty$, where μ^{-1} is a dominant inverse to μ and $\hat{b} = b^{-1} \in B(G, \mu^{-1})$ is the unique basic element [SW20, Corollary 23.3.2.], and $B(G, \mu^{-1}) \simeq B(J, \mu^{-1})$ under the inner twisting. This inverse explains the appearance of duals in the formula for the π -isotypic part.

(2) We see that, via Corollary 3.3, this, in the case that ϕ is supercuspidal, should provide us insight into the multiplicity spaces V_{π_b} appearing in Corollary 3.11, assuming compatibility of the Fargues-Scholze and Gan-Tantono/Gan-Takeda local Langlands correspondences. Namely, we will see later (Theorem 3.17 and 3.18) that $R\Gamma_c(G, b, \mu)[\rho] \simeq R\Gamma_c^b(G, b, \mu)[\rho]$ and is concentrated in middle degree 3 if ϕ_ρ^{FS} is supercuspidal. Assuming compatibility, Corollary 3.11 will therefore tell us that $R\Gamma_c(G, b, \mu)[\rho]$ will be a direct sum over representations $\pi \in \Pi_\phi(G)$ with W_L -action given by $\text{std} \circ \phi_\rho^{\text{FS}} = \text{std} \circ \phi_\rho$ decomposed as a representation of S_ϕ . The summands in the decomposed S_ϕ -representation correspond to the weight spaces appearing in the above description in the Grothendieck group.

We now wish to write out the precise formula for the ρ and π -isotypic parts, using the refined local Langlands discussed in section 2.

(1) (ϕ stable) In this case, the L -packet $\Pi_\phi(G) = \{\pi\}$ is a singleton so the RHS of the above formula for the ρ -isotypic part has one term

$$-\pi \text{Hom}_{S_\phi}(\delta_{\pi,\rho}, \text{std} \circ \phi_\rho)$$

In this case, $S_\phi = \mathbb{G}_m$ and $\delta_{\pi,\rho}$ is simply the identity representation. Thus, this Hom space gets identified with the characters of GL_4 , so the formula reduces to

$$-4\pi$$

(2) (ϕ endoscopic) In this case, the L -packet has size 2 and, as seen in section 2.1, $\Pi_\phi(G) = \{\pi^+, \pi^-\}$, where π^+ (resp. π^-) corresponds to the trivial (resp. non-trivial) character of the component group. ρ can be either of the two representations corresponding to the irreducible representation of S_ϕ given by τ_i for $i = 1, 2$ the projection to the two coordinates of S_ϕ . However, the RHS remains the same regardless of which one it corresponds to. So, without loss of generality, we assume that $\rho = \rho_1$. Then the RHS of the above formula for the ρ -isotypic part has two terms

$$\pi^+ \text{Hom}_{S_\phi}(\tau_1, \text{std} \circ \phi_\rho)$$

and

$$\pi^- \text{Hom}_{S_\phi}(\tau_1 \otimes \tau_{\pi^-} \simeq \tau_2, \text{std} \circ \phi_\rho)$$

However, writing $\text{std} \circ \phi_\rho \simeq \phi_1 \oplus \phi_2$, these get identified with

$$-\pi^+ \text{Hom}_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell^*, \phi_1)$$

and

$$-\pi^- \text{Hom}_{\overline{\mathbb{Q}}_\ell^*}(\overline{\mathbb{Q}}_\ell, \phi_2)$$

which will both be identified with characters of GL_2 . Thus, the RHS is equal to

$$-2\pi^+ - 2\pi^-$$

Similarly, for the π -isotypic part, we get that the RHS of the above formula is given by

$$-4\rho$$

in the stable case and

$$-2\rho_1 - 2\rho_2$$

in the endoscopic case.

As mentioned in section 1.2, we will now use the previous result to perform a bootstrap to the supercuspidal representations occurring in the L -packets $\Pi_\phi(G)$ (resp. $\Pi_\phi(J)$), for ϕ a mixed supercuspidal parameter. For this, we will mention one last result from the Fargues-Scholze local Langlands correspondence.

Proposition 3.14. *[FS21, Section IX.7.1] For G any connected reductive group over \mathbb{Q}_p , the action of the excursion algebra on $\text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ commutes with Hecke operators. Moreover, it is compatible with restriction to the HN-strata Bun_G^b for $b \in B(G)$ in the following sense. Given a Schur irreducible object $A \in \text{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)$ and $b \in B(G)$, if we let B be a smooth irreducible constituent of $j_b^*(A)$ then we can view it as a sheaf on the neutral strata of $\text{Bun}_{J_b}^1$ and let ϕ_B^{FS} denote its Fargues-Scholze parameter with respect to the excursion algebra on Bun_{J_b} . Then the Fargues-Scholze parameter of A is the composition*

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_B^{\text{FS}}} {}^L J_b(\overline{\mathbb{Q}}_\ell) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

where ${}^L J_b(\overline{\mathbb{Q}}_\ell) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is the twisted embedding, as defined in [FS21, Page 327].

Remark 3.11. The commutation of Hecke operators and the excursion algebra follows from the interpretation of the excursion algebra in terms of endomorphisms coming from multiplication by the ring of global functions of the stack of L -parameters, as discussed in section 3.2.

From this, we can deduce the following useful corollary.

Corollary 3.15. *For G any connected reductive group with (G, b, μ) a local Shimura datum and $\pi \in \Pi(G)$ and $\rho \in \Pi(J_b)$ smooth irreducible representations. All smooth irreducible representations occurring in the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ have Fargues-Scholze parameter equal to ϕ_π^{FS} . Similarly, all smooth irreducible representations occurring in the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ have Fargues-Scholze parameter equal to ϕ_ρ^{FS} . The same is also true for $R\Gamma_c^b(G, b, \mu)[\rho]$ and $R\Gamma_c^b(G, b, \mu)[\pi]$.*

Proof. The first part follows immediately from Proposition 3.14 and Corollary 3.3. It remains to see the same is true for the complexes $R\Gamma_c^b(G, b, \mu)[\rho]$ and $R\Gamma_c^b(G, b, \mu)[\pi]$. This can be done by writing them as $j_1^* T_\mu Rj_{b*}(\rho)$ and $j_b^* T_{\mu^{-1}} Rj_{1*}(\pi)$, as in [FS21, Section IX.7.1], where it again follows from Proposition 3.14 and Corollary 3.3. \square

We now exploit this corollary to deduce compatibility in the mixed supercuspidal case.

Corollary 3.16. *Let ϕ be an L -parameter of Howe-Piatetski-Schapiro or Saito-Kurokawa type. Then, for any $\pi \in \Pi_\phi(G)$ (resp. $\rho \in \Pi_\phi(J)$), the Fargues-Scholze and Gan-Takeda (resp. Gan-Tantono) local Langlands correspondences are compatible.*

Proof. We give the proof for the Gan-Takeda local Langlands correspondence with the proof for the Gan-Tantono correspondence being completely analogous. If ϕ is of Saito-Kurokawa type then, as seen in section 2.2, we can write $\Pi_\phi(G) = \{\pi_{sc}, \pi_{disc}\}$ and $\Pi_\phi(J) = \{\rho_{sc}, \rho_{disc}\}$, where π_{sc} (resp. ρ_{sc}) is a supercuspidal representation of G (resp. J) and π_{disc} (resp. ρ_{disc}) is a non-supercuspidal representation. We note that the Gan-Takeda (resp. Gan-Tantono) correspondences are compatible with the Fargues-Scholze correspondence for π_{disc} (resp. ρ_{disc}), by Corollary 3.12.

If we let μ be the Siegel cocharacter and $b \in B(G, \mu)$ be the unique basic element. Then the σ -centralizer J_b is isomorphic to J and we can consider the complex

$$R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]$$

of $J(\mathbb{Q}_p) \times W_L$ -representations. We then let $R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]_{sc}$ denote the direct summand of $R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]$, where $J(\mathbb{Q}_p)$ acts via a supercuspidal representation. Theorem 3.13 tells us that we can describe this complex in the Grothendieck group of admissible $G(\mathbb{Q}_p)$ -representations of finite length as

$$[R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]_{sc}] = -2\pi_{sc}$$

which tells us that π_{sc} , occurs as a non-zero sub-quotient of the complex $R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]$. By Corollary 3.15, we know that we have an equality:

$$\phi_{\rho_{disc}}^{\text{FS}} = \phi_{\pi_{sc}}^{\text{FS}}$$

However, Corollary 3.12 tells us that $\phi_{\rho_{disc}}^{\text{FS}} = \phi_{\rho_{disc}}^{\text{SS}}$, which is equal to $\phi_{\pi_{sc}}^{\text{SS}}$, so we get the desired equality. The analysis in the Howe-Piatetski-Schapiro case is the same, where one can look at the ρ -isotypic part for any of the two non-supercuspidals in $\Pi_\phi(J)$. \square

In the remaining part of this section, we address proving compatibility in the case where the parameter ϕ is supercuspidal. Before tackling the question of compatibility, we address some geometric properties of the sheaves \mathcal{F}_ρ , for ρ with supercuspidal Fargues-Scholze parameter. This will be leveraged in proving the strong form of the Kottwitz Conjecture for the ρ and π -isotypic parts in section 8, as mentioned in Remark 3.10 (2).

Now, considering again G a general connected reductive group, $b \in B(G)_{\text{basic}}$ a basic element, and a smooth irreducible representation ρ of the σ -centralizer $J_b(\mathbb{Q}_p)$. We will now address some further consequences of the Fargues-Scholze parameter ϕ_ρ^{FS} being supercuspidal. It turns out that the sheaves defined by representations with these parameters have interesting geometric properties, which were leveraged in [Han20] to prove various general results on the cohomology groups. In particular, Hansen shows the following:

Theorem 3.17. *[Han20, Theorem 1.1.] Let (G, b, μ) be a basic local Shimura datum with E the reflex field of μ as before, and let ρ be a smooth irreducible representation of $J_b(\mathbb{Q}_p)$. Suppose the following conditions hold:*

- (1) *The spaces $(\mathrm{Sht}(G, b, \mu)_K)_{K \subset G(\mathbb{Q}_p)}$ occur in the basic uniformization at p of a global Shimura variety in the sense of Definition 4.1.*
- (2) *The Fargues-Scholze parameter $\phi_\rho^{\mathrm{FS}} : W_{\mathbb{Q}_p} \rightarrow^L G(\overline{\mathbb{Q}_\ell})$ is supercuspidal.*

Then the complex $R\Gamma_c(G, b, \mu)[\rho]$ defined above is concentrated in middle degree $d = \dim(\mathrm{Sht}(G, b, \mu)_\infty) = \langle 2\rho_G, \mu \rangle$.

One of the key ideas in the argument is to exploit the behavior of the sheaf $j_{b!}(\mathcal{F}_\rho)$ under Verdier duality, where $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ is the inclusion of the open HN-strata corresponding to $b \in B(G)_{\mathrm{basic}}$. In particular, by Proposition 3.14 and Theorem 3.6 (4), one can see that the natural map $j_{b!}(\mathcal{F}_\rho) \rightarrow Rj_{b*}(\mathcal{F}_\rho)$ is an isomorphism. Namely, Proposition 3.14 implies that a non-zero restriction of $Rj_{b*}(\mathcal{F}_\rho)$ to any non-basic HN-strata must be valued in representations having Fargues-Scholze parameter ϕ_ρ^{FS} under the relevant twisted embedding, which is impossible since the σ -centralizers of non-basic elements are extended pure inner forms of proper Levi subgroups of G and, by assumption, the parameter ϕ_ρ^{FS} is supercuspidal. This implies that, if we apply Verdier duality to both sides of the isomorphism

$$j_{\mathbf{1}}^* T_\mu j_{b!}(\mathcal{F}_\rho) \simeq R\Gamma_c(G, b, \mu)[\rho][d]\left(\frac{d}{2}\right)$$

supplied by Corollary 3.3, we see that the LHS is isomorphic to

$$j_{\mathbf{1}}^* T_\mu j_{b!}(\mathcal{F}_{\rho^*}) \simeq R\Gamma_c(G, b, \mu)[\rho^*][d]\left(\frac{d}{2}\right)$$

On the other hand, on the RHS we act through Verdier duality on the tower $(\mathrm{Sht}(G, b, \mu)_K)_{K \subset G(\mathbb{Q}_p)}$, which are smooth rigid spaces of dimension d . So, in particular, the dualizing object is isomorphic to $\overline{\mathbb{Q}_\ell}[2d](d)$. This allows one to deduce the following consequence for the cohomology groups $R\Gamma_c(G, b, \mu)[\rho]$.

Theorem 3.18. *[Han20, Theorem 1.3, Theorem 2.23] Fix a basic local Shimura datum (G, b, μ) and let ρ be representation of $J_b(\mathbb{Q}_p)$ with supercuspidal Fargues-Scholze parameter. Then there is a natural isomorphism*

$$R\mathcal{H}om(R\Gamma_c(G, b, \mu)[\rho], \overline{\mathbb{Q}_\ell}) \simeq R\Gamma_c(G, b, \mu)[\rho^*][2d](d)$$

as W_E -equivariant objects of $D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}_\ell})$, where ρ^ is the contragredient of ρ . In particular, we have a natural W_E -equivariant isomorphism of admissible $G(\mathbb{Q}_p)$ -representations for all $0 \leq i \leq 2d$*

$$H^i(R\Gamma_c(G, b, \mu)[\rho])^* \simeq H^{2d-i}(R\Gamma_c(G, b, \mu)[\rho^*])(d)$$

Remark 3.12. As noted in Remark 3.4, the LHS of the above formula is isomorphic to $R\Gamma_c^b(G, b, \mu)[\rho^*][2d](d)$, so it follows by cancelling the shifts and Tate twists and relaxing contragredients that one has an isomorphism

$$R\Gamma_c^b(G, b, \mu)[\rho] \simeq R\Gamma_c(G, b, \mu)[\rho]$$

as $J_b(\mathbb{Q}_p) \times W_E$ -representations for all such ρ .

Now we turn our attention to the question of showing compatibility for supercuspidal parameters assuming Proposition 1.4. So again let L/\mathbb{Q}_p be a finite extension and let $G := \text{Res}_{L/\mathbb{Q}_p}(\text{GSp}_4)$ be the restriction of scalars of GSp_4 and $J := \text{Res}_{L/\mathbb{Q}_p} \text{GU}_2(D)$ the unique non-split inner form as before. As we will see in section 8, it essentially follow from Theorem 3.13 and Corollary 3.15 that showing compatibility for $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter implies the corresponding statement for $\pi \in \Pi(G)$ with supercuspidal Gan-Takeda parameter. So we fix such a ρ and assume that the Gan-Tantono parameter ϕ_ρ is endoscopic supercuspidal with the stable case being strictly easier. We will write $\text{std} \circ \phi_\rho \simeq \phi_1 \oplus \phi_2$ for ϕ_i distinct irreducible 2-dimensional representations of W_L and let μ be the Siegel cocharacter. The shtuka space $\text{Sht}(G, b, \mu)_\infty$ in this case will have dimension $d := \langle 2\rho_G, \mu \rangle = 3$. We will assume for the rest of this section that Proposition 1.4 is true.

Proposition 3.19. *Let ϕ be a supercuspidal parameter with associated L -packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

is concentrated in middle degree 3 and admits a non-zero W_L -stable sub-quotient with W_L -action given by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$.

First, we combine this with the following lemma.

Lemma 3.20. *Let ϕ be a supercuspidal parameter then all representations in the L -packet $\Pi_\phi(J)$ have the same Fargues-Scholze parameter.*

Proof. We choose a $\pi \in \Pi_\phi(G)$ and then apply Corollary 3.15 to deduce that all representations occurring in the cohomology of $R\Gamma_c^\flat(G, b, \mu)[\pi]$ have Fargues-Scholze parameter equal to ϕ_π^{FS} . However, by Theorem 3.13, we have that all representations in $\rho \in \Pi_\phi(J)$ occur in the cohomology of $R\Gamma_c^\flat(G, b, \mu)[\pi]$, so their Fargues-Scholze parameters are the same as desired. \square

With this in hand, we are ready to prove the key consequence of Proposition 1.4 using the results on the spectral action obtained in section 3.2.

Corollary 3.21. *Assume that L/\mathbb{Q}_p is an unramified extension and that $p > 2$ and that Proposition 1.4 is true. Then, for ρ a smooth irreducible representation of $J(\mathbb{Q}_p)$ with supercuspidal Gan-Tantono parameter ϕ , the Gan-Tantono and Fargues-Scholze correspondences coincide.*

Proof. As mentioned in the introduction, the key will be the isomorphism

$$\bigoplus_{\rho' \in \Pi_\phi(J)} j_1^* T_\mu j_{b!}(\mathcal{F}_{\rho'}) \simeq \bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho'] [3] \left(\frac{3}{2}\right)$$

supplied by Corollary 3.3. Now Proposition 1.4 tells us that one of the summands on the RHS admits a sub-quotient with W_L -action given by ϕ_1 and one of them admits a sub-quotient with

W_L -action given by ϕ_2 . Applying Lemma 3.20 and 3.10 therefore tells us that $\text{std} \circ \phi_\rho^{\text{FS}}$ admits a sub-quotient isomorphic to ϕ_1 or $\phi_1(1)$ and a sub-quotient isomorphic to ϕ_2 or $\phi_2(1)$. This gives four possibilities for what the parameter $\text{std} \circ \phi_\rho^{\text{FS}}$ is. Since ϕ_ρ^{FS} is a GSp_4 -valued parameter only two of these are possible; namely, $\phi_1 \oplus \phi_2$ or $(\phi_1 \oplus \phi_2)(1)$. However, the second possibility can be ruled out since the similitude character of $\text{std} \circ \phi_\rho^{\text{FS}}$ must coincide with the central character of ρ by Theorem 3.6 (2), which agrees with the similitude character of $\text{std} \circ \phi_\rho = \phi_1 \oplus \phi_2$. Therefore, we conclude that $\text{std} \circ \phi = \text{std} \circ \phi_\rho^{\text{FS}}$ and, by the aforementioned equality of similitude characters of these two parameters and [GT11, Lemma 6.1], this is enough to conclude that $\phi_\rho = \phi_\rho^{\text{FS}}$, as conjugacy classes of GSp_4 -valued parameters. \square

4. BASIC UNIFORMIZATION

In this section, we will briefly review what basic uniformization of the generic fiber of a global Shimura variety means, following [Han20, Section 3.1]. Then we will apply it to our particular case and derive an analogue of Boyer's trick, providing useful consequences for the proof of Proposition 1.4.

4.1. A Review of Basic Uniformization. We now recall briefly what basic uniformization means. Let \mathbf{G}/\mathbb{Q} be a connected reductive group over \mathbb{Q} and let (\mathbf{G}, X) be a Shimura datum, with associate conjugacy class of Hodge cocharacters $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$. Fix a prime p , and set $G := \mathbf{G}_{\mathbb{Q}_p}$. Using our fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$, we can and do regard μ as a conjugacy class of cocharacters $\mu : \mathbb{G}_{m, \overline{\mathbb{Q}_p}} \rightarrow G_{\overline{\mathbb{Q}_p}}$. This allows us to consider the μ -admissible locus $B(G, \mu)$ in the Kottwitz set of G . Let \mathbb{A} (resp. \mathbb{A}_f) denote the adèles (resp. finite adèles) of \mathbb{Q} and \mathbb{A}_f^p denote the finite adèles away from p . For any compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathcal{S}(\mathbf{G}, X)_K$ be the associated rigid analytic Shimura variety over \mathbb{C}_p of level K . We let $K = K^p K_p$, where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$ are open compact subgroups. We set

$$\mathcal{S}(\mathbf{G}, X)_{K^p} := \lim_{K_p \rightarrow \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}.$$

If (\mathbf{G}, X) is of pre-abelian type, this is (up to completing the structure sheaf) representable by a perfectoid space and in general it can be described as a diamond. By the results of [Han16], there exists a canonical $G(\mathbb{Q}_p)$ -equivariant Hodge-Tate period map

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)_{K^p} \rightarrow \mathcal{F}\ell_{G, \mu^{-1}},$$

where $\mathcal{F}\ell_{G, \mu^{-1}} := (G_{\mathbb{C}_p}/P_{\mu^{-1}})^{ad}$ is the adic space associated to the flag variety defined by the parabolic $P_{\mu^{-1}} \subset G_{\mathbb{C}_p}$ given by μ^{-1} via the dynamical method. By the $G(\mathbb{Q}_p)$ -equivariance, π_{HT} descends to a map:

$$\pi_{HT, K_p} : \mathcal{S}(\mathbf{G}, X)_{K^p K_p} \rightarrow [\mathcal{F}\ell_{G, \mu^{-1}}/K_p].$$

We let $b \in B(G, \mu)$ be the unique basic element, and let $\mathcal{F}\ell_{G, \mu^{-1}}^b$ be the basic Newton stratum. This parametrizes, for S a perfectoid space in characteristic p , modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}$ of type μ^{-1} between the trivial G -bundle \mathcal{E}_0 on the relative Fargues-Fontaine curve X_S and \mathcal{E} a bundle isomorphic to the G -bundle \mathcal{E}_b corresponding to $b \in B(G)$ after pulling back to a geometric point of S . (G, b, μ) defines a local Shimura datum, as in section 3.1, so we may consider the infinite level Shimura variety/shtuka space $\text{Sht}(G, b, \mu)_\infty$ and its base change $\text{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p}$. By pulling back along

π_{HT} , we get an open subspace $\mathcal{S}(\mathbf{G}, X)_{K^p}^b \subset \mathcal{S}(\mathbf{G}, X)_{K^p}$, which descends to an open subspace $\mathcal{S}(\mathbf{G}, X)_K^b$, for $K \subset \mathbf{G}(\mathbb{A}_f)$ an open compact. We now have the key definition.

Definition 4.1. We say a global Shimura datum (\mathbf{G}, X) satisfies basic uniformization at p if there exists (in fact this is always true see [Han20, Proposition 3.1]) a unique up to isomorphism \mathbb{Q} -inner form \mathbf{G}' of \mathbf{G} satisfying

- $\mathbf{G}'_{\mathbb{A}_f^p} \simeq \mathbf{G}_{\mathbb{A}_f^p}$ as algebraic groups over \mathbb{A}_f^p ,
- $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$, where J_b is the inner form of G given by the σ -centralizer of the basic element $b \in B(G, \mu)$,
- $\mathbf{G}'(\mathbb{R})$ is compact modulo center,

and a $\mathbf{G}(\mathbb{A}_f)$ -equivariant isomorphism of diamonds over \mathbb{C}_p

$$\lim_{K^p \rightarrow \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p}^b \simeq (\underline{\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)} \times_{\mathrm{Spd}(\mathbb{C}_p)} \mathrm{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p}) / \underline{J_b(\mathbb{Q}_p)}, \quad (2)$$

where $J_b(\mathbb{Q}_p)$ acts diagonally, such that, under the identification $\mathcal{F}\ell_{G, \mu^{-1}}^b \simeq \mathrm{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p} / \underline{J_b(\mathbb{Q}_p)}$, the morphism

$$\pi_{HT} : \lim_{K^p \rightarrow \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p}^b \rightarrow \mathcal{F}\ell_{G, \mu^{-1}}^b$$

identifies with the projection

$$(\underline{\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)} \times_{\mathrm{Spd}(\mathbb{C}_p)} \mathrm{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p}) / \underline{J_b(\mathbb{Q}_p)} \rightarrow \mathrm{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p} / \underline{J_b(\mathbb{Q}_p)},$$

where $\mathbf{G}(\mathbb{A}_f) \simeq \mathbf{G}'(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$ acts on the RHS via the natural action of $\mathbf{G}'(\mathbb{A}_f^p)$ on $\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)$ and $G(\mathbb{Q}_p)$ acts on $\mathrm{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p}$. Moreover, if the reflex field of the cocharacter $\mu : \mathbb{G}_{m, \overline{\mathbb{Q}}_p} \rightarrow G_{\overline{\mathbb{Q}}_p}$ is E/\mathbb{Q}_p then this isomorphism descends to an isomorphism of diamonds over $\check{E} := E\check{\mathbb{Q}}_p$.

We now mention some consequences of uniformization which will be key to us in what follows. Let $\mathcal{H}(J_b) := C_c^\infty(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ be the usual smooth Hecke algebra. We fix an algebraic representation of \mathbf{G}/\mathbb{Q} , denoted \mathcal{V}_ξ , of some regular highest weight ξ . The isomorphism $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ then determines a $\overline{\mathbb{Q}}_\ell$ -local system \mathcal{L}_ξ on the Shimura variety $\mathcal{S}(G, X)_{K^p}$. We now consider the space of algebraic automorphic forms valued in \mathcal{V}_ξ :

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) := \mathrm{colim}_{K^p \rightarrow \{1\}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p K_p, \mathcal{L}_\xi)$$

in the sense of Gross [Gro99]. Namely, it is the space of all continuous functions $\phi : \mathbf{G}'(\mathbb{A}_f) \rightarrow \mathcal{V}_\xi(\overline{\mathbb{Q}}_\ell)$ with respect to the pro-finite topology on the source and the discrete topology on the target such that, for all $k \in K^p$, $\gamma \in G(\mathbb{Q})$, and $g \in \mathbf{G}'(\mathbb{A}_f)$, we have that $\phi(gk) = \phi(g)$ and $\phi(\gamma g) = \gamma\phi(g)$, where γ acts via \mathcal{V}_ξ . The isomorphism (2) will then allow us to deduce an isomorphism

$$R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \xrightarrow{\sim} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}^b, \mathcal{L}_\xi)$$

of $G(\mathbb{Q}_p) \times W_E$ -modules, which when composed with the morphism

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}^b, \mathcal{L}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

coming from excision with \mathbb{Z}_ℓ -coefficients with respect to the open strata $\mathcal{S}(\mathbf{G}, X)_{K^p}^b \hookrightarrow \mathcal{S}(\mathbf{G}, X)_{K^p}$, gives rise to the uniformization map mentioned in the introduction. We show this now.

Proposition 4.1. *Assume that (\mathbf{G}, X) satisfies basic uniformization at p , then there exists a $G(\mathbb{Q}_p) \times W_E$ -equivariant map*

$$\Theta : R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

functorial in the level K^p .

Proof. We will want to use excision in this argument with respect to the Newton stratification. However, with $\overline{\mathbb{Q}_\ell}$ -coefficients there are some subtleties if one just does this naively, as noted at the end of the introduction of [Han20] (See also [Hub98, Proposition 2.6, Example 2.7] for examples in more classical language). To remedy this, as in [Han20] we first construct a map $\hat{\Theta}$ with \mathbb{Z}_ℓ -coefficients, where we use the formalism of adic sheaves defined in [Sch18, Section 26], which defines a full six functor formalism with the usual excision triangles. We then deduce the statement with $\overline{\mathbb{Q}_\ell}$ -coefficients from this. We write $\hat{\mathcal{L}}_\xi$ for the \mathbb{Z}_ℓ -local system defined by fixing a K^p -stable \mathbb{Z}_ℓ -lattice in the \mathbb{Q}_ℓ -realization of the algebraic representation \mathcal{V}_ξ considered above, as in [Han20, Proposition 2.9]

We fix a level $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ and write $\pi_{HT, K_p} : \mathcal{S}(\mathbf{G}, X)_{K^p K_p} \rightarrow [\mathcal{F}\ell_{G, \mu^{-1}} / \underline{K}_p]$ for the induced Hodge-Tate period map. Then we have an isomorphism:

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \hat{\mathcal{L}}_\xi) \simeq \operatorname{colim}_{K_p \rightarrow \{1\}} R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}} / \underline{K}_p], R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi)),$$

where the LHS is defined as the colimit of the cohomology at finite level (or alternatively, since we are working with compactly supported cohomology which is in particular the colimit over the cohomology of the qcqs opens by definition, the cohomology at infinite level). Write $j_{K_p} : [\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p] \hookrightarrow [\mathcal{F}\ell_{G, \mu^{-1}} / \underline{K}_p]$ for the open inclusion of the basic locus quotiented out by K_p . Now, we claim that, under this identification, the desired map is given by the map

$$R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p], j_{K_p}^* R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi)) \rightarrow R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}} / \underline{K}_p], R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi))$$

coming from applying excision with respect to j_{K_p} , for varying K_p . There is a natural map

$$q_{K_p}^b : [\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p] \rightarrow [\operatorname{Spd}(\mathbb{C}_p) / \underline{J}_b(\mathbb{Q}_p)],$$

which is cohomologically smooth as in [Han20, Proposition 2.16]. Since we are working with the category of \mathbb{Z}_ℓ -adic sheaves, the category of sheaves on the target is not identified with the unbounded derived category of smooth representations of $J_b(\mathbb{Q}_p)$, but rather the unbounded derived category of smooth ℓ -complete representations by [Han20, Proposition 2.6 (1)] (cf. Remark 3.1). However, since $\hat{\mathcal{L}}_\xi$ was defined by fixing a K^p -stable lattice inside the \mathbb{Q}_ℓ -realization of \mathcal{V}_ξ , the representation $\Pi := \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \hat{\mathcal{L}}_\xi)$ defined by the ℓ -adic completion of the space of \mathbb{Z}_ℓ -algebraic automorphic forms as defined above (with respect to the discrete topology on $V_\xi(\mathbb{Z}_\ell)$) has the structure of such a $J_b(\mathbb{Q}_p)$ -representation, and we write \mathcal{F}_Π for this sheaf. By [Han20, Corollary 3.7], the fact that we know basic uniformization at p implies we have a natural in K^p isomorphism:

$$(q_{K_p}^{b*})(\mathcal{F}_\Pi) \simeq j_{K_p}^* R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi).$$

Applying $R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p], -)$, we obtain

$$R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p], (q_{K_p}^{b*})(\mathcal{F}_\Pi)) \simeq R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{K}_p], j_{K_p}^* R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi)),$$

but now, by [Han20, Proposition 2.17], we have an isomorphism

$$R\Gamma_c(\text{Sht}(G, b, \mu)_{K_p}, \mathbb{Z}_\ell) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \hat{\mathcal{L}}_\xi) \simeq R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / K_p], (q_{K_p}^{b*})(\mathcal{F}_\Pi))$$

natural in K_p . However, by proper base-change, we have an identification

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p K_p}^b, \hat{\mathcal{L}}_\xi) \simeq R\Gamma_c([\mathcal{F}\ell_{G, \mu^{-1}}^b / K_p], j_{K_p}^* R\pi_{HT, K_p!}(\hat{\mathcal{L}}_\xi)),$$

and by composing the previous isomorphism with the natural map

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p K_p}^b, \hat{\mathcal{L}}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p K_p}, \hat{\mathcal{L}}_\xi)$$

coming from excision applied to the adic sheaf $\hat{\mathcal{L}}_\xi$, we obtain a map of the desired form. Since all these constructions are natural in K_p , if we take the colimit over K_p we obtain a uniformization map:

$$\hat{\Theta} : R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \hat{\mathcal{L}}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \hat{\mathcal{L}}_\xi).$$

However, using that both sides of $\hat{\Theta}$ are admissible $G(\mathbb{Q}_p)$ -representations (i.e its invariants under varying K_p give rise to a perfect complex), we can obtain the desired map with $\overline{\mathbb{Q}}_\ell$ -coefficients by taking smooth vectors with respect to $G(\mathbb{Q}_p)$ and tensoring by $\overline{\mathbb{Q}}_\ell$ (cf. the equivalence [Han20, Proposition 2.3] and Definition 3.2). \square

4.2. Boyer's Trick. We will now be interested in applying uniformization to the situation we are interested in, proving an analogue of Boyer's trick [Boy99] and deducing some relevant consequences. The relevant results are due to Shen.

Theorem 4.2. [She20] *If (\mathbf{G}, X) is a Shimura datum of abelian type and $p > 2$ is a prime where G is unramified then (\mathbf{G}, X) satisfies basic uniformization at p .*

Remark 4.1. To see a full proof of this exact statement for the full integral model at hyperspecial level, one can look at the proof of [Li-22, Theorem D].

Now, consider a Shimura datum (\mathbf{G}, X) , where \mathbf{G} is a \mathbb{Q} -inner form of $\mathbf{G}^* := \text{Res}_{F/\mathbb{Q}} \text{GSp}_4$, with F/\mathbb{Q} a totally real field such that p is totally inert and $F_p \simeq L$, a fixed unramified extension of \mathbb{Q}_p and assume that $\mathbf{G}_{\mathbb{Q}_p} \simeq \text{Res}_{L/\mathbb{Q}_p} \text{GSp}_4 = G$. We fix a level $K = K_p K^p \subset \mathbf{G}(\mathbb{A}_f)$ as before, and assume from now on that (\mathbf{G}, X) is such that the corresponding cocharacter μ is the Siegel cocharacter. Therefore, the unique basic $b \in B(G, \mu)$ will have σ -centralizer given by $\text{Res}_{L/\mathbb{Q}_p} \text{GU}_2(D)$, with D/L the quaternionic division algebra. Since L/\mathbb{Q}_p is unramified and $p > 2$, we can apply Theorem 4.2 to deduce basic uniformization at p . Let \mathbf{G}' be the \mathbb{Q} -inner form defined above. Now we prove the following result, which plays a similar role to Boyer's trick [Boy99] in the study of the cohomology of the Lubin-Tate/Drinfeld towers.

Lemma 4.3. *For $b \in B(G, \mu)$ non-basic there exists a proper parabolic $P = LU \subset G$, a dominant cocharacter μ_L of the Levi L , and an element $b_L \in B(L, \mu_L)$, such that the adic Newton strata $\mathcal{F}\ell_{G, \mu^{-1}}^b$ is parabolically induced as a space with $G(\mathbb{Q}_p)$ -action from the flag variety $\mathcal{F}\ell_{L, \mu_L^{-1}}^{b_L}$, where $P(\mathbb{Q}_p)$ acts via the surjection $P(\mathbb{Q}_p) \rightarrow L(\mathbb{Q}_p)$.*

Proof. We recall [RV14, Definition 4.28] that we say $b \in B(G, \mu)$ is Hodge-Newton reducible if there exists a proper Levi subgroup L together with a basic element $b_L \in B(L, \mu_L)$ mapping to $b \in B(G, \mu)$ under the natural map $B(L) \rightarrow B(G)$, where μ_L is a choice of representative for μ as a geometric dominant cocharacter of L . Given such a b , we let P be the standard parabolic of G with respect to a choice of Borel such that its Levi factor is L , and fix μ_L to be the conjugacy class of dominant cocharacters of L that is dominant respect to B . We will also assume that b_L can always be chosen so that $J_{b_L} \xrightarrow{\cong} J_b$ via the map induced by $B(L) \rightarrow B(G)$.

We recall that, if \mathcal{J}_b denotes the group diamond parametrizing automorphisms of the bundle corresponding to $b \in B(G)$, this has a semi-direct product decomposition

$$\mathcal{J}_b \simeq \underline{J_b(\mathbb{Q}_p)} \rtimes \mathcal{J}_b^U$$

given by the splitting of the HN-filtration of \mathcal{E}_b [FS21, Proposition III.5.1]. The flag variety $\mathcal{F}\ell_{G, \mu^{-1}}^b$ identifies with the moduli space $\mathcal{E} \dashrightarrow \mathcal{E}_0$ parametrizing modifications on X_S of meromorphy μ , where \mathcal{E}_0 denotes the trivial G -bundle and \mathcal{E} is a bundle which is isomorphic to \mathcal{E}_b after pulling back to a geometric point of S . We then have the local Hodge-Tate period map

$$\pi_{\text{HT}}^b : \text{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p} \rightarrow \mathcal{F}\ell_{G, \mu^{-1}}^b,$$

which is the \mathcal{J}_b -torsor defined by rigidifying $\mathcal{E} \simeq \mathcal{E}_b$. The space $\text{Sht}(G, b, \mu)_{\infty, \mathbb{C}_p}$ has an action of $G(\mathbb{Q}_p)$ and \mathcal{J}_b coming from the automorphisms of the G -bundles \mathcal{E}_0 and \mathcal{E}_b , respectively. It now follows by [GI16, Proposition 4.13] that $P(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ stabilizes a subspace $\mathcal{C}_{b_L}^{\mu_L} \subset \text{Sht}(G, b, \mu)_{\infty}$, and that we have a $L(\mathbb{Q}_p)$ -equivariant isomorphism of diamonds:

$$\mathcal{C}_{b_L}^{\mu_L} \times^{P(\mathbb{Q}_p)} G(\mathbb{Q}_p) \simeq \text{Sht}(G, b, \mu)_{\infty}.$$

By [GI16, Proposition 4.24] and our assumption that $J_{b_L} \xrightarrow{\cong} J_b$, the space $\mathcal{C}_{b_L}^{\mu_L}$ is isomorphic to

$$\text{Sht}(L, b_L, \mu_L)_{\infty} \times \mathcal{J}_b^U.$$

Now, by quotienting out by the $\mathcal{J}_b \simeq J_b(\mathbb{Q}_p) \rtimes \mathcal{J}_b^U$ -action on $\text{Sht}(G, b, \mu)_{\infty}$, we deduce that

$$[\text{Sht}(G, b, \mu)_{\infty} / \mathcal{J}_b] \simeq \mathcal{F}\ell_{G, \mu^{-1}}^b$$

is parabolically induced from

$$[\text{Sht}(L, b_L, \mu_L)_{\infty} \times \mathcal{J}_b^U / J_b(\mathbb{Q}_p) \rtimes \mathcal{J}_b^U] \simeq [\text{Sht}(L, b_L, \mu_L)_{\infty} / J_{b_L}(\mathbb{Q}_p)] \simeq \mathcal{F}\ell_{L, \mu_L^{-1}}^{b_L}$$

where $P(\mathbb{Q}_p)$ acts on $\mathcal{F}\ell_{G, \mu_L^{-1}}^{b_L}$ via the natural surjection $P(\mathbb{Q}_p) \rightarrow L(\mathbb{Q}_p)$ and the last isomorphism is given by the local Hodge-Tate period map for the local Shimura datum (L, b_L, μ_L) .

Therefore, the claim will follow from checking that the non-basic elements in $B(G, \mu)$ are Hodge-Newton reducible in such a way that the basic element b_L can be chosen in such a way that $J_{b_L} \simeq J_b$. This follows from the classification of this condition in [GHN19, Theorem 3.5]. For clarity, we recall how this works in our case. There are two non-basic elements $b \in B(G, \mu)$. The μ -ordinary element, defined by the maximal element $b^{\text{max}} \in B(G, \mu)$ with respect to the partial ordering on $B(G, \mu)$, and the intermediate strata corresponding to the element lying between the basic element and b^{max} with respect to the partial ordering on $B(G)$. The element b^{max} admits a reduction to the unique element $b_T \in B(T, \mu_T)$, where T is the maximal torus inside GSp_4/L . If we take the image of

b_T inside the Kottwitz set defined by the Levi of the Siegel parabolic then we can arrange that $J_{b_L} \simeq J_b$, as well. Similarly, the element $b \in B(G, \mu)$ corresponding to the intermediate strata admits a reduction to the unique basic element $b_L \in B(L, \mu_L)$ such that $J_{b_L} \xrightarrow{\simeq} J_b$, where L is the Levi factor of the Klingen parabolic of GSp_4 . \square

We now consider some irreducible representation defined by a highest weight ξ as in the previous section, and look at the uniformization map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

furnished by Proposition 4.1 and Theorem 4.2. We let $R\Gamma_c(G, b, \mu)_{sc}$ and $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$ be the direct summands where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then we have the following key consequence of the previous lemma, which justifies why we are referring to this as Boyer's trick.

Proposition 4.4. *The uniformization map Θ induces an isomorphism*

$$\Theta_{sc} : R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi) \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

on the summand where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation.

Proof. As we will want to apply excision, we will show this by first proving a claim in the setting of adic \mathbb{Z}_ℓ -complete sheaves, as in the proof of Proposition 4.1. We will use the notation introduced there. We let $\mathcal{F}\ell_{G, \mu^{-1}}^{nbas}$ denote the closed complement of $\mathcal{F}\ell_{G, \mu^{-1}}^b$ for $b \in B(G, \mu)$ the unique basic element. The space $\mathcal{F}\ell_{G, \mu^{-1}}^{nbas}$ is stratified by $\mathcal{F}\ell_{G, \mu^{-1}}^b$ for $b \in B(G, \mu)$ non-basic. For varying $K_p \subset G(\mathbb{Q}_p)$, we set $[\mathcal{F}\ell_{G, \mu^{-1}}^{nbas} / \underline{K_p}]$ to be the v -stack quotient. We let $\mathcal{S}(\mathbf{G}, X)_{K^p K_p}^{nbas}$ denote the preimage of $[\mathcal{F}\ell_{G, \mu^{-1}}^{nbas} / \underline{K_p}]$ under the map $\pi_{HT, K_p} : \mathcal{S}(\mathbf{G}, X)_{K^p K_p} \rightarrow [\mathcal{F}\ell_{G, \mu^{-1}} / \underline{K_p}]$. It follows from the proof of Proposition 4.1 that the cone of the \mathbb{Z}_ℓ -complete uniformization map $\hat{\Theta}$ is identified with the $G(\mathbb{Q}_p)$ -representation

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}^{nbas}, \hat{\mathcal{L}}_\xi) := \mathrm{colim}_{K_p \rightarrow \{1\}} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p K_p}^{nbas}, \hat{\mathcal{L}}_\xi).$$

We now consider the Hodge-Tate period morphism quotiented out by $G(\mathbb{Q}_p)$ restricted to this locus

$$[\mathcal{S}(\mathbf{G}, X)_{K^p}^{nbas} / \underline{G(\mathbb{Q}_p)}] \rightarrow [\mathcal{F}\ell_{G, \mu^{-1}}^{nbas} / \underline{G(\mathbb{Q}_p)}],$$

which we will denote as π_{HT}^{nbas} .

We write $h_G^{\rightarrow} : [\mathcal{F}\ell_{G, \mu^{-1}}^{nbas} / \underline{G(\mathbb{Q}_p)}] \rightarrow [*/\underline{G(\mathbb{Q}_p)}]$ for the natural map remembering the isomorphism class of the trivial G -bundle. We have an identification:

$$h_G^{\rightarrow} R\pi_{HT!}^{nbas}(\hat{\mathcal{L}}_\xi) \simeq R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}^{nbas}, \hat{\mathcal{L}}_\xi)$$

of objects in $\hat{D}(G(\mathbb{Q}_p), \mathbb{Z}_\ell)$ the derived category of \mathbb{Z}_ℓ -complete representations, as in [Han20, Definition 2.5] (See also [Han20, Proposition 2.6 (1)]). Here we again recall that compactly supported cohomology respects taking limits of spaces since it is defined in terms of the colimit over the qcqs opens (however, this fails for regular cohomology unless the spaces in the limit are qcqs). Now we can apply excision to the sheaf $R\pi_{HT!}^{nbas}(\hat{\mathcal{L}}_\xi)$, and the stratification of $[\mathcal{F}\ell_{G, \mu^{-1}}^{nbas} / \underline{G(\mathbb{Q}_p)}]$ by $[\mathcal{F}\ell_{G, \mu^{-1}}^b / \underline{G(\mathbb{Q}_p)}]$ for $b \in B(G, \mu)$, which is well-defined since the adic Newton stratification is

$G(\mathbb{Q}_p)$ -equivariant. We set $A_b := j_b^* R\pi_{HT!}^{nbas}(\hat{\mathcal{L}}_\xi)$. It follows by Lemma 4.3 and the definition of being parabolically induced that we have an isomorphism of v -stacks

$$[\mathcal{F}\ell_{G,\mu^{-1}}^b/\underline{G(\mathbb{Q}_p)}] \simeq [\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}/\underline{P(\mathbb{Q}_p)}],$$

and therefore A_b gives rise to a sheaf on $\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}$ with $P(\mathbb{Q}_p)$ -equivariant structure. Moreover, since the $P(\mathbb{Q}_p)$ -action on $\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}$ factors through the surjection $P(\mathbb{Q}_p) \rightarrow L(\mathbb{Q}_p)$, we can assume that A_b is pulled back from a complex \bar{A}_b on the v -stack $[\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}/\underline{L(\mathbb{Q}_p)}]$ along the natural map $[\mathcal{F}\ell_{L,\mu_L^{-1}}^b/\underline{P(\mathbb{Q}_p)}] \rightarrow [\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}/\underline{L(\mathbb{Q}_p)}]$. We consider the commutative diagram

$$\begin{array}{ccc} [\mathcal{F}\ell_{G,\mu^{-1}}^b/\underline{G(\mathbb{Q}_p)}] & \xrightarrow{h_G^{\rightarrow,b}} & [\mathrm{Spd}(\mathbb{C}_p)/\underline{G(\mathbb{Q}_p)}] \\ \simeq \uparrow & & \uparrow p \\ [\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}/\underline{P(\mathbb{Q}_p)}] & \longrightarrow & [\mathrm{Spd}(\mathbb{C}_p)/\underline{P(\mathbb{Q}_p)}] \\ \downarrow & & \downarrow q \\ [\mathcal{F}\ell_{L,\mu_L^{-1}}^{bL}/\underline{L(\mathbb{Q}_p)}] & \xrightarrow{h_M^{\rightarrow}} & [\mathrm{Spd}(\mathbb{C}_p)/\underline{L(\mathbb{Q}_p)}], \end{array}$$

where the bottom square is Cartesian. Here $h_G^{\rightarrow,b} := h_G^{\rightarrow} \circ j_b$, where $j_b : [\mathcal{F}\ell_{G,\mu^{-1}}^b/\underline{G(\mathbb{Q}_p)}] \hookrightarrow [\mathcal{F}\ell_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$ is the natural inclusion of the locally closed strata. Therefore, we have identifications

$$h_{G!}^{\rightarrow} j_b! j_b^* R\pi_{HT!}^{nbas}(\hat{\mathcal{L}}_\xi) = h_{G!}^{\rightarrow} j_b!(A_b) \simeq h_{G!}^{\rightarrow,b}(A_b),$$

as \mathbb{Z}_ℓ -complete $G(\mathbb{Q}_p)$ -representations. By an application of proper base-change to the bottom Cartesian square, we deduce that this is isomorphic to

$$p_! q^* h_{M!}^{\rightarrow}(\bar{A}_b),$$

but it is easy to see that $p_! q^* : \hat{D}(L(\mathbb{Q}_p), \mathbb{Z}_\ell) \rightarrow \hat{D}(G(\mathbb{Q}_p), \mathbb{Z}_\ell)$ identifies with the parabolic induction functor.

In summary, we have constructed a filtration on the cone of $\hat{\Theta}$ as a \mathbb{Z}_ℓ -complete $G(\mathbb{Q}_p)$ -representation whose graded pieces are parabolically induced. If we now take smooth vectors and tensor by $\bar{\mathbb{Q}}_\ell$ noting as before the admissibility of the complexes appearing then this is enough to deduce desired claim. \square

This result will be the key tool in allowing us to describe the W_L -action on the local Shimura variety by global methods. We start this global analysis by constructing strong transfers between GSp_4 and its inner forms over a number field F and proving a strong multiplicity one result.

5. EXISTENCE OF STRONG TRANSFERS AND A STRONG MULTIPLICITY ONE RESULT

In this section, we will show the existence of strong transfers of certain automorphic representations of an inner form of GSp_4 over a number field F , using the analysis of the trace formula similar to that of [KS23, Section 6]. We will then combine this with analysis of the simple twisted trace formula of Kottwitz-Shelstad [KS99], to deduce a kind of strong multiplicity one result for inner forms of GSp_4 .

5.1. The Simple Trace Formula and Existence of Strong Transfers. In order to describe the Galois action on the global Shimura variety, we will need to construct strong transfers for inner forms of GSp_4/F over a totally real field F . This will allow us to compute the traces of Frobenius on the global Shimura variety in terms of the Langlands parameters of the strong transfer. The construction of strong transfers will be accomplished by applying the elliptic part of the stable trace formula with respect to the Lefschetz functions constructed by Kret-Shin [KS23] at the Steinberg/Infinite places and pseudo-coefficients at some finite number places where the representation has supercuspidal L -parameter, applying the character identities of Chan-Gan [CG15] to conclude equality of the orbital integrals at these latter places. First, we recall the key results of Kret-Shin on the trace formula with fixed central character. For now, we will work generally. Let \mathbf{G} denote a connected reductive group over a number field F with center Z , write A_Z for the maximal \mathbb{Q} -split torus of $\mathrm{Res}_{F/\mathbb{Q}}Z$, and set $A_{Z,\infty} = A_Z(\mathbb{R})^0$ to be the connected component of the identity. Let \mathbb{A}_F be the adèles of F and write $\mathbf{G}(\mathbb{A}_F)^1$ for a choice of subgroup so that $\mathbf{G}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A}_F)^1 \times A_{Z,\infty}$, as in [Art81, Page 11]. We consider a closed subgroup $\mathfrak{X} \subset Z(\mathbb{A}_F)$ which contains $A_{Z,\infty}$ such that $Z(F)\mathfrak{X}$ is closed inside $Z(\mathbb{A}_F)$ and a continuous character $\chi : (\mathfrak{X} \cap Z(F)) \backslash \mathfrak{X} \rightarrow \mathbb{C}^*$.

We write Pl_F for the set of places of F , and, for $v \in \mathrm{Pl}_F$, let $\mathfrak{X}_v \subset Z(F_v)$ denote a closed subgroup. We let $\chi_v : \mathfrak{X}_v \rightarrow \mathbb{C}^*$ be a smooth character. Write $\mathcal{H}(\mathbf{G}(F_v), \chi_v^{-1})$ for the space of smooth compactly supported functions modulo center on $\mathbf{G}(F_v)$ which transform under \mathfrak{X}_v via χ_v^{-1} . We also require the functions to be K_v -finite for some maximal compact subgroup K_v of $\mathbf{G}(F_v)$ if v is archimedean. We now assume that \mathfrak{X} as above can be written as restricted product $\prod'_{v \in \mathrm{Pl}_F} \mathfrak{X}_v$ in $Z(\mathbb{A}_F)$, for $\mathfrak{X}_v \subset Z(F_v)$ a closed subgroup. Given a semisimple element $\gamma_v \in \mathbf{G}(F_v)$ and an admissible representation π_v of $\mathbf{G}(F_v)$ with central character χ_v on \mathfrak{X}_v , we define the orbital integral and trace character for $f_v \in \mathcal{H}(\mathbf{G}(F_v), \chi_v^{-1})$ as follows. Let I_{γ_v} denote the connected centralizer of γ_v in \mathbf{G} . We have

$$O_{\gamma_v}(f_v) := \int_{I_{\gamma_v}(F_v) \backslash \mathbf{G}(F_v)} f_v(x^{-1}\gamma_v x) dx$$

and

$$\mathrm{tr}(f_v | \pi_v) := \mathrm{tr} \left(\int_{\mathbf{G}(F_v)/Z(F_v)} f_v(g) \pi_v(g) dg \right)$$

where we have fixed compatible choices of Haar measure on \mathbf{G} and Z throughout. We note that this operator is well defined since the above operator is of finite rank if v is finite and is of trace class if v is infinite, by the K_v -finiteness assumption.

We define the adelic Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, as well as the global orbital integrals by taking

a restricted tensor product over the local Hecke algebras defined above and products of the local integrals. Write $\Gamma_{ell}(\mathbf{G})$ to be the set of F -elliptic conjugacy classes in $\mathbf{G}(F)$. Let $\mathbb{A}_{f,F}$ denote the finite adeles of F . For our purposes, it will suffice to consider a central character datum (\mathfrak{X}, χ) , where $\mathfrak{X} = Z(\mathbb{A}_F)$, and we write $\chi = \bigotimes_{v \in \text{Pl}_F} \chi_v$ for smooth characters $\chi_v : Z(F_v) \rightarrow \mathbb{C}^*$ and all $v \in \text{Pl}_F$. We let $L_{disc,\chi}^2(\mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F))$ denote the space of functions on $\mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)$ transforming under \mathfrak{X} by χ and square-integrable on $\mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)^1 / \mathfrak{X}(\mathbb{A}_F) \cap G(\mathbb{A}_F)^1$. Write $\mathcal{A}_{cusp,\chi}(\mathbf{G})$ for the set of isomorphism classes of cuspidal automorphic representations of $\mathbf{G}(\mathbb{A}_F)$ whose central characters restricted to \mathfrak{X} are χ . For $f \in \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$, define the invariant distributions $T_{ell,\chi}^{\mathbf{G}}$ and $T_{disc,\chi}^{\mathbf{G}}$ by

$$T_{ell,\chi}^{\mathbf{G}}(f) := \sum_{\gamma \in \Gamma_{ell}(\mathbf{G})} i(\gamma)^{-1} \text{vol}(I_\gamma(F) \backslash I_\gamma(\mathbb{A}_F) / \mathfrak{X}(\mathbb{A}_F)) O_\gamma(f)$$

$$T_{disc,\chi}^{\mathbf{G}}(f) := \text{tr}(f | L_{disc,\chi}^2(\mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)))$$

where $i(\gamma)$ is the number of connected components in the centralizer of γ . Analogously, we define $T_{cusp,\chi}^{\mathbf{G}}(f)$ by taking the trace on the space of square-integrable cusp forms whose central character restricted to \mathfrak{X} is χ . Let \mathbf{G}^* denote the quasi-split inner form of \mathbf{G} over F , with a fixed inner twist $\mathbf{G}^* \simeq \mathbf{G}$ over \bar{F} . Since Z is canonically identified with the center of \mathbf{G}^* , we may view the character χ as a central character datum for \mathbf{G}^* . We then let f^* denote a Langlands-Shelstad transfer of f to \mathbf{G}^* . One can construct such a transfer by lifting f along the surjection $\mathcal{H}(\mathbf{G}(\mathbb{A}_F)) \rightarrow \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$ applying the transfer with trivial central character due to Waldspurger and then taking the image along the analogous surjection for $\mathbf{G}^*(\mathbb{A}_F)$. We let $\Sigma_{ell,\chi}(\mathbf{G}^*)$ denote the set of $Z(\mathbb{A}_F)$ orbits of stable F -elliptic conjugacy classes in $\mathbf{G}^*(F)$. We define the stable elliptic distribution

$$ST_{ell,\chi}^{\mathbf{G}^*}(f^*) := \tau(\mathbf{G}^*) \sum_{\gamma \in \Sigma_{ell,\chi}(\mathbf{G}^*)} \tilde{i}(\gamma)^{-1} SO_{\gamma,\chi}^{\mathbf{G}^*}(f)$$

where $SO_{\gamma,\chi}^{\mathbf{G}^*}(f^*)$ denotes the stable orbital integral of f^* at γ , $\tau(\mathbf{G}^*)$ is the Tamagawa number of \mathbf{G}^* , and $\tilde{i}(\gamma)$ is the number of Galois fixed connected components of the centralizer of γ in \mathbf{G}^* . Let ξ be an irreducible representation of \mathbf{G}_{F_∞} . Denote by $\chi_\xi : Z(F_\infty) \rightarrow \mathbb{C}^*$ the restriction of ξ to $Z(F_\infty)$. Write $f_\xi^{\mathbf{G}} \in \mathcal{H}(\mathbf{G}(F_\infty), \chi_\xi^{-1})$ for a Lefschetz function associated with ξ . In other words, a function such that $\text{tr}(f_\xi^{\mathbf{G}} | \pi_\infty)$ computes the Euler-Poincaré characteristic for the relative Lie algebra cohomology of $\pi_\infty \otimes \xi$ for every irreducible admissible representation π_∞ of $\mathbf{G}(F_\infty)$ with central character χ_ξ . It follows by the Vogan-Zuckerman classification [VZ84] that, if ξ is regular, this will be non-zero if and only if π_∞ is an (essentially) discrete series representation cohomological of regular weight ξ . For a finite place v_{st} of F , we let $f_{v_{st}}^{\mathbf{G}} := f_{Lef,v_{st}}^{\mathbf{G}} \in \mathcal{H}(\mathbf{G}(F_{v_{st}}))$ denote a Lefschetz function at v_{st} . Morally, this should be characterized by the property that $\text{tr}(f_{v_{st}}^{\mathbf{G}} | \pi_{v_{st}})$ computes the Euler-Poincaré characteristic of the continuous group cohomology of $\mathbf{G}(F_{v_{st}})$ valued in $\pi_{v_{st}}$. In the case that the center is anisotropic, it follows from the computations in [BW00, Theorem XI.3.9] that this means that the trace of $f_{Lef,v_{st}}^{\mathbf{G}}$ will only be non-zero if π_v is 1-dimensional or an unramified twist of the Steinberg representation, for all irreducible admissible unitary π_v . For finite places $v = v_{st}$, they were originally constructed by Kottwitz; however, these results do not apply for the desired application, since the center of GSp_4 is not compact. For the construction of these functions in this

case and the proof of the property that their traces detect when a representation is 1-dimensional or an unramified twist of Steinberg, see [KS23, Appendix A].

We now assume for simplicity that the center Z of \mathbf{G} is split, which will be the case in all our applications. We consider a character $\eta : \mathbf{G}(F) \rightarrow \mathbb{C}^*$ such that $\eta|_{Z(F_{v_{st}})} = \chi_{v_{st}}$. We can define a function $f_{Lef,\eta}^{\mathbf{G}} \in \mathcal{H}(\mathbf{G}(F), \chi_{v_{st}}^{-1})$ as $f_{Lef,\eta}^{\mathbf{G}}(g) := \eta^{-1}(g)f_{Lef}^{G/Z}(\bar{g})$, where $\bar{g} \in G(F_{v_{st}})/Z(F_{v_{st}})$ denotes the image of g under the quotient map. This (up to scaling by a non-zero constant) forms a pseudo-coefficient for the Steinberg representation twisted by η , as constructed in [Kaz86; SS97]. In particular, by [KS23, Corollary A.8 (2)], we have, for $\pi_{v_{st}}$ an irreducible (essentially) unitary representation of $\mathbf{G}(F_{v_{st}})$ with central character $\chi_{v_{st}}$, that $\text{tr}(f_{Lef,\eta}^{\mathbf{G}}|\pi_{v_{st}}) \neq 0$ if and only if $\pi_{v_{st}}$ is isomorphic to η or Steinberg twisted by η .

Now we have the key lemma, which tells us that with respect to these choices of test functions we get the following "simple trace formula".

Lemma 5.1. *[KS23, Lemma 6.1, 6.2] Fix a central character datum $(Z(\mathbb{A}_F), \chi)$ with $\chi_\xi = \chi|_{Z(F_\infty)}$ as above for χ_ξ attached to some regular weight ξ . For an element $f \in \mathcal{H}(\mathbf{G}(\mathbb{A}_F), \chi^{-1})$ in the global Hecke algebra as above, assume that $f_\infty = f_\xi^{\mathbf{G}} \in \mathcal{H}(\mathbf{G}(F_\infty), \chi^{-1})$ is a Lefschetz function at ∞ and assume that $f_{v_{st}} = f_{Lef,\eta}^{\mathbf{G}}$ is the Lefschetz function described above at v_{st} for a character η of $\mathbf{G}(F_{v_{st}})$ such that $\eta|_{Z(F_{v_{st}})} = \chi_{v_{st}}$. Then we have an equality:*

$$ST_{ell,\chi}^{\mathbf{G}^*}(f^*) = T_{ell,\chi}^{\mathbf{G}}(f) = T_{disc,\chi}^{\mathbf{G}}(f) = T_{cusp,\chi}^{\mathbf{G}}(f)$$

Proof. Strictly speaking, the cited Lemmas only prove this in the case where the central character datum is $(Z(F_\infty), \chi_\xi)$; however, the result in this case easily follows using [KS23, Corollary A.8 (3)] (cf. the proof of [KS23, Corollary 8.5]). \square

Let S_{st} and S_{sc} be disjoint finite sets of finite places of F , and let S_0 be a finite set of places contained in $S_{st} \cup S_{sc}$. Let S_∞ denote the infinite places of F . Set S to be a finite set of places containing $S_{st} \cup S_{sc} \cup S_\infty$. We assume that the inner twist \mathbf{G}^* of \mathbf{G} is trivialized away from S_0 and S_∞ (i.e for all $v \notin S_0 \cup S_\infty$, the inner twisting gives rise to an isomorphism $\mathbf{G}_{F_v} \simeq \mathbf{G}_{F_v}^*$). In particular, \mathbf{G} is unramified outside S and we can fix a reductive model for \mathbf{G} and \mathbf{G}^* over $\mathcal{O}_F[1/S]$. By abuse of notation, we write \mathbf{G} and \mathbf{G}^* for the integral models of these groups. The inner twist gives an isomorphism $\mathbf{G}_{F_v}^* \simeq \mathbf{G}_{F_v}$ and isomorphisms $\mathbf{G}_{\mathcal{O}_{F_v}}^* \simeq \mathbf{G}_{\mathcal{O}_{F_v}}$ of the hyperspecial subgroups determined by this model for the finite places $v \notin S$. The notion of unramified local representation on either side will be defined with respect to this fixed choice of hyperspecial level.

For the rest of the section, we will assume that $\mathbf{G}^* = \text{GSp}_4$. We assume throughout that π is a global cuspidal automorphic representation of the group $\mathbf{G}(\mathbb{A}_F)$ satisfying the following properties:

- (1) π is cohomological of some regular weight ξ at infinity.
- (2) π_v is unramified at all finite places $v \notin S$.
- (3) π_v has supercuspidal L -parameter for $v \in S_{sc}$.
- (4) π_v is an unramified twist of Steinberg at all finite places $v \in S_{st}$.

Note, as in Definition 2.1, we can further partition S_{sc} into S_{ssc} , where $\text{std} \circ \phi_{\pi_v}$ is irreducible (ssc = stable supercuspidal) and S_{esc} , where $\text{std} \circ \phi_{\pi_v}$ is reducible (esc = endoscopic supercuspidal).

The main result of this section shows the existence of strong transfers from \mathbf{G} to \mathbf{G}^* at the places in $S_{sc} \cup S_{st} \cup S_{\infty}$ for a certain class of automorphic representations of \mathbf{G} . It is essentially a more refined version of [KS23, Proposition 6.3] in the particular case that $\mathbf{G}^* = \text{GSp}_4$.

Theorem 5.2. *Suppose that S_{st} is non-empty. Given a π as above, there exists a cuspidal automorphic representation τ of $\mathbf{G}^*(\mathbb{A}_F)$ satisfying the following:*

- $\tau^S \simeq \pi^S$.
- At all $v \in S_{sc} \cup S_{st} \cup S_{\infty}$, τ_v has the same Langlands parameter as π_v .

Moreover, we can choose τ to be globally generic. For the first part, the same is true with the roles of τ and π reversed.

Proof. First off note that, since π is an unramified twist of Steinberg at some finite place, τ , if it exists, is automatically (essentially) tempered at all places (cf. Remark 5.1). It follows, by [GT19, Remark 7.4.7], that the global L -packet of τ therefore contains a globally generic representation. So, if we can show the existence of some τ with the desired properties, that means we can find τ globally generic with the same properties. For the former, we now apply the trace formula.

Let $\mathfrak{X} = Z(\mathbb{A}_F)$ and χ be the central character of π . We set $f = \bigotimes_{v \in \text{Pl}_F} f_v$ to be a test function on $\mathbf{G}(\mathbb{A}_F)$ satisfying the following:

- (1) $f_{\infty} = f_{\xi}^{\mathbf{G}}$ is a Lefschetz/Euler-Poincaré function of weight ξ of $\mathbf{G}(F_{\infty})$.
- (2) At $v \in S_{st}$, $f_v = f_{\text{Lef}, \eta_v}^{\mathbf{G}}$ is a Lefschetz function for $\mathbf{G}(F_v)$, where η_v is the unique character such that π_v is the Steinberg twisted by η_v .
- (3) At $v \in S_{ssc}$, $f_v = f_{\pi_v}$ is the pseudo-coefficient of π_v , as constructed in [Kaz86; SS97].
- (4) At $v \in S_{esc}$, $f_v = f_{\pi_v^+} + f_{\pi_v^-}$, where $\{\pi_v^+, \pi_v^-\}$ is the L -packet over ϕ_{π_v} and $f_{\pi_v^{\pm}}$ is the pseudo-coefficient of π_v^{\pm} .
- (5) At the finite places $v \notin S$, f_v is an arbitrary element of the unramified Hecke algebra.
- (6) For $v \in S \setminus S_{st} \cup S_{\infty} \cup S_{sc}$, choose f_v to be an arbitrary function such that $\text{tr}(f_v | \pi_v) > 0$.

Given such a f , we choose a test function $f^* = \bigotimes_{v \in \text{Pl}_F} f_v^*$ on $\mathbf{G}^*(\mathbb{A}_F)$ satisfying the following:

- (1) $f_{\infty}^* = f_{\xi}^{\mathbf{G}^*}$ is a Lefschetz/Euler-Poincaré for the representation ξ of $\mathbf{G}^*(F_{\infty})$.
- (2) At $v \in S_{st}$, $f_v^* = f_{\text{Lef}, \eta_v}^{\mathbf{G}^*}$ is the Lefschetz function for $\mathbf{G}^*(F_v)$.
- (3) At $v \in S_{ssc}$, $f_v^* = f_{\tau_v}$ is a pseudo-coefficient of τ_v , where τ_v is the unique supercuspidal representation of $\mathbf{G}^*(F_v)$ with Langlands parameter ϕ_{π_v} .
- (4) At $v \in S_{esc}$, $f_v^* = f_{\tau_v^+} + f_{\tau_v^-}$, where $\{\tau_v^+, \tau_v^-\}$ is the L -packet over ϕ_{π_v} of $\mathbf{G}^*(F_v)$ and $f_{\tau_v^{\pm}}$ is the pseudo-coefficient of τ_v^{\pm} .
- (5) At the finite places $v \notin S$, $f_v^* = f_v$ is the same element of the unramified Hecke algebra.
- (6) For $v \in S \setminus S_{st} \cup S_{\infty} \cup S_{sc}$, choose $f_v^* = f_v$.

Now we wish to check that f and f^* are matching up to a non-zero constant c , in the sense of [KS23, Section 5.5]. We can check this place by place. For the finite places $v \notin S_{st} \cup S_{sc} \cup S_{\infty}$, this is tautological. For all $v \in S_{\infty} \cup S_{st}$, this follows from [KS23, Lemma A.4, A.11]. For $v \in S_{sc}$, this

follows from the character identities of Chan-Gan [CG15, Proposition 11.1]. Namely, recall for a regular semisimple elliptic element γ the orbital integrals of the pseudo-coefficients of a discrete series representation π is given by the Harish-Chandra character of π evaluated at γ , and it vanishes if γ is non-elliptic [Kaz86, Theorem K] [KST20, Proposition 3.2]. It follows that the orbital integrals for a sum of pseudo-coefficients over an L -packet of GSp_4 or its inner form is already a stable distribution, by [CG15, Main Theorem] combined with [CG15, Proposition 11.1 (1)] for the inner form. Therefore, the desired equality of stable orbital integrals reduces to showing an equality of the sum of Harish-Chandra characters over the L -packets of supercuspidal parameters, and this is precisely [CG15, Proposition 11.1 (1)].

Since the test functions are matching and $S_{st} \neq \emptyset$, we can apply Lemma 5.1 to conclude:

$$T_{\mathrm{cuspidal}, \chi}^{\mathbf{G}^*}(f^*) = ST_{\mathrm{ell}, \chi}^{\mathbf{G}^*}(f^*) = cT_{\mathrm{cuspidal}, \chi}^{\mathbf{G}}(f)$$

Then, by linear independence of characters, we have a relationship

$$\sum_{\substack{\Pi' \in \mathcal{A}_{\mathrm{cuspidal}, \chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') \mathrm{tr}(f_S^* | \Pi'_S) = c \cdot \sum_{\substack{\Pi \in \mathcal{A}_{\mathrm{cuspidal}, \chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) \mathrm{tr}(f_S | \Pi_S)$$

where $m(\Pi)$ (resp. $m(\Pi')$) denotes the multiplicity of Π (resp. Π') in the cuspidal spectrum of \mathbf{G} (resp. \mathbf{G}^*). Now at the infinite places, as soon as $\mathrm{tr}(f_v | \Pi_v) \neq 0$ at $v \mid \infty$ the regularity condition on ξ implies that Π_v is an (essentially) discrete series representation cohomological of regular weight ξ and that $\mathrm{tr}(f_v | \Pi_v) = (-1)^{q(\mathbf{G}_v)}$ by the Vogan-Zuckerman classification of unitary cohomological representations, where $q(\mathbf{G})$ is the F -rank of the derived group of \mathbf{G} . At $v_{st} \in S_{st}$ it follows by [KS23, Corollary A.8] that $\Pi_{v_{st}}$ is either the $\eta_{v_{st}}$ -twist of the Steinberg or trivial representation. If $\Pi_{v_{st}}$ were one-dimensional then the global representation would also be one-dimensional by a strong-approximation argument [KST20, Lemma 6.2], implying that Π_∞ cannot be tempered, which would contradict the fact Π_∞ is an (essentially) discrete series representation. Therefore, $\Pi_{v_{st}}$ is always the $\eta_{v_{st}}$ twist of the Steinberg representation. At the remaining $v \in S_{ssc}$ (resp. $v \in S_{esc}$), it follows from the definition of pseudo-coefficients that, if $\mathrm{tr}(f_v | \Pi_v) > 0$, we have $\Pi_v \simeq \pi_v$ (resp. $\Pi_v \in \{\pi_v^+, \pi_v^-\}$). Similar considerations apply for $\Pi' \in \mathcal{A}_{\mathrm{cuspidal}, \chi}(\mathbf{G}^*)$ occurring non-trivially in the LHS. In summary, by the above analysis, we can deduce that the RHS of the previous equation is non-zero for the term corresponding to π and that all the non-trivial terms on the RHS have the same sign. Therefore, the LHS is also non-zero, and we see, by choosing any non-zero term, that we obtain the desired τ . The converse direction works similarly, where the role of \mathbf{G} and \mathbf{G}^* , are swapped. \square

Remark 5.1. We note that we crucially used at the places $v \in S_{sc}$ that ϕ_{π_v} was supercuspidal. Otherwise, $\mathrm{tr}(f_v | \Pi_v) \neq 0$ wouldn't necessarily imply that Π_v lies in the L -packet over ϕ_{π_v} without assuming that Π_v is tempered. However, by [KS23, Lemma 2.7] any representation of GSp_4 that is Steinberg at some non-empty finite set of places is tempered at all places. Therefore, we can relax this assumption at least for the forward direction of Theorem 5.2 to just assuming that τ_v is a discrete series representation at all $v \in S_{sc}$.

5.2. The Stable and σ -twisted Simple Trace Formula. For the proof of strong multiplicity one, we will need some more refined analysis of trace formulae. Namely, we will be interested in the discrete part of the stable trace formula in the particular case of $\mathbf{G}^* = \mathrm{GSp}_4/F$, as discussed

in [CG15, Section 7.1]. To this end, fix a central character datum $(\mathfrak{X}, \chi) = (Z(\mathbb{A}_F), \chi)$ as before. We recall that the unique elliptic proper endoscopic group of GSp_4/F is $C = \mathrm{GSO}_{2,2} \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(t, t^{-1}) | t \in \mathrm{GL}_1\}$. Then, for a test function f^* on $\mathbf{G}^*(\mathbb{A}_F)$ as above, the discrete part of the stable trace formula is an equality:

$$I_{disc, \chi}^{\mathbf{G}^*}(f^*) = ST_{disc, \chi}^{\mathbf{G}^*}(f^*) + \frac{1}{4} ST_{disc, \chi}^C(f^C)$$

for f^C a matching test function on C . Here

$$I_{disc, \chi}^{\mathbf{G}^*}(f^*) = \sum_M |W(G, M)|^{-1} \cdot \sum_{s \in W(M, G)_{reg}} |\det(s-1)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \cdot \mathrm{tr}(M_P(s, 0) \cdot I_{disc, \chi}^P(0, f^*))$$

is a sum indexed over classes of standard Levi subgroups of \mathbf{G}^* . The precise definition of the terms will not be important for our purposes, but the interested reader can look at [Art02, Section 3]. We simply note that the term corresponding to $M = \mathbf{G}^*$ is precisely equal to $T_{disc, \chi}^{\mathbf{G}^*}(f^*)$, as defined in section 5.1. $ST_{disc, \chi}^{\mathbf{G}^*}$ is a stable distribution on \mathbf{G}^* , similar to $ST_{ell, \chi}^{\mathbf{G}^*}$, and $ST_{disc, \chi}^C$ is the analogous stable distribution on $C(\mathbb{A}_F)$. However, since C has no proper elliptic endoscopic group, we have

$$ST_{disc, \chi}^C(f^C) = I_{disc, \chi}^C(f^C) = T_{disc, \chi}^C(f^C) + (\text{other terms})$$

with the other terms indexed by proper standard Levi subgroups of C as above.

We will be interested in combining this with the elliptic part of the twisted trace formula as described by Kottwitz-Shelstad [KS99] for the particular group $\tilde{\mathbf{G}} := \mathrm{GL}_4 \times \mathrm{GL}_1/F$ with respect to involution

$$\sigma : (g, e) \mapsto (J^t g^{-1} J^{-1}, e \det(g))$$

where

$$J := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

We can enumerate the elliptic σ -twisted endoscopic groups as follows.

- (1) $\mathbf{G}^* = \mathrm{GSp}_4$
- (2) $C_E = \mathrm{Res}_{E/F} \mathrm{GL}'_2 := \{(g_1, g_2) \in \mathrm{Res}_{E/F} \mathrm{GL}_2 | \det(g_1) = \det(g_2)\}$
- (3) $C_+^E = (\mathrm{GL}_2 \times \mathrm{Res}_{E/F} \mathrm{GL}_1)/\mathrm{GL}_1$

where E is an étale quadratic F -algebra and E is not split in case (3). The simple stable twisted trace formula says that if \tilde{f} is a test function on $\tilde{\mathbf{G}}$ whose twisted orbital integral is supported on the regular elliptic set at at least 3 finite places, then we have an identity

$$I_{disc, \chi}^{\tilde{\mathbf{G}}, \sigma}(\tilde{f}) = \frac{1}{2} ST_{disc, \chi}^{\mathbf{G}^*}(f^*) + \frac{1}{4} \sum_E ST_{disc, \chi}^{C_E}(f^{C_E}) + \frac{1}{8} \sum_{E \neq F^{\oplus 2}} ST_{disc, \chi}^{C_+^E}(f^{C_+^E})$$

where

- \sum_E is a sum over étale quadratic F -algebras E ,
- \tilde{f} , f^* , f^{C_E} , and $f^{C_+^E}$ are matching test functions,

- $ST_{disc,\chi}^{\mathbf{G}^*}$, $ST_{disc,\chi}^{CE}$, and $ST_{disc,\chi}^{CE+}$ are the stable distributions appearing in the discrete part of the stable trace formula, as described above,
- $I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}$ is the invariant distribution which is the twisted analogue of $I_{disc,\chi}^{\mathbf{G}^*}$. It is given by [LW13, Theorem 14.3.1 and Proposition 14.3.2] and has the form

$$I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}(\tilde{f}) = \sum_M |W(G, M)|^{-1} \cdot \sum_{s \in W(M, G)_{reg}} |\det(s-1)_{\mathfrak{a}_M/\mathfrak{a}_G}|^{-1} \cdot \text{tr}(M_P(s, 0)) \cdot I_{disc,\chi}^P(0, \tilde{f}) I_{P, disc}(\sigma)$$

where the sum runs over standard Levi subgroups M of G .

Now we want to apply these trace formulae with respect to appropriately chosen test functions. We will assume that S_{st} is a finite set of places such that $|S_{st}| \geq 3$. Then, for all $v \in S_{st}$, we let f_v^* be a pseudo-coefficient for the unramified twist of Steinberg by η_v , where η_v is a character of $\mathbf{G}(F_v)$ such that the restriction to $Z(F_v)$ is χ_v . In particular, we will take f_v^* to be the Lefschetz function $f_{Lef,\eta_v}^{\mathbf{G}^*} \in \mathcal{H}(\mathbf{G}^*(F_v), \chi_v^{-1})$ considered in the previous section which is a pseudo-coefficient for Steinberg up to scaling. It follows, by [CG15, Corollary 10.8], that we can choose the local constituent of the matching function \tilde{f} at v to be the σ -twisted pseudo-coefficient of the Steinberg twisted by η_v , as defined in [MW18]. The orbital integrals of these functions are supported on the regular elliptic set (See the construction in [MW18, Section 7.2]) and therefore we can apply the simple twisted trace formula. Moreover, the twisted orbital integral of \tilde{f}_v is a stable function, and hence the κ -orbital integral of \tilde{f}_v is zero for all $\kappa \neq 1$. Therefore, it follows that the transfers of f_v to all elliptic twisted endoscopic groups $(\tilde{\mathbf{G}}_{F_v}, \sigma)$ vanish, except possibly for $\mathbf{G}_{F_v}^*$. Thus, the simple twisted trace formula simplifies giving an equality:

$$I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}(\tilde{f}) = \frac{1}{2} ST_{disc,\chi}^{\mathbf{G}^*}(f^*)$$

Now we apply the discrete part of the stable trace formula for \mathbf{G}^* to the RHS this gives us an equality:

$$I_{disc,\chi}^{\mathbf{G}^*}(f^*) - \frac{1}{4} ST_{disc,\chi}^C(f^C) = ST_{disc,\chi}^{\mathbf{G}^*}(f^*)$$

However, for any $v \in S_{st}$, the orbital integral of f_v^* is again stable, so we see that its transfer to the endoscopic group C is zero. Hence, the second term vanishes. All in all, we obtain the following lemma.

Lemma 5.3. *For S_{st} a finite set of finite places with $|S_{st}| \geq 3$, f^* and \tilde{f} matching test functions on \mathbf{G}^* and $\tilde{\mathbf{G}}$, respectively, such that f_v^* is a pseudo-coefficient for the (essentially) discrete series Steinberg representation twisted by an unramified character η_v such that $\eta_v|_{Z(F_v)} = \chi_v$ and \tilde{f}_v is the σ -twisted pseudo-coefficient for the Steinberg representation of $\tilde{\mathbf{G}}_{F_v}$ twisted by η_v , we have an equality:*

$$\frac{1}{2} I_{disc,\chi}^{\mathbf{G}^*}(f^*) = I_{disc,\chi}^{\tilde{\mathbf{G}},\sigma}(\tilde{f})$$

relating spectral information on \mathbf{G}^* to $\tilde{\mathbf{G}}$.

5.3. Strong Multiplicity One. We now would like to combine the analysis of sections 5.1 and 5.2 to deduce a strong multiplicity one result for $\mathbf{G}^* = \mathrm{GSp}_4/F$ and certain inner forms. Our analysis is very similar to [CG15, Sections 10.5 and 10.6] and benefited from reading the proofs of [RW21, Proposition 10.1 and Theorem 11.4] in a paper of Rosner and Weissauer, where they prove a similar multiplicity one result using Weselmann's topological twisted trace formula [Wes12] instead of the simple twisted trace formula of Kottwitz-Shelstad. Let S_{st} and S_{sc} be disjoint finite sets of finite places. Let S_∞ denote the set of infinite places. Set $S_0 \subset S_{st} \cup S_{sc}$ and $S_{sc} \cup S_{st} \cup S_\infty \subset S$ to be finite sets of places as before. We let \mathbf{G} be an inner form over F , as in Theorem 5.2, trivialized outside of $S_0 \cup S_\infty$. We have the following.

Proposition 5.4. *Assume that $|S_{st}| \geq 3$. Let π be a cuspidal automorphic representation of $\mathbf{G}^* = \mathrm{GSp}_4/F$ or the above inner form \mathbf{G} satisfying the following:*

- (1) π is cohomological of regular weight ξ at infinity,
- (2) π is unramified outside of S ,
- (3) π is an unramified twist of Steinberg at all places in S_{st} ,
- (4) π has supercuspidal L -parameter at all places in S_{sc} .

If π' is a cuspidal automorphic representation of \mathbf{G}^ satisfying (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in S agrees with π . If π' is a cuspidal automorphic representation of \mathbf{G} satisfying conditions (1), (3), and $\pi'^S \simeq \pi^S$ then its Langlands parameter at all places in $S_{st} \cup S_{sc} \cup S_\infty$ agrees with π .*

Proof. Set χ to be the central character of π . We apply the above trace formulae with central character datum $(Z(\mathbb{A}_F), \chi)$. If π is a representation of $\mathbf{G}^* = \mathrm{GSp}_4/F$, we take τ to be a globally generic member of the global L -packet of π , as in the proof of Theorem 5.2. If π is a representation of the inner form \mathbf{G} then, using Theorem 5.2, we take τ to be a globally generic strong transfer τ of π to a cuspidal automorphic representation of \mathbf{G}^* , with Langlands parameter equal to ϕ_{τ_v} at all places in $v \in S_{sc} \cup S_{st} \cup S_\infty$. Now we apply [GT11, Section 13] to τ to deduce the existence of a strong transfer to a globally generic automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, denoted $\tilde{\tau}$. It satisfies the following:

- (1) $\tilde{\tau}$ is a global theta lift of τ .
- (2) $\tilde{\tau}^\vee \otimes \chi \simeq \tilde{\tau}$.
- (3) For all places v , we have that $\phi_{\tilde{\tau}_v} = \mathrm{std} \circ \phi_{\tau_v}$ as conjugacy classes of parameters.
- (4) Its form falls into the two cases:
 - (a) $\tilde{\tau}$ is cuspidal
 - (b) $\tilde{\tau} = \sigma \boxplus \sigma'$ for $\sigma \neq \sigma'$ a cuspidal automorphic representation of GL_2 .

In the latter case, τ is the theta lift of a cusp form $\sigma \otimes \sigma'$ on $C = \mathrm{GSO}_{2,2}$.

To distinguish these two cases, we say that $\tilde{\tau}$ is a stable or endoscopic lift. We note that actually, since $v \in S_{st}$, we have that $\phi_{\tilde{\tau}_v} = \mathrm{std} \circ \phi_{\tau_v}$, and ϕ_{τ_v} is an unramified twist of the Steinberg parameter by assumption. Therefore, $\tilde{\tau}$ is necessarily a stable lift. We choose matching test functions \tilde{f} and f^* on \mathbf{G} and \mathbf{G}^* , respectively, such that, for $v \in S_{st}$, they are pseudo-coefficients for Steinberg twisted by η_v , as in Lemma 5.3, where η_v the unramified character that $\pi_{v_{st}}$ is a twist of Steinberg of, where we can, up to scaling, take this to be the Lefschetz function $f_{Lef, \eta_{v_{st}}}^{\mathbf{G}^*}$ considered in section 5.1. We let f_∞^* be a Lefschetz function for the discrete series L -packet given by ξ as before. Lemma 5.3 then

gives us an equality:

$$\frac{1}{2} I_{disc, \chi}^{\mathbf{G}^*}(f^*) = I_{disc, \chi}^{\tilde{\mathbf{G}}, \sigma}(\tilde{f})$$

We first treat the case where $\tilde{\tau}$ is a stable lift, and consider the part of the RHS corresponding to the cuspidal representation $\tilde{\tau}$ constructed above. By using linear independence of the unramified characters and the strong multiplicity one property for $\tilde{\mathbf{G}}$, the above identity implies an equality

$$c_1 \cdot \sum_{\Pi' \simeq \pi^S} m(\Pi') \text{tr}(f_S^* | \Pi'_S) = \text{tr}_\sigma(\tilde{\tau}_S | \tilde{f}_S) \quad (3)$$

where c_1 is a non-zero constant. Here $\text{tr}_\sigma(\tau_S | \tilde{f}_S)$ is the σ -twisted trace, as defined in [CG15, Section 5.16]. The LHS runs over automorphic representations satisfying the following:

- (1) Π' has non-zero contribution to the discrete part of the trace formula $I_{disc, \chi}^{\mathbf{G}^*}$.
- (2) The coefficient $m(\Pi')$ is the coefficient associated with the trace of Π' in $I_{disc, \chi}^{\mathbf{G}^*}$.

We can further simplify the LHS of (3) by noting that non-discrete spectrum representations which intervene in $I_{disc, \chi}^{\mathbf{G}^*}$ are parabolically induced from the discrete spectrum of proper Levi subgroups of \mathbf{G}^* . By [CG15, Section 5.8], we know that parabolically induced representations of \mathbf{G}^* lift to parabolically induced representations of $\tilde{\mathbf{G}}$. Therefore, since $\tilde{\tau}$ is a stable lift and therefore cuspidal, all terms occurring in in the LHS must all come from the discrete spectrum $T_{disc, \chi}^{\mathbf{G}^*}(f^*)$, by strong multiplicity one for $\tilde{\mathbf{G}}$. Moreover, the coefficients $m(\Pi')$ must then be the multiplicities of Π' in the discrete spectrum. However, as in Lemma 5.1, we have an equality:

$$T_{disc, \chi}^{\mathbf{G}^*}(f^*) = T_{cusp, \chi}^{\mathbf{G}^*}(f^*)$$

In other words, we may assume that the sum on the LHS of (3) ranges over $\Pi' \in \mathcal{A}_{cusp, \chi}(\mathbf{G}^*)$, and that $m(\Pi')$ denotes the multiplicity in the cuspidal automorphic spectrum. In other words, we can rewrite the LHS as

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp, \chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') \text{tr}(f_S^* | \Pi'_S)$$

Now, for the RHS, we apply the local character identities of Chan-Gan [CG15, Proposition 9.1], this tells us that we have an equality:

$$\text{tr}_\sigma(\tilde{\tau}_S | \tilde{f}_S) = c_2 \prod_{v \in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v}^*)} \text{tr}(f_v^* | \pi'_v)$$

for some non-zero constant c_2 , where we have used property (3) of the representation $\tilde{\tau}$. In summary, we have concluded

$$\sum_{\substack{\Pi' \in \mathcal{A}_{cusp, \chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') \text{tr}(f_S^* | \Pi'_S) = c \cdot \prod_{v \in S} \sum_{\pi'_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v}^*)} \text{tr}(f_v^* | \pi'_v)$$

for some non-zero constant c . If π was a representation of \mathbf{G}^* , we know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S$, so, by linear independence of characters at the places $v \in S \setminus S_\infty \cup S_{st}$, this tells us that the local constituents of some Π' occurring in the LHS with non-zero trace at $S_\infty \cup S_{st}$

are described by members of the L -packet over ϕ_{π_v} occurring with some multiplicity. Since the representation π' is by assumption cohomological of regular weight ξ and an unramified twist of Steinberg at all places in S_{st} , by arguing as in proof of Theorem 5.2, we have that $\text{tr}(f_{S_{st} \cup S_\infty} | \pi'_{S_{st} \cup S_\infty}) \neq 0$, and this gives us the desired claim for $\mathbf{G}^* = \text{GSp}_4$. Now, if π' is a representation of the inner form, we apply the character identities of Chan-Gan [CG15, Proposition 11.1]. This tells us that the RHS of the previous equation is equal to

$$c_3 \prod_{v \in S} \sum_{\rho_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} \text{tr}(f_v | \rho_v)$$

for some non-zero constant c_3 . Now, to rewrite the LHS, we apply the trace formula as in the proof of Theorem 5.2. By linear independence of characters, we obtain a relationship

$$\sum_{\substack{\Pi' \in \mathcal{A}_{\text{cusp}, \chi}(\mathbf{G}^*) \\ \Pi'^S \simeq \pi^S}} m(\Pi') \text{tr}(f_S^* | \Pi'_S) = c_4 \cdot \sum_{\substack{\Pi \in \mathcal{A}_{\text{cusp}, \chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) \text{tr}(f_S | \Pi_S)$$

for some non-zero constant c_4 . All in all, we obtain that

$$\sum_{\substack{\Pi \in \mathcal{A}_{\text{cusp}, \chi}(\mathbf{G}) \\ \Pi^S \simeq \pi^S}} m(\Pi) \text{tr}(f_S | \Pi_S) = c' \prod_{v \in S} \sum_{\rho_v \in \Pi_{\phi_{\tau_v}}(\mathbf{G}_{F_v})} \text{tr}(f_v | \rho_v)$$

for some non-zero constant c' . We know by our choice of τ that $\phi_{\tau_v} = \phi_{\pi_v}$ for all $v \in S_{sc} \cup S_{st} \cup S_\infty$. From here, the claim follows. \square

Remark 5.2. Strong multiplicity one for globally generic automorphic representations of GSp_4 has been proven by Jiang-Soudry [JS07]. So, in the particular case that π and π' are representations of GSp_4 , we could have just assumed that S_{st} is non-empty and then applied their results to a globally generic member in the global L -packets of π and π' to deduce the desired claim.

6. GALOIS REPRESENTATIONS IN THE COHOMOLOGY OF SHIMURA VARIETIES

We now would like to combine the results of the previous section with results of Sorensen [Sor10] on the Galois representations associated to automorphic representations of $\mathbf{G}^* = \text{GSp}_4/F$ to say something about the Galois action of the global Shimura varieties occurring in basic uniformization. Let F/\mathbb{Q} be a totally real field and $\mathbb{A}_{f,F}$ the finite adeles of F . Throughout, we will assume that τ is a cuspidal automorphic representation of \mathbf{G}^* satisfying the same properties as in the previous section.

- (1) τ_∞ is cohomological of some regular weight ξ of $\mathbf{G}^*(F_\infty)$.
- (2) τ_v is unramified at all finite places outside of S .
- (3) τ_v is an unramified twist of Steinberg at some finite set of finite places S_{st} .

We have the following key result of Sorensen.

Theorem 6.1. [Sor10, Theorem A] *Fix a globally generic τ as above such that S_{st} is non-empty. Then there exists, a unique (after fixing the isomorphism $i : \overline{\mathbb{Q}_\ell} \xrightarrow{\simeq} \mathbb{C}$) irreducible continuous*

representation $\rho_\tau : \text{Gal}(\overline{F}/F) \rightarrow \text{GSp}_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F , we have

$$i\text{WD}(\rho_\tau|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where $(-)^{F-s.s}$ denotes the Frobenius semisimplification and ϕ_{τ_v} is the Gan-Takeda parameter of τ_v .

Now let us fix τ with associated ρ_τ as above and assume that τ is a strong transfer of some cuspidal automorphic representation π of \mathbf{G} , as in Theorem 5.2. We assume that S_{st} contains q an odd inert prime in the number field F and choose the inner form \mathbf{G} to be of the following form, as in Kret-Shin [KS23, Section 8],

- $\mathbf{G}(\mathbb{R}) \simeq \text{GSp}_4(\mathbb{R}) \times \text{GU}_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$,
- $\mathbf{G}_{F_v} \simeq \text{GSp}_4/F_v$ at all finite places v if $[F:\mathbb{Q}]$ is odd,
- $\mathbf{G}_{F_v} \simeq \text{GSp}_4/F_v$ at all but the finite place q if $[F:\mathbb{Q}]$ is even,

where \mathbb{H} is the Hamilton quaternions. Let $A(\pi)$ be the set of isomorphism classes of cuspidal automorphic representations Π of \mathbf{G} such that, for all $v \in S_{st}$, Π_v is an unramified twist of Steinberg, Π_∞ is ξ cohomological, and, for all $v \notin S_\infty \cup S_{st}$, $\Pi_v \simeq \pi_v$. Our main task now is to show that ρ_τ is realized in the π^∞ isotypic component of the Shimura variety associated to a Shimura datum (\mathbf{G}, X) , where X is as in the discussion before [KS23, Lemma 7.1]. Let $\text{Sh}(\mathbf{G}, X)_{K, \overline{F}}$ be the associated Shimura variety over \overline{F} which we recall is 3-dimensional. We set \mathcal{L}_ξ to be the $\overline{\mathbb{Q}}_\ell$ local system associated to a irreducible representation of \mathbf{G} over F of highest weight ξ on it as before, and let $H_c^i(\text{Sh}(\mathbf{G}, X)_{\overline{F}}, \mathcal{L}_\xi)_{ss}$ denote the semi-simplification of the colimit of the i th-cohomologies over all compact opens $K \subset \mathbf{G}(\mathbb{A}_{f,F})$ as a $\mathbf{G}(\mathbb{A}_{f,F}) \times \Gamma_F$ -module. We note this is non-zero if and only if $0 \leq i \leq 6$.

Let S_{bad} denote the set of prime numbers p for which either $p = 2$, the group \mathbf{G} is ramified, or $K_p = \prod_{v|p} K_v$ is not hyperspecial. Then we define the virtual Galois representation

$$\rho_{shim}^\pi := (-1)^3 \sum_{\Pi \in A(\pi)} \sum_{i=0}^6 (-1)^i [Hom_{\mathbf{G}(\mathbb{A}_{f,F})}(\Pi^\infty, H_c^i(\text{Sh}(\mathbf{G}, X)_{\overline{F}}, \mathcal{L}_\xi)_{ss})] \in K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F)) \quad (4)$$

where $K_0(\overline{\mathbb{Q}}_\ell(\Gamma_F))$ denotes the Grothendieck group of continuous $\Gamma_F := \text{Gal}(\overline{F}/F)$ -representations with coefficients in $\overline{\mathbb{Q}}_\ell$. We now define the rational number

$$a(\pi) := (-1)^3 N_\infty^{-1} \sum_{\Pi \in A(\pi)} m(\Pi) \cdot ep(\Pi_\infty \otimes \xi)$$

where,

- (1) $m(\Pi)$ is the multiplicity of Π in the automorphic spectrum of \mathbf{G} ,
- (2) $N_\infty = |\Pi_\xi^{G(F_\infty)}| \cdot |\pi_0(\mathbf{G}(F_\infty)/Z(F_\infty))| = 4$, where $\Pi_\xi^{G(F_\infty)}$ denotes the discrete series L -packet of representations of $G(F_\infty)$ cohomological of weight ξ ,
- (3) $ep(\Pi_\infty \otimes \xi) := \sum_{i=0}^\infty (-1)^i \dim(H^i(\text{Lie}(\mathbf{G}(F_\infty)), K_\infty; \Pi_\infty \otimes \xi))$.

Then we have the following proposition of Kret-Shin.

Proposition 6.2. *[KS23, Proposition 8.2] With notation as above, for almost all finite F -places v not dividing a prime number in S_{bad} and all sufficiently large integers j , we have:*

$$\text{tr}(\rho_{\text{shim}}^\pi(\text{Frob}_v^j)) = a(\pi)q_v^{j\frac{3}{2}}\text{tr}(\text{std} \circ \phi_{\pi_v})(\text{Frob}_v^j).$$

Moreover, the virtual representation ρ_{shim}^π is a true representation. In particular, the only non-zero term appearing in the above alternating sum occurs in middle degree ($= 3$).

Remark 6.1. The claim about it occurring in middle degree is part of the proof of the Proposition not the statement. (See the discussion after equation (8.13) in [KS23])

For π as above, we define the π^∞ -isotypic part of $R\Gamma_c(\text{Sh}(\mathbf{G}, X)_{\overline{F}}, \mathcal{L}_\xi)$ to be the complex of $\overline{\mathbb{Q}}_\ell$ -vector spaces with Γ_F -action:

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbf{G}(\mathbb{A}_{f,F})}(\pi^\infty, H_c^i(\text{Sh}(\mathbf{G}, X)_{\overline{F}}, \mathcal{L}_\xi)_{\text{ss}})[-i],$$

as in the beginning of [KS23, Section 8]. If $K \subset \mathbf{G}(\mathbb{A}_{f,F})$ is a sufficiently small compact open subgroup such that π^∞ has a non-zero K -invariant vector then we note that this is identified with

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{H}(\mathbf{G}(\mathbb{A}_{f,F}))/K}((\pi^\infty)^K, H_c^i(\text{Sh}(\mathbf{G}, X)_{K,\overline{F}}, \mathcal{L}_\xi)_{\text{ss}})[-i],$$

where $\mathcal{H}(\mathbf{G}(\mathbb{A}_{f,F}))/K$ is the Hecke algebra of K bi-invariant compactly supported functions, and $H_c^i(\text{Sh}(\mathbf{G}, X)_{K,\overline{F}}, \mathcal{L}_\xi)_{\text{ss}}$ is the semi-simplification of the i th cohomology of the Shimura variety at level K as a $\mathcal{H}(\mathbf{G}(\mathbb{A}_{f,F}))/K \times \Gamma_F$ -module.

We now use the previous discussion to deduce the following Corollary.

Corollary 6.3. *The π^∞ -isotypic component of $R\Gamma_c(\text{Sh}(\mathbf{G}, X)_{\overline{F}}, \mathcal{L}_\xi)$ is concentrated in degree 3 and has Γ_F -action given (up to multiplicity) by $\text{std} \circ \rho_\tau$.*

Proof. The first part follows immediately from the previous Proposition, and the second part follows from the identification of the traces. In particular, by the Brauer-Nesbitt Theorem, Chebotarev density theorem, and the condition characterizing ρ_τ , we can identify (up to multiplicity) the Galois representation ρ_{shim}^π with the Galois representation $\text{std} \circ \rho_\tau$. \square

7. PROOF OF THE KEY PROPOSITION

We will now combine the results of the previous three sections to deduce some key consequences that will be used to derive Proposition 1.4. For this, using Krasner's lemma, we now fix a totally real number field F with two odd totally inert primes p and q such that $F_p \simeq L$ the fixed unramified extension of \mathbb{Q}_p . We fix the \mathbb{Q} -inner form \mathbf{G} of $\mathbf{G}^* = \text{Res}_{F/\mathbb{Q}}\text{GSp}_4$ defined in section 6, and let \mathbf{G}' be the inner form of \mathbf{G} seen in Definition 4.1. We take (\mathbf{G}, X) to be the Shimura datum considered in section 6. We note that this forces the associated geometric dominant cocharacter μ of $G := \mathbf{G}_{\mathbb{Q}_p}$ to be the Siegel cocharacter; in particular, we can apply the results of section 4.2. Set ξ to be a regular weight of an algebraic representation of \mathbf{G} over \mathbb{Q} . Let $K^p \subset \mathbf{G}(\mathbb{A}^{p^\infty})$ be an open compact subgroup. We set $S_{\text{sc}} = \{p\}$ and S_{st} to be a disjoint finite set of finite places of \mathbb{Q} containing q . We consider the uniformization map

$$\Theta : R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi) \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi) \quad (5)$$

supplied by Theorem 4.2 and Proposition 4.1. Now fix a smooth irreducible supercuspidal representation ρ of $J(\mathbb{Q}_p) = \text{Res}_{L/\mathbb{Q}_p}(\text{GU}_2(D))(\mathbb{Q}_p) = \text{GU}_2(D)(L)$. We have the following lemma.

Lemma 7.1. *Suppose ρ is a supercuspidal representation of $J(\mathbb{Q}_p)$, then, for sufficiently regular ξ and sufficiently small K^p , we can find a lift Π' to a cuspidal automorphic representation of \mathbf{G}' , such that Π'^∞ occurs as a $J(\mathbb{Q}_p)$ -stable direct summand of $\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f)) / K^p, \mathcal{L}_\xi$. Moreover, for all places in S_{st} , we can assume that the local constituents at $v \in S_{st}$ are unramified twists of the Steinberg representation.*

Proof. This follows from an argument using the simple trace formula. See for example [Han20, Proposition 2.9] or [Shi12]. We note in particular that cuspidality is vacuous, since $\mathbf{G}'(\mathbb{R})$ is compact modulo center by construction. \square

So let Π' be a globalization of a fixed supercuspidal ρ to a cuspidal automorphic representation of \mathbf{G}' for some sufficiently regular ξ and sufficiently small K^p . We can and do regard $\Pi'^{p\infty}$ as a representation of $\mathbf{G}(\mathbb{A}_f^p) \simeq \mathbf{G}'(\mathbb{A}_f^p)$. We set $K^p = K_S^p K^S$, where $K^S \subset \mathbf{G}(\mathbb{A}_f^S)$ is an open compact in the finite adeles away from S , for $S \subset \text{Pl}_F$ some finite set of places of \mathbb{Q} containing $S_{st} \cup \{p\} \cup \{\infty\}$, as in section 5. We assume that S is sufficiently large such that outside of S the automorphic representation Π' is unramified, so, in particular, the subgroup $K^S \subset \mathbf{G}(\mathbb{A}_f^S)$ is a product of hyperspecial subgroups away from S . We consider the abstract commutative Hecke algebra

$$\mathbb{T}^S := \mathcal{H}(\mathbf{G}(\mathbb{A}_f^S) // K^S)$$

of bi-invariant compactly supported smooth functions on $\mathbf{G}(\mathbb{A}_f^S)$. We regard both sides of (5) as \mathbb{T}^S -modules and consider the maximal ideal \mathfrak{m} defined by the Hecke eigenvalues of Π'^S . We then localize both sides of (5) at \mathfrak{m} to obtain a map:

$$\Theta_{\mathfrak{m}} : (R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

We would like to apply Propositions 4.4 and 5.4 to the representations occurring on both sides of this map. However, to apply these results we need to make some more modifications. In particular, the automorphic representations of \mathbf{G}' occurring in the LHS (resp. RHS) of $\Theta_{\mathfrak{m}}$ are not necessarily unramified twists of Steinberg at all places in S_{st} .

To remedy this, we set $K^p = K_{\{p\} \cup S_{st}} K^{\{p\} \cup S_{st}}$, where $K^{\{p\} \cup S_{st}} \subset \mathbf{G}(\mathbb{A}_f^{\{p\} \cup S_{st}}) \simeq \mathbf{G}'(\mathbb{A}_f^{\{p\} \cup S_{st}})$ is an open compact subgroup. Then we consider the colimits

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{\{p\} \cup S_{st}}}, \mathcal{L}_\xi) := \text{colim}_{K_{\{p\} \cup S_{st}} \rightarrow \{1\}} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K_{\{p\} \cup S_{st}} K^{\{p\} \cup S_{st}}}, \mathcal{L}_\xi)$$

and

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{\{p\} \cup S_{st}}, \mathcal{L}_\xi) := \text{colim}_{K_{\{p\} \cup S_{st}} \rightarrow \{1\}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K_{\{p\} \cup S_{st}} K^{\{p\} \cup S_{st}}, \mathcal{L}_\xi)$$

Since $S_{st} \subset S$, the map $\Theta_{\mathfrak{m}}$ gives rise to a map

$$(R\Gamma_c(G, b, \mu) \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}} \rightarrow R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}}$$

By Proposition 4.4, we obtain an isomorphism

$$(R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}} \xrightarrow{\cong} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)_{\mathfrak{m}, sc}$$

Now, for all $v \in S_{st}$, we can project to the summand of the LHS where $\mathbf{G}(F_v) \simeq \mathbf{G}'(F_v)$ acts via an unramified twist of Steinberg, noting that the LHS and hence the RHS is semisimple as $\mathbf{G}(F_v)$ representation.

This gives an isomorphism:

$$\Theta_{\mathfrak{m},sc}^{st} : (R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}}^{st} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi})_{\mathfrak{m},sc}^{st}$$

We now apply Proposition 5.4 to obtain the following.

Proposition 7.2. *Let $\rho \in \Pi(J)$ be a representation with supercuspidal Gan-Tantono parameter ϕ . Assume that $|S_{st}| \geq 3$. Then, for Π' a choice of globalization of ρ as in Lemma 7.1, unramified outside S with associated maximal ideal $\mathfrak{m} \subset \mathbb{T}^S$ in the Hecke algebra defined by the Hecke eigenvalues of Π'^S , the LHS of the map*

$$\Theta_{\mathfrak{m},sc}^{st} : (R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}}^{st} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi})_{\mathfrak{m},sc}^{st}$$

breaks up as a direct sum of $R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \bar{\Pi}'^{\{p,\infty\} \cup S_{st}}$ of $G(\mathbb{Q}_p) \times W_L$ -representations, for $\bar{\Pi}'$ a cuspidal automorphic representation of $\mathbf{G}'(\mathbb{A})$ satisfying the following:

- $\bar{\Pi}'^S \simeq \Pi'^S$,
- $\bar{\Pi}'$ is cohomological of regular weight ξ at ∞ ,
- $\bar{\Pi}'$ is unramified twist of Steinberg at all $v \in S_{st}$,
- $\bar{\Pi}'$ has local constituent at p with associated L -parameter ϕ .

Proof. The fact that the cohomology breaks up as a direct sum of $R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \bar{\Pi}'^{\{p,\infty\} \cup S_{st}}$ for $\bar{\Pi}'$ an automorphic representation of $\mathbf{G}'(\mathbb{A})$ follows from the semi-simplicity of the space of algebraic automorphic forms. We localized at \mathfrak{m} corresponding to Π'^S and are considering algebraic automorphic representations of \mathbf{G}' valued in the algebraic representation defined by ξ . Therefore, it is clear that any representation $\bar{\Pi}'$ giving rise to such a summand is a cuspidal automorphic form of \mathbf{G}' satisfying (1) and (2). Here cuspidality is automatic since $\mathbf{G}'(\mathbb{R})$ is compact modulo center. Moreover, by construction, it follows that $\mathbf{G}'(\mathbb{Q}_v)$ acts on the LHS via representations which are an unramified twist of Steinberg for all $v \in S_{st}$. This allows us to apply proposition 5.4, since $|S_{st}| \geq 3$ by assumption. Proposition 5.4 applied to the inner form \mathbf{G}' of \mathbf{G}^* and the cuspidal automorphic representation Π' of \mathbf{G}' tells us that $\bar{\Pi}'$ must have Langlands parameter at $\{p\} = S_{sc}$ given by ϕ , which was the desired claim. \square

We now combine this with Corollary 6.3 to deduce the following.

Corollary 7.3. *With notation as above, the map*

$$\Theta_{\mathfrak{m},sc}^{st} : (R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \mathcal{A}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi))_{\mathfrak{m}}^{st} \xrightarrow{\simeq} R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}, \mathcal{L}_\xi})_{\mathfrak{m},sc}^{st}$$

is an isomorphism of complexes of $G(\mathbb{Q}_p) \times W_L$ -modules concentrated in degree 3 with W_L -action given, up to multiplicity and semi-simplification as a W_L -module, by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. Proposition 7.2 tells us that the LHS of $\Theta_{\mathfrak{m},sc}^{st}$ breaks up as a direct sum of $G(\mathbb{Q}_p) \times W_L$ -modules of the form

$$R\Gamma_c(G, b, \mu)_{sc} \otimes_{\mathcal{H}(J_b)}^{\mathbb{L}} \bar{\Pi}'^{\{p,\infty\} \cup S_{st}}$$

for $\overline{\Pi}'$ a cuspidal automorphic representation of \mathbf{G}' that has L -parameter ϕ at p , and is also cohomological of regular weight ξ at infinity and an unramified twist of Steinberg at all places in S_{st} . It suffices to prove the claim for each one of these summands. This summand will map to the $\overline{\Pi}'^{\{p,\infty\} \cup S_{st}}$ -isotypic part of the RHS by construction, where this is defined analogously to Section 6. Here the semi-simplification of the RHS as a Hecke module is not required since this is semi-simple with respect to the $\mathbf{G}'(\mathbb{A}_f^{S_{st} \cup \{p,\infty\}}) \simeq \mathbf{G}(\mathbb{A}_f^{S_{st} \cup \{p,\infty\}})$ -action. Let τ denote a strong transfer of Π' to a cuspidal automorphic representation of \mathbf{G}^* given by Theorem 5.2, with associated Galois representation ρ_τ given by Theorem 6.1. Applying Theorem 5.2 again, we consider a strong transfer of τ to \mathbf{G} given by Π . We note, by Corollary 6.3, that the $\overline{\Pi}'^{\{p,\infty\} \cup S_{st}} \simeq \Pi^{\{p,\infty\} \cup S_{st}}$ -isotypic part will be concentrated in degree 3 and have W_L -action given (up to multiplicity and semi-simplification) by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$, by the property characterizing ρ_τ and the fact that τ was a strong transfer. \square

With this in hand, we are finally ready to conclude our key Proposition.

Proposition 7.4. *Let ϕ be a supercuspidal parameter with associated L -packet $\Pi_\phi(J)$. Then the direct summand of*

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']$$

where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

is concentrated in middle degree 3 and admits a non-zero W_L -stable sub-quotient with W_L -action given by $\text{std} \circ \phi \otimes |\cdot|^{-3/2}$.

Proof. This is an immediate consequence of Proposition 7.2 and Corollary 7.3. \square

In particular, using Corollary 3.21, we can deduce the following.

Corollary 7.5. *If $p > 2$ and L/\mathbb{Q}_p is an unramified extension, then, for all $\rho \in \Pi(J)$ with supercuspidal Gan-Tantono parameter ϕ_ρ , the Fargues-Scholze and Gan-Tantono correspondences are compatible.*

8. APPLICATIONS

We will now apply Corollary 7.5 to deduce some applications to the strong form of the Kottwitz conjecture and conclude the proof of Theorem 1.1. We begin with the latter.

Theorem 8.1. *The following is true.*

- (1) *For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal, we have that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*
- (2) *If L/\mathbb{Q}_p is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal, that the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.*

Proof. Part (1) follows by Corollary 3.12 and Corollary 3.16. Part (2) for the Gan-Tantono local Langlands is precisely Corollary 7.5. It remains to show that for L/\mathbb{Q}_p unramified and $p > 2$, π a smooth irreducible representation of GSp_4/L with supercuspidal Gan-Takeda ϕ_π parameter that the two correspondences are compatible. To show this, we consider the complex

$$R\Gamma_c^b(G, b, \mu)[\pi]$$

of $J(\mathbb{Q}_p) \times W_L$ -representations. We know, by Theorem 3.13, that this admits sub-quotients as a $J(\mathbb{Q}_p)$ -module given by ρ , for all ρ whose Gan-Tantono parameter ϕ_ρ is equal to the Gan-Takeda parameter ϕ_π of π . However, by Corollary 3.15, we know that these representations must have Fargues-Scholze parameter equal to ϕ_π^{FS} . Therefore, we get a chain of equalities

$$\phi_\pi^{\mathrm{FS}} = \phi_\rho^{\mathrm{FS}} = \phi_\rho = \phi_\pi$$

where we have used compatibility of the Gan-Tantono and the Fargues-Scholze correspondence for the middle equality. \square

Now, with this out of the way, we turn our attention to proving some strong forms of the Kottwitz conjecture for these representations, verifying Theorem 1.3.

Theorem 8.2. *Let L/\mathbb{Q}_p be an unramified extension with $p > 2$. Let π (resp. ρ) be members of the L -packet over a supercuspidal parameter $\phi : W_L \rightarrow \mathrm{GSp}_4(\overline{\mathbb{Q}_\ell})$. Then the complexes*

$$R\Gamma_c(G, b, \mu)[\pi]$$

and

$$R\Gamma_c(G, b, \mu)[\rho]$$

are concentrated in middle degree 3.

- (1) *If ϕ is stable supercuspidal, with singleton L -packets $\{\pi\} = \Pi_\phi(G)$ and $\{\rho\} = \Pi_\phi(J)$, then the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho \boxtimes (\mathrm{std} \circ \phi)^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$ -module, and the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi \boxtimes \mathrm{std} \circ \phi \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$ -module.

- (2) *If ϕ is an endoscopic parameter, with L -packets $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$, the cohomology of $R\Gamma_c(G, b, \mu)[\pi]$ in middle degree is isomorphic to*

$$\rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

or

$$\rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}$$

as a $J(\mathbb{Q}_p) \times W_L$ -module. Similarly, the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ in middle degree is isomorphic to

$$\pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

or

$$\pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}$$

as a $G(\mathbb{Q}_p) \times W_L$ -module. Here we write $\text{std} \circ \phi_\rho \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct irreducible 2-dimensional representations of W_L and $\det(\phi_1) = \det(\phi_2)$.

Moreover, both possibilities for the cohomology of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) in the endoscopic case occur for some choice of representation $\rho \in \Pi_\phi(J)$ (resp. $\pi \in \Pi_\phi(G)$). In particular, knowing the precise form of either $R\Gamma_c(G, b, \mu)[\rho]$ or $R\Gamma_c(G, b, \mu)[\pi]$ for some $\rho \in \Pi_\phi(J)$ or $\pi \in \Pi_\phi(G)$ determines the precise form of the cohomology in all other cases.

Proof. We show the proof in the endoscopic case, with the stable case being strictly easier. We first note that, since $\phi = \phi_\rho^{\text{FS}}$ by Theorem 8.1, it follows by assumption that the Fargues-Scholze parameter ϕ_ρ^{FS} of ρ is supercuspidal. Therefore, by Remark 3.12, we have an isomorphism

$$R\Gamma_c(G, b, \mu)[\rho] \simeq R\Gamma_c^b(G, b, \mu)[\rho]$$

of $G(\mathbb{Q}_p) \times W_L$ -modules. Moreover, by Theorem 3.17, we see that both are concentrated in middle degree 3. Applying Theorem 3.13, we get the following chain of equalities in the Grothendieck group of admissible $G(\mathbb{Q}_p)$ -representations of finite length

$$-[H^3(R\Gamma_c(G, b, \mu)[\rho])] = [R\Gamma_c(G, b, \mu)[\rho]] = [R\Gamma_c^b(G, b, \mu)[\rho]] = - \sum_{\pi \in \Pi_\phi(G)} \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, \text{std} \circ \phi_\rho) \pi$$

Now we saw in the discussion preceding Theorem 3.13 that the RHS takes the form:

$$-2\pi^+ - 2\pi^-$$

We set p_1 and p_2 to be the two representations of $S_\phi \simeq \{(a, b) \in \overline{\mathbb{Q}}_\ell^* \times \overline{\mathbb{Q}}_\ell^* \mid a^2 = b^2\} \subset (\text{GL}_2(\overline{\mathbb{Q}}_\ell) \times \text{GL}_2(\overline{\mathbb{Q}}_\ell))^0$ given by projecting to the first and second $\overline{\mathbb{Q}}_\ell^*$ -factor, respectively. By Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism of $G(\mathbb{Q}_p) \times W_L$ -modules

$$R\Gamma_c(G, b, \mu)[\rho] \simeq \text{Act}_{p_1}(\rho)[-3] \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \text{Act}_{p_2}(\rho)[-3] \boxtimes \phi_2 \otimes |\cdot|^{-3/2}$$

where $\text{Act}_{p_1}(\rho)$ and $\text{Act}_{p_2}(\rho)$ are a priori direct sum of shifts of supercuspidal representations of $G(\mathbb{Q}_p)$ with Fargues-Scholze (= Gan-Takeda) parameter equal to ϕ . However, since we know that the LHS is a complex concentrated in middle degree 3, this implies, by the above description in the Grothendieck group, that one of the $\text{Act}_{p_1}(\rho)$ and $\text{Act}_{p_2}(\rho)$ is isomorphic to π^+ and the other is isomorphic to π^- . Without loss of generality, assume that

$$\text{Act}_{p_1}(\rho_1) \simeq \pi^+$$

and

$$\text{Act}_{p_2}(\rho_1) \simeq \pi^-$$

We let p_+ and p_- be the representation of S_ϕ determined by the trivial and non-trivial characters of the component group, respectively. Now, given two representations of S_ϕ , denoted W and W' , it follows from Remark 3.7 (3) that we have an isomorphism:

$$\text{Act}_W \circ \text{Act}_{W'}(\cdot) \simeq \text{Act}_{W \otimes W'}(\cdot)$$

In turn, we get

$$\text{Act}_{p_1^\vee}(\pi^+) \simeq \text{Act}_{p_1^\vee} \circ \text{Act}_{p_1}(\rho_1) \simeq \text{Act}_{p_+}(\rho_1) \simeq \rho_1$$

where the last isomorphism follows since p_+ is the trivial representation. Similarly, depending on the values of $\text{Act}_{p_1}(\rho_2)$ and $\text{Act}_{p_2}(\rho_2)$ we can deduce that $\text{Act}_{p_2^\vee}(\pi^+)$ is isomorphic to ρ_1 or ρ_2 . Now by Corollary 3.3, Corollary 3.11, and Theorem 8.1, we have an isomorphism

$$R\Gamma_c(G, b, \mu)[\pi^+] \simeq \text{Act}_{p_1^\vee}(\pi^+)[-3] \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \text{Act}_{p_2^\vee}(\pi^+)[-3] \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}$$

Since $\text{Act}_{p_1^\vee}(\pi^+) \simeq \rho_1$ it therefore follows, by Theorem 3.13 and Remark 3.12, that $\text{Act}_{p_2^\vee}(\pi^+)$ must be isomorphic to ρ_2 . Moreover, we know that $\text{Act}_{p_-} \circ \text{Act}_{p_1}(\rho_1) \simeq \text{Act}_{p_2}(\rho_1)$ and $\text{Act}_{p_-} \circ \text{Act}_{p_2}(\rho_1) \simeq \text{Act}_{p_1}(\rho_1)$. Therefore, we obtain that $\text{Act}_{p_-}(\pi^+) \simeq \pi^-$ and $\text{Act}_{p_-}(\pi^-) \simeq \pi^+$. This allows us to determine that

$$\text{Act}_{p_1^\vee}(\pi^-) \simeq \text{Act}_{p_2^\vee} \circ \text{Act}_{p_-}(\pi^-) \simeq \text{Act}_{p_2^\vee}(\pi^+) \simeq \rho_2$$

and

$$\text{Act}_{p_2^\vee}(\pi^-) \simeq \text{Act}_{p_1^\vee} \circ \text{Act}_{p_-}(\pi^-) \simeq \text{Act}_{p_1^\vee}(\pi^+) \simeq \rho_1$$

which will determine the cohomology of $R\Gamma_c(G, b, \mu)[\pi^-]$. It only remains to show that the value of $R\Gamma_c(G, b, \mu)[\rho_2]$ is determined. However, this follows since

$$\text{Act}_{p_2}(\rho_2) \simeq \text{Act}_{p_2} \circ \text{Act}_{p_2^\vee}(\pi^+) \simeq \pi^+$$

and

$$\text{Act}_{p_1}(\rho_2) \simeq \text{Act}_{p_1} \circ \text{Act}_{p_1^\vee}(\pi^-) \simeq \pi^-$$

□

To conclude this section, we use Theorem 8.1 to deduce compatibility with the local Langlands correspondence for Sp_4 and its unique non quasi-split inner form $\text{SU}_2(D)$. These correspondences are described in the papers [GT10] and [Cho17] by Gan-Takeda and Choiy, respectively. For Sp_4 , this is described as the unique correspondence which sits in the commutative diagram:

$$\begin{array}{ccc} \Pi(\text{GSp}_4) & \xrightarrow{\text{LLC}_{\text{GSp}_4}} & \Phi(\text{GSp}_4) \\ \downarrow & & \downarrow \alpha \\ \Pi(\text{Sp}_4) & \xrightarrow{\text{LLC}_{\text{Sp}_4}} & \Phi(\text{Sp}_4) \end{array}$$

Here, the left vertical arrow is not a map at all, it is a correspondence defined by the subset of $\Pi(\text{GSp}_4) \times \Pi(\text{Sp}_4)$ consisting of pairs (π, ω) such that ω is a constituent of the restriction of π to Sp_4 , and the right vertical arrow is the map on L -parameters induced by the natural map $\text{GSpin}_5(\mathbb{C}) \rightarrow \text{SO}_5(\mathbb{C})$. One has a similar characterization of the local Langlands correspondence for $\text{SU}_2(D)$. With this description of the correspondence, compatibility for Sp_4 and $\text{SU}_2(D)$ follows from Theorem 8.1 and Theorem 3.6 (7).

Corollary 8.3. *For π (resp. ρ) a smooth irreducible representation of Sp_4/L (resp. $\text{SU}_2(D)$), with associated Gan-Takeda (resp. Choiy) parameter $\phi_\pi : W_L \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_5(\mathbb{C})$ (resp. ϕ_ρ) we have that:*

- (1) *The Fargues-Scholze and Gan-Takeda (resp. Choiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is not supercuspidal.*

- (2) If L/\mathbb{Q}_p is unramified and $p > 2$ then the Fargues-Scholze and Gan-Takeda (resp. Choiiy) local Langlands correspondences are compatible for any representation π (resp. ρ) such that ϕ_π (resp. ϕ_ρ) is supercuspidal.

Remark 8.1. We note that Corollary 3.11, Theorem 3.18, Remark 3.11, and [HKW21, Theorem 1.0.2] apply to a triple (G, b, μ) , where μ is any cocharacter and $b \in B(G, \mu)$ is the unique basic element. Therefore, by applying the same kind of analysis as in the proof of Theorem 8.2, we can prove the analogue of Theorem 8.2 in the Grothendieck group of finite length admissible representations with a smooth action of W_E (cf. [HKW21, Conjecture 1.0.1]) for the cohomology of the local shtuka spaces defined by the triple (G, b, μ) . By Corollary 8.3, this works even in the case when $G = \mathrm{Sp}_4$ and there are no Shimura varieties that these spaces uniformize. Moreover, in the case that $G = \mathrm{GSp}_4$, one can also deduce from Theorem 8.2 that it is concentrated in middle degree $\langle 2\rho_G, \mu \rangle$, using the monoidal property of the Act-functors and Corollary 3.11, for μ any cocharacter. This in particular will imply some form of Fargues' Conjecture for these groups (See e.g. [BHN22, Pages 37-40], for this worked out in the more complicated case of $G = \mathrm{U}_n$).

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