Course Notes for Math 256: Dynamics, Stability, and Noncommutative Algebra

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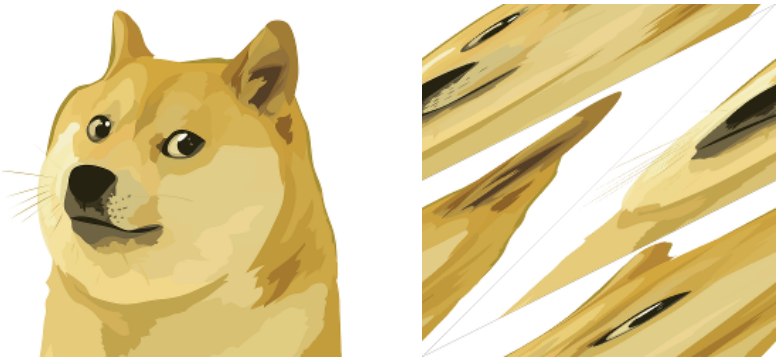
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Lecture 1: A preview of the course

1.1 An elementary example

You have probably come across the group $SL(2,\mathbb{Z})$, of 2 by 2 matrices with integer coefficients and determinant one, at some point in your mathematical life. These are maps which preserve the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, thus induce a map on the quotient $\mathbb{R}^2/\mathbb{Z}^2$, which is a torus. Here is a visualization of how such a map acts on the torus. The example is V. Arnold's map, $(x, y) \mapsto (2x + y, x + y)$.



As one can tell from the picture, there is a certain amount of "mixing" or "distortion" going on. Quantitatively, this is measured by the *dynamical entropy*. This is an invariant which has definitions in various contexts — measure theoretic, topological, homological, ... — which we will explore in the course. For a linear map, T, it is just the logarithm of the spectral radius

$$\begin{aligned} h(T) &= \max\{\log|\lambda|, \lambda \text{ is eigenvalue of } T\} \\ &= \lim_{n\to\infty} \frac{1}{n} \log\|T^n\| \end{aligned}$$

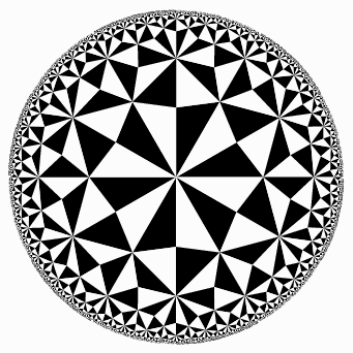
In Arnold's example we have positive entropy, $h(T) > 0$, and more generally for those $T \in SL(2,\mathbb{Z})$ with $|\mathrm{Tr}(T)| > 2$, so this case is the typical one.

As it turns out, the idea of dynamical entropy is quite general, and can be applied in such exotic contexts as endofunctors of triangulated categories. The spectral radius remains, however, as a useful source for lower bounds on dynamical entropy, as we will see.

From the above we have a very rough classification of elements of $SL(2,\mathbb{Z})$, depending on whether $h(T) > 0$ or not. To get a more detailed picture, we let it act on the upper half plane $H = \{z \mid \mathrm{Im}(z) > 0\}$ by Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \tag{1}$$

with integers a, b, c, d. These transformations also act on the closure $\overline{H} = H \cup \mathbb{R} \cup \{\infty\}$ in the Riemann sphere, which is topologically a disk (think of the Poincaré disk). Elements of



$SL(2,\mathbb{Z})$ fix at least one point in $\overline{H}$, by the famous Brouwer fixed-point theorem, and it turns out they can be classified by which points they fix. Those with $h(T) > 0$ fix two points on the boundary, and no points in the interior.

Thurston, in celebrated work, extended all this from the torus to surfaces of higher genus. The generalizations of $SL(2,\mathbb{Z})$ and H are called the *mapping class group* and *Teichmüller space*, respectively.

1.2 Modular groups

The mapping class group of a closed oriented surface X, $\mathrm{MCG}(X)$, is the quotient of the group of orientation preserving diffeomorphisms $\phi : X \to X$ by the subgroup of diffeomorphisms isotopic to the identity (i.e. we identify diffeomorphisms which continuously deform into one another). Mapping class groups are also called *modular groups*. The simplest non-trivial case is $\mathrm{MCG}(T^2) = SL(2,\mathbb{Z})$, which you can study using the methods of linear algebra. For higher genus there is a beautiful kind of analogue of linear algebra:

Linear algebra	Surface topology
$SL(n,\mathbb{Z})$	$\mathrm{MCG}(X)$
vector	measured foliation/train track
elementary transformation	Dehn twist
eigenvector	invariant foliation
eigenvalue	stretch factor
inner product	geometric intersection
Jordan normal form	Nielsen–Thurston classification

The group $MCG(X)$ acts on Teichmüller space, the space of conformal structures on X up to isotopy. Understanding this action is a key step in the above mentioned classification. Mapping class groups will provide us with examples and intuition, before we plunge ahead into more abstract settings.

1.3 Dynamical entropy

Suppose we have a random variable x taking values in a finite set $\{x_1, \ldots, x_n\}$. A measure of the "unpredictability" of x is the (Shannon) *entropy*

$$H(x) = \sum_{i=1}^{n} P(x_i) \log \frac{1}{P(x_i)}. \tag{2}$$

If you have never seen this before, I encourage you compute it for some simple distributions! The formula is very similar to the one in thermodynamics, hence the terminology.

In ergodic theory the basic object of study is a probability space $(X, \mathcal{B}, \mu)$, i.e. a set X with σ-algebra $\mathcal{B}$, and measure μ with $\mu(X) = 1$, together with a measure preserving map $f : X \to X$, i.e. we require that $\mu(f^{-1}A) = \mu(A)$ for $A \in \mathcal{B}$. We think of X as the set of values of a random variable, and μ its probability distribution. The map f represents some discrete evolution in time.

A nice example is the $(p_1, \ldots, p_n)$-*Bernoulli shift*. Take $Y = \{1, \ldots, n\}$ with probability distribution given by the p_i, and $X = \prod_{i=1}^{\infty} Y$ the product probability space. A measure preserving map is given by the shift

$$f(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots). \tag{3}$$

A basic problem is to classify measure preserving maps $X \to X$ up to conjugacy, i.e. up to automorphisms of X. Is the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift essentially the same as the $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-shift? An invariant which can distinguish them is given by the measure-theoretic (Kolmogorov–Sinai) entropy. Suppose there is some finite partition $P = A_1, \ldots, A_n$ of X. Define

$$H(P) = \sum_{i=1}^{n} \mu(A_i) \log \frac{1}{\mu(A_i)} \tag{4}$$

$$h(f, P) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i} P) \tag{5}$$

$$h(f) = \sup_{P} h(f, P) \tag{6}$$

where the join of partitions is just the collection of intersections:

$$P \vee Q = \{A \cap B \mid A \in P, B \in Q\}. \tag{7}$$

The quantity $h(f)$ is the *dynamical entropy* of f. Intuitively, it measures the amount of information required to encode a typical orbit $x, f(x), f^2(x), \ldots$ of f.

There is also a related version of entropy for topological spaces. As we will see, the definition follows a very similar pattern.

1.4 Noncommutative algebraic geometry

By noncommutative algebraic geometry I mean the study of triangulated categories and their enhancements, dg- and A_∞-categories. If you have never heard of these, don't worry, they will be reviewed in the course. The facts which make the subject appealing are

1. Triangulated categories appear in many subjects, such as algebraic geometry, symplectic topology, and representation theory, and mathematical physics.
2. Many of the concepts and tools in geometry generalize to this setting. Examples are smoothness, compactness, (co)homology, duality, Cartan calculus, ...
3. New phenomena: Fourier–Mukai transform, semi-orthogonal gluing, ...
4. Homological mirror symmetry (conjecture)

Briefly, a triangulated category $\mathcal{C}$ is an additive category with shift functor $X \mapsto X[1]$ and a collection of exact triangles, which are diagrams of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \tag{8}$$

which have to satisfy some axioms. The basic operation in a triangulated category is taking the cone of a morphism $f : X \to Y$ which produces an exact triangle $X \to Y \to \mathrm{Cone}(f) \to X[1]$.

An object $G \in \mathcal{C}$ is a *generator* if every other object in $\mathcal{C}$ is obtained from G by applying the shift and cone operations a finite number of times, and taking direct summands. More precisely, we require that there are $F_0, \dots, F_n \in \mathcal{C}$ with triangles $F_{i-1} \to F_i \to G[k_i]$ such that $F_0 = 0$ and E is a summand of F_n. We call the minimal number n of copies of G necessary to construct an object E as above the *complexity* of E (relative to G), denoted $\delta(G, E)$. Given an exact (i.e. structure-preserving) functor $F : \mathcal{C} \to \mathcal{C}$, the limit

$$h(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta(G, F^n G) \tag{9}$$

turns out to be independent of G, and is called the *entropy* of F.

1.5 Stability structures on categories

Going into even deeper territory, a certain analogue of Teichmüller space in String Theory led to Bridgeland's definition of the *space of stability structures* (of a triangulated category), a complex manifold with wall and chamber structure. This is still a largely mysterious object, and illuminating some parts of it will be one of the goals of this course.

The data of a stability condition on $\mathcal{C}$ is a collection of *semistable* objects in $\mathcal{C}$ for each phase $\phi \in \mathbb{R}$, together with a *central charge* $Z : K_0(\mathcal{C}) \to \mathbb{C}$. There is a list of axioms, one of which implies that every object in $\mathcal{C}$ is an extension of semistable ones. Fixing a stability condition, one can define the *mass*, $m(E) \geq 0$, of any object, and it turns out to be closely related to the complexity defined above. The analogue of the mapping class group in this setting is the group of autoequivalences (symmetries) of $\mathcal{C}$.

Lecture 2: Measure theoretic entropy, Part I

The material here is adapted from P. Walters: *An Introduction to Ergodic Theory.*

2.1 Measure preserving transformations

We begin with some definitions. Recall that a **probability space** is a triple $(X, \mathcal{B}, \mu)$ where X is a set, $\mathcal{B}$ is a σ-algebra, and μ is a measure on $(X, \mathcal{B})$ with $\mu(X) = 1$. A map $T : X \to Y$ between probability spaces is **measurable** if preimages of measurable sets are measurable. It is **measure preserving** if furthermore

$$\mu_X(T^{-1}A) = \mu_Y(A) \tag{10}$$

for all measurable subsets $A \subset Y$. We will be particularly interested in measure preserving maps $T : X \to X$ from the space to itself. Some examples are:

1. Let G be a compact Hausdorff topological group together with its unique Haar probability measure. By definition, the left translations $T_g(x) = gx$ for $g \in G$ are measure preserving transformations.

2. With G as before, suppose that $f : G \to G$ is a surjective endomorphism. Then f preserves the Haar measure, m. To see this, define another probability measure $\mu(E) = m(f^{-1}E)$ on Borel subsets of G. Regularity of μ follows from regularity of m, and
$$\mu(f(x)E) = m(f^{-1}(f(x)E)) = m(xf^{-1}E) = \mu(E) \tag{11}$$
so μ is left-invariant, since f was assumed to be onto. Thus, $\mu = m$ by uniqueness of the Haar measure.

3. The $(p_1, \ldots, p_n)$-Bernoulli shift (see Lecture 1). There is also a two-sided invertible variant where we consider sequences indexed by $\mathbb{Z}$.

When do we consider two measure preserving transformations, T_1 on $(X_1, \mathcal{B}_1, \mu_1)$ and T_2 on $(X_2, \mathcal{B}_2, \mu_2)$, essentially the same? The first notion is that of isomorphism. We say that T_1 is **isomorphic** to T_2 if there are $M_1 \in \mathcal{B}_1$, $M_2 \in \mathcal{B}_2$, and an invertible measure preserving $f : M_1 \to M_2$ such that

- $\mu_i(M_i) = 1$, $T_i(M_i) \subset M_i$, and
- $f \circ T_1|_{M_1} = T_2|_{M_2} \circ f$.

This is an equivalence relation.

A weaker notion is based on the σ-algebras, instead of the spaces themselves. Let $(X, \mathcal{B}, \mu)$ be a probability space. Define an equivalence relation on $\mathcal{B}$ by $A \sim B$ iff $\mu(A \triangle B) = 0$, and let $\widetilde{\mathcal{B}}$ be set of equivalence classes. $\widetilde{\mathcal{B}}$ is a Boolean σ-algebra, and μ descends to a well defined $\widetilde{\mu} : \widetilde{\mathcal{B}} \to \mathbb{R}_{\geq 0}$. The pair $(\widetilde{\mathcal{B}}, \widetilde{\mu})$ is called a *measure algebra.* A measure preserving map

$f : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ induces a homomorphism of measure algebras $\tilde{f}^{-1} : \widetilde{B}_2 \to \widetilde{B}_1$. We say that measure preserving transformation T_i on $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, are **conjugate** if there is a measure algebra isomorphism $f : (\widetilde{\mathcal{B}}_1, \widetilde{\mu}_1) \to (\widetilde{\mathcal{B}}_2, \widetilde{\mu}_2)$ such that $f \circ \widetilde{T}_1^{-1} = \widetilde{T}_2^{-1} \circ f$.

It is clear that isomorphism implies conjugacy. The following simple example shows that the converse is not true: Take $X_1 = \{1\}$, $X_2 = \{1, 2\}$ with the minimal (two-element) σ-algebras and T_i the identity maps, then they are conjugate but not isomorphic.

Another important definition related to the measure algebra $\widetilde{\mathcal{B}}$: For sub-σ-algebras $\mathcal{C}, \mathcal{D}$ of $\mathcal{B}$ we write $\mathcal{C} \overset{\circ}{\subset} \mathcal{D}$ if for every $C \in \mathcal{C}$ there is a $D \in \mathcal{D}$ with $\mu(C \triangle D) = 0$. This is equivalent to $\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{D}}$ in $\widetilde{\mathcal{B}}$. Also write $\mathcal{C} \overset{\circ}{=} \mathcal{D}$ if $\widetilde{\mathcal{C}} = \widetilde{\mathcal{D}}$ in $\widetilde{\mathcal{B}}$.

2.2 Entropy of partitions

A **partition** of a probability space $(X, \mathcal{B}, \mu)$ is a collection of disjoint measurable subsets, $(A_i)_{i \in I}$, whose union is X. We will mostly consider finite partitions in this subsection. They correspond to the finite sub-σ-algebras of $\mathcal{B}$. If $\alpha = (A_1, \ldots, A_m)$ and $\beta = (B_1, \ldots, B_n)$ are finite partitions, then α is a **refinement** of β, written $\beta \leq \alpha$, if every B_i is a union of A_j's. The **join** of α and β, denoted $\alpha \vee \beta$, is the common refinement

$$\alpha \vee \beta = \{A_i \cap B_j \mid A_i \in \alpha, B_j \in \beta\}. \tag{12}$$

If $T : X \to X$ is a measure preserving transformation, we can define the pullback partition

$$T^{-1}\alpha = \{T^{-1}E \mid E \in \alpha\} \tag{13}$$

and this is compatible with the refinement relation and join.

We can interpret the elements $A_1, \ldots, A_m$ of a partition, α, as the (finite) set of values of a random variable. A measure for the unpredictability of this random variable is the **entropy**

$$H(\alpha) = -\sum_{i=1}^{m} \mu(A_i) \log \mu(A_i) \tag{14}$$

where one takes $0 \log 0 = 0$. Of course, we could just as well have taken the logarithm to some other base. One can justify this definition by coming up with a list of properties that a reasonable measure of "unpredictability" should satisfy, then proving that only constant multiples of $H(\alpha)$ satisfy them. From a more positive point of view, $H(\alpha)$ is the amount of information gained by performing an experiment with outcomes $\{A_1, \ldots, A_m\}$. The following proposition tells is that H is maximized strictly for the uniform distribution.

Proposition 1. *If $\alpha = (A_1, \ldots, A_n)$ is a partition of a probability space, then $H(\alpha) \leq \log(n)$ with equality if and only if $\mu(A_i) = 1/n$ for $i = 1, \ldots, n$.*

Proof. The idea is to use strict convexity of the function $x \log x$. We leave the details as an exercise. □

There is also a conditional version of entropy. Let $\alpha = (A_1, \ldots, A_m)$ and $\beta = (B_1, \ldots, B_n)$ be partitions of a probability space, then the **conditional entropy** of α given β is

$$H(\alpha \mid \beta) = -\sum_{i=1}^{n} \mu(B_i) \sum_{j=1}^{m} \frac{\mu(A_j \cap B_i)}{\mu(B_i)} \log \frac{\mu(A_j \cap B_i)}{\mu(B_i)} \tag{15}$$

$$= -\sum_{i,j} \mu(A_j \cap B_i) \log \frac{\mu(A_j \cap B_i)}{\mu(B_i)} \tag{16}$$

where we can omit terms with $\mu(B_i) = 0$ in the sum. We recover the entropy of a single partition $H(\alpha) = H(\alpha \mid \{X\})$ as a special case.

Proposition 2. *Let $(X, \mathcal{B}, \mu)$ be a probability space, α, β, γ finite partitions of X, and $T : X \to X$ measure preserving. We have:*

1. $H(\alpha \vee \beta \mid \gamma) = H(\alpha \mid \gamma) + H(\beta \mid \alpha \vee \gamma)$
2. $\alpha \leq \beta \implies H(\alpha \mid \gamma) \leq H(\beta \mid \gamma)$
3. $\beta \leq \gamma \implies H(\alpha \mid \beta) \geq H(\alpha \mid \gamma)$
4. $H(\alpha \vee \beta \mid \gamma) \leq H(\alpha \mid \gamma) + H(\beta \mid \gamma)$
5. $H(T^{-1}\alpha \mid T^{-1}\beta) = H(\alpha \mid \beta)$

Proof. Let us show the first equality. For the partitions $\alpha = \{A_i\}$, $\beta = \{B_j\}$, $\gamma = \{C_k\}$ we may assume, without loss generality, that all sets have positive measure. We compute

$$H(\alpha \vee \beta \mid \gamma) = -\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \log \frac{\mu(A_i \cap B_j \cap C_k)}{\mu(C_k)} \tag{17}$$

$$= -\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \log \frac{\mu(A_i \cap C_k)}{\mu(C_k)}$$

$$-\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \log \frac{\mu(A_i \cap B_j \cap C_k)}{\mu(A_i \cap C_k)} \tag{18}$$

$$= -\sum_{i,k} \mu(A_i \cap C_k) \log \frac{\mu(A_i \cap C_k)}{\mu(C_k)} + H(\beta \mid \alpha \vee \gamma) \tag{19}$$

$$= H(\alpha \mid \gamma) + H(\beta \mid \alpha \vee \gamma). \tag{20}$$

Note it is possible that $\mu(A_i \cap C_k) = 0$, but then $\mu(A_i \cap B_j \cap C_k) = 0$, so we are good. The next property is a formal consequence of the first, if $\alpha \leq \beta$, then:

$$H(\beta \mid \gamma) = H(\alpha \vee \beta \mid \gamma) = H(\alpha \mid \gamma) + H(\beta \mid \alpha \vee \gamma) \geq H(\alpha \mid \gamma) \tag{21}$$

We omit the proof of the remaining properties. □

2.3 Kolmogorov–Sinai entropy

As before we have a measure preserving transformation, T, on a probability space $(X, \mathcal{B}, \mu)$. We define the **entropy of T with respect to a partition** α as the limit

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right). \tag{22}$$

To show that the limit really exists, we use the following elementary lemma.

Lemma 1. *For a sequence $\{a_n\}_{n\geq 1}$ of real numbers with $a_{m+n} \leq a_m + a_n$ the limit $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_n a_n/n$.*

Now if we let $a_n = H(\bigvee_{i=0}^{n-1} T^{-i}\alpha)$ then

$$\begin{aligned} a_{m+n} &= H\left(\bigvee_{i=0}^{m+n-1} T^{-i}\alpha\right) && (23)\\ &\leq H\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha\right) + H\left(\bigvee_{i=m}^{m+n-1} T^{-i}\alpha\right) && (24)\\ &= a_m + a_n && (25) \end{aligned}$$

so $H(T, \alpha)$ is a well defined non-negative real number. Finally, the Kolmogorov–Sinai **entropy** of T is

$$h(T) = \sup_{\alpha} h(T, \alpha) \tag{26}$$

where α ranges over all finite partitions of X. It is clear that $h(T)$ depends only on the induced map $\widetilde{T}^{-1}$ on $(\widetilde{\mathcal{B}}, \widetilde{\mu})$, and is thus a conjugacy invariant.

We continue with some basic properties of $h(T, \alpha)$ and $h(T)$.

Proposition 3. *If T is measure preserving and α, β are finite partitions, then*

1. $h(T, \alpha) \leq H(\alpha)$
2. $h(T, \alpha \vee \beta) \leq h(T, \alpha) + h(T, \beta)$
3. $\alpha \leq \beta \implies H(T, \alpha) \leq H(T, \beta)$
4. $h(T, \alpha) \leq h(T, \beta) + H(\alpha \mid \beta)$
5. $h(T^k) = kh(T)$ *for* $k \geq 1$
6. $h(T^{-1}) = h(T)$ *for T invertible*

Proof. We show $h(T^k) = kh(T)$. Observe that

$$h\left(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha\right) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-kj}\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right)\right) \tag{27}$$

$$= \lim_{n\to\infty} \frac{k}{nk} H\left(\bigvee_{i=0}^{nk-1} T^{-i}\alpha\right) \tag{28}$$

$$= kh(T, \alpha). \tag{29}$$

Hence,

$$kh(T) = k \sup_{\alpha} h(T, \alpha) = \sup_{\alpha} h\left(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha\right) \tag{30}$$

$$\leq \sup_{\alpha} h(T^k, \alpha) = h(T^k) \tag{31}$$

but also $h(T^k, \alpha) \leq h(T^k, \bigvee_{i=0}^{k-1} T^{-i}) = kh(T, \alpha)$, so we must have equality, $h(T^k) = kh(T)$.

The remaining statements are left as an exercise. □

The main result used in computing $h(T)$ is the following.

Theorem 1 (Kolmogorov–Sinai Theorem)**.** *Let T be an invertible measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Suppose γ is a finite partition generating a subalgebra $\mathcal{G}$ with $\bigvee_{n=-\infty}^{\infty} T^{-n}\mathcal{G} \doteq \mathcal{B}$, then $h(T) = h(T, \gamma)$.*

Lemma 2. *Fix an integer $n \geq 1$, then for each $\epsilon > 0$ there is a $\delta > 0$ such that for any two partitions $\alpha = \{A_1, \ldots, A_n\}$, $\beta = \{B_1, \ldots, B_n\}$ of X into n subsets with $\sum_i \mu(A_i \cap B_i) < \delta$ we have $H(\alpha \mid \beta) + H(\beta \mid \alpha) < \epsilon$.*

Proof. Let $\epsilon > 0$ and choose $\delta \in (0, \frac{1}{4})$ such that

$$-n(n-1)\delta \log \delta - (1-\delta)\log(1-\delta) < \frac{\epsilon}{2}. \tag{32}$$

Let γ be the partition into sets $A_i \cap B_j$, $i \neq j$ and $\bigcup_i (A_i \cap B_i)$. For $i \neq j$ we have $A_i \cap B_j \subset \bigcup_k (A_k \triangle B_k)$ therefore $\mu(A_i \cap B_j) < \delta$. Also $\mu(\bigcup_i (A_i \cap B_i)) > 1 - \delta$, hence $H(\gamma) < -n(n-1)\delta \log \delta - (1-\delta)\log(1-\delta) < \epsilon/2$. We conclude

$$H(\beta) + H(\alpha \mid \beta) = H(\alpha \vee \beta) = H(\beta \vee \gamma) \leq H(\beta) + H(\gamma) < H(\beta) + \epsilon/2 \tag{33}$$

thus $H(\alpha \mid \beta) < \epsilon/2$ and by symmetry also $H(\beta \mid \alpha) < \epsilon/2$. □

Lecture 3: Measure theoretic entropy, Part II

Lemma 3. *Let $(X, \mathcal{B}, \mu)$ be a probability space and $\mathcal{B}_0 \subset \mathcal{B}$ a subalgebra such that $\overline{\mathcal{B}_0} = \mathcal{B}$, where $\overline{\mathcal{B}_0}$ is the σ-algebra generated by $\mathcal{B}_0$. Let $\mathcal{C} \subset \mathcal{B}$ be a finite subalgebra and $\epsilon > 0$, then there is a finite algebra $\mathcal{D} \subset \mathcal{B}_0$ such that $H(\mathcal{D} \mid \mathcal{C}) + H(\mathcal{C} \mid \mathcal{D}) < \epsilon$.*

Proof. Let $\mathcal{C}$ be generated by the partition $\{C_1, \ldots, C_n\}$. By the previous lemma it suffices to show that for each $\epsilon > 0$ we can find a partition $\{D_1, \ldots, D_n\}$ with $D_i \in \mathcal{B}_0$ and $\mu(C_i \triangle D_i) \leq \epsilon$, $i = 1, \ldots, n$. For this choose $\lambda > 0$ with

$$\lambda(n-1)(1 + n(n-1)) < \epsilon \tag{34}$$

and $B_i \in \mathcal{B}_0$ for each i with

$$\mu(B_i \triangle C_i) < \lambda. \tag{35}$$

Then for $i \neq j$ we have $B_i \cap B_j \subset (B_i \triangle C_i) \cup (B_j \triangle C_j)$ therefore $\mu(B_i \cap B_j) < 2\lambda$. Moreover, for $N = \bigcup_{i \neq j}(B_i \cap B_j)$ we have $\mu(N) < n(n-1)\lambda$. Consider the partition $\{D_1, \ldots, D_n\}$ with $D_i = B_i \setminus N$ for $i < n$ and $D_n = X \setminus \bigcup_{i=0}^{n-1} D_i$, in particular $D_i \in \mathcal{B}_0$. If $i < n$ then $D_i \triangle C_i \subset (B_i \triangle C_i) \cup N$, hence $\mu(D_i \triangle C_i) < \lambda(1 + n(n-1)) < \epsilon$. On the other hand, $D_n \triangle C_n \subset \bigcup_{i=1}^{n-1}(D_i \triangle C_i)$, hence $\mu(D_n \triangle C_n) < (n-1)\lambda(1 + n(n-1)) < \epsilon$. □

Corollary 1. *Suppose $\{A_n\}$ is an increasing sequence of finite sub-algebras of $\mathcal{B}$ and $\mathcal{C}$ is a finite sub-algebra with $\mathcal{C} \overset{\circ}{\subset} \bigvee_n \mathcal{A}_n$, then $H(\mathcal{C} \mid \mathcal{A}_n) \to 0$ as $n \to \infty$.*

Proof. Write $\mathcal{B}_0 = \bigcup_n \mathcal{A}_n$, then $\mathcal{C} \overset{\circ}{\subset} \overline{\mathcal{B}_0}$ by assumption. For $\epsilon > 0$ there exists a finite sub-algebra $\mathcal{D}_\epsilon \subset \mathcal{B}_0$ with $H(\mathcal{C} \mid \mathcal{D}_\epsilon) < \epsilon$. Since $\mathcal{D}_\epsilon \subset \mathcal{A}_N$ for some N we get

$$H(\mathcal{C} \mid \mathcal{A}_n) \leq H(\mathcal{C} \mid \mathcal{A}_N) \leq H(\mathcal{C} \mid \mathcal{D}_\epsilon) < \epsilon, \tag{36}$$

hence $H(\mathcal{C} \mid \mathcal{A}_n) \to 0$. □

Proof of Theorem 1. We want to show that $h(T, \mathcal{C}) \leq h(T, \mathcal{G})$ for $\mathcal{C} \subset \mathcal{B}$ a finite subalgebra. For $n \geq 1$ let $\mathcal{G}_k = \bigvee_{i=-k}^{k} T^i \mathcal{G}$. We compute

$$\begin{aligned}
h(T, \mathcal{G}_k) &= h\left(T, \bigvee_{i=0}^{2k} T^{-i}\mathcal{G}\right) && (37)\\
&= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-j}\left(\bigvee_{i=0}^{2k} T^{-i}\mathcal{G}\right)\right) && (38)\\
&= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{2k+n-1} T^{-i}\mathcal{G}\right) && (39)\\
&= \lim_{n\to\infty} \left(\frac{2k+n-1}{n}\right) \frac{1}{2k+n-1} H\left(\bigvee_{i=0}^{2k+n-1} T^{-i}\mathcal{G}\right) && (40)\\
&= h(T, \mathcal{G}). && (41)
\end{aligned}$$

Thus,

$$h(T, \mathcal{C}) \leq h(T, \mathcal{G}_k) + H(\mathcal{C} \mid \mathcal{G}_k) = h(T, \mathcal{G}) + H(\mathcal{C} \mid \mathcal{G}_k) \tag{42}$$

but by the previous corollary we have $H(\mathcal{C} \mid \mathcal{G}_k) \to 0$, $k \to \infty$, completing the proof. □

There is a variant of the previous theorem for T not necessarily invertible. The proof is very similar.

Theorem 2. *Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Suppose γ is a finite partition generating a subalgebra $\mathcal{G}$ with $\bigvee_{n=0}^{\infty} T^{-n}\mathcal{G} \doteq \mathcal{B}$, then $h(T) = h(T, \gamma)$.*

The following result tells us that the hypothesis of Theorem 2 will not be satisfied for invertible transformations, unless the entropy is zero, so we really need Theorem 1 as well. It can also be used to prove that $h(T) = 0$ in some cases.

Proposition 4. *Suppose T is an invertible measure preserving transformation of a probability space $(X, \mathcal{B}, \mu)$ and there exists a finite subalgebra $\mathcal{G} \subset \mathcal{B}$ with $\bigvee_{i=0}^{\infty} T^{-i}\mathcal{G} \doteq \mathcal{B}$, then $h(T) = 0$.*

Proof. One can show by induction that for a finite partition α

$$H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) = H(\alpha) - \sum_{j=1}^{n-1} H\left(\alpha \mid \bigvee_{i=1}^{j} T^{-i}\alpha\right) \tag{43}$$

hence, dividing by n and taking the limit,

$$h(T, \alpha) = \lim_{n\to\infty} H\left(\alpha \mid \bigvee_{i=1}^{n} T^{-i}\alpha\right) \tag{44}$$

since the Cesaro limit of a convergent sequence equals the ordinary limit. Combining this with Theorem 2 we get

$$h(T) = \lim_{n\to\infty} H(\mathcal{G} \mid \mathcal{G}_n) \tag{45}$$

where $\mathcal{G}_n = \bigvee_{i=1}^{n} T^{-i}\mathcal{G}$. Now since this is an increasing sequence of finite subalgebras with $\bigvee_{i=1}^{\infty} \mathcal{G}_n \doteq T^{-1}\mathcal{B} \doteq \mathcal{B}$ we have $H(\mathcal{G} \mid \mathcal{G}_n) \to 0$, so $h(T) = 0$. □

3.1 Examples

We now have everything to compute the entropy in some non-trivial examples.

Circle Rotation. First, consider the circle $X = U(1)$ with Borel σ-algebra $\mathcal{B}$ and the standard probability measure. We claim that for any $g \in U(1)$ the rotation $T_g(x) = gx$ has $h(T_g) = 0$. First, if g is a root of unity, say $g^n = 1$, then $nh(T_g) = h(T_g^n) = h(\mathrm{id}) = 0$. This argument applies to any periodic map. If g is not a root of unity, then $\{g^n \mid n \in \mathbb{Z}\}$ is dense in $U(1)$. Let $\alpha = \{A_1, A_1\}$ be a partition of $U(1)$ into the upper and lower semicircle of $U(1) = S^1$. By density, any semi-circle belongs to $\bigvee_{n=0}^{\infty} T^{-n}\alpha$, thus any arc, since this is a

σ-algebra. As arcs generate $\mathcal{B}$, we conclude that $\bigvee_{n=0}^{\infty} T^{-n}\alpha = \mathcal{B}$. Proposition 4 implies that $h(T) = 0$.

Circle covering. Consider again the circle $U(1) \subset \mathbb{C}$, but with non-invertible measure preserving map $T(z) = z^2$. If we identify $U(1)$ with $[0,1)$ as measure spaces using the map $[0,1) \to U(1)$, $x \mapsto \exp(2\pi i x)$, then T corresponds to $x \mapsto 2x \bmod 1$. Up to sets of measure zero, $[0,1)$ is identified with

$$\{0,1\}^{\mathbb{Z}_{>0}} \tag{46}$$

by taking the binary expansion. The sets of measure zero which we remove are numbers with eventually constant binary expansion in $[0,1)$ and eventually constant sequences in $\{0,1\}^{\mathbb{Z}_{>0}}$, respectively. The map $x \mapsto 2x \bmod 1$ corresponds to left shift. Thus we find that $z \mapsto z^2$ on $U(1)$ is isomorphic to the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift. Of course this generalizes to $z \mapsto z^n$. We will compute the entropy of the Bernoulli shift below. In this particular example it turns out to be $\log(2)$.

Bernoulli shift. In this example $Y = \{1, \ldots, k\}$ with the σ-algebra of all subsets and any probability measure, and $X = \prod_{i=1}^{\infty} Y$ with T being the left shift. Consider the partition $\gamma = \{G_1, \ldots, G_k\}$ with $G_i = \{(x_j) \mid x_1 = i\}$, i.e. G_i is the set of sequences with first term i. By definition of the product σ-algebra $\mathcal{B}$ on X we have

$$\bigvee_{i=0}^{\infty} T^{-i}\alpha = \mathcal{B} \tag{47}$$

so $h(T) = h(T, \alpha)$ by Theorem 2. Note the $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ is the partition of X into subsets of sequences with specified first n terms, i.e. of the form

$$G_{i_1,\ldots,i_n} = \{(x_i) \mid x_1 = i_1, \ldots, x_n = i_n\} \tag{48}$$

which has measure $p_{i_1} \cdots p_{i_n}$, where $p_j = \mu_Y(\{j\})$. Hence

$$h(T) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-1}\alpha\right) \tag{49}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(-\sum p_{i_1} \cdots p_{i_n} \log(p_{i_1} \cdots p_{i_n})\right) \tag{50}$$

$$= -\lim_{n\to\infty} \sum_{i_1} \log p_{i_1} \sum_{i_2,\ldots,i_n} p_{i_1} \cdots p_{i_n} \tag{51}$$

$$= -\sum_{i=1}^{k} p_i \log p_i. \tag{52}$$

In fact, it turns out that entropy is a complete invariant for Bernoulli shifts by a deep result of D. Ornstein.

Theorem 3 (Ornstein)**.** *Let T_1, T_2 be Bernoulli shifts with $h(T_1) = h(T_2)$, then T_1 is isomorphic to T_2.*

Infinite entropy. Let us give an example with $h(T) = \infty$. Let $Y = [0, 1]$ with standard probability measure, $X = \prod_{i \geq 1} Y$, and T the left shift on X. Let $\alpha = \{A_1, \ldots, A_n\}$ be a partition of Y into n intervals of length $\frac{1}{n}$ and $\beta = \{A_i \times Y \times Y \times \ldots\}$ the pullback of α under the projection to the first factor. The same calculation as for the Bernoulli shift shows that $h(T, \beta) = \log(n)$, so $h(T) = \infty$ since n was arbitrary.

3.2 Topological entropy: First definition

Instead of a probability space with measure preserving map, we will now consider a compact Hausdorff space X with continuous map $T : X \to X$. Again it is possible to define a dynamical entropy $h(T)$. The first definition of $h(T)$, due to Adler–Konheim–McAndrew, is formally very similar to the definition of measure theoretic entropy. Bowen later gave an equivalent definition which a priori depends on a metric on X, but is better suited for computing $h(T)$.

Instead of partitions we consider open covers of X. To exclude pathological cases, we assume $X \neq \emptyset$. As for partition, we have (finite) joins $\alpha \vee \beta$, the refinement relation $\alpha \leq \beta$ (meaning that every $U \in \beta$ is a subset of some $V \in \alpha$), and pullbacks along continuous maps. If α is an open cover, we define its **entropy**

$$H(\alpha) = \min_{\beta \subset \alpha} \log |\beta| \tag{53}$$

where β ranges over all finite subcovers and $|\beta|$ is the cardinality. Since X is compact by assumption, there is at least one such β. We make the following observations

1. $H(\alpha) \geq 0$ and $H(\alpha) = 0$ iff $X \in \alpha$
2. $\alpha \leq \beta \implies H(\alpha) \leq H(\beta)$
3. $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$
4. If $T : X \to X$ is continuous, then $H(T^{-1}\alpha) \leq H(\alpha)$. If T is furthermore surjective, then $H(T^{-1}\alpha) = H(\alpha)$.

We define the entropy of a continuous map $T : X \to X$ relative to an open cover α by

$$h(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right). \tag{54}$$

The proof that the limit exists is the same as in the measure theoretic case. Finally, define the **topological entropy** of T as

$$h(T) = \sup_{\alpha} h(T, \alpha) \tag{55}$$

where α ranges over all (or just finite) open covers of X. Basic properties of topological entropy are collected in the following proposition.

Proposition 5. *Suppose X is a compact Hausdorff space and $T : X \to X$ continuous.*

1. *If $A \subset X$ is closed with $T(A) = A$, then $h(T|A) \leq h(T)$.*
2. *If T is a homeomorphism, then $h(T^{-1}) = h(T)$.*

3.3 Topological entropy: Second definition

Suppose (X, d) is a metric space and $T : X \to X$ a uniformly continuous map. If $n \geq 1$ then

$$d_n(x, y) = \max_{0 \leq i < n} d(T^i(x), T^i(y)) \tag{56}$$

defines a new metric on X. For a compact subset $K \subset X$ and $\epsilon > 0$ there is a notion of the size (or volume) of K given by the number $V_{n,\epsilon}(K)$ of closed balls of radius ϵ, with respect to the metric d_n, needed to cover K. In other words, we say that $F \subset X$ is an (n, ϵ)**-spanning set** for K if for every $x \in K$ there is a $y \in F$ with $d_n(x, y) \leq \epsilon$, and $V_{n,\epsilon}(K)$ is the minimal cardinality of such a set F. Of course all this depends on T, which is not explicit in the notation. Define

$$h_\epsilon(T, K) = \limsup_{n \to \infty} \frac{1}{n} \log V_{n,\epsilon}(K) \tag{57}$$

$$h(T, K) = \lim_{\epsilon \to 0} h_\epsilon(K, T) \tag{58}$$

where the limit exists since $h_\epsilon(K, T)$ is increasing in $1/\epsilon$. Finally, define the **topological entropy** of T as

$$h(T) = \sup_K h(T, K) \tag{59}$$

where we take the supremum over all compact subsets $K \subset X$.

When X is compact we can simplify the definition, since $h(T) = h(T, X)$. Also any continuous map is uniformly continuous, and the choice of metric does not matter. We will see below that the definition agrees with the one using open covers.

Lecture 4: Topological entropy, continued

Instead of spanning sets we could have considered *separating sets*. A subset $F \subset K$ is an (n, ϵ)**-separating set** if any two distinct elements of F have at least distance ϵ, i.e. $x \neq y \in F \implies d_n(x, y) > \epsilon$. Let $S_{n,\epsilon}(K)$ denote the maximal cardinality of such a subset. The reason this does not lead to different definition of topological entropy is that

$$V_{n,\epsilon}(K) \leq S_{n,\epsilon}(K) \leq V_{n,\epsilon/2}(K). \tag{60}$$

The first inequality is clear, since any maximal (n, ϵ)-separated subset of K is an (n, ϵ)-spanning set. If we have an (n, ϵ)-separating set, E, and an $(n, \epsilon/2)$-spanning set F of K, then we can choose a map $f : E \to F$ so that $d_n(x, f(x)) \leq \epsilon/2$ for $x \in E$. But we see that f has to be injective, and so $|E| \leq |F|$. In conclusion, $h(T)$ can be defined using either spanning or separating sets.

Proposition 6. *Let (X, d) be a metric space, $T : X \to X$ uniformly continuous. If $K, K_1, \ldots, K_m \subset X$ are compact subsets with $K \subset \bigcup_i K_i$ then*

$$h(T, K) \leq \max_i h(T, K_i). \tag{61}$$

Proof. For each $\epsilon > 0, n \geq 1$ choose $i(n, \epsilon)$ such that

$$S_{n,\epsilon}(K_{i(n,\epsilon)}) = \max_j S_{n,\epsilon}(K_j). \tag{62}$$

Hence

$$S_{n,\epsilon}(K) \leq \sum_j S_{n,\epsilon}(K_j) \leq m S_{n,\epsilon}(K_{i(n,\epsilon)}) \tag{63}$$

which implies

$$\log S_{n,\epsilon}(K) \leq \log m + \log S_{n,\epsilon}(K_{i(n,\epsilon)}). \tag{64}$$

We can find a sequence $n_j \to \infty$ such that

$$\lim_{j\to\infty} \frac{1}{n_j} \log S_{n_j,\epsilon}(K) = \limsup_{n\to\infty} \frac{1}{n} \log S_{n,\epsilon}(K) \tag{65}$$

and so that $K_{i(n_j,\epsilon)} = K_{i(\epsilon)}$ for some $i(\epsilon)$. We conclude that $h_\epsilon(T, K) \leq h_\epsilon(T, K_{i(\epsilon)})$, so there must be an i_0 with $h(T, K) \leq h(T, K_{i_0}) \leq \max_j h(T, K_j)$, which completes the proof. □

The next goal is to show that the two definitions of topological entropy agree. We will temporarily denote by $h'(T)$ the entropy defined using open covers.

Theorem 4. *Let (X, d) be a compact metric space, $T : X \to X$ continuous, then $h'(T) = h(T)$, i.e. the topological entropy computed from open covers coincides with the one using the metric d.*

Proof. The main point is the following. Suppose (X, d) is a compact metric space, α an open cover of X. First, one says that $\delta > 0$ is a *Lebesgue number* for α, if every $E \subset X$ with $\operatorname{diam}(E) < \delta$ is contained in some element of α. Second, if we define

$$\operatorname{diam}(\beta) = \sup_{U \in \beta} \operatorname{diam}(U) \tag{66}$$

then if $\operatorname{diam}(\beta)$ is a Lebesgue number for α we have $\alpha \leq \beta$, i.e. β is a refinement of α. In particular, if we have a sequence of covers β_n with $\operatorname{diam}(\beta_n) \to 0$, then

$$h'(T) = \lim_{n \to \infty} h(T, \beta_n) \tag{67}$$

since the diameter of β_n will eventually be smaller than a Lebesgue number for an arbitrary cover α, thus $\alpha \leq \beta_n$ for n big.

Now if α is an open cover with Lebesgue number δ, then

$$H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leq \log V_{n,\delta/2}(X). \tag{68}$$

To see this note that if E is an $(n, \delta/2)$-spanning set for X, then

$$X = \bigcup_{x \in E} \bigcap_{i=0}^{n-1} T^{-i}\overline{B(T^i x, \delta/2)} \tag{69}$$

and each $\overline{B(T^i x, \delta/2)}$ is a subset of some member of α. On the other hand, if α is an open cover with $\operatorname{diam}(\alpha) \leq \epsilon$, then

$$\log S_{n,\epsilon}(X) \leq H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \tag{70}$$

since if E is (n, ϵ)-separating, then no element of the cover $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ can contain more than one element of E. Combining these two estimates gives the statement of the theorem. □

Theorem 5. *Let $T : S^1 \to S^1$ be a homeomorphism, then $h(T) = 0$.*

Proof. Choose a metric d on $X = S^1$ to that the circumference is 1. We claim that

$$V_{n,\epsilon}(X) \leq n\lceil 1/\epsilon \rceil \tag{71}$$

which implies the statement of the theorem. To see this, choose for each n a finite subset $E_n \subset X$ which cuts X into intervals of length no more than ϵ with respect to the metric $d(T^n x, T^n y)$. Distributing points evenly, we may assume that

$$|E_n| \leq \lceil 1/\epsilon \rceil. \tag{72}$$

For $y \in X$, $n \geq 0$ we can find $x_n \in E_n$ such that the clockwise path from y to x_n has length $< \epsilon$ with respect to the metric $d(T^n x, T^n y)$. Fixing n, one of the points $x_0, \ldots, x_{n-1}$, say x_{i_n}, will come first as we go clockwise around X starting at y. But then $d_n(y, x_{i_n}) \leq \epsilon$, so $\bigcup_{i=0}^{n-1} E_i$ is an (n, ϵ)-spanning set for X. □

4.1 Topological vs. measure-theoretic entropy

As it turns out, there is an interesting relation between topological and measure-theoretic entropy. Suppose (X, d) is a compact metric space with Borel σ-algebra $\mathcal{B}$. We denote by $M(X)$ the space of probability measures on $\mathcal{B}(X)$. Any such measure turns out to be regular. We have an injection

$$M(X) \to C(X)^*, \qquad \mu \mapsto \left(f \mapsto \int_X f d\mu \right) \tag{73}$$

where $C(X)$ is the space of continuous functions on X. The image is exactly the set of positive functionals $J : C(X) \to \mathbb{R}$ with $J(1) = 1$. If we give $C(X)$ the weak*-topology, then $M(X)$ becomes a compact convex subset $C(X)^*$.

Let $T : X \to X$ be a continuous maps, and let

$$M(X, T) = \{\mu \in M(X) \mid \mu \circ T^{-1} = \mu\} \tag{74}$$

be the set of T-invariant probability measures.

Proposition 7. *$M(X, T) \subset M(X)$ is a non-empty, convex, and compact subset. Moreover, $\mu \in M(X)$ is an extreme point if and only if T is ergodic on $(X, \mathcal{B}, \mu)$.*

For $\mu \in M(X, T)$ we write $h_\mu(T)$ for the corresponding measure theoretic entropy. This gives a map $M(X, T) \to [0, \infty]$ which is affine, but usually not continuous, or even semi-continuous.

Theorem 6 (Variational principle)**.** *For a compact metric space X and continuous $T : X \to X$ we have*

$$h(T) = \sup\{h_\mu(T) \mid \mu \in M(X, T)\} \tag{75}$$

where $h(T)$ is the topological entropy.

Proof. We will not give a complete proof, but note some of the ideas that go into it. First, if $\alpha = \{A_1, \ldots, A_n\}$ is a partition of X and $\epsilon > 0$, then by regularity we find compact $K_i \subset A_i$ with $\mu(A_i \setminus K_i) < \epsilon$. On the other hand, the K_i given an open cover by sets $X \setminus \bigcup_{j \neq i} K_j$, $i = 1, \ldots, n$. This is used to show that $h_\mu(T) \leq h(T)$ for any measure μ.

To get the other inequality we find for $\epsilon > 0$ a $\mu \in M(X, T)$ with $h_\mu(T) \geq S_{n,\epsilon}(T, X)$. To construct μ, let σ_n be a probability measure which is evenly concentrated at the points of an (n, ϵ)-separating set of maximal cardinality, and

$$\mu_n = \frac{1}{n} \sum_i \sigma_n \circ T^{-i} \tag{76}$$

then μ_n has a convergent subsequence by compactness, and the limit μ lies in $M(X, T)$. $\square$

4.2 Entropy as volume growth

Let M be an m-dimensional smooth Riemannian manifold, and $f : M \to M$ a C^∞ map. It turns out that the topological entropy, $h(f)$ measures growth of volumes of submanifolds of M under mapping by f^n. We make the following definition: If $1 \leq k \leq m$ and $\sigma : [0,1]^k \to M$ is a C^∞ map from the k-dimensional unit cube, we let $\mathrm{Vol}(\sigma)$ be the k dimensional volume of the image of σ in M, counted with multiplicity. Let

$$V_k(f) = \sup_{\sigma:[0,1]^k \to M} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}(f^n \circ \sigma) \tag{77}$$

and

$$V(f) = \max_k V_k(f). \tag{78}$$

The following result is due to Y. Yomdin.

Theorem 7. $V(f) = h(f)$.

This can fail if f is only C^k, but there are still bounds in that case. A related result proves a version of the *Entropy Conjecture*.

Theorem 8 (Yomdin)**.** *For M a compact smooth manifolds, $f : M \to M$ a smooth map:*

$$\log \rho(H_*(f, \mathbb{R})) \leq h(f) \tag{79}$$

where $H_*(f, \mathbb{R}))$ is the induced map on homology with coefficients in $\mathbb{R}$ and ρ is the spectral radius of a linear map.

There are indeed C^∞ maps for which the inequality is strict. An interpretation of this is that entropy can come from both local and global complexity of f, and $\log \rho(H_*(f, \mathbb{R}))$ only detects the global, homological part.

The definition of entropy of exact functors, given in the first lecture and below, is formally similar to the one of $V(f)$. Instead of maps $[0,1]^k \to M$ one considers objects $E \in \mathcal{C}$ in the category, and the analogue of volume is the "length" in the homological sense. There is also a version of Theorem 8 for compact and smooth dg-categories, where Hochschild homology replaces singular homology.

4.3 Abelian categories

Abelian categories provide a general setting in which to talk about kernels, images, exact sequences, and cohomology theories. They were introduced by Buchsbaum and Grothendieck in the 50's. A standard textbook reference is *Abelian Categories* by P. Freyd. As usual, we start with the definitions.

An **additive category** is a category $\mathcal{A}$ with the structure of an abelian group on each Hom-space $\mathrm{Hom}(A, B)$ such that

1. composition is bilinear, i.e. $\mathcal{A}$ is enriched over the category of abelian groups

2. $\mathcal{A}$ has biproducts, i.e. the natural map

$$A_1 \sqcup \ldots \sqcup A_n \to A_1 \times \ldots \times A_n \tag{80}$$

is an isomorphism for $n \geq 0$.

The **direct sum** of A and B, $A \oplus B$, is $A \sqcup B = A \times B$, and the **zero object** is the empty product, the unique object up to isomorphism with $\mathrm{Hom}(0, A) = \mathrm{Hom}(A, 0) = 0$. Additive categories allow us to work with matrices: An element in $\mathrm{Hom}(A_1 \oplus \ldots \oplus A_m, B_1 \oplus \ldots \oplus B_n)$ is given by a collection of elements $a_{ij} \in \mathrm{Hom}(A_j, B_i)$, and multiplication of such matrices corresponds to composition of morphisms.

An **abelian category** is an additive category $\mathcal{A}$ such that kernels and cokernels exist, and every monomorphism (resp. epimorphism) is a kernel (resp. cokernel) of some morphism. Recall that kernel an image are special cases of limits and colimits for a diagram with two parallel arrows, where one of them is zero:

$$\mathrm{Ker}(f) \longrightarrow A \underset{0}{\overset{f}{\rightrightarrows}} B \qquad\qquad A \underset{0}{\overset{f}{\rightrightarrows}} B \longrightarrow \mathrm{Coker}(f) \tag{81}$$

If we let $\mathrm{Im}(f) = \mathrm{KerCoker}(f)$ and $\mathrm{Coim}(f) = \mathrm{CokerKer}(f)$, then the axioms ensure that the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

There are plenty of examples of abelian categories:

- Vector spaces over a field
- Finite dimensional vector spaces over a field
- Modules over a ring, not necessarily commutative
- Sheaves of abelian groups over a topological space
- Coherent sheaves over a projective variety
- ...

The central notion in an abelian category is that of an **exact sequence**, a finite or infinite sequence of objects and maps

$$\ldots \to X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} X_{i+1} \to \ldots \tag{82}$$

such that the image of f_i is the kernel of f_{i+1} for all i.

All is not well, however, in the world of abelian categories. Almost all the functors between such categories which appear naturally, such as $\mathrm{Hom}(X, _)$, $\mathrm{Hom}(_, X)$, $X \otimes _$, f_*, do not preserve exactness. For example, if $A \in \mathcal{A}$ and $0 \to X \to Y \to Z \to 0$ is exact, then

$$0 \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z) \tag{83}$$

is exact, but the map $\mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z)$ is not surjective in general. This is one reason for introducing left/right *derived functors*, but these are not structure preserving either. This could be taken as a hint that one should find a way to remove structure from an abelian category, which leads to the construction of passing to the *derived category*.

Lecture 5: Towards derived categories

5.1 Examples of derived equivalent abelian categories

We have seen that there are "structural" reasons for passing from abelian to derived categories, namely standard functors being only left/right exact. To further strengthen the point we will discuss examples of pairs of abelian categories which look like they should be pieces of the same larger structure. This structure will turn out to be their derived category. The examples will come from representations of quivers and coherent sheaves on projective varieties.

A **quiver**, Q, is just a directed graph. This means we have a set Q_0 of vertices, a set Q_1 of arrows, and maps $\mathrm{in}, \mathrm{out} : Q_1 \to Q_0$ mapping an arrow to its source and target vertex. Lets fix a ground field k and consider vector spaces over k. A representation of Q is the data of a finite-dimensional vector space V_i for each vertex $i \in Q_0$ together with linear maps $T_\alpha : V_{\mathrm{in}(\alpha)} \to V_{\mathrm{out}(\alpha)}$ for each arrow $\alpha \in Q_1$. A morphism of representations $f : V = (\{V_i\}, \{S_\alpha\}) \to W = (\{W_i\}, \{T_\alpha\})$ is given by a collection of linear maps $f_i : V_i \to W_i$ which intertwines the S_α with the T_α, i.e. the diagram

$$\begin{array}{ccc} V_{\mathrm{in}(\alpha)} & \xrightarrow{S_\alpha} & V_{\mathrm{out}(\alpha)} \\ {\scriptstyle f_{\mathrm{in}(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{out}(\alpha)}} \\ W_{\mathrm{in}(\alpha)} & \xrightarrow{T_\alpha} & W_{\mathrm{out}(\alpha)} \end{array} \tag{84}$$

commutes for every $\alpha \in Q_1$.

The category of all representations of Q over k is abelian and is denoted $\mathrm{Rep}_k(Q)$. Kernels and cokernels are just computed pointwise, i.e. for each vertex separately. There is a variant where we fix some polynomial relations among the arrows and then require the T_α to satisfy these same relations. In this way we get a full subcategory of $\mathrm{Rep}_k(Q)$.

For notational reasons we assume that $Q_0 = \{1, \ldots, n\}$. To each representation V we can assign its dimension vector

$$\overrightarrow{\dim} V = (\dim V_1, \ldots, \dim V_n) \in \mathbb{Z}^n_{\geq 0}. \tag{85}$$

If Q is acyclic, i.e. arrows do not form any oriented cycles, then $\overrightarrow{\dim} V$ has an abstract meaning as the class of V in the Grothendieck group $K_0(\mathcal{A}) \cong \mathbb{Z}^n$. A basic problem is to understand the quotient stack

$$\prod_\alpha \mathrm{Hom}(V_{\mathrm{in}(\alpha)}, V_{\mathrm{out}(\alpha)}) \Big/ \prod_i Gl(V_i) \tag{86}$$

of isomorphism classes of representations with fixed dimension vector $(\dim V_1, \ldots, \dim V_n)$. Here $Gl(V_i)$ acts on $\mathrm{Hom}(V_{\mathrm{in}(\alpha)}, V_{\mathrm{out}(\alpha)})$ whenever α starts or ends at i. The complexity of that problem depends very much on Q and the the magnitude of the $\overrightarrow{\dim} V$.

Let us look at $\mathrm{Rep}_k(Q)$ for the simplest Q. When Q has no vertices, $\mathrm{Rep}_k(Q)$ is the trivial abelian category with only the zero object. For $Q_1 = \bullet$, the quiver with a single vertex and no arrows, $\mathrm{Rep}_k(Q_1)$ is the category of finite-dimensional vector spaces over k. More

generally, when Q has no arrows then $\mathrm{Rep}_k(Q)$ is a semisimple category, i.e. every object is a direct sum of simple ones. Next, consider $Q_2 = \bullet \to \bullet$. A representation of Q_2 is just an arbitrary linear map $T : V_1 \to V_2$ between finite dimensional vector spaces. It decomposes (non-canonically) into a sum of representations $T : \mathrm{Ker}(T) \to 0$, $V_1/\mathrm{Ker}(T) \to \mathrm{Im}(T)$, and $0 \to \mathrm{Coker}(T)$. It follows that every representation is direct sum of

$$k \to 0, \qquad k \xrightarrow{1} k, \qquad 0 \to k \tag{87}$$

i.e. these are exactly the indecomposable representations, up to isomorphism.

Only slightly more complicated is the case $Q_3 = \bullet \to \bullet \to \bullet$. The indecomposable representations are

$$\begin{gathered} k \xrightarrow{1} k \xrightarrow{1} k \\ k \xrightarrow{1} k \to 0 \qquad 0 \to k \xrightarrow{1} k \\ k \to 0 \to 0 \qquad 0 \to k \to 0 \qquad 0 \to 0 \to k \end{gathered}$$

This can be proven as before by simple linear algebra. In fact, you may recognize the dimension vectors of these representations as the positive roots in a root system of type A_3, whose Dynkin diagram is exactly the unoriented version of Q, i.e. $\bullet - \bullet - \bullet$. This is a general phenomenon for quivers with underlying graph a Dynkin diagram of type ADE.

Let's reverse the direction of one of the arrows to get $Q_3' = \bullet \leftarrow \bullet \to \bullet$. The indecomposables are

$$\begin{gathered} k \xleftarrow{1} k \xrightarrow{1} k \\ k \xleftarrow{1} k \to 0 \qquad 0 \leftarrow k \xrightarrow{1} k \\ k \leftarrow 0 \to 0 \qquad 0 \leftarrow k \to 0 \qquad 0 \leftarrow 0 \to k \end{gathered}$$

but still $\mathrm{Rep}_k(Q_3') \ncong \mathrm{Rep}_k(Q_3)$, i.e. the two abelian categories are not equivalent. In fact the stronger statement is true that we can recover Q from $\mathcal{A} = \mathrm{Rep}_k(Q)$. One way to see this is that vertices of Q correspond to *simple objects* in $\mathcal{A}$, i.e. those $E \in \mathcal{A}$ which have exactly two subobjects, up to isomorphism, 0 and E itself. Explicitly, the simple representation for a vertex i, S_i, has $V_i = k$, and $V_j = 0$, $i \neq j$. We claim that the number of arrows from i to j is $\dim \mathrm{Ext}^1(S_i, S_j)$, which we may verify later.

Exercise: Verify the claim for Q_3. Some hints on how to do this: First identify the projective representations by checking the universal property. Then find projective resolutions of the simple representations to compute Ext^1.

It turns out though that there is an equivalence of derived categories (which we have not yet defined)

$$D^b(\mathrm{Rep}_k(Q_3)) \cong D^b(\mathrm{Rep}_k(Q_3')). \tag{88}$$

In particular, we cannot recover the quiver from the derived category. Indeed, the concept of subobject, and thus simple object, does not make sense in derived categories.

We return to $Q_3 = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ but add the relation $\beta\alpha = 0$, i.e. we restrict to the category $\mathrm{Rep}_k(Q, \beta\alpha)$ of representations with $T_\beta T_\alpha = 0$. The list of indecomposables is the same, except we drop $k \xrightarrow{1} k \xrightarrow{1} k$. Thus $\mathrm{Rep}_k(Q_3, \beta\alpha)$ cannot be equivalent to either of the previous abelian categories, simply because the number of indecomposable objects is not the same. Still, it turns out that we get the same derived category. The missing sixth indecomposable turns out to be an object in $D^b(\mathrm{Rep}_k(Q_3, \beta\alpha))$ given by a complex of two representations.

We have already discussed the abelian group $\mathbb{Z}^{Q_0}$ containing the dimension vectors as a cone. There is a natural integral bilinear form on this group, the **Euler form**, given by

$$\chi(v, w) = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W) \tag{89}$$

where V and W are any representations with $\overrightarrow{\dim} V = v$, $\overrightarrow{\dim} W = w$. Explicitly, in the standard basis we have

$$\chi_{ij} = \dim \mathrm{Hom}(S_i, S_j) - \dim \mathrm{Ext}^1(S_i, S_j) = -\#\text{ of arrows } i \to j. \tag{90}$$

In the case of Dynkin-type quiver it is a theorem that there is at most one indecomposable object for each dimension vector, and the dimension vectors of the indecomposables are exactly the positive roots of χ, i.e. those $v \in \mathbb{Z}_{\geq 0}^{Q_0}$ with $\chi(v, v) = 1$.

The next example we consider is the Kronecker quiver $Q = \bullet \rightrightarrows \bullet$. For simplicity, we assume that ground field k is algebraically closed. The obvious indecomposable representations are the two simple ones and

$$k \underset{y}{\overset{x}{\rightrightarrows}} k \tag{91}$$

parametrized by $[x : y] \in \mathbb{P}^1_k$. Thus, in contrast to the Dynkin case, we get a *family* of indecomposable objects depending on a parameter. The complete answer turns out to be the following: For each dimension vector (n, n), $n \geq 1$, we have a $\mathbb{P}^1_k$ family of indecomposable objects, while for each dimension vector $(n + 1, n)$ and $(n, n + 1)$, $n \geq 0$ we have a single indecomposable representation. These vectors are just the positive vectors with $\chi(v, v) = 0$ and $\chi(v, v) = 1$, respectively.

An abelian category of different origin is the category $\mathrm{Coh}(\mathbb{P}^1_k)$ of coherent sheaves on the projective line over k. Coherent sheaves on $\mathbb{P}^1_k$, and algebraic curves more generally, are vector bundles and torsion sheaves (supported on some discrete subset), and their direct sums. On $\mathbb{P}^1_k$ more specifically, the indecomposable coherent sheaves are

- Skyscraper sheaves $\mathcal{O}_{np}$, $p \in \mathbb{P}^1_k$, $n \geq 1$,
- Line bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}$.

The category of coherent sheaves over a projective variety has X has a more algebraic and elementary description in terms of graded modules. Indeed, if R is the $\mathbb{Z}$-graded homogeneous coordinate ring of X, then

$$\mathrm{Coh}(X) = \mathrm{Mod}(R)/\mathrm{Tors}(R) \tag{92}$$

where $\mathrm{Mod}(R)$ is the category of finitely generated graded modules over R and $\mathrm{Tors}(R)$ is the category of finite dimensional graded modules over R.

The K-theory class of a coherent sheaf X is given by the pair $(\deg(X), \mathrm{rank}(X)) \in \mathbb{Z}\times\mathbb{Z}_{\geq 0}$. If we plot the classes of indecomposable objects in the previous two examples in the xy-plane, and centrally reflect all points at the origin, we get almost the same picture in both cases, just rotated by 45°. This is due to the existence of an equivalence of derived categories

$$D^b(\mathrm{Rep}_k(\bullet\vec{\to}\bullet)) \cong D^b(\mathrm{Coh}(\mathbb{P}^1_k)) \tag{93}$$

with a generalization to $\mathbb{P}^n_k$ due to Beilinson.

5.2 The derived category: Motivation

To any abelian category, $\mathcal{A}$, one can assign its derived category $D(\mathcal{A})$, as well as bounded variants $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$. We will see that the category $D(\mathcal{A})$ has additional structure making it a *triangulated category*.

We will give the complete construction below, but for now lets see how one could arrive at it. As was mentioned, the functor $\mathrm{Hom}(A, _)$ is not exact, only left-exact. However, it is exact if (and only if!) A happens to be *projective*. For general A we can find a presentation, also known as a *resolution*, in terms of projectives, assuming that there is a good supply of them.

$$\ldots \to P_{n+1} \xrightarrow{d_n} P_n \to \ldots \to P_1 \xrightarrow{d_0} P_0 \to A \to 0 \tag{94}$$

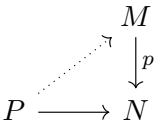
To be a resolution, we require the above sequence to be exact. The meaning of this sequence is that P_0 is a set of generators for A (exactness at A), P_1 is a set of generators for the relations among these generators (exactness at P_0), P_2 is a set of generators for the "relations among the relations", and so on. If we regard (P_i, d) as a complex, then its homology is $P_0/\mathrm{Im}(d_0) \cong A$, so from the points of view of homology, the complex (P_i, d) and $(A, 0)$ can be regarded as the same. Thus, we seek to define a category which includes all complexes as objects and so that maps which induce an isomorphism in cohomology are invertible. Since there may not be very many projectives, one allows complexes of arbitrary objects.

Lecture 6: Resolutions

6.1 Projective objects

A generalization of free modules which is useful in computations in homological algebra is the notion of a projective object. The property of free modules over a ring R which we single out is the following: If F is free over R with basis $(b_i)_{i\in I}$, and $p : M \to N$ is surjective, then any $f : F \to N$ lifts to M. This means that there is a $g : F \to M$ with $pg = f$. To construct g we choose for each $i \in I$ an $a_i \in M$ with $p(a_i) = f(b_i)$ and set $g(b_i) = a_i$, then extend to all of F by linearity.

An object P in an abelian category $\mathcal{A}$ is **projective** if in any diagram



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with p an epimorphism, the dotted arrow exists, making the diagram commute. This is equivalent to saying that the functor $\mathrm{Hom}(P, _)$ is *exact*, i.e. for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ we get a short exact sequence of abelian groups

$$0 \to \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y) \to \mathrm{Hom}(P, Z) \to 0. \tag{96}$$

In general, sums and summands of projective objects are projective. The argument in the previous paragraph shows that a free module over a ring R is projective in the category $\mathrm{Mod}(R)$ of R-modules. In fact, the projectives in $\mathrm{Mod}(R)$ are exactly the summands of free modules.

When R is functions on a space X then projectives modules over R are nothing but vector bundles. This gives some geometric intuition for projective modules over a more general ring.

Theorem 9 (Serre–Swan)**.** *1. If X is a compact smooth manifold, then vector bundles over X correspond to finitely generated projective $C^\infty(X)$-modules.*

2. *If X is a compact Hausdorff spaces, then vector bundles over X correspond to finitely generated projective $C(X)$-modules.*

3. *If X is an affine variety, then vector bundles over X correspond to finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules.*

Next, we will determine the projectives in $\mathrm{Rep}_k(Q)$, the category of linear representations of a quiver Q. For this it is helpful to reinterpret $\mathrm{Rep}_k(Q)$ as a category of modules over the *path algebra* of Q. Let us assume that Q has a finite number of vertices for simplicity. A **path** in Q is either a composable sequence of arrows $\alpha_1 \cdots \alpha_n$, i.e. $\mathrm{in}(\alpha_{i+1}) = \mathrm{out}(\alpha_i)$, pictorially

$$\bullet \xleftarrow{\alpha_1} \bullet \leftarrow \cdots \leftarrow \bullet \xleftarrow{\alpha_n} \bullet \tag{97}$$

or a constant path, e_i, at some vertex $i \in Q_0$. For a path $a = \alpha_1 \cdots \alpha_n$ we define $\mathrm{in}(a) = \mathrm{in}(\alpha_n)$ and $\mathrm{out}(a) = \mathrm{out}(\alpha_1)$, and $\mathrm{in}(e_i) = \mathrm{out}(e_i) = i$. The **path algebra**, kQ, of Q over a ring k is the free module with basis the paths in Q. Composition of non-constant paths $a = \alpha_1 \cdots \alpha_m$ and $b = \beta_1 \cdots \beta_n$ is by concatenation

$$ab = \begin{cases} \alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_n & \text{if } \mathrm{out}(\beta_1) = \mathrm{in}(\alpha_m) \\ 0 & \text{else} \end{cases} \tag{98}$$

and composition with e_i is defined similarly so that they become orthogonal idempotents. Note that the unit in kQ is $\sum e_i$. As an example, for $Q = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ the path algebra kQ has basis $e_1, e_2, e_3, \alpha, \beta, \beta\alpha$ and is isomorphic to the algebra of 3-by-3 lower triangular matrices. If Q has a single vertex and $n \geq 1$ arrows, then kQ is the free associative algebra on n letters, and is infinite-dimensional.

Proposition 8. *The category* $\mathrm{Rep}_k(Q)$ *is equivalent to the category of finite-dimensional left* kQ*-modules.*

If we drop the condition of finite-dimensionality in the definition of $\mathrm{Rep}_k(Q)$, then the category is equivalent to the category of all left kQ-modules.

Proof. Given a representation (V_i, T_α) the corresponding module is $M = \bigoplus_i V_i$ with action on $x = x_1 + \ldots + x_n$, $x_i \in V_i$ given by $\alpha x = T_\alpha(x_{\mathrm{in}(\alpha)})$, and $e_i x = x_i$. We leave the details as an exercise. □

We can consider $A = kQ$ as a left or right module over itself, it is the free module of rank 1. For each vertex i there is a left module Ae_i with basis given by the paths starting at i. Since the e_i are orthogonal idempotents with $\sum e_i = 1$, we get a direct sum decomposition $A = \bigoplus_i Ae_i$ of A as a left-module. In particular, each Ae_i is a projective module, and one can show that we get all indecomposable projectives in this way. For example, if $Q = \bullet \to \bullet$ then $Ae_1 = k \to k$ and $Ae_2 = 0 \to k$ are the projectives. The representation $k \to 0$ is not projective.

6.2 Resolving by projectives

As was already pointed out, the functor $\mathrm{Hom}(X, _)$ is ill-behaved when X is not projective. The solution to this is to present X in terms of projectives, i.e. find a projective resolution

$$\ldots \to P_{n+1} \xrightarrow{d_n} P_n \to \ldots \to P_1 \xrightarrow{d_0} P_0 \to X \to 0 \tag{99}$$

and regard the complex of P_i's as an avatar for X. To find the "correct" $\mathrm{Hom}(X, Y)$ we then apply $\mathrm{Hom}(_, Y)$ to this complex and get a complex of abelian groups

$$\mathrm{Hom}(P_0, Y) \to \mathrm{Hom}(P_1, Y) \to \ldots \to \mathrm{Hom}(P_n, Y) \to \mathrm{Hom}(P_n, Y) \to \tag{100}$$

the zeroth cohomology of which is $\mathrm{Hom}(X,Y)$. However, there can be cohomology groups in any (positive) degree, which are usually denoted by $\mathrm{Ext}^i(X,Y)$. These groups do not depend on the choice of resolution.

Let us see how this works in an example. Consider the ring $R = k[x]$ of polynomials in one variable over a field. For any $\lambda \in R$ there is a simple module T_λ with $\dim_k T_\lambda = 1$ so that x acts by multiplication by λ. There is a free resolution

$$0 \to R \xrightarrow{(x-\lambda)\cdot} R \to T_\lambda \to 0. \tag{101}$$

Thus we replace T_λ by the complex $R \xrightarrow{(x-\lambda)\cdot} R$ which is concentrated in degrees -1 and 0. (We use cohomological conventions, so the differential has degree 1.) Applying $\mathrm{Hom}(_, R)$ to this complex, instead of T_λ, we get essentially the same complex but now concentrated in degrees 0, 1. It is still a complex of R-modules, not just abelian groups, since R is commutative. The zeroth cohomology, i.e. the kernel of the map $(x-\lambda)\cdot$, vanishes, reflecting the fact that $\mathrm{Hom}(T_\lambda, R) = 0$, but the first cohomology, the cokernel of $(x-\lambda)\cdot$, is non-trivial and isomorphic to T_λ itself. Thus, $\mathrm{Ext}^1(T_\lambda, R) \cong T_\lambda$, and all other Ext-groups vanish.

As another example let us consider the elementary quiver $Q = \bullet \to \bullet$ with representations

$$P_1 = k \to k, \qquad P_2 = 0 \to k, \qquad S_1 = k \to 0 \tag{102}$$

fitting into an exact sequence

$$0 \to P_2 \to P_1 \to S_1 \to 0 \tag{103}$$

which gives a projective resolution of the non-projective S_1. Applying $\mathrm{Hom}(_, P_2)$ to the complex $P_2 \to P_1$ we get a complex which is k in degree 1 and zero in other degrees, i.e. $\mathrm{Ext}^1(S_1, P_2) = k$.

More generally for any quiver Q, one can show that every kQ-module has a two-step resolution as above. In particular, we have $\mathrm{Ext}^i(X,Y) = 0$ for $i \geq 2$ in $\mathrm{Mod}(kQ)$. However, this is not true if we impose relations among the arrows. Take for example $Q = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ with the relation $\beta\alpha = 0$. The projectives are $P_1 = k \to k \to 0$, $P_2 = 0 \to k \to k$, and $P_3 = 0 \to 0 \to k$. We claim $\mathrm{Ext}^2(S_1, P_3) = k$, where $S_1 = k \to 0 \to 0$. To verify this, one can use a projective resolution of S_1 of the form

$$0 \to P_3 \to P_2 \to P_1 \to S_1 \to 0. \tag{104}$$

It can also happen that there is no finite-length projective resolution at all. Consider $R = k[x]/x^2$ and the module $k = R/(x)$. There is an infinite free resolution

$$\ldots \to R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \to k \to 0 \tag{105}$$

which gives $\mathrm{Ext}^i(k,k) = k$, $i \geq 0$, i.e. non-vanishing Ext-groups in every degree.

6.3 The derived category

A more conceptual definition of the Ext-groups, and derived functors in general, comes from the derived category. The current goal is to define it. The first step is to construct a category $K(\mathcal{A})$ of complexes. An object in $K(\mathcal{A})$ is a sequence of objects and maps in $\mathcal{A}$

$$\ldots \to X^{i-1} \xrightarrow{d^{i-1}} X_i \xrightarrow{d^i} X^{i+1} \to \ldots \tag{106}$$

with $d^i d^{i-1} = 0$ for all i. Here i runs over all integers. A **chain map** from a complex (X, d) to a complex (Y, d) is a sequence of morphisms $f^i : X^i \to Y^i$ such that the diagrams

$$\begin{array}{ccc} X^i & \xrightarrow{d^i} & X^{i+1} \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ Y^i & \xrightarrow{d^i} & Y^{i+1} \end{array} \tag{107}$$

commute, i.e. $fd = df$ in short. Note that we can add and compose chain maps. One says that f, g are **homotopic** if there is a sequence of maps $H^i : X^i \to Y^{i-1}$ such that

$$f^i - g^i = d^{i-1} \circ H^i + H^{i+1} d^i. \tag{108}$$

The morphisms in $K(\mathcal{A})$ are defined to be equivalence classes of homotopic chain maps.

Lecture 7: The derived category

There are three basic constructions with complexes which are used all the time.

Shift. Let $C = (C^i, d^i)$ be a complex, $n \in \mathbb{Z}$ an integer, then the shifted complex, $C[n]$, is defined by $(C[n])^i = C^{i+n}$ with differential $(-1)^n d$. Note that for positive n, $[n]$ has the effect of shifting the complex *down*. Also, $[n]$ is a functor on chain complexes and chain maps if we define $(f[n])^i = f^{i+n}$, in fact an autoequivalence, and induces one likewise on $K(\mathcal{A})$ and $D(\mathcal{A})$.

Cone. This construction takes a chain map $f : C \to D$ and produces a complex $\mathrm{Cone}(f)$, the cone of f. It is defined by

$$(\mathrm{Cone}(f))^i = C^{i+1} \oplus D^i, \qquad d_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_C & 0 \\ f & d_D \end{pmatrix} \tag{109}$$

which is a "twisted" version of $C[1] \oplus D$. The cone simultaneously replaces the kernel and cokernel, which do not exist in $K(\mathcal{A})$ or $D(\mathcal{A})$. The categorical properties of $\mathrm{Cone}(f)$ will lead to the axioms of a triangulated category.

Cylinder. Finally, the cylinder, $\mathrm{Cyl}(f)$, of a chain map $f : C \to D$ is defined by

$$\mathrm{Cyl}(f)^i = C^{i+1} \oplus D^i \oplus C^i, \qquad d_{\mathrm{Cyl}(f)} = \begin{pmatrix} -d_C & 0 & 0 \\ f & d_D & 0 \\ -1 & 0 & d_C \end{pmatrix} \tag{110}$$

and turns out to be canonically isomorphic to D in $K(\mathcal{A})$ via the inclusion and projection maps. By construction we also have sequences of complexes and chain maps

$$0 \to D \to \mathrm{Cone}(f) \to C[1] \to 0 \tag{111}$$

$$0 \to C \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to 0 \tag{112}$$

which are exact in each degree.

The cohomology of a complex (X, d) is a sequence of objects in $\mathcal{A}$ defined by

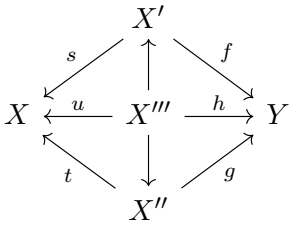
$$H^i(X, d) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}). \tag{113}$$

Any chain map induces morphisms on cohomology groups, and homotopic chain maps induce the same morphism. However, it can happen that a chain map which induces an isomorphism in cohomology is not invertible in $K(\mathcal{A})$. But if we wish to identify an object $X \in \mathcal{A}$ with its projective resolution, we need to formally invert such chain maps, called *quasi-isomorphisms*, giving the derived category $D(\mathcal{A})$.

There is a general construction for inverting a class of morphisms in a category, called Gabriel-Zisman localization, which simplifies in the case of $D(\mathcal{A})$. The objects of $D(\mathcal{A})$ are the same as those of $K(\mathcal{A})$. A morphism in $\mathrm{Hom}_{D(\mathcal{A})}$ is an equivalence class of diagrams

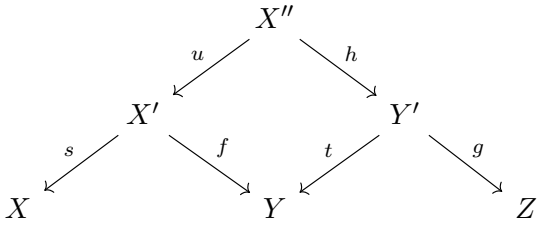
$$X \xleftarrow{s} X' \xrightarrow{f} Y \tag{114}$$

where s is a quasi-isomorphism and f is arbitrary. Two such diagrams are identified if there is a third which dominates them both, i.e. if there is a commutative diagram



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Composition is defined by finding a commutative diagram



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with u a quasi-isomorphism, then taking $X \xleftarrow{su} X'' \xrightarrow{gh} Z$. For this we need the following lemma.

Lemma 4. *Given a diagram $X \xrightarrow{f} Y \xleftarrow{s} Y'$ in $K(\mathcal{A})$ with s a quasi-isomorphism, there is a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \tag{117}$$

with t a quasi-isomorphism.

Proof. We take

$$X' = \mathrm{Cone}(X \to \mathrm{Cone}(s))[-1] \tag{118}$$

which is explicitly

$$(X')^i = X^i \oplus (Y')^i \oplus Y^{i-1}, \qquad d_{X'} = \begin{pmatrix} d_X & 0 & 0 \\ 0 & d_{Y'} & 0 \\ -f & -s & -d_Y \end{pmatrix} \tag{119}$$

and the chain maps $-t$ and g are just the projections to X and Y' respectively. The chain homotopy H given by the degree -1 projection from X' to Y then satisfies

$$ft - sg = d_Y H + H d_{X'} \tag{120}$$

so that the required square commutes in $K(\mathcal{A})$.

It remains to show that t is a quasi-isomorphism. First, from the exact sequence of complexes

$$0 \to Y \to \mathrm{Cone}(s) \to Y'[1] \to 0 \tag{121}$$

and the fact that s is a quasi-isomorphisms, it follows, using the long exact sequence, that the cohomology of $\mathrm{Cone}(s)$ is vanishes. But then the long exact sequence for the sequence of complexes

$$0 \to \mathrm{Cone}(s)[-1] \to X' \xrightarrow{t} X \to 0 \tag{122}$$

shows that t is a quasi-isomorphism. □

This completes the definition of $D(\mathcal{A})$ *as a category*. We will see that there is more structure hidden in $D(\mathcal{A})$ coming from the cone construction, making it a *triangulated category*. In fact, already $K(\mathcal{A})$ has this structure. We note also that there are variants of $K(\mathcal{A})$ and $D(\mathcal{A})$ where we impose boundedness condition on the complexes. The notations $K^+(\mathcal{A})/K^-(\mathcal{A})/K^b()$ are used for the subcategories of chain complexes which vanish in almost all negative/positive/both negative and positive degrees, respectively. Similarly for $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$.

Exercise: Show that $D(\mathcal{A})$ is additive by defining a composition of diagrams of the form 114. Hint: Find a "common denominator" using the previous lemma.

Proposition 9. *Let $\mathcal{A}$ be an abelian category, then the functor $\mathcal{A} \to D(\mathcal{A})$ sending an object to the complex concentrated in degree 0 is fully faithful, so we can regard $\mathcal{A}$ as a subcategory of $\mathcal{D}(A)$. Moreover, for $X, Y \in \mathcal{A}$ and $n \geq 0$ we have*

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y[n]) = \mathrm{Ext}^n(X, Y) \tag{123}$$

while $\mathrm{Hom}_{D(\mathcal{A})}(X, Y[n]) = 0$ *for $n < 0$. To be precise, the first statement makes sense if $\mathcal{A}$ has enough projective or injectives. If not, then we can use this as a definition of* Ext^n.

A proof can be found in any textbook on derived categories, e.g. Gelfand–Manin, *Methods of Homological Algebra*.

A **triangle** in $K(\mathcal{A})$ or $D(\mathcal{A})$ is a sequence of objects and morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]. \tag{124}$$

An morphism of triangles is given by a triple of maps (u, v, w) making a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{125}$$

In particular, it makes sense to talk about isomorphic triangles. An **exact triangle** is a triangle which is isomorphic to one of the form

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to Z[1] \tag{126}$$

Note that since $\mathrm{Cyl}(f)$ is homotopy equivalent to Y we could also have taken

$$X \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to Z[1] \tag{127}$$

where the first three terms form an exact sequence of complexes. More generally, one can show that any short exact sequence of complexes $0 \to C \to D \to E \to 0$ (i.e. giving an exact sequence in $\mathcal{A}$ in every degree) gives an exact triangle $C \to D \to E \to C[1]$, where the last map is the connecting map δ.

There is a somewhat subtle point above, in that to define $\mathrm{Cone}(f)$ we need an actual chain map, not just a homotopy class. This is not a problem for the moment, since homotopic maps give isomorphic triangles, in particular the isomorphism class of $\mathrm{Cone}(f)$ only depends on the morphism in $K(\mathcal{A})$ or $D(\mathcal{A})$. However, the isomorphism of cones depends on the homotopy between the chain maps. A consequence of this is that **Cone is not a functor on $K(\mathcal{A})$ or $D(\mathcal{A})$!**. This is an issue for certain constructions, and the main reason to work with enhanced, and also more involved, versions of $D(\mathcal{A})$.

7.1 Triangulated categories

As was noted above, the categories $K(\mathcal{A})$ and $D(\mathcal{A})$ are in fact triangulated categories, the axioms of which reflect the key properties of the cone construction for complexes.

A **triangulated category** is an additive category $\mathcal{C}$ with

1. An autoequivalence $T : \mathcal{C} \to \mathcal{C}$, the **shift** functor. One usually writes $X[n] := T^n(X)$ and $X[f] := T^n(f)$.
2. A collection of triangles $X \to Y \to Z \to X[1]$, the **exact triangles**.

These have to satisfy the following axioms.

1. Any triangle of the form $X \xrightarrow{1_X} X \to 0 \to X[1]$ is exact, triangles isomorphic to exact triangles are exact, and any morphism $X \xrightarrow{f} Y$ can be completed to a triangle $X \xrightarrow{f} Y \to Z \to X[1]$. We will see that Z is determined by f up to (non-canonical) isomorphism, and we can think of it as the cone of f.
2. The triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is exact if and only if the triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is exact. This means we can "rotate" a triangle, and that in the infinite sequence

   $$\ldots \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to Z[1] \to X[2] \to \ldots \tag{128}$$

   any three consecutive morphisms form an exact triangle.
3. For a solid arrow diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{129}$$

where the left square commutes, the dotted arrow, w, exists such that the other two squares commute.

4. This axioms roughly states that given composable morphisms u, v, then the cones of u, v, vu fit into an exact triangle. More precisely, given exact triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} \tag{130}$$

$$Y \xrightarrow{v} Z \xrightarrow{l} X' \xrightarrow{i} \tag{131}$$

$$X \xrightarrow{vu} Z \xrightarrow{m} Y' \xrightarrow{n} \tag{132}$$

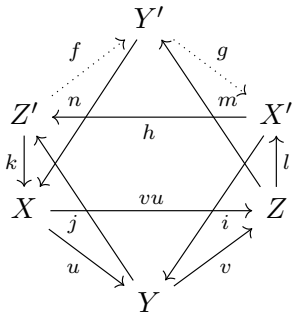
there exists an exact triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} \tag{133}$$

such that

$$l = gm, \qquad k = nf, \qquad h = j[1]i, \qquad ig = u[1]n, \qquad fj = mv. \tag{134}$$

This axiom is called the "octahedral axiom" since there is a suggestive diagram



(135)

in which four of the faces are exact triangles, the other four are commutative triangles, and the two upright squares inside the octahedron commute.

Remark. From the modern point of view (see e.g. Lurie's book Higher Algebra), triangulated categories are an approximation to *stable* $(\infty, 1)$*-categories*. More precisely, if $\mathcal{C}$ is a stable $(\infty, 1)$-category, then its homotopy category $H(\mathcal{C})$, which has the same objects and morphisms

$$\mathrm{Hom}_{H(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \tag{136}$$

is a triangulated category, with exact triangles coming from fibration (equivalently cofibration) sequences. A linear version of $(\infty, 1)$-categories are dg-categories, which we will discuss later.

Lecture 8: More on triangulated categories

The slogan is that *behind every long exact sequence in (co)homology there is an exact triangle.* The following proposition illustrates this in the case of Ext. We use the notation $\mathrm{Ext}^i(X,Y) = \mathrm{Hom}(X, Y[i])$ for any $i \in \mathbb{Z}$ and pair of objects X, Y in a triangulated category.

Proposition 10. *Let $\mathcal{C}$ be a triangulated category, $A \to B \to C \to A[1]$ an exact triangle in $\mathcal{C}$, and $X \in \mathcal{C}$ any object. Then the sequences*

$$\ldots \to \mathrm{Ext}^{i-1}(X,C) \to \mathrm{Ext}^i(X,A) \to \mathrm{Ext}^i(X,B) \to \mathrm{Ext}^i(X,C) \to \mathrm{Ext}^{i+1}(X,A) \to \ldots \tag{137}$$

$$\ldots \to \mathrm{Ext}^{i-1}(A,X) \to \mathrm{Ext}^i(C,X) \to \mathrm{Ext}^i(B,X) \to \mathrm{Ext}^i(A,X) \to \mathrm{Ext}^{i+1}(C,X) \to \ldots \tag{138}$$

are exact.

Proof. By duality, we only need to prove exactness of the first sequence. In fact it suffices to show exactness at $\mathrm{Hom}(X,B)$ by rotation invariance of exactness of triangles.

First we check that for any exact triangle $A \xrightarrow{u} B \xrightarrow{v} C \to A[1]$ we have $vu = 0$. This follows from the third axiom applied to the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{1_A} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle u} & & \vdots & & \downarrow \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \longrightarrow & A[1] \end{array} \tag{139}$$

which shows that vu factors through 0. Thus also $v_*u_* : \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,C)$ is zero.

Suppose that $f \in \mathrm{Hom}(X,B)$ with $v_*(f) = vf = 0$. We want to show that there is a $g \in \mathrm{Hom}(X,A)$ with $u_*(g) = ug = f$, which we get by applying the third axiom to the diagram

$$\begin{array}{ccccccc} X[-1] & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{1_X} & X \\ \downarrow{\scriptstyle f[-1]} & & \downarrow & & \vdots{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B[-1] & \xrightarrow{v[-1]} & C[-1] & \longrightarrow & A & \xrightarrow{u} & B \end{array} \tag{140}$$

□

Exercise: Show that for any pair of objects A, B the triangle $A \to A \oplus B \to B \xrightarrow{0} A[1]$ is exact.

An abelian category, $\mathcal{A}$, has (homological) dimension n if $\mathrm{Ext}^i(X,Y) \neq 0$ for $i > n$, but $\mathrm{Ext}^n(X,Y)$ is non-trivial for some pair of objects. It turns out that when $\mathcal{A}$ has dimension ≤ 1, then $D(\mathcal{A})$ is not too different from $\mathcal{A}$ in a sense.

Proposition 11. *Let $\mathcal{A}$ be an abelian category of dimension ≤ 1, then every $X \in D(\mathcal{A})$ is isomorphic to its cohomology $H(X)$, i.e. a complex with vanishing differential.*

Proof. By vanishing of $\mathrm{Ext}^2(H^nX, \mathrm{Ker}(d^{n-1}))$ we get a diagram with exact rows

$$\begin{array}{ccccccc} X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^nX & \longrightarrow & X^{n-1}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Im}(d^{n-1}) & \longrightarrow & \mathrm{Ker}(d^n) & \longrightarrow & H^nX & \longrightarrow & \mathrm{Im}(d^{n-1})[1] \end{array} \tag{141}$$

By the long exact sequence in cohomology, $H^i(E^n) = 0$ for $i \neq 0$, so $E^n \in \mathcal{A}$ and the rows in the diagram above come from short exact sequences in $\mathcal{A}$.

Putting everything together, we get a diagram

$$\begin{array}{ccccccc} \ldots & \longrightarrow & H^{n-1}X & \xrightarrow{0} & H^nX & \longrightarrow & \ldots \\ & & \uparrow & & \uparrow & & \\ \ldots & \longrightarrow & X^{n-1} \oplus E^{n-1} & \longrightarrow & X^n \oplus E^n & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \longrightarrow & \ldots \end{array} \tag{142}$$

where the differential of the complex in the second row is the sum of two-step complexes $X^{n-1} \to E^n$ and both vertical chain maps are quasi-isomorphisms. □

In particular, under the assumptions of the proposition, every indecomposable object in $D^b(\mathcal{A})$ is a shift of an object in $\mathcal{A}$.

A large class of abelian categories of dimension ≤ 1 are categories of representations of quivers (without relations!). When Q is a quiver with path algebra $A = kQ$, then we can view A as a bimodule over A (called the *diagonal bimodule*), i.e. as a module over $A^{\mathrm{op}} \otimes A$. As such, it has a projective resolution

$$0 \to \bigoplus_{\alpha \in Q_1} Ae_{\mathrm{out}(\alpha)} \otimes_k e_{\mathrm{in}(\alpha)}A \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes_k e_iA \xrightarrow{g} A \to 0 \tag{143}$$

where f sends $a \otimes b \in Ae_{\mathrm{out}(\alpha)} \otimes_k e_{\mathrm{in}(\alpha)}A$ to $a\alpha \otimes b - a \otimes \alpha b$ and $g(a \otimes b) = ab$. A proof of exactness of this sequence can be found in Crawley-Boevey's excellent notes *Lectures on Representations of Quivers*. Now if M is any left module over A, we can tensor it with this short exact sequence of bimodules to obtain a two-step projective resolution of M. As a consequence, $\mathrm{Ext}^i(M, N) = 0$ for $i > 1$ and any pair of A-modules, so $\mathrm{Rep}_k(Q)$ has dimension ≤ 1. It has dimension 0 if and only if there are no arrows. This allows us to completely and explicitly understand the structures of some simple triangulated categories.

Let us consider the simplest examples.

1. When $\mathcal{A}$ is the trivial abelian category with only the zero object, then the same is true for $D(\mathcal{A})$.

2. Let $\mathcal{A}$ be the category of finite dimensional vector spaces over a field k, which has dimension 0. Every indecomposable object in $D^b(\mathcal{A})$ is isomorphic to $k[n]$ for some

$n \in \mathbb{Z}$. As an additive category, $D^b(\mathcal{A})$ can be identified with the category of finite-dimensional $\mathbb{Z}$-graded vector spaces and degree 0 maps. Similarly for variants with finiteness conditions dropped.

3. As a small generalization of the previous example we can consider representations of a quiver without arrows, leading to doubly-graded vector spaces.

4. We consider the quiver $Q = \bullet \to \bullet$ and its category $A = \mathrm{Rep}_k(Q)$ of representations. There are three indecomposable representations fitting into a short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0. \tag{144}$$

The indecomposables in $D^b(A)$ fit into a long sequence

$$\ldots \to E_3[-1] \to E_1 \to E_2 \to E_3 \to E_1[1] \to E_2[1] \to E_3[1] \to E_1[2] \to \ldots \tag{145}$$

in which any three consecutive morphisms form an exact triangle. In fact this suggest a certain symmetry of $D^b(A)$, an autoequivalence F of $D^b(\mathcal{A})$ sending each object in this sequence to the next one, so that $F^3 = [1]$. So all the objects E_i play the same role in $D^b(A)$, which is in contrast to $\mathcal{A}$, where E_1 and E_2 are the projective ones and E_1 and E_3 are simple ones.

Lecture 9: Entropy of functors

9.1 Generators

We fix a triangulated category $\mathcal{C}$. An object $G \in \mathcal{C}$ is a **generator** if the smallest triangulated subcategory containing G, and which is closed under direct summands, is equivalent to $\mathcal{C}$. This is equivalent to: For every object $E \in \mathcal{C}$ there is a tower of exact triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{n-1} \longrightarrow E_n = E \oplus E' \\ & \searrow & \downarrow & \searrow & \downarrow & & & \searrow & \downarrow \\ & & G[k_1] & & G[k_2] & & & & G[k_n] \end{array} \tag{146}$$

where dotted arrows indicate degree one morphisms, the triangles $E_{i-1} \to E_i \to G[k_i] \to E_{i-1}[1]$ are exact, $k_i \in \mathbb{Z}$, and $E' \in \mathcal{C}$ is some object.

For example, when $\mathcal{C} = D^b(k)$ is the bounded derived category of complexes over k with finite total dimension, then k itself is a generator. Indeed we can get every other object from k using just direct sums and shift alone, as we have seen. In fact, any non-zero object in $\mathcal{C}$ is a generator. This would not be true if we did not include summands in the definition above.

As another example, when Q is a quiver then $A = kQ$ is a generator of the bounded derived category of finitely generated A modules, as the resolution of the diagonal (143) shows. When Q is acyclic, then finitely generated modules are the same as finite-dimensional modules.

Even though not every triangulated category has a generator, resonable notions of *compactness* and *finite type* exist for (enhanced) triangulated categories, and they both imply existence of a generator. That being said, the existence of a generator is often a very non-trivial property to establish in geometric examples.

9.2 Complexity in triangulated categories

Our next task is to define a notion of *size* or *complexity* of an object in a triangulated category $\mathcal{C}$. Let $A, B \in \mathcal{C}$ be any pair of objects. Suppose B is contained in the subcategory generated by A, then by definition there is a tower

$$\begin{array}{ccccccccc} 0 = B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \dots & \longrightarrow & B_n = B \oplus B' \\ & \searrow & \downarrow & \searrow & \downarrow & & & \searrow & \downarrow \\ & & A[k_1] & & A[k_2] & & & & A[k_n] \end{array} \tag{147}$$

and we can choose it so that n, i.e. the number of copies of A needed to build B, is minimal. Define the **complexity of B relative to A**, $\delta(A, B)$, to be this minimal n, and $\delta(A, B) = +\infty$ if there is no such tower, i.e. if B is not contained in the subcategory generated by A.

If we want to take into account the shifts k_i also, we can assign to each tower as above, not necessarily minimal, the function $\mathbb{R} \to \mathbb{R}_{>}0$ given by

$$\sum_{i=1}^{n} \exp(k_i t) \tag{148}$$

and let $\delta_t(A, B) : \mathbb{R} \to [0, \infty]$ be infimum of all these functions. Clearly, $\delta_0(A, B) = \delta(A, B)$. We collect some fundamental properties of $\delta_t(A, B)$ in the following proposition.

Proposition 12. *Let $\mathcal{T}$ be a triangulated category, then*

- $\delta_t(A, C) \leq \delta_t(A, B)\delta_t(B, C)$ *for all* $A, B, C \in \mathcal{T}$.
- $\delta_t(A, C) \leq \delta_t(A, B) + \delta_t(A, D)$ *for all* $A, B, C, D \in \mathcal{T}$ *such that there is an exact triangle* $B \to C \to D \to B[1]$.
- $\delta_t(F(A), F(B)) \leq \delta_t(A, B)$ *for any* $A, B \in \mathcal{C}$ *and exact functor* $F : \mathcal{T} \to \mathcal{T}'$.
- *If* $\mathcal{T} = D^b(k)$, *the category of finite-dimensional complexes over a field* k, *then*

$$\delta_t(k, A) = \sum_n \dim(H^n A) e^{-nt}. \tag{149}$$

  which is the Poincaré polynomial of A *in* e^{-t}.

Proof. The proof of the first two claims comes down to repeated application of the octahedral axiom. The following notation is helpful. Suppose $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ are full subcategories. We write $\langle \mathcal{A}, \mathcal{B} \rangle$ for the full subcategory of objects E such that there is an exact triangle

$$A \to E \oplus E' \to B \to \tag{150}$$

for some $A \in \mathcal{A}$, $B \in \mathcal{B}$. We claim that $\langle \mathcal{A}, \langle \mathcal{B}, \mathcal{C} \rangle \rangle = \langle \langle \mathcal{A}, \mathcal{B} \rangle, \mathcal{C} \rangle$. To see this, suppose $E \in \langle \mathcal{A}, \langle \mathcal{B}, \mathcal{C} \rangle \rangle$, then there are exact triangles

$$A \to E \oplus E' \to F \to, \qquad B \to F \oplus F' \to C \to \tag{151}$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$. The composite map $E \oplus E' \to F \to F \oplus F' \to C$ fits into a triangle

$$G \to E \oplus E' \to C \to \tag{152}$$

and the octahedral axiom implies the existence of a triangle

$$A \to G \oplus F' \to B \to \tag{153}$$

showing that $E \in \langle \langle \mathcal{A}, \mathcal{B} \rangle, \mathcal{C} \rangle$. The reverse inclusion is similar. As a consequence it makes sense to use the notation $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$, and with it

$$\delta_t(A, B) = \inf \left\{ \sum_i e^{k_i t} \mid B \in \langle A[k_1], \ldots, A[k_n] \rangle \right\}. \tag{154}$$

With this, the first and second statements are easily verified.

The third statement is clear, since exact functors preserve triangles and direct sums. For the last statement we use the fact that any $A \in D^b(k)$ is isomorphic to its cohomology $H(A)$ which is a direct sum of $\dim H(A)$ copies of shifts of k, showing the inequality "$\leq$". On the other hand, $A \in \langle k[n_1], \ldots, k[n_m] \rangle$ if and only if there is a complex (C, d) so that $C = \bigoplus k[-n_i]$ as a graded vector space, and $H(A)$ is a summand in (C, d). Therefore, $\dim C^i \geq \dim H^i(A)$ and

$$\sum_i e^{n_i t} \geq \sum \dim H^n(A) e^{-nt}. \tag{155}$$

□

9.3 Entropy of exact endofunctors

Assume that $\mathcal{T}$ has a generator G, so $\delta_t(G, E) < \infty$ for all $E \in \mathcal{T}$. Define the dynamical **entropy**, $h_t : \mathbb{R} \to [-\infty, \infty)$, of an exact endofunctor $F : \mathcal{T} \to \mathcal{T}$ as

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n(G)). \tag{156}$$

First, the limit exists since

$$\delta_t(G, F^{m+n}G) \leq \delta_t(G, F^m G)\delta_t(F^m G, F^{m+n}G) \leq \delta_t(G, F^m G)\delta_t(G, F^n G) \tag{157}$$

and hence $\log \delta_t(G, F^n(G))$ is subadditive in n. Second, if G, G' are generators, then

$$\delta_t(G', F^n G') \leq C_t \delta_t(G, F^n G) \tag{158}$$

with $C_t = \delta_t(G', G)\delta_t(G, G')$, showing that $h_t(F)$ is indeed independent of G, as the notation suggests. Note that it follows immediately from the definition that

$$h_t(F^n) = n h_t(F). \tag{159}$$

The first example is the shift functor. We need some assumptions on the category to guarantee $h_t([n]) = nt$.

Proposition 13. *Suppose $\mathcal{C}$ is a triangulated category such that if $X, Y \in \mathcal{C}$ is any pair of objects then $\mathrm{Ext}^i(X, Y) = 0$ for all but finitely many i. Then for all integers n and any $0 \neq X \in \mathcal{C}$ we have $\delta_t(X, X[n]) = e^{nt}$, hence $h_t([n]) = nt$ if $\mathcal{C}$ has a non-zero generator.*

Proof. Fix $t \in \mathbb{R}$. The claim $\delta_t(X, X[n]) = e^{nt}$ is clear for $t = 0$. We assume $t > 0$, the case of $t < 0$ being similar. Since $X[n] \in \langle X[n] \rangle$, we have $\delta_t(X, X[n]) \leq e^{nt}$. Suppose $X[n] \in \langle X[n_1], \ldots, X[n_p] \rangle$ for some $n_i < n$. Then there are triangles $X_{i-1} \to X_i \to X[n_i] \to$, $1 \leq i \leq p$ with $X_0 = 0$ and $X_p = X[n] \oplus X'$ By assumption there is a minimal integer q so that $\mathrm{Ext}^{q+n}(X, X) \neq 0$. We claim that $\mathrm{Ext}^q(X, X_i) = 0$ which follows by induction from the exactness of

$$\mathrm{Ext}^q(X, X_{i-1}) \to \mathrm{Ext}^q(X, X_i) \to \mathrm{Ext}^{q+n_i}(X, X) \tag{160}$$

thus also $\mathrm{Ext}^q(X, X[n]) \subset \mathrm{Ext}^q(X, X_p) = 0$, a contradiction. □

It will be convenient to write functors in terms of the tensor product of complexes, which is defined as follows. If R is a commutative ring and $X = (X^i, d_X^i)$ and $Y = (Y^i, d_Y^i)$ are complexes of R-modules, then define a complex of R-modules $X \otimes_R Y$ with

$$(X \otimes_R Y)^n = \bigoplus_{i+j=n} X^i \otimes_R Y^j \tag{161}$$

with differential

$$d_{X\otimes_R Y}(x \otimes y) = (d_X x) \otimes y + (-1)^i x \otimes (d_Y y), \qquad x \otimes y \in X^i \otimes_R Y^j \tag{162}$$

which just expresses the graded Leibniz rule. There are some variants of this, for example if A is a k-algebra, not necessarily commutative, and X a complex of right A-modules, and Y is a complex of left A-modules, then $X \otimes_A Y$ makes sense as a complex of k-modules.

Let us look at some examples. First, consider $\mathcal{C} = D^b(k)$, the category of finite-dimensional complexes over k. Any exact functor $F : \mathcal{C} \to \mathcal{C}$ is determined by the object $F(k) = K$, since it sends a complex C to $K \otimes C$. Also, we may assume that $K = H(K)$ is a complex with zero differential. It has a Poincaré (Laurent) polynomial

$$P_K(z) = \sum_{n\in\mathbb{Z}} \dim K^n z^n. \tag{163}$$

We compute

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(k, K^{\otimes n}) \tag{164}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log(P_K(e^{-t}))^n \tag{165}$$

$$= \log P_K(e^{-t}). \tag{166}$$

Thus, in coordinates $(x, y) = (e^t, e^h) \in \mathbb{R}^2_{>0}$ entropy is given by the graph of a Laurent polynomial with integer coefficients. In particular $h_0(F) = \log \dim K$.

Generalizing the previous example, we can consider the category $\mathcal{C} = D^b(k^m)$ where $A = k^m$ is the semisimple algebra which is a product of m copies of k. Objects in $\mathcal{C}$ are just m-tuples of complexes. An object $K \in D^b(A \otimes_k A)$ is given by vector spaces

$$K_{ij}^n, \qquad 1 \le i, j \le m, n \in \mathbb{Z}, \qquad \dim\left(\bigoplus_{i,j,n} K_{ij}^n\right) < \infty \tag{167}$$

together with a differential, which we may assume to be zero. Such K defines a functor $F_K : \mathcal{C} \to \mathcal{C}$ by convolution, $F_K(X) = K \otimes_A X$. In fact, every endofunctor of $\mathcal{C}$ is of this form, which is a very special case of Morita theory, identifying functors with bimodules. We associate with K the $m \times m$ matrix of Laurent polynomials, $P_K(z)$, with

$$(P_K(z))_{ij} = \sum_n \dim K_{ij}^n z^n \tag{168}$$

which satisfies $P_{K\otimes L} = P_K P_L$. Taking A itself as a generator, the complexity of an object $B = (B_1, \ldots, B_m) \in \mathcal{C}$ is

$$\delta_t(A, B) = \sum_n e^{-nt} \max\{\dim H^n(B_i) \mid 1 \leq i \leq m\} \tag{169}$$

so we get

$$\delta_t(A, F_K^l A) = \|(P_K(e^{-t}))^l\| \tag{170}$$

where we use the matrix norm $\|A\| = \max_i \sum_j |a_{ij}|$. By Gelfand's formula,

$$h_t(F_K) = \lim_{l\to\infty} \frac{1}{l} \log \|(P_K(e^{-t}))^l\| = \log \rho(P_K(e^{-t})) \tag{171}$$

where $\rho(A)$ the spectral radius of a matrix A. Here we have used the fact that all entries of $P_K(e^{-t})$ are non-negative. For any non-negative matrix, M, the spectral radius $\rho(M)$ is in fact an eigenvalue of M, not just the absolute value of an eigenvalue. This is a consequence of Perron–Frobenius theory, which deals with matrices with positive or non-negative entries. Hence, if we consider the spectral curve

$$C = \{(x, y) \in \mathbb{R}_{>0} \times \mathbb{R} \mid \det(P_K(x) - y) = 0\} \tag{172}$$

then the graph of $\rho(P_K(x)) = \log h_{\log x}(F_K)$ is the top-most branch of C, i.e.

$$\rho(P_K(x)) = \max\{y \mid (x, y) \in C\}. \tag{173}$$

In particular, $\exp(h_0(F_K))$ is an algebraic integer of degree at most m.

Lecture 10: Stability — The big picture

10.1 Entropy of functors — Outlook

In order to get general results on the entropy of a functor $F : \mathcal{C} \to \mathcal{C}$, we need to make some assumptions on the category $\mathcal{C}$. The first is *compactness*. A triangulated category $\mathcal{C}$ defined over a field k is **compact** if it has a generator $G \in \mathcal{C}$ with $\dim_k \operatorname{Ext}(G, G) < \infty$. Here $\operatorname{Ext}(X, Y) = \bigoplus_i \operatorname{Ext}^i(X, Y)$ is the total Ext-group. These assumptions then imply $\dim \operatorname{Ext}(X, Y) < \infty$ for all $X, Y \in \mathcal{C}$.

The second is (homological) *smoothness*. The definition of this property does not make sense for a triangulated categories, but only for their dg enhancements. The basic issue is that given triangulated categories $\mathcal{C}, \mathcal{D}$, the functor category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ and the tensor product $\mathcal{C} \otimes \mathcal{D}$ have no natural triangulated structure. The solution is to work with dg-categories, which are categories "enriched" over complexes in the sense that each $\operatorname{Hom}(X, Y)$ is a complex and composition satisfies the Leibniz rule. The homotopy category, $H(\mathcal{C})$, of a dg-category, $\mathcal{C}$, is the (linear) category with morphisms

$$\operatorname{Hom}_{H(\mathcal{C})}(X, Y) = H^0(\operatorname{Hom}(X, Y)). \tag{174}$$

Assuming that $\mathcal{C}$ admits arbitrary shifts and cones, the category $H(\mathcal{C})$ will be triangulated. One says that $\mathcal{C}$ is a dg-enhancement of the triangulated category $H(C)$. For dg-categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C} \otimes \mathcal{D}$ is again a dg-category. A dg-category $\mathcal{C}$ is **smooth** if the diagonal bimodule, $\mathcal{C}$, over $\mathcal{C}^{op} \otimes \mathcal{C}$ is in the subcategory generated by the free bimodule $\mathcal{C}^{op} \otimes \mathcal{C}$.

Examples of compact and smooth dg-categories are

1. $D^b(\operatorname{Rep}_k(Q))$, the bounded derived category of representations of an acyclic quiver Q with finitely many vertices and arrows, and
2. $D^b(\operatorname{Coh}(X))$, the bounded derived category of coherent sheaves on a smooth projective variety over a field k.

Suppose C the homotopy category of a compact and smooth dg-category, G is a generator of C, and $F : \mathcal{C} \to \mathcal{C}$ is an exact functor which comes from a functor of dg-categories. The first results provides a simplified way of computing the entropy.

Theorem 10.

$$h_t(F) = \lim_{l \to \infty} \frac{1}{l} \log \sum_{n \in \mathbb{Z}} \dim \operatorname{Ext}^n(G, F^l G) e^{-nt} \tag{175}$$

The point is that there are estimates

$$C_1(t)\delta_t(G, E) \leq \sum_n \dim \operatorname{Ext}^n(G, E) e^{-nt} \leq C_2(t)\delta_t(G, E) \tag{176}$$

for some functions $C_1, C_2 : \mathbb{R} \to [0, \infty)$ which depend only on G, not on E.

To any (small) dg-category, $\mathcal{C}$, one can assign its Hochschild homology $HH_*(C)$. When $\mathcal{C}$ is compact and smooth, $HH_*(\mathcal{C})$ is finite-dimensional over k. A functor $F : \mathcal{C} \to \mathcal{C}$ induces a linear map, $HH_*(F)$, on $HH_*(\mathcal{C})$.

For a $\mathbb{Z}/2$-graded vector space $V = V_+ \oplus V_-$ over $\mathbb{C}$, consider the following generic condition, (NR), on a homogeneous endomorphism $A = A_+ \oplus A_-$: The maps A_+ and A_- do not have the same eigenvalues, with multiplicity, on the circle $\{|z| = \rho(A)\}$. For instance, if $\rho(A_+) \neq \rho(A_-)$, then (NR) is satisfied.

Theorem 11. *Let $\mathcal{C}$ be a compact and smooth dg-category over $\mathbb{C}$, $F : \mathcal{C} \to \mathcal{C}$ a functor so that $HH_*(F)$ satisfies (NR). Then*

$$\log \rho(HH_*(F)) \leq h_0(F). \tag{177}$$

More details and example computations can be found in the paper "Dynamical Systems and Categories" by Dimitrov–Haiden–Katzarkov–Kontsevich. Let us mention one open question raised there. In a wide variety of examples, $h_0(F)$ is the logarithm of an algebraic integer. Is this a general phenomenon, and if so what are sufficient conditions to ensure it?

10.2 Stability and special metrics

Suppose X is a compact Riemann surface. A Hermitian metric, h, on X is choice of Hermitian inner product on each tangent space T_pX which varies smoothly with p. The famous Uniformization Theorem implies that among all these metrics there is a unique one, up to rescaling, which has constant curvature. On the universal covering of X, which is $\mathbb{C}P^1$, $\mathbb{C}$, or $\mathbb{H}$, it is the standard parabolic, flat, or hyperbolic metric, respectively.

A good generalization of the constant curvature condition to higher dimensional Kähler manifolds is the Einstein condition

$$\mathrm{Ric} = \mu g \tag{178}$$

where $\mu \in \mathbb{R}$ is a constant and Ric is the Ricci curvature tensor of g. For manifolds with $c_1 \leq 0$ there is a unique Kähler–Einstein metric, as was shown independently by Aubin and Yau in the case $c_1 < 0$ and in celebrated work by Yau for $c_1 = 0$. The situation is more complicated when $c_1 > 0$. Fairly recently it was shown by Chen–Donaldson–Sun that the existence of Kähler–Einstein metrics is equivalent to an algebro-geometric criterion called K-stability.

Hermitian Yang–Mills. One can also consider metrics not on the manifold itself, i.e. TX, but instead on arbitrary holomorphic bundles $E \to X$. Given a holomorphic bundle E over a complex manifold together with a Hermitian metric h, that is a Hermitian inner product on each fiber, there is a unique connection $D : \Gamma(E) \to \Gamma(T^*X \otimes E)$ such that

1. $d(h(s,t)) = h(Ds,t) + h(s,Dt)$ (compatibility with the metric),
2. $D''s = \overline{\partial}s$, where D'' is the $(0,1)$-part of D (compatibility with holomorphic structure).

In local coordinates, the connection 1-form of D is $A = (\partial h)h^{-1}$, and the curvature of D is $F = dA - A \wedge A$, which is a section of $\mathrm{End}(E) \otimes \Lambda^2 T^*X$.

Choosing a Kähler metric g on X, contraction with g defines a bundle map

$$\mathrm{tr}_g : \mathrm{End}(E) \otimes \Lambda^2 T^*X \to \mathrm{End}(E) \tag{179}$$

with which the Hermitian–Yang–Mills condition on the metric h on E can be written as

$$\mathrm{tr}_g F = \mu 1_E \tag{180}$$

where 1_E is the identity operator on E and $\mu \in \mathbb{R}$ is constant. When E is the holomorphic tangent bundle and $g = h$, this is just the Kähler–Einstein condition from before. The factor μ is determined by the topology of E, and computed as

$$\mu = \mu(E) = \frac{1}{\mathrm{Vol}(X)} \frac{\deg_\omega E}{\mathrm{rank} E} \tag{181}$$

where ω is the Kähler form and

$$\deg_\omega E = \int_X c_1(E) \wedge *\omega. \tag{182}$$

The *slope* $\mu(E)$ makes sense for arbitrary coherent analytic sheaves on X. A holomorphic vector bundle E is said to be **stable** if every coherent subsheaf $0 \neq F \subsetneq E$ satisfies $\mu(F) < \mu(E)$. One says that E is **polystable**, if it is a direct sum of stable bundles of the same slope. Note that these notions depend on the class of ω.

Theorem 12. *A holomorphic vector bundle E admits a Hermitian–Yang–Mills metric iff it is polystable.*

This was proven for curves by Narasimhan–Seshadri, for algebraic surfaces by Donaldson, and in the general case by Uhlenbeck–Yau.

Quiver representations. There is a finite-dimensional analogue of the above story where holomorphic vector bundles are replaced by representations of a quiver over $\mathbb{C}$. Fix a quiver Q and choose a complex number $Z_i \in \mathbb{C}$ with $\mathrm{Im}(Z_i) > 0$ for each vertex i. Then for any representation E of Q there is a complex number

$$Z(E) = \sum_i Z_i \dim E_i \tag{183}$$

and the **phase** of a representation $0 \neq E$ is given by $\phi(E) = \arg(Z(E)) \in (0, \pi)$. Then, as above, an object $0 \neq E$ is **stable** if every $0 \neq F \subsetneq E$ satisfies $\phi(F) < \phi(E)$. A polystable representation is by definition a direct sum of stable ones of the same phase. Note that all of this makes sense in an arbitrary abelian category $\mathcal{A}$, where $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is additive. Also we have not yet used the assumption that the representations are defined over $\mathbb{C}$.

The following result is due to A. King.

Theorem 13. *A representation $E = (V_i, T_\alpha)$ is polystable iff there are Hermitian metrics h_i on V_i so that*

$$\arg\left(Z + \sum [T_\alpha^*, T_\alpha]\right) = \phi 1_V \tag{184}$$

where we consider T_α as endomorphisms of $V = \bigoplus_i V_i$, and $Z = \sum z_i 1_{V_i}$.

The point of considering stable representations, as opposed to all representations, is that their moduli form algebraic varieties instead of stacks. This is a special case of Geometric Invariant Theory, which is used to construct moduli spaces of algebro-geometric nature by throwing out certain *unstable* points. Furthermore, an arbitrary representation is obtained from stable ones by extension in an essentially unique way. First, call a representation E **semistable** if $\phi(F) \leq \phi(E)$ for all subrepresentations $0 \neq F \subset E$. Equivalently, E is an extension of stable representations of the same phase. One shows that for any representation E there is a filtration $0 = E_0 \subset \ldots \subset E_n = E$ with $A = E_i/E_{i-1}$ semistable such that $\phi(A_1) > \ldots > \phi(A_n)$. This filtration is called the **Harder–Narasimhan filtration** in analogy with the case vector bundles over a smooth projective curve, where the same result holds.

Lecture 11: Stability — The big picture and some examples

Let's see how this works in some examples. First, we take Q to be the A_2-type quiver $\bullet \to \bullet$. Recall from before that the indecomposable representations are

$$S_1 : \mathbb{C} \to 0, \qquad E : \mathbb{C} \to \mathbb{C}, \qquad S_2 : 0 \to \mathbb{C} \tag{185}$$

which fit into an exact sequence

$$0 \to S_2 \to E \to S_1 \to 0. \tag{186}$$

We first note that, very generally, any stable representation is indecomposable, and any simple representation is stable. Thus, S_1, S_2, which are simple in $\mathrm{Rep}_{\mathbb{C}}(Q)$, are stable for any choice of Z. The stability of E, on the other hand, depends on the order of $\phi(S_1)$ and $\phi(S_2)$.

- $\underline{\phi(S_1) > \phi(S_2)}$: The only non-trivial subobject of E is S_2, and $\phi(S_2) < \phi(E)$, so E is stable. A metric on E is determined by $h_1, h_2 \in (0, \infty)$ which specify the inner product on the first and second copy of $\mathbb{C}$, respectively. The special metric is only determined up to scalar multiple. If $\lambda = h_2/h_1$, then the condition is that $\arg(Z(S_1) + \lambda) = \arg(Z(S_2) - \lambda)$ which is solved by

  $$\lambda = \frac{\mathrm{Im}\langle z_1, z_2 \rangle}{\mathrm{Im}(z_1 + z_2)} \tag{187}$$

  where $Z(S_1) = z_1$, $Z(S_2) = z_2$.

- $\underline{\phi(S_1) = \phi(S_2)}$: In this case, E is an extension of stable objects of the same phase, thus semistable, but not stable.

- $\underline{\phi(S_1) < \phi(S_2)}$: We have $\phi(S_2) > \phi(E)$, so E has a subobject of bigger slope and is not stable or semistable. The Harder-Narasimhan filtration of E is just $0 \subset S_2 \subset E$.

We see that the space of all Z, which is $\{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Im}(z_i) > 0\}$, is divided into two chambers separated a "wall" on which $\arg(z_1) = \arg(z_2)$.

The second example we look at is the Kronecker quiver $Q = \bullet \overrightarrow{\to} \bullet$. Recall that for each dimension vector $(n, n+1)$ or $(n+1, n)$, $n \geq 0$ there is a single indecomposable representation, while for dimension vectors of the form (n, n), $n > 0$ there is a $\mathbb{P}^1$ of indecomposable representations. To specify Z we choose z_1, z_2 in the upper half-plane. Let's again look at the various cases.

1. $\underline{\arg(z_1) < \arg(z_2)}$: The only stable objects are the simple ones. They have dimension vectors $(1, 0)$ and $(0, 1)$.

2. $\underline{\arg(z_1) = \arg(z_2)}$: All representations are semistable with the same phase. The stable representations are again the simple ones.

3. $\underline{\arg(z_1) > \arg(z_2)}$: The stable representations are the indecomposable representations with dimension vectors $(n, n+1)$, $(n+1, n)$, or $(1,1)$. Representations with dimension vector $(1,1)$ are of the form

$$\mathbb{C} \underset{b}{\overset{a}{\rightrightarrows}} \mathbb{C} \tag{188}$$

with $(a,b) \in \mathbb{C}^2$. However, two such representations are isomorphic if they are related by a "gauge transformation" $(a,b) \mapsto (\lambda a, \mu^{-1} b)$, $\lambda, \mu \in \mathbb{C}^*$. Thus, the stack of such representations is $\mathbb{C}^2 // (\mathbb{C}^*)^2$. To get a good quotient, we need to throw out the unstable representation corresponding to the origin $a = b = 0$, giving $(\mathbb{C}^2 \setminus \{0\})/(\mathbb{C}^*)^2 = \mathbb{C}P^1$ as moduli of stable representations.

Special Lagrangians. String theory predicts a "Mirror symmetry dual" to the (deformed) Hermitian–Yang–Mills equation, which is the special Lagrangian condition. The mathematical picture, developed by Thomas, Yau, Joyce, Kontsevich, and others, is largely conjectural for the moment.

We consider a Calabi–Yau n-fold M with Kähler form ω and holomorphic $(n,0)$-form Ω, so

$$\omega^n / n! = (-1)^{n(n-1)/2} (i/2)^n \Omega \wedge \overline{\Omega}. \tag{189}$$

In particular, (M, ω) is a symplectic manifold, and a smooth submanifold $L \subset M$ is **Lagrangian** if L has real dimension n and $\omega|_L = 0$. If $L \subset M$ is Lagrangian, then $\Omega|_L$ is a complex-valued n-form on L with $|\Omega|_L| = 1$. An oriented Lagrangian is **special** of phase $\phi \in \mathbb{R}$ if

$$\Omega_L = e^{i\phi} d\mathrm{Vol}_L \tag{190}$$

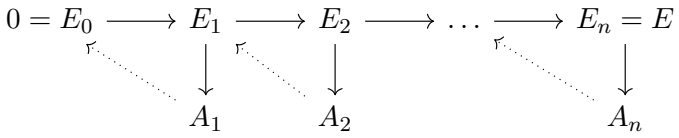
where $d\mathrm{Vol}_L$ is the volume form obtained from the Kähler metric on M, restricted to L. Special Lagrangians are minimal submanifolds in the sense of Riemannian geometry.

Lagrangian submanifolds, together with some extra data and conditions, are objects in the Fukaya category $\mathcal{F}(M)$ of M. Lagrangians which are related by Hamiltonian isotopy give the same object in $\mathcal{F}(M)$. In analogy with the DUY theorem, we would like to find an algebraic condition on an object $E \in \mathcal{F}(M)$ which is equivalent to the existence of a special Lagrangian representative, possibly immersed and with singularities. In trying to mimic the case of vector bundles and quiver representations, a fundamental obstacle immediately becomes apparent: The category $\mathcal{F}(M)$ is not Abelian but *triangulated*, and so the notion of subobject does not make sense.

It turns out that one can still make sense of stability in a triangulated category $\mathcal{C}$ by an axiomatic approach due to Bridgeland. A **stability structure** on $\mathcal{C}$ is given by a full additive subcategory $\mathcal{C}^\phi \subset \mathcal{C}$ of **semistable objects** for each phase $\phi \in \mathbb{R}$ and a **central charge** $Z : K_0(\mathcal{C}) \to \mathbb{C}$, factoring through a fixed map $\mathrm{cl} : K_0(\mathcal{C}) \to \Gamma$ where Γ is a finite rank abelian group. The data is required to satisfy the following axioms.

1. $\mathcal{C}^\phi[1] = \mathcal{C}^{\phi+\pi}$
2. If $\phi_1 > \phi_2$, $E \in \mathcal{C}^{\phi_1}$, $F \in \mathcal{C}^{\phi_2}$, then $\mathrm{Hom}(E, F) = 0$.

3. For every $E \in \mathcal{C}$ there is a tower of triangles



(191)

with $0 \neq A_k \in \mathcal{C}^{\phi_k}$ for some $\phi_1 > \ldots > \phi_n$.

4. For $0 \neq A \in \mathcal{C}^{\phi}$ one has $Z(E) \in \mathbb{R}_{>0} e^{i\phi}$.

5. There is a norm on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and a $C > 0$ so that

$$\|\gamma\| \leq C|Z(\gamma)| \tag{192}$$

for $\gamma \in \Gamma$ the class of a semistable object.

You should compare these axioms to the properties of semistable representations or vector bundles over a curve.

Conjecturally, if one takes as class of semistable objects the special Lagrangians and as central charge the integration over Ω, then this data should satisfy the axioms above. The heuristic idea is that a non-special Lagrangian will converge towards a union of special Lagrangians under mean-curvature flow with surgery. From what is known, the details seem to be quite intricate and one needs to take into account "instanton corrections", i.e. pseudoholomorphic disks with boundary on L, to prevent blowup of the long-term solution. For details see Joyce's *Conjectures on Bridgeland stability for Fukaya categories of Calabi–Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow.*

In the simplest case $\dim_{\mathbb{C}} M = 1$ the special Lagrangians are essentially just geodesics with respect to a flat metric. There, the conjecture has been verified by Haiden–Katzarkov–Kontsevich also for non-compact M and geometries with potential $f : M \to \mathbb{C}$.

11.1 Some ideas from GIT

Moduli spaces in algebraic geometry are often obtained as quotients X/G, where X is a variety and G is an algebraic group. The quotient exists as a set, even topological space, but it may fail to be Hausdorff, and so cannot be an algebraic variety.

As an example, take $Y = \mathbb{C}^{n+1}/\mathbb{C}^*$, with $\mathbb{C}^*$ acting by rescaling. As a set, $Y = \mathbb{C}P^n \sqcup \{0\}$, and 0 is in the closure of every non-empty set in the quotient topology. The solution in this case seems to be to first remove the origin $0 \in \mathbb{C}^{n+1}$, then pass to the much better behaved quotient $\mathbb{C}P^n$. This also checks out from the birational point of view: The rational functions on $\mathbb{C}^{n+1}$ which are invariant under the $\mathbb{C}^*$ action are the same as the rational functions on $\mathbb{C}P^n$. The question is how to do this systematically, i.e. which points to remove in the general case. It turns out that this is not completely canonical, one needs to choose a G-equivariant *polarizing line bundle* L on X.

Lecture 12: Geometric Invariant Theory

Before we dive into the more modern *geometric* reformulation of the theory, let's look at some example from *classical* invariant theory of the 19th century from a geometric perspective.

The elementary symmetric polynomials express the coefficients of a monic polynomial in terms of the roots.

$$\prod_{i=1}^{n}(t-x_i) = t^n - e_1(x_1,\ldots,x_n)t^{n-1} + e_2(x_1,\ldots,x_n)t^{n-2} - \ldots + (-1)^n e_n(x_1,\ldots,x_n) \quad (193)$$

Since the left hand side is invariant under permutation of the x_i's, so are the e_i, which we can write as $e_i \in \mathbb{C}[x_1,\ldots,x_n]^{S_n}$ where

$$R^G = \{a \in R \mid g \cdot a = a \text{ for all } g \in G\} \quad (194)$$

where R is a ring and G is a group acting on R by homomorphisms. The main result on symmetric polynomials is that the ring $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$ is in fact just the polynomial ring generated by the e_i, i.e.

$$\mathbb{C}[x_1,\ldots,x_n]^{S_n} = \mathbb{C}[e_1,\ldots,e_n]. \quad (195)$$

This is a very nice result — a priori $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$ is just some algebra which may not even be finitely generated. More geometrically, we can think of symmetric polynomials as functions on the quotient $\mathbb{C}^n/S_n$ of affine space. The e_i's provide coordinates on the quotient $\mathbb{C}^n/S_n \cong \mathbb{C}^n$, which in this case is not just a set or topological space, but an affine algebraic variety.

The next example we consider is $G = SL(n,\mathbb{C})$ acting on the space of square matrices, $X = \mathrm{Mat}(n\times n,\mathbb{C})$. We expect that $\dim X/G = \dim X - \dim G = 1$. An invariant polynomial on $X = \mathrm{Mat}(n\times n,\mathbb{C})$ is given by the determinant, det, and all other invariant polynomials are polynomials in det, i.e.

$$\mathbb{C}[\mathrm{Mat}(n\times n,\mathbb{C})]^{SL(n,\mathbb{C})} = \mathbb{C}[\det]. \quad (196)$$

In contrast to the previous example though, det does not give a coordinate on X/G, which has points

$$G \cdot \mathrm{diag}(\lambda,1,\ldots,1), \qquad \lambda \in \mathbb{C}^* \quad (197)$$

$$G \cdot \mathrm{diag}(\underbrace{1,\ldots,1}_{k \text{ times}},0,\ldots,0), \qquad 0 \le k < n \quad (198)$$

The map $\det : X/G \to \mathbb{C}$, though generically 1-on-1, sends n different points to 0. These are distinguished by the rank, an invariant function which is not algebraic or even continuous. Also, X/G is not so well behaved as a topological space, since it is not Hausdorff. The moral is that if we want X/G to be an affine variety, we need to identify some of its points. These points are special, in this example, in that their stabilizer subgroup $H \subset G$ has positive dimension.

The general setup of Invariant Theory is that of an algebraic group, G, acting linearly on an affine space V. The object of study is then the ring of G-invariant polynomials on V. Recall that an algebraic group is a group in the category of algebraic varieties, e.g. a matrix group defined by polynomial relations, an Abelian variety, or a finite group. A nice class of algebraic groups are the reductive ones. A linear algebraic group over an algebraically closed field k is **reductive** if its radical, i.e. the smallest closed normal subgroup $H \subset G$ such that G/H is semisimple, contains no unipotent elements except the identity. When $\mathrm{char}(k) = 0$, this is equivalent to the property that all linear representations of G are semisimple, i.e. direct sums of simple representations.

Historical Notes. The following foundational theorem was established by Hilbert around 1890.

Theorem 14. *Let $G = SL(n, \mathbb{C})$ act linearly on $X = \mathbb{C}^n$, then the algebra of invariants, $\mathbb{C}[x_1, \ldots, x_n]^G$, is finitely generated over $\mathbb{C}$.*

The proof uses the existence of a projection $p : \mathbb{C}[X] \to \mathbb{C}[X]^G$ which is a homomorphism of $\mathbb{C}[X]^G$-modules, as well as Hilbert's basis theorem. Problem 14 on Hilbert's famous list from 1900 asks if this is true for more general algebraic groups, to which the answer is no, which was not shown until 1958 by Nagata. In fact, activity in invariant theory was greatly reduced after Hilbert's contributions, and only revived by the influence of Mumford and his Geometric Invariant Theory. The focus of this "geometric" theory is less on the rings of invariants themselves, but more on constructing quotients (e.g. moduli spaces) as (quasi)projective varieties.

GIT. For simplicity, we will restrict to the case when $X = V$ is a vector space and G is a reductive group acting linearly, both define over an algebraically closed field k. We allow $G \to GL(V)$ to have kernel N. Line bundles over V are trivial, but specifying a G-equivariant structure amounts to giving a character $\chi : G \to k^*$. The action of G on the total space of $L = V \times k$ is then

$$g \cdot (v, x) = (g \cdot v, \chi(g)x). \tag{199}$$

An equivariant section of L^n is given by a function f on V such that

$$f(g \cdot v) = \chi(g)^n f(v) \tag{200}$$

for all $g \in G$, $v \in V$. Write $k[V]^{G,\chi^n}$ for the space of such sections. One way of defining the GIT quotient $V/\!\!/_\chi G$ is

$$V/\!\!/_\chi G = \mathrm{Proj}\left(\bigoplus_{n \geq 0} k[V]^{G,\chi^n}\right) \tag{201}$$

which is projective over the ring $k[V]^G$ of G-invariant functions on V. In particular, if $k[V]^G = k$, then $V/\!\!/\chi G$ is a projective variety.

The points of the quotient $V/\!\!/_\chi G$ are described as follows. Define $x \in V$ to be **χ-semistable** if there is a $f \in k[V]^{G,\chi^n}$ for some $n > 0$ with $f(x) \neq 0$, and **χ-stable** if in

addition $\dim G \cdot x = \dim G/N$ and the G-action on $\{x \in V \mid f(x) \neq 0\}$ is closed (i.e. orbits are closed). The geometric description of $V/\!/_\chi G$ is then as the quotient of the set V^{ss} of χ-semistable points by the equivalence relation $x \sim y$ if the orbit closures $\overline{G \cdot x}$ and $\overline{G \cdot y}$ intersect in V^{ss}.

An easier way of determining when a point $x \in V$ is (semi)stable is provided by the *Mumford Numerical Criterion*. To state it, we note that when $\lambda : k^* \to G$ is a one-parameter subgroup and $\chi : G \to k^*$ a character, the composition is $\chi(\lambda(t)) = t^n$ for some integer n, and define $\langle \chi, \lambda \rangle = n$.

Proposition 14. *A point $x \in V$ is χ-semistable iff $\chi(N) = \{1\}$ and every one parameter subgroup λ of G for which $\lim_{t\to 0} \lambda(t) \cdot x$ exists satisfies $\langle \chi, \lambda \rangle \geq 0$. A point $x \in V$ is χ-stable iff in addition all such one-parameter subgroups with $\langle \chi, \lambda \rangle = 0$ are in N.*

We return to the example of $\mathbb{C}^*$ acting on $\mathbb{C}^{m+1}$ by scalar multiplication. First, let $\chi : \mathbb{C}^* \to \mathbb{C}^*$ be the identity character, then $k[V]^{G,\chi^n}$ is homogeneous polynomials of degree n in $m+1$ variables for $n \geq 0$, and zero for $n < 0$. Hence for general $\chi(t) = t^n$ it follows that

$$\mathbb{C}^{m+1}/\!/_\chi \mathbb{C}^* = \begin{cases} \mathbb{C}P^m & n > 0 \\ \{*\} & n = 0 \\ \emptyset & n < 0 \end{cases} \tag{202}$$

A more sophisticated example is the following. We let

$$V = \left\{ \sum_{i+j+k=3} a_{ijk} x^i y^j z^k \mid a_{ijk} \in \mathbb{C} \right\} \tag{203}$$

be the 10-dimensional space of homogeneous degree 3 polynomials on $\mathbb{C}^3$. Each $x \in V \setminus \{0\}$ defines a cubic in $\mathbb{C}P^2$. The general linear group $G = GL(3, \mathbb{C})$ acts by change of coordinates. We take $\chi = \det : G \to \mathbb{C}^*$ as polarization. A one-parameter subgroup of G is of the form

$$\lambda_{abc}(t) = \begin{pmatrix} t^a & & \\ & t^b & \\ & & t^c \end{pmatrix} \tag{204}$$

up to conjugation, and $\langle \chi, \lambda_{abc} \rangle = a + b + c$. Note that $\lambda_{abc}(t)$ acts on a_{ijk} by multiplication with $t^{ai+bj+ck}$, so for given $x \in V$ the limit $\lim_{t\to 0} \lambda_{abc} \cdot x$ exists iff

$$ai + bj + ck < 0 \implies a_{ijk} = 0. \tag{205}$$

A good way to visualize these conditions is to think of (i, j, k) as barycentric coordinates, so the coefficients a_{ijk} are arranged on a triangle

$$\begin{array}{ccccc} & & a_{003} & & \\ & a_{102} & & a_{012} & \\ a_{201} & & a_{111} & & a_{021} \\ a_{300} & & a_{210} & & a_{120} \quad a_{030} \end{array}$$

and λ_{abc} a linear functional $(i,j,k) \mapsto ai + bj + ck$ on it, with value $a+b+c$ at the center of the triangle.

The next step is to translate the condition of (semi-)stability into geometric conditions on the cubic $C \subset P^2$ cut out by $x \in V \setminus \{0\}$. We will just state the result, which is:

- x is stable if C is smooth.
- x is semistable if C has at most ordinary double points. There are three such points, up to the action of G, corresponding to the cubic with 1, 2, or 3 double points respectively.
- x is unstable if C has a cusp or worse singularity.

In the GIT quotient $V/\!\!/_\chi G = \mathbb{C}P^1$ the three semistable orbits get identified. We also have $V^{ss}/G = \mathbb{C}$ with ring of functions the polynomials in the j-invariant.

Geometric Invariant Theory was developed by Mumford based on ideas of Hilbert on classical invariant theory. A standard textbook reference is *Geometric Invariant Theory* by Mumford–Fogarty.

Lecture 13: More GIT examples

An interesting example is that of configurations of n points on $\mathbb{P}^1$, more precisely effective divisors of degree n, to allow multiplicities. The vector space is that of homogeneous polynomials of degree n in two variables

$$X = \left\{ \sum_{i=0}^{n} a_i x^i y^{n-i} \mid a_i \in \mathbb{C} \right\} \tag{206}$$

on which $G = GL(2, \mathbb{C})$ acts by change of coordinates. A point in the projectivization $x \in \mathbb{P}(X)$ is identified with a divisor

$$D = n_1 z_1 + \ldots + n_k z_k \tag{207}$$

where $z_i \in \mathbb{P}^1$ are distinct points and n_i positive integers with $\sum n_i = n$. We claim that x is

- *semistable* if and only if $n_i \leq n/2$ for all i, and
- *stable* if and only if $n_i < n/2$ for all i.

The proof uses an analysis similar to the case of cubics above, and is left as an exercise.

13.1 Stability of representations

So far we have defined (semi)stability in two different contexts: quiver representations and GIT. It turns out that the former is essentially a special case of the latter.

Let Q be a quiver (finite) and fix a dimension vector $v \in \mathbb{Z}_{\geq 0}^{Q_0}$, as well as vector spaces V_i with $\dim V_i = v_i$ (over $\mathbb{C}$, say). For each element in the vector space

$$X = \prod_{\alpha \in Q_1} \mathrm{Hom}(V_{\mathrm{in}(\alpha)}, V_{\mathrm{out}(\alpha)}) \tag{208}$$

we get a representation of Q, but two such are isomorphic if they lie in the same orbit of the action by

$$G = \prod_{i \in Q_0} GL(V_i) \tag{209}$$

acting by conjugation. In other words, X/G is the set of isomorphism classes of representations with dimension vector v. Note that G is reductive and the kernel of $G \to GL(X)$ is the group $\mathbb{C}*$ of scalars.

Are there non-constant G-invariant polynomials on X? If Q has a cycle $\alpha_1 \cdots \alpha_n$ of arrows, then for example

$$\mathrm{Tr}(T_{\alpha_1} \cdots T_{\alpha_n}) \tag{210}$$

is such a polynomial, assuming $v_i > 0$ for all the vertices i on this cycle. However, if Q is *acyclic*, then for every $\alpha \in Q_1$ the subgroup $\mathbb{C}^* \subset GL(V_{\mathrm{out}(\alpha)}) \subset G$ acts non-trivially on $A_\alpha = \mathrm{Hom}(V_{\mathrm{in}(\alpha)}, V_{\mathrm{out}(\alpha)})$, unless $v_{\mathrm{out}(\alpha)} = 0$, in which case $A_\alpha = 0$ anyway. Thus, the only

G-invariant polynomials on X are the constants in this case. The story is different if we choose a non-trivial polarization.

Any character of G is of the form

$$\chi(\phi) = \prod_{i \in Q_0} (\det(\phi_i))^{n_i} \tag{211}$$

for some $n_i \in \mathbb{Z}$. Thus for each $\chi \in \mathbb{Z}^{Q_0}$ we get a GIT quotient $X/_\chi G$. These are projective over the ring of invariant polynomials, hence projective over $\mathbb{C}$ if Q is acyclic. For $X/_\chi G$ to be non-empty we need, according to the first condition in the numerical criterion, χ to restrict to the trivial character on the kernel N of $G \to GL(V)$, which is equivalent to $\chi \cdot v = 0$. Note also that for n a positive integer we have $X/_\chi G = X/_{\chi^n} G$. In fact it turns out that one can make sense of $X/_\chi G$ also for $\chi \in \mathbb{R}^{Q_0} \supset \mathbb{Z}^{Q_0}$.

Choose Z_i, $i \in Q_0$, in the upper half plane. They together define a homomorphism $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$. Fixing a phase $\phi \in (0, \pi)$ we get a homomorphism

$$\theta = \mathrm{Im}(e^{-i\phi} Z) : \mathbb{Z}^{Q_0} \to \mathbb{R} \tag{212}$$

which we assume to be integer valued, so that it gives a character, though this is really not necessary by the above remarks. We then have the following:

Theorem 15 (A. King)**.** *A representation $E \in X$ is Z-(semi)stable of phase ϕ iff it is χ-(semi)stable. Each point in $X/_\chi G$ corresponds to a unique polystable representation, up to isomorphism.*

Proof. Suppose $\lambda : \mathbb{C}^* \to G$ is a one-parameter subgroup, then it defines weight decompositions

$$V_i = \bigoplus_{w \in \mathbb{Z}} V_i^{(w)} \tag{213}$$

where $\lambda(t)$ acts on $V_i^{(w)}$ by t^w. Then λ acts on $\mathrm{Hom}\left(V_{\mathrm{in}(\alpha)}^{(r)}, V_{\mathrm{out}(\alpha)}^{(s)}\right)$ with weight $s - r$. Hence $\lim_{t \to 0} \lambda(t) \cdot T_\alpha$ exists iff $T_\alpha^{(sr)} = 0$ for $s < r$, where $T_\alpha^{(sr)}$ is the component of T_α in $\mathrm{Hom}\left(V_{\mathrm{in}(\alpha)}^{(r)}, V_{\mathrm{out}(\alpha)}^{(s)}\right)$. Thus, for $E \in A$ existence of $\lim_{t \to 0} \lambda(t) \cdot E$ is equivalent to the T_α preserving the filtrations

$$V_i^{(\geq w)} = \bigoplus_{r \leq w} V_i^{(r)} \tag{214}$$

i.e. each $E_w = \bigoplus V_i^{(w)}$ being a *subrepresentation* of E. Together they fit into a filtration $\ldots \subset E_w \subset E_{w+1} \subset \ldots$ with $E_w = E$ for $w \ll 0$ and $E_w = 0$ for $w \gg 0$. Moreover, any filtration with these properties arises from some one-parameter subgroup λ. The limit $\lim_{t \to 0} \lambda(t) \cdot E \in X$ is nothing but the associated graded

$$\bigoplus_{w \in \mathbb{Z}} M_w / M_{w+1} \tag{215}$$

with structure maps $T_\alpha^{(ww)}$. The filtration will be non-trivial in the sense that some E_w is not 0 or E, if and only if λ is not contained in the kernel N.

The next step is to compute $\langle \chi, \lambda \rangle$ under the assumption $\theta(v) = \chi \cdot v = 0$ as follows.

$$\langle \chi, \lambda \rangle = \sum_{i \in Q_0} \theta_i \sum_{w \in \mathbb{Z}} \dim(V_i^{(w)}) \tag{216}$$

$$= \sum_{w \in \mathbb{Z}} w\theta(E_w/E_{w+1}) \tag{217}$$

$$= \sum_{w \in \mathbb{Z}} \theta(E_w) \tag{218}$$

In particular, $\langle \chi, \lambda \rangle > 0$ for some λ not in N if and only if E has a proper subrepresentation with $F \subset E$ with $\theta(F) > 0$. It is easy to see that this is equivalent to stability of E.

If E is semistable, then the limit $\lim_{t \to 0} \lambda(t) \cdot E$ as above is again semistable only if $\theta(E_w) = 0$ for all w, i.e. all E_w have the same phase as E. We conclude that the orbit $G \cdot E$ is closed in X^{ss} if and only if E is a direct sum of simple representations of phase ϕ, and for general E the closure of the orbit $G \cdot E$ contains such a polystable representation.

□

Lecture 14: Moment maps

14.1 Cartan calculus

We will be doing basic differential geometry in the next sections, so this is a good time to review the relations between the fundamental operators: exterior derivative d, interior product ι_X, and Lie derivative $\mathcal{L}_X$. These are derivations on the graded commutative algebra of forms (of degrees 1, -1, and 0 respectively), meaning that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta \tag{219}$$

$$\iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota_X\beta \tag{220}$$

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta \tag{221}$$

where α, β are forms and X is a vector field. For a smooth function f we have

$$\iota_{fX} = f\iota_X, \qquad \mathcal{L}_{fX} = f\mathcal{L}_X + df \wedge \iota_X. \tag{222}$$

The various supercommutators of the operators are given by

$$\iota_X d + d\iota_X = \mathcal{L}_X \tag{223}$$

$$\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_X\mathcal{L}_Y = \mathcal{L}_{[X,Y]} \tag{224}$$

$$\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X = \iota_{[X,Y]} \tag{225}$$

$$\iota_X\iota_Y + \iota_Y\iota_X = 0. \tag{226}$$

14.2 Hamiltonian dynamics

An important class of dynamical systems are the ones of *Hamiltonian* form. The phase space is in this case a symplectic manifold, (M, ω), on which there is a smooth function $H : M \to \mathbb{R}$, the **Hamiltonian**. The time evolution on M is given by the flow of the vector field X_H such that

$$\omega(X_H, Y) = (dH)(Y) \tag{227}$$

i.e. which is dual to dH, where we use ω to get an identification $TM \cong T^*M$. The relation between H and X_H is thus formally similar to the relation between a function f and its gradient ∇f on a Riemannian manifold, except that we have an antisymmetric tensor, ω, instead of a symmetric one, g.

The motivating example comes from Newton's equations describing the motion of a particle in $\mathbb{R}^n$ with mass m subjected to a force $-\nabla V$ determined by a potential $V : \mathbb{R}^n \to \mathbb{R}$. In this case the manifold is $M = T^*\mathbb{R}^n = \mathbb{R}^{2n}$ with symplectic form

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \tag{228}$$

where $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are coordinates on $\mathbb{R}^{2n}$ describing position and momentum. The Hamiltonian is the total energy

$$H(q,p) = \frac{|p|^2}{2} + V(q). \tag{229}$$

We can generalize this by replacing $\mathbb{R}^n$ by an arbitrary Riemannian manifold. For $V = 0$, the flow generated by X_H is then just the geodesic flow. More generally, dynamics described by Lagrangian equations (least action principle) can be transformed to Hamiltonian form under suitable assumptions.

What makes Hamiltonian dynamics special? A first observation is that the flow ϕ_t generated by a Hamiltonian vector field X_H preserves the symplectic form ω, meaning that $\phi_t^*\omega = \omega$ for all $t \in \mathbb{R}$. To see this, note that

$$\mathcal{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H} d\omega = d(dH) + 0 = 0 \tag{230}$$

where by definition

$$\mathcal{L}_{X_H}\omega = \lim_{t=0} \frac{\phi_t^*\omega - \omega}{t} \tag{231}$$

is the Lie derivative. In particular, the flow preserves the volume form $\omega^{\wedge n}/n!$, where $2n = \dim M$, i.e. each ϕ_t is a *measure preserving* transformation of phase space.

An even stronger fact is *Noether's principle*, giving a correspondence

$$\text{continuous symmetries} \longleftrightarrow \text{conserved quantities.} \tag{232}$$

Let us make this precise. A continuous symmetry is an action of the Lie group $\mathbb{R}$ on M by symplectomorphisms preserving H. It is generated by a vector field ("infinitesimal symmetry") X such that

$$\mathcal{L}_X\omega = 0, \qquad X(H) = 0. \tag{233}$$

The first condition implies that X is dual to a *closed* 1-form, i.e. $d(\iota_X\omega) = 0$, by the same argument as in (230). Let us assume that $\iota_X\omega$ is *exact*, so that $X = X_f$ for some function $f : M \to \mathbb{R}$ which is determined up to a constant. By the second condition,

$$0 = X_f(H) = (dH)X_f = \omega(X_H, X_f) = -\omega(X_f, X_H) = -(df)X_H = -X_H(f) \tag{234}$$

hence f is invariant under the flow of X_H, i.e. a *conserved quantity*. Conversely, given a conserved quantity f, the corresponding Hamiltonian vector field X_f is an infinitesimal symmetry of the system and its flow a continuous symmetry, assuming it exists for all times.

One defines the Poisson bracket

$$\{f, g\} = \omega(X_f, X_g) \tag{235}$$

which defines a Lie algebra structure on $C^\infty(M, \mathbb{R})$ so that the map $f \mapsto X_f$ is a homomorphism of Lie algebras, up to sign, i.e

$$X_{\{f,g\}} = -[X_f, X_g]. \tag{236}$$

In particular, the Hamiltonian flows generated by f, g commute if and only if $\{f, g\} = 0$, as we have already seen. In coordinates where ω has the standard form (228), one has explicitly

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{237}$$

As an example, consider the dynamics of a particle in $\mathbb{R}^2$ with rotationally invariant potential $V = V(r)$. On phase space, the rotation is generated by the vector field

$$X_f = q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1} \tag{238}$$

where

$$f = q_1 p_2 - q_2 p_1 = q \times p \tag{239}$$

is the angular momentum.

So far, we have just considered actions of $\mathbb{R}$ by Hamiltonian symplectomorphisms. Suppose we have a general Lie group, G, acting on a symplectic manifold M by Hamiltonian symplectomorphisms. We can choose a basis $v_1, \ldots, v_n$ of its Lie algebra, $\mathfrak{g}$, to get corresponding Hamiltonians $f_1, \ldots, f_n : M \to \mathbb{R}$. These generate the vector space of Hamiltonians we get from various $v \in \mathfrak{g}$. A basis-free way of talking about the f_i is as follows. The derivative of the action $G \to \mathrm{Symp}(M)$ gives a Lie-algebra anti-homomorphism $\mathfrak{g} \to \Gamma(M, TM)$, with image contained in the subalgebra of Hamiltonian vector fields, by assumption. We assume that this lifts to a homomorphism of Lie algebras $H : \mathfrak{g} \to C^\infty(M, \mathbb{R})$.

$$\begin{array}{ccccccccc} & & & & & & \mathfrak{g} & & \\ & & & & & & \downarrow H & \searrow & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M,\mathbb{R}) & \longrightarrow & C^\infty(M,\mathbb{R})/\mathbb{R} \cong \mathfrak{ham}(M,\omega) & \longrightarrow & 0 \end{array} \tag{240}$$

The **moment map** of the action of G is the map $\mu : M \to \mathfrak{g}^*$ such that

$$\langle \mu(p), v \rangle = (H(v))(p), \qquad \text{for all } v \in \mathfrak{g}, p \in M \tag{241}$$

where $\langle _, _ \rangle$ is the pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. The condition that $\{H(v), H(w)\} = H([v, w])$ is equivalent to (infinitesimal) $\mathfrak{g}$-equivariance of the moment map, where $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by its coadjoint representation. The coadjoint representation is by definition just the dual, $v \mapsto -(\mathrm{ad}(v))^*$, of the adjoint representation $v \mapsto \mathrm{ad}(v) = [v, _]$.

A moment map need not exist in general, but when M is compact the sequence (240) is split by the map sending a class $f + \mathbb{R}$ to the unique representative f with vanishing total integral. These functions form an ideal in $C^\infty(M, \mathbb{R})$, since

$$\int_M \{f, g\} \omega^{\wedge n} = 0 \tag{242}$$

for any pair of functions, as a calculation shows. So when M is compact we can always choose a canonical lift in such a way. However, it is still advantageous to consider more general lifts.

We note that when $f \in \mathfrak{g}^*$ is central, i.e. annihilated by every element of $\mathfrak{g}$ under the coadjoint representation, then $\widetilde{H}(v) = H(v) + f(v)$ is also a lie algebra homorphism, hence $\mu + f$ a different choice of moment map for the same action. Conversely, the difference of two choices of moment maps is given by a constant in the center of $\mathfrak{g}^*$.

As an example, consider $U(n)$ acting on $\mathbb{C}^n$ in the usual way, then for any $c \in i\mathbb{R}$ we get a moment map

$$\mu_c : \mathbb{C}^n \to u(n)^* \cong u(n), \qquad z \mapsto \frac{i}{2} z \otimes z^* + c \tag{243}$$

where $z \otimes z^*$ is matrix with entries $a_{ij} = z_i \bar{z}_j$. Note that $U(n)$ is acting by *holomorphic* symplectomorphisms. For a compact group K acting on $\mathbb{C}^n$ by complex linear maps we can always achieve this by taking the symplectic structure given by an invariant Hermitian inner product.

14.3 Symplectic reduction

The notion of a moment map leads to a quotient construction for symplectic manifolds due to Marsden–Weinstein. Suppose a compact group G is acting on a symplectic manifold (M, ω) and $\mu : M \to \mathfrak{g}^*$ is a moment map for the action such that 0 is a regular value for μ and the action of G on $\mu^{-1}(0)$ is free and proper. Then by assumption both $\mu^{-1}(0)$ and μ^{-1}/G are smooth manifolds, of dimensions $\dim M - \dim G$ and $\dim M - 2 \dim G$ respectively. There is a unique symplectic form on μ^{-1}/G whose pullback to μ^{-1} is the restriction of ω. Thus, the quotient, which depends on the choice of μ, is again a symplectic manifold.

Perhaps the simplest example is that of $U(1)$ acting on $\mathbb{C}^n$ by multiplication. A moment map for the action is given by

$$\mu(z) = \frac{|z|^2}{2} - c \in \mathbb{R} \cong u(1)^* \tag{244}$$

so that

$$\mu^{-1}(0)/U(1) = \begin{cases} \mathbb{P}^{n-1} & c > 0 \\ \{*\} & c = 0 \\ \emptyset & c < 0 \end{cases} \tag{245}$$

where in the second case $U(1)$ is not acting freely, so $\{*\}$ is not a symplectic quotient.

Lecture 15: Kempf–Ness

15.1 Kempf–Ness principle

The link between GIT (over $\mathbb{C}$) and symplectic reduction is provided by the Kempf–Ness theorem, stating that the two constructions lead to the same quotient. Thus, we suppose that a compact group K with reductive complexification $G = K^{\mathbb{C}}$ is acting on a Hermitian vector space V by unitary transformations. A moment map for that action of K is given by

$$\mu : V \to \mathfrak{k}^*, \qquad (\mu(v))(iA) = \langle Av, v\rangle. \tag{246}$$

For simplicity we suppose that the polarization is trivial at first.

The key idea is to try to minimize that norm squared, $\|v\|^2$, on each orbit of G. This is of course preserved under the action of K, so we look at the variation in directions $A \in i\mathfrak{k}$, which is

$$\frac{d}{dt}|_{t=0}\|e^{tA}v\|^2 = 2\langle Av, v\rangle = 2(\mu(v))(iA) \tag{247}$$

so critical points of $\|v\|^2$, restricted to some orbit, are given by $\mu^{-1}(0)$. Moreover the second variation is

$$\frac{d^2}{dt^2}|_{t=0}\|e^{tA}v\|^2 = 4\|Av\|^2 \tag{248}$$

meaning that $\|v\|^2$ induces a strictly convex function on the quotient $(G \cdot v)/K$ of an orbit. Thus, any critical point is a strict minimum, and there can be at most one. The question which remains is on which orbits the minimum is actually attained. Certainly if an orbit $G \cdot v$ is closed, then $\|_\|^2$ restricts to a proper function on it, so there is a minimum. Conversely, we claim that if an orbit $G \cdot v$ is not closed, then the infimum of $\|_\|^2$ is not attained. By the Hilbert–Mumford criterion there is a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot v = v_0$ exists, but it not in $G \cdot v$. Conjugating λ if necessary, we can diagonalize it with respect to a unitary basis, so

$$\lambda(t) \cdot v = (t^{n_1}v_1, \ldots, t^{n_k}v_k), \qquad n_1, \ldots, n_l \geq 0, \qquad n_{l+1}, \ldots, n_k < 0. \tag{249}$$

Since the limit exists, we must have $v_{l+1} = \ldots = v_n = 0$, so

$$\|\lambda(t) \cdot v\|^2 = |t|^{n_1}\|v_1\|^2 + \ldots + |t|^{n_l}\|v_l\|^2 \tag{250}$$

which is strictly decreasing as $|t| \to 0$, since we must have $n_i > 0$, $v_i \neq 0$ for at least one i, as otherwise $\lambda(t) \cdot v$ would be constant. Thus we have established

Theorem 16. *$\mu^{-1}(0)$ meets exactly those G-orbits which are closed in V, hence we have*

$$\mu^{-1}(0)/K = V/\!/_0 G = \mathrm{Spec}(\mathbb{C}[V]^G). \tag{251}$$

All this extends to the case of non-trivial character $\chi : G \to \mathbb{C}^*$. Note that $\chi' : \mathfrak{g} \to \mathbb{C}$ restricts to a central element of $\mathfrak{k}^*$.

Theorem 17. *$\mu^{-1}(\chi'|_{\mathfrak{k}})$ meets exactly those G-orbits which are closed in V^{ss}, hence we have*

$$\mu^{-1}(\chi'|_{\mathfrak{k}})/K = V/\!\!/_{\chi} G \tag{252}$$

i.e. the symplectic quotient with respect to the moment map $\mu - \chi'|_{\mathfrak{k}}$ and the GIT quotient with respect to the polarization χ are the same.

The proof is similar, except that we have to consider the G-action on the total space of the equivariant line bundle $V \times \mathbb{C}$, and instead of the norm squared we consider the map

$$f : V \times \mathbb{C}^* \to \mathbb{R}, \qquad f(v, z) = \frac{1}{2}\|v\|^2 + \log|z|. \tag{253}$$

As before, f will attain a minimum precisely on the closed orbits $G \cdot (v, z)$. There is now a third possibility: f is not bounded below on an orbit $G \cdot (v, z)$. This happens precisely if v is unstable.

Let us take a look at how all this works in some examples. First, consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ with weights $1, -1$, i.e.

$$\mathbb{C}^* \to GL(2, \mathbb{C}), \qquad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}. \tag{254}$$

The orbits of the action are the generic ones: $\{(zw, z^{-1}) \mid z \in \mathbb{C}^*\}$ for $w \in \mathbb{C}^*$, and the three special ones: $\{(0,0)\}$, $\mathbb{C} \times \{0\}$, $\{0\} \times \mathbb{C}$. By the theorem the function $\|_\|^2$ attains its infimum precisely on closed orbits: the generic orbits and the origin. The GIT quotient with respect to the trivial character is just $\mathbb{C}$ with coordinate $z_1 z_2$, the three semistable orbits get identified. If we choose a positive polarization, say $\chi = 1$, then the quotient remains the same, but its construction is slightly different: The orbits $\{0, 0\}$ and $\{0\} \times \mathbb{C}$ become unstable. Similarly for negative polarization.

We can recover all this from the points of view of symplectic reduction. The canonical choice of moment map is

$$\mu : \mathbb{C}^2 \to \mathbb{R} \cong u(1)^*, \qquad \mu(z_1, z_2) = |z_1|^2 - |z_2|^2 \tag{255}$$

so $\mu^{-1}(0)$ meets exactly the closed orbits in the points which are closest to the origin. The action of $U(1)$ on $\mu^{-1}(c)$ is free as long as $c \neq 0$.

We return to the example of n points on $\mathbb{P}^1$. For $K = SU(2)$ acting on $\mathbb{P}^1$ the moment map is just the inclusion $\mathbb{P}^1 \cong S^2 \to \mathbb{R}^3 \cong su(2)^*$ of the sphere in 3-space. For $SU(2)$ acting on the symmetric power $S^n\mathbb{P}^1$, i.e. configurations of n (unmarked) points, we get the moment map by addition, so

$$\mu : S^n\mathbb{P}^1 \to \mathbb{R}^3 \cong su(2)^* \tag{256}$$

is n times the center of mass of the points. Thus, the condition $\mu = 0$ means that the points have center of mass the origin in $\mathbb{R}^3$. Acting by Möbius transformations, $SL(2, \mathbb{C})$, this can only be achieved if no $n/2$ or more points come together, or, when n is even, if $n/2$ of the points

come together and so do the other $n/2$, but at a different point in $\mathbb{P}^1$. Put differently, an orbit intersects $\mu = 0$, and thus gives a point in the symplectic quotient, exactly if all multiplicities are $< n/2$, or if the multiplicities are $(n/2, n/2)$. To prove this, one can determine the closed orbits by the Hilbert–Mumford criterion and then apply the Kempf–Ness theorem.

Another example is the space $\mathrm{Mat}(n \times n, \mathbb{C})$ of complex n-by-n matrices with the adjoint action of $GL(n, \mathbb{C})$ by conjugation. The invariant functions are the symmetric polynomials in the eigenvalues, so the invariant theory quotient is affine space $\mathbb{C}^n$. Closed orbits correspond to diagonalizable matrices. A moment map for the action of the compact group, $U(n)$ is given by $\mu(A) = [A, A^*]/2$ with zeros precisely the normal matrices. A matrix is normal iff it can be diagonalized with respect to a unitary basis. If we think of $GL(n, \mathbb{C})$ acting not on the linear transformation but on the Hermitian metric, then condition $\mu = 0$ singles out those special metrics in which the eigenspaces become orthogonal. The metric is far from unique, however, reflecting the fact that the stabilizer is at least $(\mathbb{C}^*)^n$.

More generally, the point of view of $\mu = 0$ being the equation for a special metric works when $G = GL(W)$ and V is a vector space of tensors over V, i.e. constructed from W by $\otimes, \mathrm{Hom}, _^*, S^d, \Lambda^d, \ldots$. Then G naturally acts on V as well. In the example above, $W = \mathbb{C}^n$ and $V = \mathrm{Hom}(W, W)$. This is the problem of classifying endomorphisms, i.e. representations of the quiver with a single vertex and arrow. Everything generalizes to arbitrary quivers Q, but instead of a single vector space W one has a vector space W_i for each vertex $i \in Q_0$, and $G = \prod GL(W_i)$ is a product of general linear groups.

15.2 Another point of view on the moment map

Let (M, ω) be a symplectic manifold. We have seen that $C^\infty(M, \mathbb{R})/\mathbb{R}$ is in a certain sense the (infinite dimensional) Lie algebra of the group of Hamiltonian symplectomorphisms on M. One can ask if the central extension $C^\infty(M, \mathbb{R})$ and corresponding group are also acting in some geometric way. To make this work we assume that the class of the symplectic form divided by 2π, $[w]/(2\pi) \in H^2(M, \mathbb{R})$, is integral, i.e. contained in the image of $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. We can then realize $i\omega$ as the curvature of a line bundle.

Lemma 5. *Let $\omega \in \Omega^2(M, \mathbb{R})$ be a closed 2-form in an integral cohomology class. Then there is a Hermitian line bundle L on M with metric connection D such that*

$$\frac{i}{2\pi} F_A = \omega \tag{257}$$

where $F_A = D^2$ is the curvature.

Proof. First of all, we can find a complex line bundle L with $c_1(L) = [\omega]$. To see this consider the exponential exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp(2\pi i _)} \mathbb{C}^* \to 0 \tag{258}$$

where we take sections to be *smooth* functions with values in $\mathbb{Z}, \mathbb{C}, \mathbb{C}^*$ respectively. Since the sheaf of C^∞ functions with values in $\mathbb{C}$ has no cohomology, the long exact sequence gives an

isomorphism

$$H^1(M, \mathbb{C}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \tag{259}$$

the connecting homomorphism being the first Chern class. The group $H^1(M, \mathbb{C}^*)$ classifies complex line bundle, up to isomorphism, as can be seen from the Čech definition of sheaf cohomology.

The next step is to pick any Hermitian metric and unitary connection D_0 on L. This can be done by partition of unity argument. By Chern–Weil theory, $c_1(L) = [iF_{D_0}/(2\pi)]$ so since $[\omega] = c_1(L)$ the difference $\omega - iF_{D_0}/(2\pi)$ is exact. Thus, there is a 1-form $\alpha \in \Omega^1(M, i\mathbb{R})$ with $F_{D_0} + d\alpha = -2\pi i\omega$, but this is precisely the curvature F_D of the new connection $D = D_0 + \alpha$ which is also unitary since α is purely imaginary. □

Lecture 16: Gauge theory on 2-manifolds

We fix some L over M with metric connection D such that $F_A = i\omega$, which is possible by the condition on ω and the above lemma. We then have an infinitesimal action of the Lie algebra $C^\infty(M,\mathbb{R})$ with the Poisson bracket on the total space of L given by

$$C^\infty(M,\mathbb{R}) \to \Gamma(L,TL), \qquad f \mapsto \widehat{X_f} + if \tag{260}$$

where $\widehat{X_f}$ is the horizontal lift of the Hamiltonian vector field X_f, and if is the vector field which on a fiber L_x is multiplication by $if(x)$, where we use the identification $T_z L_x \cong L_x$. In particular the constant functions generate the fiberwise action by $U(1)$ on L.

Lemma 6. *The map* (260) *is a Lie algebra anti-homomorphism, i.e.*

$$\widehat{X_{\{f,g\}}} + i\{f,g\} = -\left[\widehat{X_f} + if, \widehat{X_g} + ig\right]. \tag{261}$$

Proof. This is a local computation, so we can assume that $L = M \times \mathbb{C}$ is trivial and D is given by a connection 1-form $\alpha \in \Omega^1(M,\mathbb{C})$. The horizontal lift of a vector field X on M is given by

$$\widehat{X} = X + \alpha(X) z\partial_z \tag{262}$$

where $\partial_z = \partial/\partial z$. For $f \in C^\infty(M)$, $X \in \Gamma(M,TM)$ we have

$$\left[\widehat{X}, fz\partial_z\right] = [X, fz\partial_z] = X(f)z\partial_z \tag{263}$$

since $[X, z\partial_z] = 0$. The curvature $F_D = d\alpha$ of D can be computed in terms of horizontal lifts as

$$F_D(X,Y)z\partial_z = \left[\widehat{X}, \widehat{Y}\right] - \widehat{[X,Y]} \tag{264}$$

as one can check by direct calculation. Thus,

$$\left[\widehat{X_f} + ifz\partial_z, \widehat{X_g} + igz\partial_z\right] = \left[\widehat{X_f}, \widehat{X_g}\right] + \left[\widehat{X_f}, igz\partial_z\right] - \left[\widehat{X_g}, ifz\partial_z\right] + [ifz\partial_z, igz\partial_z] \tag{265}$$

$$= \widehat{[X_f, Y_g]} + i\omega(X_f, X_g)z\partial_z + i\left(X_f(g) - X_g(f)\right)z\partial_z + 0 \tag{266}$$

$$= -\widehat{X_{\{f,g\}}} - i\{f,g\}z\partial_z \tag{267}$$

where we use $\{f,g\} = \omega(X_f, X_g) = X_g(f)$. □

The corresponding Lie group, i.e. the one generated by vector fields $\widehat{X_f} + if$, is that of isometries of L which preserve the connection. If we have some action of a Lie group G on M by Hamiltonian symplectomorphisms, we can thus interpret the choice of moment map as a choice of lift of the action to L. Compare this to the choice of equivariant line bundle in the context of GIT.

In quantum mechanics all this goes under the name of *prequantization*, which is the first step in geometric quantization of classical phase space. The Hilbert space of L^2 sections of L is too big to be the right physical one though, one needs to choose a Lagrangian foliation on M, separating position from momentum.

16.1 Gauge theory on surfaces

A linear analog of the Kempf–Ness theorem is the simple fact that given a finite-dimensional inner product space V with subspace $W \subset V$ then each affine space $x + W \in V/W$ contains a unique vector of smallest norm, which is the one contained in the orthogonal complement $W^\perp$. Consequently, we have an identification $W^\perp \cong V/W$. When W is infinite-dimensional, on the other hand, then there is a priori no guarantee that a minimizer exists in a given class $x + W$. For example, when $V = Z^p(M)$ is the space of closed p-forms on a compact Riemannian manifold M and $W = B^p(M)$ is space of exact forms, then Hodge theory tells us that we can indeed find an element of smallest norm, the harmonic representative, and thus get an identification of the space of harmonic forms with deRham cohomology.

Gauge theory, in particular Yang–Mills theory, can be thought of as a non-abelian version of Hodge theory. Because spaces of connections and gauge groups are infinite-dimensional, we cannot apply the Kempf–Ness theorem directly, however the statement remains true, the proof of which requires significant analysis.

Unitary connections. To begin, fix a Hermitian bundle E over a smooth manifold M. A metric connection on E is an operator

$$D : \Omega^0(E) \to \Omega^1(E) \tag{268}$$

which satisfies

$$D(fs) = (df)s + fD(s), \qquad f \in C^\infty(S), s \in \Gamma(E) \tag{269}$$

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle, \qquad s_1, s_2 \in \Gamma(E). \tag{270}$$

Let $\mathcal{A}$ be the set of such connections on E. The basic fact is that this is an (infinite-dimensional) affine space over $\Omega^1(\mathfrak{g}_E)$, the space of 1-forms with values in the bundle $\mathfrak{g}_E$ of skew-adjoint endomorphisms of E, where $a \in \Omega^1(\mathfrak{g}_E)$ is acting by $(D + a)(s) = Ds + a(s)$. In particular, $\Omega_1(\mathfrak{g}_E)$ is the tangent space to $\mathcal{A}$ at any point.

The space $\mathcal{A}$ is acted on by the gauge group G, which is the group of sections of the bundle G_E with fiber over $p \in M$ the space of isometries of E_p. The action on a connection is by conjugation: $g \cdot D = g \circ D \circ g^{-1}$. The infinitesimal action

$$\rho_D : \mathrm{Lie}(S) = \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E) = T_D\mathcal{A} \tag{271}$$

is given by the connection on $\mathfrak{g}_E$ induced by D.

In the case when the base manifold $M = S$ is a compact oriented surface ($\dim_{\mathbb{R}} = 2$) we can introduce a symplectic form on $\mathcal{A}$ by

$$\Omega(a, b) = \int_S \mathrm{Tr}(a \wedge b) \tag{272}$$

where $a, b \in \Omega^1(\mathfrak{g}_E) = T_p\mathcal{A}$. The gauge group G preserves the symplectic form. At least formally, we can thus try to find a moment map $\mu : \mathcal{A} \to \Omega^0(\mathfrak{g}_E)^*$ for the action of G. We

claim that $\mu(D) = F_D$ is the curvature of the connection. Here, $F_D \in \Omega^2(\mathfrak{g}_E)$ gives an element of $\Omega^0(\mathfrak{g}_E)^*$ via the pairing

$$\langle F_D, b\rangle = \int_S \mathrm{Tr}(F_D b), \qquad b \in \Omega^0(\mathfrak{g}_E). \tag{273}$$

To check this, note first that under translation of the connection the curvature transforms as

$$F_{D+a} = F_D + d_D a + a \wedge a \tag{274}$$

where d_D is the coupled exterior derivative on $\Omega^*(\mathfrak{g}_E)$, and hence the first variation of F_D in the direction $a \in \Omega^1(\mathfrak{g}_E)$ is $d_D a$. We compute for $a \in \Omega^1(\mathfrak{g}_E)$ and $b \in \Omega^0(\mathfrak{g}_E)$

$$\langle (d\mu(D))(a), b\rangle = \langle d_D(a), b\rangle \tag{275}$$

$$= \int \mathrm{Tr}(d_D(a)b) \tag{276}$$

$$= \int \mathrm{Tr}(aD(b)) \tag{277}$$

$$= \Omega(a, \rho_D(b)) \tag{278}$$

where we used $d\mathrm{Tr}(ab) = \mathrm{Tr}(d_D ab) - \mathrm{Tr}(aDb)$.

In conclusion, the critical points of the moment map are just the flat connections, and the symplectic quotient $\mu^{-1}(0)/G$ is the set of isomorphism classes of flat connections, which can be identified with the set of unitary representations of the fundamental group of S. Of course to get $F_D = 0$ we need to require that $c_1(E) = 0$. For $c_1(E) \neq 0$ one modifies the moment map by a volume form ω on S.

We can make contact with deformation theory at this point. Suppose we have a first order deformation $D + ta$ of a flat connection D by $a \in \Omega^1(\mathfrak{g}_E)$. Then $F_{D+ta} = t d_D a + O(t^2)$, so we require $d_D a = 0$. Gauge transformation, which do not change the isomorphism class of the flat bundle, act infinitesimally by $D + t d_D b$, so first order deformations of D up to isomorphisms are given by the first cohomology of the complex

$$0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d_D} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_D} \Omega^2(\mathfrak{g}_E) \to \dots. \tag{279}$$

This is not just a complex, but a dg-Lie algebra with the usual bracket of endomorphism valued forms. It "knows about" deformations of (E, D) to any order, i.e. the formal deformations. Note that all this works over arbitrary dimensional base manifold M, but over a compact surface we have a natural pairing $\Omega^0(\mathfrak{g}_E) \times \Omega^2(\mathfrak{g}_E) \to \mathbb{R}$.

16.2 Narasimhan–Seshadri theorem

Suppose now that S is a compact Riemann surface, and E a unitary bundle as before. Then we have a complex structure on the real tangent bundle T^*S which induces a complex structure on $\Omega^1(\mathfrak{g}_E) = \Gamma(\mathfrak{g}_E \otimes T^*S)$, thus on the space $\mathcal{A}$ of connections. Using the decomposition

$\Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$, where now all forms are complex valued, we can write any connection D on E uniquely as a sum

$$d_D = \partial_D + \overline{\partial}_D. \tag{280}$$

In general, the $(0,1)$ part, $\overline{\partial}_D$, defines a holomorphic structure on E only if $\overline{\partial}_D^2 = 0$, but since S has one complex dimension, this is trivially satisfied. Conversely, given a holomorphic structure on E in the form of an operator $\overline{\partial} : \Omega^0(E) \to \Omega^{0,1}(E)$ and a Hermitian metric h on E, then there is a unique metric connection with $(0,1)$ part the given $\overline{\partial}$. This is true in any dimension. The upshot is that, fixing a Hermitian metric on E, we can identify connections and complex structures on E.

Lecture 17: Hermitian Yang–Mills

Thinking of $\mathcal{A}$ as the set of operators $\overline{\partial}$ on E, it is clear that the complexified gauge group $G^c = \Gamma(S, \mathrm{Aut}(E))$ of complex linear automorphisms of E acts by

$$g \cdot \overline{\partial} = g \circ \overline{\partial} \circ g^{-1} \tag{281}$$

for $g \in G^c$. In terms of the metric connection this amounts to

$$d_{g \cdot D} = g \circ \overline{\partial}_D \circ g^{-1} + (g^*)^{-1} \circ \partial_D \circ g^* \tag{282}$$

preserving unitarity of D. By definition, $\mathcal{A}/G^c$ is the set of isomorphism classes of holomorphic bundles with underlying C^∞ bundle E. By analogy with the Kempf–Ness theorem, we can hope for an identification

$$\mathcal{A}^s/G^c \cong \mu^{-1}(0)/G \tag{283}$$

where $\mathcal{A}^s$ is the set of "stable" holomorphic bundles in some sense.

In the special case when E is a line bundle, then it is automatically stable and we have

$$\mathcal{A}/G^c \cong \mathrm{Pic}^0(S) \cong H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(S), U(1)) \cong \mu^{-1}(0)/G. \tag{284}$$

For higher rank E, stability becomes a non-trivial condition. We can try to guess what it is by looking at one-parameter subgroups. Suppose a one-parameter subgroup of G is acting on a rank 2 bundle E with weights k, $-k$ in each fiber, i.e. we have a splitting, $E = L_1 \oplus L_2$ of E as C^∞ bundle such that $z \in \mathbb{C}^*$ acts by multiplication with z^k and z^{-k} on L_1 and L_2 respectively. The orbit of a holomorphic structure $\overline{\partial}_E$ under the $\mathbb{C}^*$-action is then

$$\begin{pmatrix} \overline{\partial}_{L_1} & z^{2k}\beta \\ z^{-2k}\gamma & \overline{\partial}_{L_2} \end{pmatrix} \tag{285}$$

so the limit as $z \to 0$ exists iff $\gamma = 0$, i.e. L_1 is a holomorphic subbundle of E. The numerical value associated with the limiting point is

$$\left\langle \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \begin{pmatrix} F(L_1) & 0 \\ 0 & F(L_2) \end{pmatrix} \right\rangle = \int_S 2kF(L_1) \tag{286}$$

$$= -2k \deg(L_1) \tag{287}$$

which should be positive, if $(E, \overline{\partial})$ is stable, i.e. we want $\deg(L_1) < 0$ for all holomorphic subbundles $L_1 \subset E$.

More generally, in the case of higher rank and arbitrary degree of E, the right condition turns out to be **slope stability**, which means that

$$\frac{\deg(F)}{\mathrm{rk}(F)} < \frac{\deg(E)}{\mathrm{rk}(E)} =: \phi(E) \tag{288}$$

for all proper subbundles $F \subset E$. The Narasimhan–Seshadri theorem is then

Theorem 18. *Let S be a compact Riemann surface with Hermitian metric. A holomorphic bundle E over S is polystable if and only if there is a unitary connection on E with constant central curvature $*F = -\frac{2\pi i}{\mathrm{Vol}(S)}\phi(E)$.*

A more differential geometric proof of this theorem was given by Donaldson.

So far we have not introduced a line bundle on $\mathcal{A}$. In fact, it is not essential, since we can define, in general, a function $\mathcal{F} : K^c \to \mathbb{R}$ for each orbit $G^c \cdot z_0$ up to constant by

$$(\delta\mathcal{F})(g) = \langle \delta g g^{-1}, \mu(g(z_0)) \tag{289}$$

which has critical points the zeros of μ. It turns out though that there is a corresponding line bundle L on $\mathcal{A}$ which has fiber

$$L_D = \mathrm{Det} H^*(E \otimes K_S^{1/2}, \overline{\partial}_D) \tag{290}$$

over $D \in \mathcal{A}$. Note that

$$\mathrm{Det}(V^{\mathrm{even}} \oplus V^{\mathrm{odd}}) = \Lambda^{\max}(V^{\mathrm{even}}) \otimes \left(\Lambda^{\max}(V^{\mathrm{odd}})\right)^{-1} \tag{291}$$

is the determinant line of a $\mathbb{Z}/2$-graded vector space. The natural metric on L is the Quillen metric and the function $\mathcal{F}$ of the metric is given by

$$\mathcal{F} = \log \det{}_\zeta \Delta_{\overline{\partial}} \tag{292}$$

where $\det_\zeta$ is the zeta-regularized determinant.

17.1 Hermitian Yang–Mills bundles

In passing from Riemann surfaces to Kähler manifolds of higher dimension, several new features appear. First, not all connections D on a Hermitian bundle E give a holomorphic structure, only those which satisfy the integrability condition $\overline{\partial}_D^2 = 0$. Furthermore, the notion of stability is not unique, but depends on the Kähler class $[\omega] \in H^{1,1}(M)$ as follows. One defines the degree of a complex bundle or analytic sheaf, E, relative to ω as

$$\deg_\omega E = \int_M c_1(E) \wedge *\omega. \tag{293}$$

and the slope

$$\mu(E) = \frac{1}{\mathrm{Vol}(M)} \frac{\deg_\omega E}{\mathrm{rank} E}. \tag{294}$$

A holomorphic vector bundle E is then said to be **stable** if every proper coherent subsheaf $0 \neq F \subsetneq E$ satisfies $\mu(F) < \mu(E)$. As always, E is **polystable** if it is a direct sum of stable bundles of the same slope.

The fact that stability is tested not just by subbundles but arbitrary subsheaves corresponds to the fact that solutions to the Hermitian Yang–Mills equations have non-trivial local

structure, as opposed to the constant curvature connections in dimension one. The equation can be formulated as follows. The Kähler metric g on M gives a bundle map

$$\mathrm{tr}_g : \mathrm{End}(E) \otimes \Lambda^2 T^*X \to \mathrm{End}(E) \tag{295}$$

which in local coordinates can be written as

$$\mathrm{tr}_g F = \sum_{j,k} g^{j\bar{k}} F_{j\bar{k}}. \tag{296}$$

The Hermitian–Yang–Mills condition on the metric h on E is then

$$\mathrm{tr}_g F = \mu(E) 1_E \tag{297}$$

where 1_E is the identity operator on E and $\mu(E) \in \mathbb{R}$ is the slope of E as above. The condition can again be formulated as a moment map problem, at least formally, with moment map on the space of connections given by $\mu(D) = F_D \wedge \omega^{n-1}$ and symplectic form

$$\Omega(a, b) = \int_M a \wedge b \wedge \omega^{n-1}. \tag{298}$$

Lecture 18: A-side vs B-side stability

The equivalence of symplectic and GIT quotient holds, as was shown by Donaldson for algebraic surfaces and Uhlenbeck–Yau for general Kähler manifolds.

Theorem 19. *A holomorphic vector bundle E on a Kähler manifold M admits a Hermitian–Yang–Mills metric if and only if it is polystable.*

For more details we refer to the original papers of Donaldson (*Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles*) and Uhlenbeck–Yau (*On the existence of Hermitian–Yang–Mills connections in stable vector bundles*).

There are also variants of the HYM condition of the form $f(F_D) = 0$ where the symplectic structure is

$$\Omega(a, b) = \int_M a \wedge b^* \wedge f'(F_D). \tag{299}$$

For example the case $f = \exp(k\omega + F_D)\mathrm{Td}(M)$ was studied by C. Leung and is related to Gieseker stability of coherent sheaves. The MMMS equation is the case $f = \mathrm{Im}\left(e^{-i\theta}(\omega + F_D)^n\right)$ which is relevant in string theory.

18.1 Mirror symmetry dictionary

Mirror symmetry is a duality between supersymmetric conformal field theories discovered in string theory. Mathematically, it relates the complex geometry of a Calabi–Yau manifold M to the symplectic geometry of its mirror partner W. On a categorical level, this was made precise by Kontsevich's Homological Mirror Symmetry (HMS) conjecture, stating roughly that the bounded derived category of coherent sheaves on M, $D^b(M)$, is equivalent to the Fukaya category, $\mathcal{F}(W)$, of the mirror. The conjecture has been established in many specific cases, and also extended beyond the Calabi–Yau setting, but is still open in general.

In fact, the equivalence of categories is only part of the story. HMS is an equivalence of the topological (twisted) theories, the B-model on M and the A-model on W. Not all objects in these categories correspond to "physical" branes, only a subset of stable ones. On the B-side these are the solutions to deformed HYM, the MMMS equations. On the A-side they are special Lagrangian submanifolds with $U(1)$ bundle with specified curvature. This is all approximate, and the whole picture is fairly conjectural except for elliptic curves.

We will not discuss mirror symmetry in any detail in these notes, but must mention it, since it provides an explanation of sorts as to why similar structures appear in complex and symplectic geometry when we discuss stability. The basic dictionary to keep in mind is the following (from the Claymath *Mirror Symmetry* monograph).

Complex geometry	Symplectic geometry
holomorphic bundles E on analytic subvarieties	Lagrangian submanifolds L with $U(1)$ bundle
$D^b(M)$	$\mathcal{F}(W)$
$\mathrm{ch}(E)\sqrt{\mathrm{Td}(M)} \in H^{2*}$	$[L] \in H^n(W)$
gauge group	$U(1)$ gauge group on L
complexified gauge group	Hamiltonian deformations
moment map $F \wedge \omega^{n-1}$	moment map $\mathrm{Im}(\Omega|_L)$
(deformed) HYM bundles	special Lagrangians

18.2 D-branes on the 2-torus

The simplest Calabi–Yau manifold, besides the trivial case of the point, is the 2-torus. To be concrete, we have

$$M = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \tag{300}$$

for a complex structure parameter τ in the upper half-plane. A holomorphic volume form is given by $\Omega = a dz$, $a \in \mathbb{C}^*$. From the Calabi–Yau condition $\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \overline{\Omega}$ we get $\omega = |a|^2 dx \wedge dy$ so the corresponding flat Hermitian metric is

$$g = |a|^2 dz \otimes d\bar{z}. \tag{301}$$

In mirror symmetry one instead considers the parameter $\rho = iA + b$ where $A > 0$ is the area of the 2-torus and $b \in \mathbb{R}$ determines the "B-field". This gives a complexified Kähler form

$$\omega^c = (A - ib)\frac{dx \wedge dy}{\mathrm{Im}(\tau)}, \qquad \int_M \omega^c = A - ib. \tag{302}$$

(Warning: There are various different conventions!) Mirror symmetry interchanges τ and ρ, so the dual 2-torus has parameters $\tau' = \rho$, $\rho' = \tau$.

B-branes. We will first discuss B-branes on M, i.e. coherent sheaves, forming a category $\mathcal{A} = \mathrm{Coh}(M)$ which can be understood rather explicitly. Since the canonical bundle (dualizing sheaf) on M is trivial, Serre duality gives isomorphisms

$$\mathrm{Hom}(E, F) \cong \mathrm{Ext}^1(F, E)^* \tag{303}$$

for any pair of objects $E, F \in \mathcal{A}$. This is the way in which the CY1 property is expressed categorically. Another consequence of Serre duality is that $\mathrm{Ext}^{\geq 2}(E, F) = 0$, which holds for coherent sheaves on any smooth curve.

The basic numerical invariants of a coherent sheaf on M are the degree and rank. These define a homomorphism

$$(\deg, \mathrm{rk}) : K_0(\mathrm{Coh}(M)) \to \mathbb{Z}^2. \tag{304}$$

Coherent sheaves of rank zero and degree one are torsion sheaves $\mathcal{O}_p$, supported at a single point $p \in M$. These satisfy $\mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_p) = \mathbb{C}$, thus $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) = \mathbb{C}$ by duality. Such an object E with $\mathrm{Ext}^*(E, E) = H^*(S^1)$ is called *spherical*. The group Ext^1 classifies the self-extension

$$0 \to \mathcal{O}_p \to \mathcal{O}_{2p} \to \mathcal{O}_p \to 0 \tag{305}$$

and iterating we get the sequence of self-extensions $\mathcal{O}_p, \mathcal{O}_{2p}, \mathcal{O}_{3p}, \ldots$. These exhaust all the torsion sheaves on a smooth curve. The remaining indecomposable sheaves are vector bundles.

The line bundles of fixed degree form an elliptic curve, $\mathrm{Pic}^d(M)$, which is isomorphic to M itself. By a classification theorem of Atiyah, the same is true for indecomposable vector bundles of higher rank, so if we fix the numerical class $(\deg, \mathrm{rk})$, $\mathrm{rk} \geq 0$, and $\deg > 0$ if $\mathrm{rk} = 0$, then the set of indecomposable sheaves with in that class is isomorphic to M itself. This fact that indecomposable objects form one-dimensional families is specific to the genus one case, on higher genus curves we get families of arbitrary dimension.

The Atiyah classification is a consequence of a braid group action on the bounded derived category $D^b(\mathcal{A})$ discovered by Lenzing–Meltzer and Seidel–Thomas. The key notion is that of a **spherical twist** by an object $E \in D^b(\mathcal{A})$ with $\mathrm{Ext}^*(E, E) = H^*(S^1)$. It is the functor

$$T_E : D^b(\mathcal{A}) \to D^b(\mathcal{A}), \qquad F \mapsto \mathrm{Cone}(\mathrm{ev}_F) \tag{306}$$

where

$$\mathrm{ev}_F : \mathrm{Hom}^*(E, F) \otimes E \to F \tag{307}$$

is the evaluation map. For this to be functorial one should consider $D^b(\mathcal{A})$ as a dg-category. It makes sense for any object E, but the condition on E to be spherical ensures that T_E is an autoequivalence. On the level of K_0, a twist functor T_E acts by

$$[F] \mapsto [F] - \langle E, F \rangle [E] \tag{308}$$

where

$$\langle E, F \rangle = \chi \mathrm{Ext}^*(E, F) = \sum (-1)^k \dim \mathrm{Ext}^k(E, F) \tag{309}$$

as is immediate from the definition.

Fix a point $x \in M$ and consider the twist functors $T_1 = T_{\mathcal{O}}$ and $T_2 = T_{\mathcal{O}_x}$. These satisfy the braid group relation $T_1 T_2 T_1 = T_2 T_1 T_2$ and act on numerical classes by the transformations

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{310}$$

which generate $SL(2, \mathbb{Z})$. Since we can map any vector $(m, n) \in \mathbb{Z}^2$ to a vector $(\gcd(m, n), 0)$ under a transformation in $SL(2, \mathbb{Z})$ by Euclid's algorithm, this means we can reduce the classification of sheaves with numerical class (m, n) to those with rank zero, which are just the torsion sheaves.

Lecture 19: Stability on the 2-torus

After this digression on the structure of the bounded derived category of an elliptic curve, we return to stability. As it turns out, every indecomposable $E \in \mathcal{A}$ is at least semistable as a consequence of the CY1 property. To see this, recall that any coherent sheaf on a smooth curve has a Harder–Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E \tag{311}$$

with $A_k = E_k/E_{k-1}$ semistable, $\mu(A_1) > \ldots > \mu(A_n)$. Suppose first that an indecomposable $E \in \mathcal{A}$ has a two-step filtration so

$$0 \to A_1 \to E \to A_2 \to 0 \tag{312}$$

is a short exact sequence. Then $\mu(A_1) > \mu(A_2)$ implies $\mathrm{Hom}(A_1, A_2) = 0$ by stability of A_2, thus $\mathrm{Ext}^1(A_2, A_1) = 0$ by the CY1 property. But this means $E = A_1 \oplus A_2$ is the trivial extension, a contradiction. By induction one shows that no indecomposable object can have a HN filtration of length greater than one, i.e. be unstable.

The indecomposable sheaves in primitive numerical classes (m, n) are in fact stable, while those in non-primitive classes are strictly semistable, since they are self-extensions of stable objects. If we pass from slope stability in $\mathcal{A}$ to Bridgeland stability on $D^b(\mathcal{A})$, then also the shifts $E[n]$, $n \in \mathbb{Z}$, of stable coherent sheaves E are stable objects.

Although the set of (semi)stable objects on $D^b(\mathcal{A})$ is predetermined, there is still a freedom of choosing the central charge $Z : K_0(\mathcal{A}) \to \mathbb{C}$ depending only on the numerical class in $\mathbb{Z}^2$. One can choose Z in terms of the complexified Kähler class by the formula

$$Z(E) = \int_M \mathrm{ch}(E) e^{i\omega^c} = \deg(E) - \rho \mathrm{rk}(E) \tag{313}$$

where ρ is the Kähler parameter in he upper half-plane as before. This determines a real 2-dimensional subspace in the 4-dimensional space of stability conditions on $D^b(\mathcal{A})$, which can be identified with the universal cover of $GL^+(2, \mathbb{R})$, as was shown by Bridgeland.

A-branes. By homological mirror symmetry for the 2-torus (Kontsevich, Polishchuk–Zaslow) there is an equivalence of categories

$$D^b(M) \cong \mathcal{F}(W) \tag{314}$$

where $\mathcal{F}(W)$ is the Fukaya category of the dual elliptic curve, depending only on the complexified Kähler form, not the complex structure. Objects in $\mathcal{F}(M)$ are closed embedded curves $L \subset M$ together with a $U(1)$ local system and grading, which is a choice of $\mathrm{Arg}(\Omega)$ along L. If we want to include the self extensions of embedded curves (objects with non-primitive numerical class), then we need to consider higher rank local systems, but since the focus is on the *stable* objects, this is not essential. The numerical class of L is its homology class $[L] \in H^1(M, \mathbb{Z}) \cong \mathbb{Z}^2$, which is necessarily primitive for embedded curves. We can deform L by Hamiltonian symplectomorphism without changing its isomorphism class in $\mathcal{F}(M)$.

The existence of the braid group action on the category is much more obvious on the A-side: $SL(2,\mathbb{Z})$ acts on the 2-torus and is its mapping class group. The universal central extensions of $SL(2,\mathbb{Z})$ acts on graded Lagrangians.

The special Lagrangian condition $\mathrm{Im}(e^{i\phi}\Omega_L) = 0$ just means that L is a straight line on M of slope $\tan(\phi)$, a closed geodesic loop with respect to the flat metric g. These appear in one-dimensional families, each foliating the torus. The $U(1)$ monodromy of the local system on L adds another real parameter, so in total we get a torus of stable objects for each primitive class in $H^1(M,\mathbb{Z})$ — just as in $D^b(M)$, as expected. The central charge is given by integration over $\Omega = adz$, so

$$Z([L]) = \int_L \Omega \tag{315}$$

meaning that $Z(m,n) = a(m+\tau n)$.

The case of the 2-torus is missing in important feature, which is wall-crossing, making it somewhat atypical. Wall-crossing refers to the change of the set of (semi)stable objects as one varies the parameters determining stability, i.e. the complexified Kähler class on the B-side or the complex structure and holomorphic volume form on the A-side.

19.1 Stability in abelian categories

After these examples, we move to a more systematic discussion, starting with the case of abelian categories. Thus let $\mathcal{A}$ be some abelian category. In all examples we care about this is an essentially small category, and we should make this assumption to not get into foundational troubles. Recall that the Grothendieck group of $\mathcal{A}$, $K_0(\mathcal{A})$, is the abelian group generated by the objects $A \in \mathcal{A}$ with relations $A - B + C = 0$ for every short exact sequence $0 \to A \to B \to C \to 0$. The notion of stability depends on a **polarization** given by a function

$$Z : K_0(\mathcal{A}) \to \mathbb{C} \tag{316}$$

which is also called the *stability function* or *central charge*. We want to impose the strong positivity assumption that

$$0 \neq E \implies Z(E) \in \left\{ re^{i\phi} \mid r \in (0,\infty), \phi \in [0,\pi) \right\} \tag{317}$$

i.e. that $Z(E)$ lies either in the upper half plane or the positive real axis for every object which is not the zero object. This ensures that for every $E \neq 0$ there is a well defined **phase** $\phi(E) = \arg(Z(E)) \in [0,\pi)$.

Exercise. The above definition works better for some categories than for others. Show that when $\mathcal{A}$ is the category of finite rank abelian groups, then there is no polarization satisfying the above condition. On the other hand, when $\mathcal{A}$ is the category of finite groups, then there is an infinite-dimensional space of them.

Having fixed a polarization in the above sense, we say that an object $0 \neq E \in \mathcal{A}$ is **semistable** (resp. **stable**) if for all proper subobjects $0 \neq F \subset E$ we have $\phi(F) \leq \phi(E)$ (resp. $\phi(F) < \phi(E)$). One can equivalently define stability using quotient objects.

Exercise. Show that morphisms between semistable objects increase phase, i.e. $\mathrm{Hom}(E, F) = 0$ for semistable E, F with $\phi(E) > \phi(F)$.

The second property we want Z to satisfy is that every object in $\mathcal{A}$ is an extension of semistables. More precisely, a *Harder–Narasimhan filtration* for an object $E \in \mathcal{A}$ is an increasing filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_n = E \tag{318}$$

with $A_i = F_i/F_{i-1}$ semistable and $\phi(A_1) > \ldots > \phi(A_n)$. Using the result of the previous exercise one can show by induction that such a filtration is unique if it exists. The polarization Z is said to satisfy the **Harder–Narasimhan property** if every object has A has a Harder–Narasimhan filtration. One way to show this property is by the following result of Bridgeland.

Proposition 15. *Suppose Z satisfies the following chain conditions: There are no infinite sequences of subobjects $E_1 \supset E_2 \supset E_3 \supset \ldots$ with $\phi(E_1) < \phi(E_2) < \ldots$, and there are are no infinite sequences of quotients $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots$ with $\phi(E_1) > \phi(E_2) > \ldots$. Then Z satisfies the Harder–Narasimhan property.*

Proof. The idea is that if $0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = E$ is a HN filtration of E, then the quotient map $E \to E/F_{n-1} = A_n$ has the universal property that any other non-zero quotient map $E \to B$ satisfies $\phi(B) \geq \phi(A_n)$ with equality if and only if it factors through $E \to A_n$ via a quotient $A_n \to B$. Using the chain conditions one shows that every $0 \neq E$ has a *maximally destabilising quotient* in this sense. The result then follows by induction and the first chain condition. □

The conditions of the proposition are satisfied, for any Z, if $\mathcal{A}$ is Artinian. By definition an abelian category is **Artinian** if every descending chain of subobjects $E_1 \supset E_2 \supset E_3 \supset \ldots$ eventually stabilizes, or equivalent every chain of quotient maps $E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \ldots$. Examples of Artinian categories are finite-dimensional modules over an algebra, but for example $\mathrm{Coh}(X)$ is not Artinian if $\dim(X) > 0$. The group $K_0(\mathcal{A})$ is also easy to understand for Artinian $\mathcal{A}$, it is just the free abelian group on the set of isomorphism classes of simple objects. This is a consequence of the Jordan–Hölder property.

19.2 Novikov field and limiting stability

We can replace $\mathbb{C}$ as the target of the stability function Z by the *Novikov field* over $\mathbb{C}$. One can think of this as a formal family Z_t of polarizations with $Z_t(E)$ approaching 0 or ∞ for some objects E.

The Novikov field over a field K is the set of formal Laurent series with real powers

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i t^{\lambda_i} \mid a_i \in K, \lambda_i \in \mathbb{R}, \lambda_i \text{ strictly increasing}, \lambda_i \to +\infty \right\} \tag{319}$$

and addition, multiplication defined in the usual manner. Note that the assumptions on λ_i are important to make the product well-defined. The field λ has a valuation $\mathrm{val} : \Lambda \to \mathbb{R} \cup \{+\infty\}$

given by $\mathrm{val}(0) = +\infty$ and

$$\mathrm{val}(a_0 t^{\lambda_0} + a_1 t^{\lambda_1} + \ldots) = \lambda_0 \tag{320}$$

provided $a_0 \neq 0$ and $\lambda_0 < \lambda_1 < \ldots$. A corresponding norm is given by

$$\|a\| = e^{-\mathrm{val}(a)} \tag{321}$$

satisfying the ultrametric inequality

$$\|a + b\| \leq \max(\|a\|, \|b\|) \tag{322}$$

and making Λ a complete non-Archimedean field. The ring of integers in Λ is the closed unit disk

$$\Lambda_{\leq 1} = \{a \in \Lambda \mid \|a\| \leq 1\} \tag{323}$$

which contains the open unit disk $\Lambda_{<1}$ of series involving only positive powers of t as a maximal ideal.

We specialize to the case $K = \mathbb{C}$. Note that Im, Re make sense on Λ if we define them coefficient-wise. The exponential function, defined as a power series, converges on the open unit disk in Λ, but since the coefficient field is $\mathbb{C}$ we can extend it to the closed unit disk by

$$\exp(a_0 + a_1 t^{\lambda_1} + \ldots) = \exp(a_0) \exp(a_1 t^{\lambda_1} + \ldots). \tag{324}$$

Here exp is defined as a power series in both cases, but $\exp(a_0)$ converges in the topology in $\mathbb{C}$ while $\exp(a_1 t^{\lambda_1} + \ldots)$ converges in the t-adic topology! Also, $\mathrm{val}(\exp(a)) = 0$, so exp maps the closed unit disk to the unit circle in Λ. An inverse (with $2\pi i\mathbb{Z}$ ambiguity) is provided by the logarithm, which can be defined by the same trick. A multi-valued argument can be defined on the unit circle by $\mathrm{Arg}(a) = \mathrm{Im}(\log(a))$, but if we think of t as a real parameter, then it makes sense to put $\mathrm{Arg}(t^\lambda a) = \mathrm{Arg}(a)$, extending it to all non-zero elements of Λ. The argument is then a number in

$$\Lambda_{\mathbb{R},\leq 1} = \{a \in \Lambda \mid \mathrm{Im}(a) = 0, \|a\| \leq 1\} \subset \Lambda \tag{325}$$

which can be totally ordered with positive elements those series with positive leading coefficient.

With the above definitions it makes sense to consider polarizations $Z : K_0(\mathcal{A}) \to \Lambda$. For any $0 \neq E \in \mathcal{A}$ the condition becomes that the leading coefficient of $Z(E)$ is in the upper half-plane or the positive real axis.

Lecture 20: t-structures

20.1 t-structures

The definition of slope stability depends crucially on the notion of a subobject, which makes sense in abelian categories, but not in triangulated categories. One way of defining stability for these categories is to first choose some abelian subcategory $\mathcal{A} \subset \mathcal{C}$. This should satisfy some basic properties, making it the heart of a *t-structure*, a notion due to Beilinson–Bernstein–Deligne.

A **t-structure** on a triangulated category $\mathcal{C}$ is a full subcategory $\mathcal{P} \subset \mathcal{C}$ with $\mathcal{P}[1] \subset \mathcal{P}$ and such that if we define the right orthogonal

$$\mathcal{P}^{\perp} = \{B \in \mathcal{C} \mid \mathrm{Hom}(A, B) = 0 \text{ for all } A \in \mathcal{P}\} \tag{326}$$

then for every $E \in \mathcal{C}$ there is a triangle $A \to E \to B \to A[1]$ with $A \in \mathcal{P}$ and $B \in \mathcal{P}^{\perp}$.

Exercise. Show that $\mathcal{P}$ is the left orthogonal to $\mathcal{P}^{\perp}$, $\mathcal{P}^{\perp}[-1] \subset \mathcal{P}^{\perp}$, and both subcategories are closed under extensions. This means that if $A \to E \to B \to$ is a triangle with $A, B \in \mathcal{P}$, then $E \in \mathcal{P}$ also. For the pedantic, $E \in \mathcal{P}$ means that E is isomorphic to an object in $\mathcal{P}$, or we simply demand all full subcategories to be closed under isomorphism.

Exercise. Show that the triangle $A \to E \to B \to$ in the definition of a t-structure is unique up to isomorphism, that the inclusion $\mathcal{P} \subset \mathcal{C}$ admits a right adjoint, and that the inclusion $\mathcal{P}^{\perp} \subset \mathcal{C}$ admits a left adjoint.

The motivating example is when $\mathcal{P}$ is the full subcategory of a derived category $D(\mathcal{A})$ of complexes C with $H^i(C) = 0$ for $i > 0$. The definition also allows for quite degenerate examples, i.e. $\mathcal{P} = \mathcal{C}$ or $\mathcal{P} = 0$. Since the objects in $\mathcal{A} \subset D(\mathcal{A})$ are characterized by having cohomology concentrated in degree zero, this suggests how to get an abelian category from a t-structure. Namely, the **heart** of a t-structure $\mathcal{P} \subset \mathcal{C}$ is the full subcategory

$$\mathcal{A} = \mathcal{P} \cap (\mathcal{P}^{\perp}[1]) \tag{327}$$

which in fact turns out to be abelian, with short exact sequences exactly the triangles $A \to B \to C \to$ with all vertices in $\mathcal{A}$. The idea to show that $\mathcal{A}$ is abelian is that if $f : A \to B$ is a morphism in $\mathcal{A}$, $C = \mathrm{Cone}(f)$, then there is an exact triangle $K \to C \to L \to$ with $K \in \mathcal{P}[1]$ and $L \in \mathcal{P}^{\perp}[1]$, and $K[-1]$ should be the kernel of f and L the cokernel.

By the previous exercise the inclusion $\mathcal{P}[-n] \subset \mathcal{C}$ has a right adjoint, $\tau_{\leq n}$, and the inclusion $\mathcal{P}^{\perp}[1-n] \subset \mathcal{C}$ has a left adjoint, $\tau_{\geq n}$, which are called truncation functors. it turns out that that there is a natural isomorphism of functors

$$\tau_{\leq m} \circ \tau_{\geq n} \cong \tau_{\geq n} \circ \tau_{\leq m} \tag{328}$$

so in particular we can define zeroth cohomology

$$H^0 := \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{C} \to \mathcal{A} \tag{329}$$

and $H^n = H^0 \circ [n]$.

We cannot expect the $H^k(E)$ defined above to tell us much about E for a general t-structure. This is ensured by the following property. A t-structure is **bounded** if

$$\mathcal{C} = \bigcup_{n \geq 0} \mathcal{P}[-n] \cap \mathcal{P}^{\perp}[n] \tag{330}$$

which means that for every object we have $E = \tau_{\leq N}(\tau_{\geq -N} E)$ for N sufficiently large, i.e. every object is supported in bounded degrees relative to the t-structure. One can show that when $\mathcal{P} \subset \mathcal{C}$ is a bounded t-structure, then $\mathcal{P}$ is the closure of the heart $\mathcal{A}$ under extension and [1], thus completely determined by $\mathcal{A}$. This is allows an alternative definition of a bounded t-structure as full additive subcategory $\mathcal{A} \subset \mathcal{C}$ such that

1. $\mathrm{Ext}^{<0}(A, B) = 0$ for $A, B \in \mathcal{A}$ and

2. for every $E \in \mathcal{C}$ there is a tower of triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow^{\cdot\cdot\cdot} & \downarrow & & & \nwarrow^{\cdot\cdot\cdot} & \downarrow & \nwarrow^{\cdot\cdot\cdot} & \downarrow \\ & & A_1 & & & & A_{n-1} & & A_n \end{array} \tag{331}$$

   with $A_i \in \mathcal{A}[k_i]$, $k_1 > \ldots > k_n$.

Exercise. Show that this definition is equivalent to the previous one.

Exercise. Verify that when $\mathcal{A} \subset \mathcal{C}$ is the heart of a bounded t-structure, then $K_0(\mathcal{A}) = K_0(\mathcal{C})$.

The tower of triangles which appears in the above definition should remind you of the HN-property. Indeed, one possible definition of a **stability structure** on a triangulated category $\mathcal{C}$ is a pair $(\mathcal{A}, Z)$ where $\mathcal{A} \subset \mathcal{C}$ is the heart of a bounded t-structure and $Z : K_0(\mathcal{C}) \to \mathbb{C}$ is a central charge with the HN-property. Note that given a stability structure we can "break up" any object $E \in \mathcal{C}$ into its cohomology components in $\mathcal{A}$, and further break up those into semistable (or stable) objects using the HN-property.

We should emphasize that if $\mathcal{A} \subset \mathcal{C}$ is the heart of a bounded t-structure, then it is usually not the case that $\mathcal{C}$ is equivalent to $D^b(\mathcal{A})$, i.e. most t-structures do not come from the construction of the bounded derived category. Thus, we need to find some other ways of finding t-structures in triangulated categories, an important one of which is tilting.

20.2 Tilting

Intuitively, tilting is a "cut-and-paste" construction for hearts of bounded t-structures. A "cut" is a choice of **torsion pair** in an abelian category $\mathcal{A}$ which is a pair of full subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$, such that

1. $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$,

2. for every $E \in \mathcal{A}$ there is a short exact sequence

$$0 \to T \to E \to F \to 0 \tag{332}$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

As a first example let $\mathcal{A}$ be the category of finitely generated abelian groups, $\mathcal{T} \subset \mathcal{A}$ the subcategory of finite abelian groups, and $\mathcal{F} \subset \mathcal{F}$ the category of free abelian groups (of finite rank). The first property, $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, is clear since free groups do not contain torsion elements. For every abelian group $A \in \mathcal{A}$ we have a short exact sequence

$$0 \to \mathrm{Tors}(A) \to A \to A/\mathrm{Tors}(A) \to 0 \tag{333}$$

and $F = A/\mathrm{Tors}(A)$ is free. In fact the sequence obviously splits, i.e. we have $E \cong \mathrm{Tors}(E) \oplus F$, but not canonically.

There is a generalization of all this when X is a smooth algebraic curve and $\mathcal{A} = \mathrm{Coh}(X)$, the category of coherent sheaves on X. (The previous example was the case $X = \mathrm{Spec}(\mathbb{Z})$.) Define $\mathcal{T} \subset \mathcal{A}$ to be the category of torsion sheaves, i.e. those supported on a finite set, and $\mathcal{F} \subset \mathcal{A}$ the category of locally free sheaves, i.e. vector bundles. Again a sequence $0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ splits, since

$$\mathrm{Ext}^1(F, T) = \mathrm{Hom}(T, F \otimes K)^* = 0 \tag{334}$$

by Serre duality. Here K denotes the canonical line bundle.

A stability function Z on an abelian category $\mathcal{A}$ satisfying the Harder–Narasimhan property gives a whole family of torsion pairs. First, if $E \in \mathcal{A}$ has HN filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ with $A_i = E_i/E_{i-1}$ semistable, $\phi(A_1) > \ldots > \phi(A_n)$, then define

$$\phi_-(E) = \phi(A_n), \qquad \phi_+(E) = \phi(A_1) \tag{335}$$

i.e. the lower and upper bound for the phases of semistable factors of E. For $\phi \in (0, \pi)$ define

$$\mathcal{T} = \{E \in \mathcal{A} \mid \phi_-(E) \geq \phi\}, \qquad \mathcal{F} = \{E \in \mathcal{A} \mid \phi_+(E) < \phi\} \tag{336}$$

which one can easily check is a torsion pair. Note that as ϕ varies, the subcategories $\mathcal{T}, \mathcal{F}$ jump whenever there is semistable object in $\mathcal{A}$ which phase ϕ.

We assume now that $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{C}$. Of course, one could take $\mathcal{C} = D^b(\mathcal{A})$, but this is not the most general case. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$ define the **left tilt**

$$\mathcal{A}^L = \left\{E \in \mathcal{C} \mid H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \text{ for } i \neq -1, 0\right\} \tag{337}$$

which contains the torsion pair $(\mathcal{F}[1], \mathcal{T}) \subset \mathcal{A}^L$. Similarly, define the **right tilt**

$$\mathcal{A}^R = \left\{E \in \mathcal{C} \mid H^0(E) \in \mathcal{F}, H^1(E) \in \mathcal{T}, H^i(E) = 0 \text{ for } i \neq 1, 0\right\} \tag{338}$$

which contains the torsion pair $(\mathcal{F}, \mathcal{T}[-1]) \subset \mathcal{A}^R$. Note that $\mathcal{A}^R[1] = \mathcal{A}^L$, so the two constructions give the same t-structure up to shift. Repeating the construction, and choosing different torsion pairs each time, we start building a graph, where the vertices are the t-structures and two of them are connected by an edge if they are related by tilting.

Lecture 21: Mutation

21.1 Cluster Mutation

We consider a special case which recovers the rules of *cluster mutation* of quivers. Let $\mathcal{C}$ be a CY3 triangulated category, linear over a field, meaning that

$$\mathrm{Hom}^k(E,F) \cong \mathrm{Hom}^{3-k}(F,E)^* \tag{339}$$

for all objects $E, F \in \mathcal{C}$. Recall that an object $S \in \mathcal{C}$ is spherical if $\mathrm{Ext}^*(E,E) = H^*(S^3)$. Suppose we have a finite collection of spherical objects $S_1, \ldots, S_n$ which generate the category. The collection is called *cluster* if for every $i \neq j$ the space $\mathrm{Ext}^k(S_i, S_j)$ is zero for $k \neq 1, 2$, and non-zero for at most one $k \in \{1, 2\}$. Ranks of Ext-groups are then recorded by the matrix of the Euler form

$$a_{ij} = -\chi \mathrm{Ext}^*(S_i, S_j) \tag{340}$$

or more pictorially the quiver Q with vertices $1, \ldots, n$ and $\dim \mathrm{Ext}^1(S_i, S_j)$ arrows from i to j. By construction, the quiver Q does not have any loops or 2-cycles.

Under the assumptions one can show that the S_i generate the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{C}$, and that $\mathcal{A}$ is Artinian with simple objects $S_1, \ldots, S_n$. In particular, $K_0(\mathcal{C}) \cong \mathbb{Z}^n$, and we get stability structures on $\mathcal{C}$ by specifying n numbers in the upper half-plane.

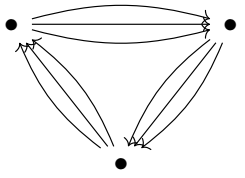
Examples of categories with cluster spherical collections come from non-compact CY 3-folds. For instance, take $Y = O_{\mathbb{P}^2}(-3)$, the total space of the canonical bundle over $\mathbb{P}^2$. The space Y is also obtained from $\mathbb{C}^3/(\mathbb{Z}/(3))$ by blowing up the origin. In fact, the derived McKay correspondence states that

$$D^b(Y) \cong D^b(\mathrm{Coh}_{\mathbb{Z}/(3)}(\mathbb{C}^3)) \tag{341}$$

where $\mathbb{Z}/(3) = \{1, \omega, \omega^2\} \subset U(1)$ is acting on $\mathbb{C}^3$ by rescaling. Here $D^b(\mathrm{Coh}_{\mathbb{Z}/(3)}(\mathbb{C}^3))$ is the category of equivariant coherent sheaves which can be described as the category of finitely generated modules over the non-commutative algebra

$$A = \mathbb{C}[x,y,z] \star \mathbb{C}[\mathbb{Z}/(3)] \tag{342}$$

which is the path algebra over the quiver



(343)

with commuting relations. The Artinian category of nilpotent modules over A corresponds to the category of sheaves supported at the origin of $\mathbb{C}^3$, which the McKay correspondence sends to complexes of sheaves with cohomology supported on the zero section of $O_{\mathbb{P}^2}(-3)$. This subcategory gives an example of a CY3 category generated by a cluster spherical collection.

Alternatively, and more directly, one can start with the quiver Q and construct a CY3 category with corresponding cluster spherical collection. For non-acyclic quivers the category depends on a choice of *potential*, which is a (possibly infinite) linear combination of cycles in Q. For details we refer to section 8 of the preprint *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations* by Kontsevich–Soibelman.

Tilting for cluster spherical collections is also known as *mutation*. We fix some S_i, then **right mutation** of the cluster collection at S_i gives a new spherical collection $(S'_1, \ldots, S'_n)$ with

$$S'_j = \begin{cases} S_j & \mathrm{Ext}^1(S_i, S_j) = 0 \\ S_i[-1] & i = j \\ T_{S_i}(S_j) & \mathrm{Ext}^1(S_i, S_j) \neq 0 \end{cases} \tag{344}$$

where

$$T_{S_i}(S_j) = \mathrm{Cone}(S_i[-1] \otimes \mathrm{Ext}^1(S_i, S_j) \to S_j) \tag{345}$$

is the spherical twist by S_i. Note that by definition there are exact triangles

$$S_j \to S'_j \to S_i \otimes \mathrm{Ext}^1(S_i, S_j) \tag{346}$$

for $i \neq j$, so all the S'_j belong to the abelian category generated by the S_j. It is possible that the S'_j do not form a cluster collection, but this is in a certain sense a non-generic phenomenon.

The relation to tilting is the following. Let $\mathcal{A}$ be the abelian category generated by $S_1, \ldots, S_n$, $\mathcal{T}$ the extension closed subcategory generated by S_i, and $\mathcal{F}$ the extension closed subcategory generated by the S_j, $i \neq j$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ and the simple objects in the right tilt $\mathcal{A}^L$ are the S'_j.

Let's look at what is happening on the level of $K_0(\mathcal{C}) = \mathbb{Z}^n$. Let $v_j = [S_j] \in \mathbb{Z}^n$, $v'_j = [S'_j] \in \mathbb{Z}^n$, then

$$v'_j = \begin{cases} v_j & a_{ij} \leq 0 \\ -v_i & i = j \\ v_j + a_{ij} v_i & a_{ij} > 0 \end{cases} \tag{347}$$

where $a_{ij} = \dim \mathrm{Ext}^1(S_i, S_j)$ as before. The corresponding matrix $a'_{jk} = -\chi \mathrm{Ext}^*(S'_j, S'_k)$ obtained from a_{jk} by this change of basis is

$$a'_{jk} = \begin{cases} a_{jk} + a_{ji} a_{ik} & a_{ji}, a_{ik} > 0 \\ -a_{jk} & j = i \text{ or } k = i \\ a_{jk} & \text{else} \end{cases} \tag{348}$$

Using the correspondence between skew-symmetric matrices and quivers without loops or 2-cycles one can describe (348) in terms of the following operation on quivers. As before, i is some fixed vertex of Q.

1. For every pair of arrows $\alpha : j \to i, \beta : i \to k$ add an arrow $j \to k$.

2. Cancel any 2-cycles $j \to k \to j$ created in the previous step.

3. Switch the orientation of all arrows which start or end at i.

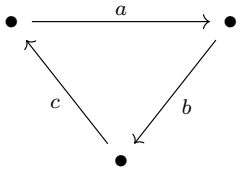
Exercise. Show that mutation is an involution on the level of quivers.

The set of quivers obtained from a given one by mutation can be finite or infinite. As an example of the former, consider the quiver $\bullet \to \bullet \to \bullet$ of type $A2$. The mutation equivalent quivers are the orientations of the $A2$ Dynkin diagram (three up to isomorphism), as well as the triangular quiver



(349)

An example where we get infinitely many quivers is (343). The quivers we obtain are of the form



(350)

where the positive integers a, b, c indicate the number of arrows and must satisfy the cubic equation

$$a^2 + b^2 + c^2 = abc. \tag{351}$$

Bernhard Keller has written an online applet to automate quiver mutation which can be found on his webpage.

We will later see that cluster spherical collections give open chambers in the space $\mathrm{Stab}(\mathcal{C})$ of stability structures. The walls between these chambers are *walls of the second kind* where by definition the central charge of some stable object is a real number. Crossing the wall from one chamber to the next is precisely mutation of the spherical collection.

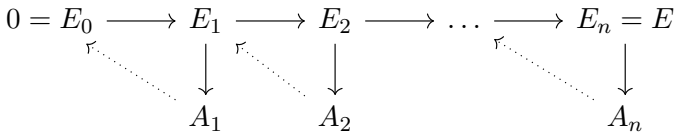
21.2 The space of stability conditions

We fix a triangulated category $\mathcal{C}$ and an additive map $\mathrm{cl} : K_0(\mathcal{C}) \to \Gamma$ where Γ is a finite rank abelian group. A **stability structure** for the pair $(\mathcal{C}, \Gamma)$ is given by a full additive subcategory $\mathcal{C}^\phi \subset \mathcal{C}$ of **semistable objects** for each phase $\phi \in \mathbb{R}$ and a **central charge** $Z : \Gamma \to \mathbb{C}$. The data is required to satisfy the following axioms.

1. $\mathcal{C}^\phi[1] = \mathcal{C}^{\phi+\pi}$

2. If $\phi_1 > \phi_2$, $E \in \mathcal{C}^{\phi_1}$, $F \in \mathcal{C}^{\phi_2}$, then $\mathrm{Hom}(E, F) = 0$.

3. (Harder–Narasimhan property) For every $E \in \mathcal{C}$ there is a tower of triangles



(352)

with $0 \neq A_k \in \mathcal{C}^{\phi_k}$ for some $\phi_1 > \ldots > \phi_n$.

4. For $0 \neq A \in \mathcal{C}^{\phi}$ one has $Z(E) := Z(\mathrm{cl}([E])) \in \mathbb{R}_{>0} e^{i\phi}$.

5. (Support property) There is a norm on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and a $C > 0$ so that

$$\|\gamma\| \leq C|Z(\gamma)| \tag{353}$$

for $\gamma \in \Gamma$ the class of a semistable object.

This is slightly more restrictive than Bridgeland's original definition, which does not include the lattice Γ or the support property (but something similar to it). In examples, these stronger axioms usually hold though, and are reasonable assumptions in general.

Lemma 7. *For each $\phi \in R$ the subcategory $\mathcal{C}^{\phi} \subset \mathcal{C}$ is abelian and finite length.*

By definition, the **stable** objects of phase ϕ are the simple objects in $\mathcal{C}^{\phi}$. We define $\phi_-(E)$ and $\phi_+(E)$ as the minimum and maximum, respectively, of phases of semistable components of $E \in \mathcal{C}$, as before. Let

$$\mathcal{C}^{\geq \phi} := \{E \in \mathcal{C} \mid \phi_-(E) \geq \phi\} \tag{354}$$

$$\mathcal{C}^{\leq \phi} := \{E \in \mathcal{C} \mid \phi_+(E) \leq \phi\} \tag{355}$$

and similarly $\mathcal{C}^{>\phi}$ and $\mathcal{C}^{<\phi}$.

Lemma 8. *For each ϕ the subcategory $\mathcal{A} = \mathcal{C}^{\geq \phi} \cap \mathcal{C}^{<\phi+\pi}$ is the heart of a bounded t-structure on $\mathcal{C}$. Same for $\mathcal{C}^{>\phi} \cap \mathcal{C}^{\leq \phi+\pi}$.*

In particular the lemma provides the link to the other definition of a stability structure based on t-structures. Warning: The subcategory $\mathcal{C}^{\geq \phi_1} \cap \mathcal{C}^{<\phi_2}$ is in general not abelian.

Lecture 22: The space of stability structures

Let's denote by $\mathrm{Stab}(\mathcal{C},\Gamma)$ the set of all stability structures for the pair $(\mathcal{C},\Gamma)$. A topology is defined on the set of stability structures so that $\sigma_k \to \sigma$ if

$$Z_{\sigma_k} \to Z_\sigma \text{ in } \mathrm{Hom}(\Gamma,\mathbb{C}) \tag{356}$$

$$\phi_{\pm,\sigma_k}(E) \to \phi_{\pm,\sigma}(E) \text{ uniformly in } E \in \mathcal{C}. \tag{357}$$

The topology is Hausdorff. What is not so obvious is that $\mathrm{Stab}(\mathcal{C},\Gamma)$ is a manifold, which follows from the main result of Bridgeland.

Theorem 20. *The map*

$$p : \mathrm{Stab}(\mathcal{C},\Gamma) \to \mathrm{Hom}(\Gamma,\mathbb{C}) \tag{358}$$

which assigns to a stability structure its central charge, is a local homeomorphism. Thus, $\mathrm{Stab}(\mathcal{C},\Gamma)$ *can be given the structure of a complex manifold of dimension* $\mathrm{rk}(\Gamma)$.

Intuitively, the theorem states that if we deform Z a little bit, then there is uniquely determined new collection of semistable objects which satisfies the axioms. The new collection is close to the original one in the sense that $\phi_\pm$ change by some bounded amount. The idea of the proof is to "recompute" semistability using slope stability for the deformed central charge in a suitable subcategory $\mathcal{C}^{>\phi-\epsilon}\cap\mathcal{C}^{<\phi+\epsilon}$ which is contained in an abelian category, so subobjects make sense.

Note that $z\in\mathbb{C}$ acts on $\mathrm{Stab}(\mathcal{C},\Gamma)$ by

$$Z \mapsto \exp(z)Z, \qquad \mathcal{C}^\phi \mapsto \mathcal{C}^{\phi+\mathrm{Im}(z)} \tag{359}$$

i.e. shifting phases of semistable objects and rescaling the central charge. With a bit more work one can extend this to an action of the universal cover of $GL^+(2,\mathbb{R})$. Also the group $\mathrm{Aut}(\mathcal{C})$ of (exact) autoequivalences of $\mathcal{C}$ up to natural isomorphism acts on $\mathrm{Stab}(\mathcal{C},\Gamma)$, assuming that $\mathrm{cl} : K_0(\mathcal{C}) \to \Gamma$ is surjective and its kernel preserved by $\mathrm{Aut}(\mathcal{C})$.

22.1 Stability structures on $D^b(A_n)$

Let Q be an orientation of the A_n Dynkin diagram, say

$$\bullet \to \bullet \to \ldots \to \bullet \tag{360}$$

for concreteness. Fixing a field $\mathbb{K}$ we can consider the bounded derived category, $\mathcal{C}_n = D^b(\mathbb{K}Q)$, of the path algebra, $\mathbb{K}Q$, of Q. The choice of $\mathbb{K}$ itself will be irrelevant in the following. Also changing the orientation on Q leads to the same triangulated category, up to equivalence. The categories $\mathcal{C}_n$ are the simplest triangulated categories, but still $\mathrm{Stab}(\mathcal{C}_n)$ is a non-trivial space with intricate wall-and-chamber structure.

We note first that $\mathcal{C}_n$ has only finitely many (isomorphism classes of) indecomposable objects, up to shift. Namely, an elementary linear algebra proof shows that an indecomposable representation of Q is of the form

$$0 \to \ldots \to 0 \to \mathbb{K} \xrightarrow{1} \ldots \xrightarrow{1} \mathbb{K} \to 0 \to \ldots \to 0 \tag{361}$$

so there are $n(n+1)/2$ of them. Since $\mathrm{Rep}(\mathbb{K}Q)$ is of cohomological dimension one, any indecomposable object in $\mathcal{C}_n$ is a representation up to shift.

A nice way of presenting this classification geometrically is to consider a disk D with $n+1$ marked points on the boundary. Indecomposable objects up to shift in $\mathcal{C}_n$ then correspond to arcs (unoriented embedded curves) in D connecting distinct points on the boundary, up to isotopy. If we consider these arcs together with an orientation and a choice of argument of the slope (assuming $D \subset \mathbb{C}$, say), then we get rid of the shift ambiguity. Moreover what is nice about this is that intersection points between arcs correspond to morphisms in $\mathcal{C}_n$ and arcs forming a triangle in D also form an exact triangle in $\mathcal{C}_n$!. The higher level reason for this is that $\mathcal{C}_n$ is the Fukaya category, $\mathcal{F}(D)$, of D. To rigorously set up the definition of the Fukaya category, even for surfaces, requires a considerable amount of technical details, so we will content ourselves with the hand-waving explanation, at least for now.

A consequence of the classification is that if $\mathcal{A} \subset \mathcal{C}_n$ is the heart of a bounded t-structure, then $\mathcal{A}$ has only finitely many indecomposable objects, so it must be Artinian. Moreover, since $K_0(\mathcal{A}) \cong K_0(\mathcal{C}_n) \cong \mathbb{Z}^n$, the number of simple objects in $\mathcal{A}$ must be exactly n. The n simple objects, $S_1, \ldots, S_n$, correspond to some arcs in D, but not all collections of n arcs can come from a t-structure. Using the fact that $\mathrm{Ext}^{\leq 0}(S_i, S_j) = 0$ for $i \neq j$ one sees that the S_i form a tree in D_n. Since it has n edges, it must include all the $n+1$ marked points on the boundary as vertices. Taking into account the grading one gets the following.

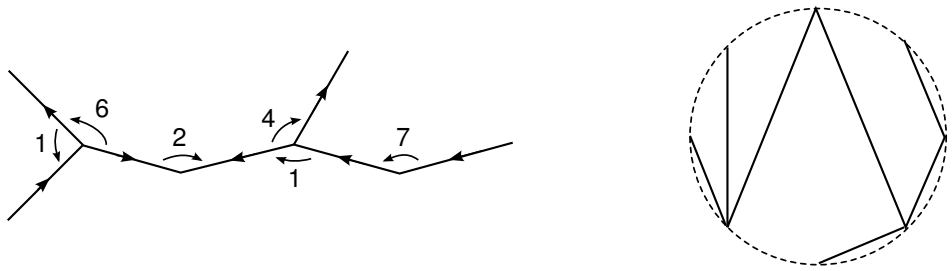


Figure 2: Tree describing a t-structure in $\mathcal{C}_7$.

Theorem 21. *Bounded t-structures on $\mathcal{C}_n$, up to autoequivalence, are in one-to-one correspondence with trees with n edges together with a linear order on the edges meeting a given vertex and an integer $d \geq 1$ for every pair of consecutive edges meeting a vertex.*

The t-structures with all $d = 1$ are those which come from $\mathrm{Mod}(\mathbb{K}Q) \subset \mathcal{C}_n$ for various orientations of Q. In the opposite extreme, if all $d > 1$, then the heart is semisimple. We will later see a more geometric interpretation of t-structures as certain foliations on D.

Since stability structures are given by bounded t-structures together with a central charge Z, the above theorem also gives one description of all stability structures on $\mathcal{C}_n$. Namely, $\mathrm{Stab}(\mathcal{C}_n)$ is obtained by gluing together copies of H^n, one for each bounded t-structure, along walls of the second kind. Two such chambers are glued if the corresponding t-structures are related by tilting. What is not so clear from this description is the wall-and-chamber structure for walls of the *first* kind, as well as the global structure of $\mathrm{Stab}(\mathcal{C}_n)$ as a complex manifold.

Lecture 23: Quadratic differentials and stability structures – A_n example

Let's continue by first stating the main result for $\mathrm{Stab}(\mathcal{C}_n)$.

Theorem 22. *There is an isomorphism of complex orbifolds*

$$\mathrm{Stab}(\mathcal{C}_n)/\mathrm{Aut}(\mathcal{C}_n) = \left\{\exp(P(z))dz^2 \mid \deg P = n+1\right\}/\mathrm{Aut}(\mathbb{C}) \tag{362}$$

$$= \left\{\exp\left(z^{n+1} + a_{n-1}z^{n-1} + \ldots + a_0\right)dz^2\right\}/\mathbb{Z}/(n+1)\mathbb{Z} \tag{363}$$

for $n > 1$. For $n = 1$ the right hand side has an extra $\mathbb{Z}/2$ stabilizer.

Without passing to quotients one can also show similarly that $\mathrm{Stab}(\mathcal{C}_n) = \mathbb{C}^n$ as complex manifold.

Let us explain the result. The expression $\exp P(z)dz^2$ denotes a quadratic differential, i.e. a section of the square of the holomorphic cotangent bundle. It is clearly non-vanishing on $\mathbb{C}$ and has a transcendental singularity at infinity. Generally, if $\varphi \in \Gamma(X, (T^*X)^{\otimes 2})$ is a non-vanishing quadratic differential on a Riemann surface X, then $|\varphi|$ defines a flat (locally Euclidean) metric on X. The locally define indefinite integrals

$$\int \sqrt{\varphi} \tag{364}$$

which are determined up to constant and choice of square-root, give local coordinates in which $\varphi = dz^2$, so the flat metric is the standard on, $g = dx^2 + dy^2$, $z = x + iy$. Note that transition functions between these coordinates are of the form $w = \pm z + c$. In the present case, the integral is globally defined and of the form

$$\int \exp\left(z^{n+1} + a_{n-1}z^{n-1} + \ldots + a_0\right)dz \tag{365}$$

which is a kind of generalized Gaussian integral.

Let us write S_{sm} for $\mathbb{C}$ together with the metric $|\varphi|$. It turns out that S_{sm} is not complete as a metric space, and $S \setminus S_{sm}$, where $S = \overline{S_{sm}}$ is the metric completion, consists of $n+1$ points, which are so called infinite-cone angle singularities of the flat metric. To understand what is going on, we take $n = 0$, where (365) reduces to the exponential function $\mathbb{C} \to \mathbb{C}^*$. Thus S can be described, as a metric space, as the universal cover of punctured Euclidean $\mathbb{R}^2$. The completion, S, adds a single point p over the puncture. A geodesic starting at p projects to a straight line in $\mathbb{R}^2$ starting at the puncture. In particular, the angle between two such geodesic can be an arbitrary positive real number. We call this an **infinite angle singularity** of the metric. The general case $n > 0$ is more complicated, since the integral cannot be computed explicitly, but the local structure of the metric is the same.

To understand the global structure of S, we look at **saddle connections**, which are smooth geodesics connecting two of the conical points. Generally, there is a unique geodesic between any two points in $\overline{S}$, but that geodesic may be only piecewise smooth, breaking at

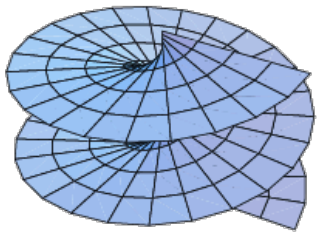


Figure 3: Approximate picture of an infinite-angle singularity.

one or more singular points of the metric. We can go one step further and form the convex hull of the $n+1$ singular points $S \setminus S_{sm}$, called the **core** of S. Here are some examples of what the core can look like in the case $n = 3$.

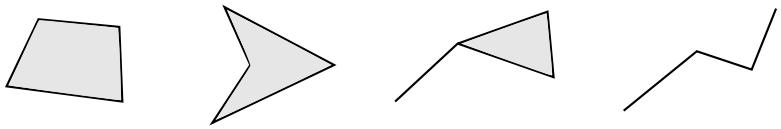


Figure 4: Possible topologies of the core when $n = 3$ for a generic choice of φ. The number of saddle connections is 6, 5, 4, 3 respectively.

Although S is not homeomorphic to a closed disk, it is clear we can correspond objects in $\mathcal{C}_n$ to embedded arcs in S with endpoints on the singular points in much the same way. Each such arc α defines a homology class

$$[\alpha] \in H_1(S, S \setminus S_{sm}; \mathbb{Z}) \cong \mathbb{Z}^n \tag{366}$$

which can be identified with the class in $[E] \in K_0(\mathcal{C}_n) \cong \mathbb{Z}^n$ of the corresponding object, E, in $\mathcal{C}_n$. The path integral over $\sqrt{\varphi}$ then defines a map

$$Z : H_1(S, S \setminus S_{sm}; \mathbb{Z}) \to \mathbb{C}), \qquad [\alpha] \mapsto \int_\alpha \sqrt{\varphi}. \tag{367}$$

This gives the data of a central charge on $\mathcal{C}_n$. If we let the stable objects correspond to saddle connections, i.e. those arcs which admit a *smooth* geodesic representative, then this data can be shown to satisfy Bridgeland's axioms. The proof of the HN property goes roughly as follows: It suffices to consider indecomposable objects. Each indecomposable object E corresponds to some arc α, which is isotopic to a concatenation of saddle connections in S. These saddle connections then give the stable components of E.

The t-structure of the stability condition constructed from φ can be seen as follows. First, every quadratic differential defines a **horizontal foliation** of tangent vectors v with $\varphi(v, v) \geq 0$. For the quadratic differentials we are considering the leaf space of the horizontal foliation is a tree with vertices corresponding to the $n+1$ singular points of the metric. The

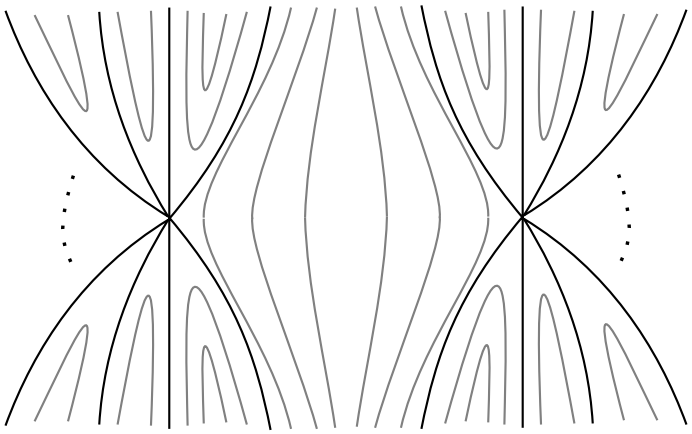


Figure 5: Illustration of the horizontal foliation for $\varphi = \exp(z^2)dz^2$.

local structure of the infinite-angle singularities shows that there are a countable number of edges meeting every vertex.

All but finitely many of those are however *open edges* which do not connect to a vertex at the other end and are of infinite length in the induced metric. The remaining edges, those of finite length, form a subtree which describes the simple objects in the heart of the t-structure as above.

Lecture 24: Flat metrics and stability structures

The correspondence between stability structures on Fukaya categories of surfaces and quadratic differentials extends as follows.

Geometry. There are two points of view. From the first we consider pairs (C, φ) where C is a compact Riemann surface and φ is a quadratic differential which is meromorphic or, more generally, also allowed to have singularities of the form

$$\exp(f(z))g(z)dz^2 \tag{368}$$

where f has a pole at $z = 0$ and g is meromorphic. These are called **exponential type** singularities of φ. The pair (C, φ) depends on both discrete (topological) and continuous parameters. Discrete parameters are for example the genus of C and the number and types of singularities of φ. Continuous parameters are locally (on the moduli space) given by the periods of $\sqrt{\varphi}$, i.e. integrals

$$\int_\alpha \sqrt{\varphi} \tag{369}$$

where α is a closed loop or path ending at singularities of φ such that the integral converges. The integral only depends on the class of α in a suitable homology group.

Given a pair (C, φ) we can puncture C at the singularities of φ, then consider it as a topological surface S_{sm} with flat metric $|\varphi|$ and take the metric completion $S = \overline{S_{sm}}$, which is a flat surface with conical singularities with cone angles $\{\pi, 3\pi, 4\pi, \ldots, \infty\}$. The relation between the special points of φ and the geometry of S goes as follows.

Quadratic differential	Flat metric
simple pole	conical point, angle π
zero of order n	conical point, angle $(n+2)\pi$
second order pole	infinite cylindrical end
higher order pole	end with infinite area
exponential singularity $\exp(z^{-n})g(z)dz^2$	n ∞-angle conical points n ends with infinite area

Conversely one can show that any complete flat surface with conical singularities as above and finite topological type (finite rank of H_1 and finite number of singularities) comes from some pair (C, φ).

Stable objects. In the general case there are two types of finite-length (maximal) geodesics.

1. **Saddle connections**, which are geodesic (straight) paths $[0, 1] \to S$ ending at conical points. The starting and ending points may coincide. These kind of geodesics are *rigid*, i.e. they have no continuous deformations as geodesics.

2. **Geodesic loops**, which are closed (periodic) geodesics $S^1 \to S$. The flat surface S looks locally like a cylinder near the image of the loop, so one can parallel transport it until one hits a conical point. In particular, geodesic loops deform by one continuous real parameter.

To make these objects of the Fukaya category one needs to choose also an argument of the slope, i.e. an argument of $\varphi(v, v)$ where v is a vector tangent to the geodesic.

Category. The relevant category is what could be called the *partially wrapped Fukaya category* of the surface X which is the complement of the singular points of φ. In general, one should choose a symplectic form ω on S. When X is compact the symplectic structure is determined by the total area, up to symplectomorphism. Note that in our case X can only be compact if $g = 1$, since there are no non-vanishing holomorphic quadratic differentials on curves of genus $\neq 1$. When X is non-compact, on the other hand, the symplectic form is exact, $\omega = d\theta$, and one requires X to be modeled on $S^1 \times (R, \infty)$ near the punctures. Thus, the symplectic area is infinite and there is only one ω on a given punctured oriented surface, up to symplectomorphism. The upshot is that in the punctured case one can, for many purposes, forget about the symplectic structure entirely.

The Fukaya category is a priori only $\mathbb{Z}/2$-graded, meaning that $\mathrm{Hom}^n(X, Y)$ is 2-periodic in n. This is an issue for stability, since for any X, Y in the heart of a bounded t-structure we should have $\mathrm{Ext}^{<0}(X, Y) = 0$. To get a $\mathbb{Z}$-graded version of the Fukaya category one needs to choose a foliation on X, i.e. a section of the projectivized tangent bundle. The category then depends, up to equivalence, only on connected component of the space of all such sections, i.e. on a discrete parameter. As an example, when $X = T^*S^1$ is a cylinder then the foliation is essentially determined by its winding number $k \geq 0$ around the zero section. The corresponding Fukaya category may be described as modules over the algebra $\mathbb{K}[x^{\pm}]$ where the degree of x is k. In the context of quadratic differentials or flat surfaces the foliation is taken to be the horizontal one.

The Fukaya category $\mathcal{F}(S)$ of a surface S can be defined either based on Floer theory (disk counting) or as global sections of a cosheaf of categories on a ribbon graph. The first approach is more useful for present purposes. Let us assume also that we have chosen a 1-form θ with $d\theta = \omega$ non-degenerate. Objects in $\mathcal{F}(S)$ are then represented by properly immersed curves $\alpha : D \to S$ defined on an open interval $(0, 1)$ or S^1 such that θ pulls back to an exact form. This condition is trivially satisfied when the domain is $(0, 1)$, and can be achieved for curves with domain S^1 after applying an isotopy. There are two additional pieces of data.

1. A **grading**, which is a continuous choice of argument of the quadratic differential along α, i.e. a choice of $\arg \Omega(v, v)$ for vectors v tangent to α. If the domain is S^1, there is a topological obstruction to the existence of such a grading. Clearly, choices of grading form a $\mathbb{Z}$-torsor.

2. A **local system** of finite-dimensional vector spaces (over the fixed ground field $\mathbb{K}$) over α. When the domain is $(0, 1)$, this amounts to giving a vector space over $\mathbb{K}$. A

local system on S^1, on the other hand, is essentially a vector space V together with an automorphism $A : V \to V$, the monodromy of the local system.

One can also allow collections of curves as above, they corresponding to direct sums of objects.

If A, B are objects as above, i.e. immersed curves with grading and local system, then $\mathrm{Hom}(A, B)$ is a direct sum of vector spaces, one for each intersection point between $\phi(A)$ and B where ϕ is a Hamiltonian isotopy of S which wraps curves infinitely many times around the punctures. This means that the Hamiltonian is arbitrary, except near the punctures where it grows quadratically towards $+\infty$ in a coordinate which identifies it with the standard symplectic cylinder. The vector space attached to an intersection point p is the space of linear maps from the fiber of the local system over A at p to the fiber of the local system over B at p. The grading depends on the choices of arguments, $\phi_p(A), \phi_p(B) \in \mathbb{R}$, at p and is given by

$$\lceil \phi_p(A) - \phi_p(B) \rceil \in \mathbb{Z}. \tag{370}$$

The A_∞-structure on $\mathcal{F}(S)$, i.e. the differential, composition, and higher compositions, is determined by counting immersed disks in S with boundary on several immersed curves and $n \geq 2$ transverse intersection points.

A priori the category $\mathcal{F}(S)$ is not closed under cones or direct summands, so one should formally enlarge it. It turns out though that this is not necessary.

Theorem 23. *$\mathcal{F}(S)$ is equivalent to its category $\mathrm{Perf}(\mathcal{F}(S))$ of perfect complexes, i.e. closed under shift, cones, and direct summands.*

This simple looking result amounts to a classification of perfect complexes over $\mathcal{F}(S)$ in terms of immersed curves, essentially a linear problem, but a very non-trivial one.

Stability. First, a quadratic differential gives a stability condition on $\mathcal{F}(S)$.

Theorem 24. *Let (C, φ) be a compact Riemann surface together with a quadratic differential with zeros, poles, and exponential singularities, and $\mathcal{F}$ the corresponding Fukaya category. Then there is a stability structure on F with stable objects of phase ϕ the objects given by curves with constant argument ϕ of φ (thus straight lines in the flat metric) and central charge given by the integral over $\sqrt{\varphi}$.*

Suppose $\mathcal{M}$ is a moduli space of pairs (C, φ) where we fix the topological parameters: the genus of C, the number and types of poles of φ, and the homotopy class of the horizontal foliation of φ. We also want elements of $\mathcal{M}$ to be *marked surfaces* in the sense of Teichmüller space, so that a suitably defined mapping class group acts on $\mathcal{M}$. There is essentially a single Fukaya category $\mathcal{F}$ attached to $\mathcal{M}$, i.e. independent of the particular choice of element in $\mathcal{M}$. The main result is the following.

Theorem 25. *There is a canonical embedding $\mathcal{M} \to \mathrm{Stab}(\mathcal{F})$ with image a union of connected components. This means that every component of the space of flat surfaces is identified with a component of the space of stability conditions on a Fukaya category of a surface.*

It is plausible that the image is in fact everything, and this can be shown in some examples, but proving this in general seems to require new ideas.