LOW DEGREE POINTS ON CURVES

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1. Introduction

Let C be a nice (smooth, projective and geometrically integral) curve over a number field k. In this paper we investigate the sets

$$C_e := \left\{ \bigcup_{[F:k]\leqslant e} C(F) \right\}$$

of algebraic points on C with residue degree bounded by e.

When $e = 1$, this is the set of k-rational points on C. If the genus of C is 0 or 1, then there is always a finite extension K/k of the base field over which $(C_K)_1 = C(K)$ is infinite. On the other hand, if the genus of C is at least 2, then for all finite extensions K/k, Faltings' theorem guarantees that the set $(C_K)_1$ is finite [5]. While understanding the set of rational points is an interesting and subtle problem, here we will be primarily concerned with the infinitude of the sets C_e as e varies. Define the arithmetic degree of irrationality to be

$$\mathrm{a.irr}_k(C) := \min(e : C_e \text{ is infinite}).$$

This invariant is *not* preserved under extension of the ground field, so we also define

$$\mathrm{a.irr}_{\overline{k}}(C) := \min(e : \text{there exists a finite extension } K/k \text{ with } (C_K)_e \text{ infinite}).$$

As is implicit in the notation, this notion depends only upon the $\overline{k}$-isomorphism class of C, see Remark 3.2. The situation for k-points can therefore be summarized as:

$$\mathrm{a.irr}_{\overline{k}}(C) = 1 \Leftrightarrow \text{genus of } C \leqslant 1$$

For $e \geqslant 2$, the situation for higher genus curves is more interesting. Recall that the k-gonality of C/k,

$$\mathrm{gon}_k(C) := \min(e : \text{there exists a dominant map } C \to \mathbb{P}^1_k \text{ of degree } e),$$

is a measure of the "geometric degree of irrationality" of C. This notion is also not invariant under extension of the base field (e.g., a genus 0 curve has k-gonality 1 if and only if it has a k-point). For that reason we also define the geometric gonality $\mathrm{gon}_{\overline{k}}(C) := \mathrm{gon}_{\overline{k}}(C_{\overline{k}})$ to be value stable under algebraic extensions. If $f\colon C \to \mathbb{P}^1_k$ is dominant of degree at most e, then $f^{-1}(\mathbb{P}^1(k)) \subset C_e$. Therefore we always have the upper bound

$$\mathrm{a.irr}_k(C) \leqslant \mathrm{gon}_k(C). \tag{1}$$

This bound need not always be sharp: if $f\colon C \to E$ is a dominant map of degree at most e onto a positive rank elliptic curve E, then $f^{-1}(E(k)) \subset C_e$ is also infinite. When $e = 2$ (resp. $e = 3$) then Harris–Silverman [8] (resp. Abramovich–Harris [1]) showed

$$\mathrm{a.irr}_{\overline{k}}(C) = e \Leftrightarrow e \text{ is minimal such that } C_{\overline{k}} \text{ is a degree } e \text{ cover of a curve of genus} \leqslant 1.$$

Debarre–Fahlaoui [2] gave examples of curves lying on projective bundles over an elliptic curve that show the analogous result is *false* for all $e \geqslant 4$. The arithmetic degree of irrationality is therefore a subtle invariant of a curve, capturing more information than only low degree maps.

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Implicit in the work of Abramovich–Harris [1] and explicit in a theorem of Frey [7], is the fact that Faltings' Theorem implies that if C_e infinite, then C admits a map of degree at most $2e$ onto $\mathbb{P}^1_k$. Therefore we have an inequality in both directions

$$\operatorname{gon}_k(C)/2 \leqslant \operatorname{a.irr}_k(C) \leqslant \operatorname{gon}_k(C). \tag{2}$$

In this paper, we develop and apply geometric techniques to compute $\operatorname{a.irr}_k(C)$ and $\operatorname{gon}_k(C)$ when C lies on a smooth auxiliary surface S. The first result in this direction is that the inequalities in (2) are sharp.

Theorem 1. *Given any number field k and a pair of integers $\alpha, \gamma \geqslant 1$, there exists a nice curve C/k such that*

$$\operatorname{a.irr}_k(C) = \operatorname{a.irr}_{\overline{k}}(C) = \alpha, \qquad \operatorname{gon}_k(C) = \operatorname{gon}_{\overline{k}}(C) = \gamma \tag{3}$$

if and only if $\gamma/2 \leqslant \alpha \leqslant \gamma$. In fact, for $\gamma \geqslant 4$, the equalities (3) *are satisfied for all smooth curves in numerical class (γ, α) on $S = E \times \mathbb{P}^1_k$, where E/k is a positive-rank elliptic curve.*

Using these geometric techniques, we next describe classes of curves where the arithmetic and geometric degrees of irrationality agree; that is, where there are as few points as allowed by the gonality.

The first explicit examples of this kind were given by Debarre and Klassen for smooth plane curves C/k of degree d sufficiently large. Max Noether calculated the gonality for $d \geqslant 2$:

(1) If $C(k) \neq \varnothing$, then $\operatorname{gon}_k(C) = d-1$, and all minimal degree maps are projection from a k-point of C, and
(2) If $C(k) = \varnothing$, then $\operatorname{gon}_k(C) = d$.

For smooth plane curves of degree $d \geqslant 8$, Debarre–Klassen [3] prove an arithmetic strengthening of this result:

(1) If $C(k) \neq \varnothing$, then C_{d-2} is finite, and so $\operatorname{a.irr}_k(C) = \operatorname{gon}_k(C) = d-1$. Furthermore, all but finitely many points of degree $d-1$ come from intersecting C with a line over k through a k-point of C.
(2) If $C(k) = \varnothing$, then C_{d-1} is finite, and so $\operatorname{a.irr}_k(C) = \operatorname{gon}_k(C) = d$.

We first generalize this result to smooth curves lying on arbitrary smooth surfaces of geometric Picard rank 1.

Theorem 2. *Suppose that C embeds in a smooth surface S/k having* $\operatorname{Pic}(S_{\overline{k}}) = \mathbb{Z} \cdot \mathcal{O}_S(1)$*, with $\mathcal{O}_S(1)$ very ample and $C \in |\mathcal{O}_S(\alpha)|$ for $\alpha \geqslant 10$. Then*

$$\operatorname{a.irr}_k(C) = \operatorname{gon}_k(C).$$

In particular, by Corollary 2.6, $(\alpha-1)\mathcal{O}_S(1)^2 \leqslant \operatorname{gon}_{\overline{k}}(C) \leqslant \operatorname{gon}_k(C)$, and so there are finitely many points of degree strictly less than $(\alpha-1)\mathcal{O}_S(1)^2$ on C_K for any finite extension K/k.

Using [9, Theorem 3.1], this result about geometric Picard rank 1 surfaces is enough to deduce the analogous result for most complete intersection curves in $\mathbb{P}^n_k$, generalizing Debarre and Klassen's result when $n = 2$.

Corollary 3. *Let C/k be a smooth complete intersection curve in $\mathbb{P}^n_k$, $n \geqslant 3$, of type $10 \leqslant d_1 < d_2 \leqslant \cdots \leqslant d_{n-1}$. Then*

$$\operatorname{a.irr}_k(C) = \operatorname{gon}_k(C).$$

In particular, by Lazarsfeld's computation of the minimal gonality of such a curve [11, Exercise 4.12]*, there are finitely many points of degree strictly less than $(d_1-1)d_2 \cdots d_{n-1} \leqslant \operatorname{gon}_{\overline{k}}(C) \leqslant \operatorname{gon}_k(C)$.*

In fact, Theorem 2 generalizes to sufficiently ample curves lying on any surface with discrete geometric Picard group, i.e. $h^1(S, \mathcal{O}_S) = 0$. The effective condition $\alpha \geqslant 10$ translates into the exclusion of finitely many classes from any closed subcone of the ample cone.

Theorem 4. *Let S/k be a smooth projective surface with $h^1(S, \mathcal{O}_S) = 0$. For any closed subcone $N \subseteq \mathrm{Amp}(S)$, there exists a finite set $\mathrm{Exc}(N)$ of integral classes in N such that if $C \subset S$ is a smooth curve with class $[C] \in N \smallsetminus \mathrm{Exc}(N)$, then*

$$\mathrm{a.irr}_k(C) = \mathrm{gon}_k(C).$$

Remark 5. The proof of Theorem 4 is constructive; in Appendix A we show that if N is rational polyhedral, the set $\mathrm{Exc}(N)$ is effectively computable in terms of the divisor structure on the surface S; we direct the interested reader there for precise statements.

Given a particular surface S, our techniques are very amenable to explicit computations of this exceptional set $\mathrm{Exc}(N)$, and even of an exceptional set for the entire ample cone. For example:

Theorem 6. *Let C be a nice curve of type (d_1, d_2), with $1 \leqslant d_1 \leqslant d_2$, on $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Then if $(d_1, d_2) \neq (2,2)$ or $(3,3)$, we have that $\mathrm{a.irr}_k(C) = \mathrm{gon}_k(C) = d_1$. In particular, this let's us compute:*

$$\mathrm{a.irr}_{\overline{k}}(C) = \begin{cases} 1 & : \ d_1 \leqslant 1 \ \text{or} \ (d_1, d_2) = (2,2), \\ 2 & : \ d_1 = 2 \ \text{and} \ d_2 \geqslant 3, \ \text{or} \ (d_1, d_2) = (3,3) \ \text{and} \ C \ \text{bielliptic}, \\ d_1 & : \ \text{otherwise}. \end{cases}$$

Remark 7. We say that a point $P \in C(\overline{k})$ is sporadic if $[k(P) : k] < \mathrm{a.irr}_k(C)$. From the perspective of the arithmetic of elliptic curves, there is much interest in understanding sporadic points on modular curves, e.g. the classical $X_1(N)$, since these indicate "usual" level structure. As $X_1(N(\mathbb{Q}) \neq \varnothing$, whenever $\#J_1(N)(\mathbb{Q})$ is finite, Faltings' Theorem implies that $\mathrm{a.irr}_{\mathbb{Q}}(X_1(N)) = \mathrm{gon}_{\mathbb{Q}}(X_1(N))$. In particular this holds for $N \leqslant 55$ and $N \neq 37, 43, 53$; moreover, Derickx and van Hoeij compute the gonality (and therefore the arithmetic degree of irrationality when $N \neq 37$) for all $N \leqslant 40$ [4]. It is our hope that the geometric techniques we develop here might prove useful for specific curves of arithmetic interest.

As in previous work, the proofs of these results begin by translating the problem of understanding degree e points on C to understanding rational points on $\mathrm{Sym}^e C =: C^{(e)}$, which is a parameter space for effective divisors of degree e on C. There is a natural map

$$C^{(e)} \to \mathrm{Pic}^e C,$$

sending an effective divisor D to the class of the line bundle $\mathcal{O}(D)$. We denote the image of this map $W_e(C)$. We now have two problems: understand the infinitude of rational points on the fibers of $C^{(e)} \to \mathrm{Pic}^e C$ (which is related to the dimension of the space of sections of the corresponding line bundle), and understand the infinitude of rational points on the image $W_e(C)$ (which, by Faltings' Theorem, is related to positive-dimensional abelian varieties in $W_e(C)$).

The majority of this paper is therefore consumed proving purely geometric results over $\mathbb{C}$ about nonexistence of positive-dimensional abelian varieties in $W_e(C)$ for appropriate e. Using the theory of stability conditions on vector bundles, we show that such an abelian variety in $W_e C$ forces the existence of a certain type of effective divisor on S. Given a particular surface S, we can often use the geometry of S to obtain a contradiction; this is how we proceed with Theorem 1. When the surface is not explicitly given, the fact that such a divisor class does not move in a positive dimensional family (from $h^1(S, \mathcal{O}_S) = 0$) allows us to construct an embedding of the abelian variety into $W_f C$ for smaller f and eventually obtain a contradiction.

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2. Abelian Varieties in W_eC

In this section we prove purely geometric results (Theorems 2.7, 2.9, and 2.10) about nonexistence of abelian subvarieties that will imply our main theorems. Therefore the basefield is assumed to be $\mathbb{C}$, unless otherwise noted, and $\operatorname{gon}(C) := \operatorname{gon}_{\overline{k}}(C)$ denotes the geometric gonality.

The proofs of these results will proceed by contradiction: the existence of a positive-dimensional abelian variety $A \subset W_eC$ will force the existence of a family of effective divisors of moderately low degree moving in basepoint-free pencils. We will then use a geometric lemma proved in the Section 2.1 to produce interesting effective divisors on an auxiliary surface containing the curve. Then we may directly use the geometry of a specific surface to obtain a contradiction, or use this divisor to inductively produce such a family of effective line bundles on C of even lower degree that will force a contradiction for all but finitely many possible starting classes of curves C.

The first step in this procedure relies on the following observation, due originally to Abramovich and Harris [1, Lemma 1], and whose consequence for the gonality of C was noted by Frey [7]. To set notation, assume that $A \subset W_eC$ is an abelian variety of dimension at least 1 and $A \not\subset x + W_{e-1}(C)$ for any x. Let A_2 denote the image of $A \times A$ under the addition map $W_e(C) \times W_e(C) \to W_{2e}(C)$. Note that A_2 is (noncanonically) isomorphic to A: a choice of basepoint in A induces an isomorphism $\operatorname{Pic}^e C \simeq \operatorname{Jac}_C$, under which the addition map on W_eC agrees with the group law on Jac_C and $A \subset \operatorname{Jac}_C$ is an abelian subvariety.

Lemma 2.1. *The line bundle L_p corresponding to a point $p \in A_2 \subseteq W_{2e}C \subseteq \operatorname{Pic}^{2e} C$ is basepoint-free and has*

$$r(L_p) := h^0(C, L_p) - 1 \geqslant \dim A.$$

Proof. Let $C^{(e)}$ denote the eth symmetric power of C. We have the following commutative diagram

$$\begin{array}{ccc} C^{(e)} \times C^{(e)} & \xrightarrow[\text{dom.}]{\text{finite}} & C^{(2e)} \\ \downarrow & & \downarrow \\ W_e(C) \times W_e(C) & \longrightarrow & W_{2e}(C) \\ \cup\!| & & \cup\!| \\ A \times A & \longrightarrow & A_2 \end{array}$$

Given a point $p \in W_{2e}(C)$, the fibers of the map $C^{(2e)} \to W_{2e}(C)$ are of dimension $r(L_p)$. As the fibers of the bottom map $A \times A \to A_2$ are $(\dim A)$-dimensional, we see that if $p \in A_2$, the fiber of $C^{(2e)} \to W_{2e}(C)$ over p must be at least this large. Furthermore, if $x \in C$ is in the base locus of this $(\dim A)$-dimensional linear system, then it would necessarily be the case that x is always in the linear system parameterized by the points of A. This is impossible, as we assumed that A is not contained in a translate of $W_{e-1}(C)$. ☐

2.1. Linear series of low degree.

In this section we prove the key geometric input on linear series of moderately low degrees on curves C whose class is ample on a surface S. This is a purely geometric result over an algebraically closed field of characteristic 0.

We first recall some of the basic theory of torsion-free coherent sheaves on varieties over $\mathbb{C}$. Let F be a torsion-free coherent sheaf on a smooth variety $X/\mathbb{C}$ of dimension m. Given an ample class H on X, we define the *slope* $\mu_H(F)$ of F by the equation

$$\mu_H(F) := \frac{c_1(F) \cdot H^{m-1}}{\operatorname{rk}(F)}.$$

In what follows, we leave the reliance on H implicit, and just refer to the slope as $\mu(F)$.

The sheaf F is called μ-semistable (with respect to H) if there is no inclusion of coherent sheaves $E \to F$ such that

$$\mu(E) > \mu(F).$$

Otherwise we say that F is μ-unstable (with respect to H).

The μ-semistable sheaves are the building blocks of torsion-free coherent sheaves on X. More precisely, for F any torsion-free coherent sheaf, by [10, Theorem 1.6.7] there exists a unique *Harder-Narasimhan filtration* of F,

$$0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$$

which is defined to satisfy the properties

(1) Each quotient
$$G_i := \frac{F_i}{F_{i-1}}$$
is a torsion free μ-semistable sheaf.

(2) If $1 \leqslant i < j \leqslant n$, then $\mu(G_i) > \mu(G_j)$.

In particular, we will use the fact that given an unstable torsion free coherent sheaf F, there is a unique nonzero subsheaf $E \subset F$ such that F/E is semistable and torsion free, and $\mu(E)$ is maximal among subsheaves of F. We call this E the *maximal destabilizing subsheaf* of F.

We also define the discriminant of a coherent sheaf F on a smooth complex projective surface in terms of Chern characters as the quantity

$$\Delta(F) := 2\operatorname{ch}_0(F)\operatorname{ch}_2(F) - \operatorname{ch}_1(F)^2.$$

The following fundamental theorem of Bogomolov [10, Theorem 3.4.1] implies that the property of μ-stability of sheaves on surfaces is numerical.

Theorem 2.2 (Bogomolov Inequality). *Let S be a smooth complex projective surface. If F is a μ-semistable torsion-free coherent sheaf on S with respect to some ample class, then $\Delta(F) \leqslant 0$.*

We now apply Bogomolov's Inequality to prove a geometric result that will ultimately produce the bounds we desire.

Proposition 2.3. *Let S be a smooth projective surface and $C \subset S$ a smooth curve such that $\mathcal{O}_S(C)$ is ample. If Γ is a divisor on C that moves in a basepoint-free pencil, satisfying*

$$\deg \Gamma < C^2/4,$$

then there exists a divisor D on S satisfying the following four conditions

(1) $h^0(S, D) \geqslant 2$,

(2) $C \cdot D < C^2/2$,

(3) $\deg \Gamma \geqslant D \cdot (C - D)$.

(4) If E is any divisor on S such that

$$h^0(\mathcal{O}_C(E|_C - \Gamma)) = 0, \qquad \text{and} \qquad E \cdot C < C^2, \tag{4}$$

then $h^0(\mathcal{O}_S(E - D)) = 0$. In particular, $h^0(\mathcal{O}_C(D|_C - \Gamma)) > 0$.

Proof. As Γ moves in a basepoint-free pencil, there is a choice of two sections generating the line bundle and hence giving a surjection $\mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(\Gamma)$. This map fits into an exact sequence

$$0 \to \mathcal{O}_C(-\Gamma) \to O_C^{\oplus 2} \to \mathcal{O}_C(\Gamma) \to 0. \tag{5}$$

Let $i\colon C \hookrightarrow S$ be the inclusion map. Then $i_*\mathcal{O}_C(\Gamma)$ is a torsion sheaf on S. We define the coherent sheaf F on S via the exact sequence of coherent sheaves

$$0 \to F \to \mathcal{O}_S^{\oplus 2} \to i_*\mathcal{O}_C(\Gamma) \to 0, \tag{6}$$

where the right map factors through the surjection $\mathcal{O}_S^{\oplus 2} \to \mathcal{O}_C^{\oplus 2}$.

We first observe that F is locally free of rank 2. This is clear away from C. And given a point $x \in C$, we have the exact sequence

$$0 \to F_x \to \mathcal{O}_{S,x}^{\oplus 2} \to \mathcal{O}_{C,x} \to 0.$$

Let C be locally cut out by $f \in \mathcal{O}_{S,x}$. We have that the morphism $O_{S,x}^{\oplus 2} \to \mathcal{O}_{C,x}$ above is given by

$$(s_1, s_2) \mapsto c_1 s_1 + c_2 s_2$$

where c_1, c_2 are two elements of $\mathcal{O}_{C,x}$. Since $O_{S,x}^{\oplus 2} \to \mathcal{O}_{C,x}$ is surjective, we can assume without loss of generality that c_1 is a unit. Choose lifts $\overline{c}_1, \overline{c}_2$ of c_1, c_2 to $\mathcal{O}_{S,x}$. We have that $\overline{c}_1$ is a unit. Then the kernel F_x of the map $O_{S,x}^{\oplus 2} \to \mathcal{O}_{C,x}$ consists of pairs s_1, s_2 such that $\overline{c}_1 s_1 + \overline{c}_2 s_2$ is a multiple of f. Rewriting this condition as

$$s_1 = fs - \frac{\overline{c}_2}{\overline{c}_1} s_2$$

for some $s \in \mathcal{O}_{S,x}$, we se that $F_x \cong \mathcal{O}_{S,x}^{\oplus 2}$ with the inclusion $F_x \cong \mathcal{O}_{S,x}^{\oplus 2} \to \mathcal{O}_{S,x}^{\oplus 2}$ given by the map

$$(s, s_2) \mapsto \left(fs - \frac{\overline{c}_2}{\overline{c}_1} s_2, s_2 \right).$$

Therefore, F_x is locally free.

We set $e := \deg \Gamma$. The vector bundle F has Chern character $\mathrm{ch}(F) = (2, -[C], \frac{C^2}{2} - e)$ and hence has discriminant $\Delta(F) = C^2 - 4e$. Therefore, by assumption, $\Delta(F) > 0$, so F is μ-unstable with respect to any ample class on S; we will use C as the ample class on S.

Take L to be the maximal destabilizing subsheaf of F. L must be rank 1 as a nontrivial proper subsheaf of a rank 2 vector bundle. Moreover, since F is locally free, L must be locally free, as otherwise, the inclusion $L^{\vee\vee} \to F$ would induce an inclusion $L^{\vee\vee}/L \to F/L$ of a torsion sheaf into the quotient F/L, contradicting the torsion-freeness of F/L. So L is a line bundle. Write $L \cong \mathcal{O}(-D)$, where D is some divisor on S. We show this D satisfies properties (1)–(4) of the proposition.

- **Property (2): $C \cdot D < C^2/2$:**
  By the definition of the maximal desabilizing subsheaf, we have
  $$(-D) \cdot C > (-C) \cdot C/2 \qquad \Rightarrow \qquad C \cdot D < C^2/2,$$
  so D satisfies condition (2).
- **Property (3): $e \geqslant D \cdot (C - D)$:**
  In the exact sequence
  $$0 \to \mathcal{O}(-D) \to F \to Q \to 0, \tag{7}$$
  the quotient Q is μ-semistable with respect to C. Then $\Delta(Q) \leqslant 0$, from which we derive
  $$e \geqslant D \cdot (C - D).$$
- **Property (1): $h^0(S, D) \geqslant 2$:** Dualizing the inclusion $\mathcal{O}(-D) \to \mathcal{O}_S^{\oplus 2}$, we must show that the map $H^0(S, \mathcal{O}_S^{\oplus 2}) \to H^0(S, \mathcal{O}(D))$ is injective. If not, then we may assume that one map $H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}(D))$ is zero, and hence (since $\mathcal{O}_S$ and $\mathcal{O}(-D)$ are reflexive), the original inclusion must factor through $\mathcal{O}(-D) \to \mathcal{O}_S \to \mathcal{O}_S^{\oplus 2}$, and so D is effective.

  Because $C \cdot D < C^2/2$ and C is integral, we have that $D \cap C$ must be zero-dimensional. But then the cokernel $\mathcal{O}_D \oplus \mathcal{O}_S$ of the inclusion $\mathcal{O}(-D) \to \mathcal{O}_S^{\oplus 2}$ does not surject onto $\mathcal{O}_C(\Gamma)$ because $\mathcal{O}_D$ admits no maps onto $\mathcal{O}_C(\Gamma)$ and any single map $\mathcal{O}_S \to \mathcal{O}_C(\Gamma)$ has zeroes and hence cannot be surjective. So we have $h^0(S, \mathcal{O}(D)) \geqslant 2$.

- **Property (4): A divisor E satisfying equation (4) also satisfies $h^0(\mathcal{O}_S(E-D)) = 0$:** Let E be a divisor on S such that

$$h^0(\mathcal{O}_C(E|_C - \Gamma)) = 0, \qquad \text{and} \qquad E \cdot C < C^2.$$

By the projection formula we have

$$\mathcal{O}_S(E) \otimes i_*\mathcal{O}_C(\pm\Gamma) \simeq i_*\mathcal{O}_C(E|_C \pm \Gamma).$$

We therefore have the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F \otimes \mathcal{O}_S(E) & \longrightarrow & \mathcal{O}_S(E)^{\oplus 2} & \longrightarrow & i_*\mathcal{O}_C(\Gamma) \otimes \mathcal{O}_S(E) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \text{res}} & & \| & & \\ 0 & \longrightarrow & i_*\mathcal{O}_C(E|_C - \Gamma) & \longrightarrow & i_*\mathcal{O}_C(E|_C)^{\oplus 2} & \longrightarrow & i_*\mathcal{O}_C(E|_C + \Gamma) & \longrightarrow & 0 \end{array}$$

By assumption we have $E \cdot C < C^2$; therefore as C is ample, $h^0(\mathcal{O}_S(E - C)) = 0$, and so the vertical map res is injective on global sections. Combined with the assumption that $h^0(C, E|_C - \Gamma) = 0$, we have that $h^0(S, F \otimes \mathcal{O}_S(E)) = 0$. Tensoring (7) with $\mathcal{O}_S(E)$ and taking global sections, this implies $h^0(S, E - D) = 0$ as desired.

Since $C \cdot D < C^2/2 < C^2$, if $h^0(C, D|_C - \Gamma) = 0$, then we could take $E = D$ and obtain the contradiction $h^0(S, \mathcal{O}_S) = 0$. Hence we must have that $\mathcal{O}_C(D|_C - \Gamma)$ *is* effective. □

Remark 2.4. The use of Bogomolov's inequality as a tool for proving the existence of divisors satisfying nice positivity properties originates in Reider's proof [13] of Reider's theorem, and has been developed by Lazarsfeld [11] and others. In particular, see [9, 12] for recent applications in the Picard rank 1 case.

We now make the previous lemma effective in the case that S has Picard rank 1.

Proposition 2.5. *Let $S/\mathbb{C}$ be a smooth projective surface with $\operatorname{Pic} S \simeq \mathbb{Z} \cdot H$ for some ample effective class H. Let $C \in |\mathcal{O}_S(\alpha H)|$ be a smooth curve on S, and suppose that Γ is an effective divisor on C moving in a basepoint-free pencil. If*

$$\deg \Gamma < k(\alpha - k) \deg S \tag{8}$$

*for some $0 < k < \alpha$, and $\mathcal{O}_C((r-1)H - \Gamma)$ is **not** effective for some $r < \alpha$, then*

$$\deg \Gamma \geqslant r(\alpha - (k-1)) \deg S.$$

In particular, subject to (8),

(a) $\mathcal{O}_C((k-1)H - \Gamma)$ is always effective.
(b) As $\mathcal{O}_C(-\Gamma)$ is never effective,

$$\deg \Gamma \geqslant (\alpha - (k-1)) \deg S.$$

Proof. It follows from the arithmetic-geometric mean inequality that $k(\alpha - k) \leqslant (\alpha/2)^2$ and so $4 \leqslant \left(\frac{\alpha^2}{k(\alpha-k)}\right)$. Therefore (8) implies that

$$\deg \Gamma < \alpha^2 H^2/4 = C^2/4,$$

and so we can apply Proposition 2.3 to conclude the existence of $D = aH$ for some $a \in \mathbb{N}_{>0}$ satisfying

2.3(2): $\alpha a < \alpha^2/2$, which implies $2a < \alpha$, and
2.3(3): $\deg \Gamma \geqslant a(\alpha - a) \deg S$.

By inequality 2.3(3) and our original assumption (8), we have that

$$a(\alpha - a) < k(\alpha - k), \qquad \Rightarrow \qquad \alpha(a - k) < (a - k)(a + k).$$

If $a \geqslant k$, then it is in fact strictly greater than k, since the above inequality is strict. But then $\alpha < a + k$ and $\alpha > 2a$ together imply that $a < k$, which is a contradiction. Therefore $a \leqslant k - 1$.

We will take $E := (r-1)H$. By assumption we have that $H^0(\mathcal{O}_C((r-1)H - \Gamma)) = 0$ and since $\alpha > 0$, $r < \alpha$ gives $E \cdot C < C^2$ as well. Therefore Proposition 2.3 implies that

$$H^0(\mathcal{O}_S((r-1-a)H)) = 0, \qquad \Rightarrow a \geqslant r.$$

Putting together the two inequalities on a, we have

$$r \leqslant a \leqslant k - 1. \tag{9}$$

In particular, the maximal possible r such that $\mathcal{O}_C((r-1)H - \Gamma)$ is *not* effective is at most $k-1$; therefore $\mathcal{O}_C((k-1)H - \Gamma)$ is necessarily effective. Using 2.3(3) together with the upper and lower bounds on a provided by (9) we get

$$\deg \Gamma \geqslant r(\alpha - (k-1)) \deg S.$$

As $\mathcal{O}_C(-\Gamma)$ is never effective, we always have

$$\deg \Gamma \geqslant (\alpha - (k-1)) \deg S.$$ □

As an easy consequence, we obtain information about the gonality of curves lying in ample classes on surfaces of Picard rank 1.

Corollary 2.6. *With the same assumptions on S, C, and Γ as above, if $\deg \Gamma < \deg_{\beta H} C = \alpha\beta H^2$, and $\alpha \geqslant (\beta+1)^2$, then $\mathcal{O}_C(\beta H|_C - \Gamma)$ is effective and*

$$\deg \Gamma \geqslant (\alpha - \beta) H^2.$$

In particular, if βH is very ample, then any smooth curve $C \in |\alpha H|$ with $\alpha \geqslant (\beta+1)^2$ has gonality at least $(\alpha - \beta)H^2 = \deg C - \deg \beta H$.

Proof. Under the assumption $\alpha \geqslant (\beta+1)^2$, we have that

$$\deg \Gamma < \alpha\beta H^2 \leqslant (\beta+1)(\alpha - (\beta+1))H^2,$$

and so Γ satisfies the requirements of Proposition 2.5 with $k = \beta + 1$. Furthermore, we always have the upper bound $\operatorname{gon} C < C \cdot P$ for any very ample divisor P (in particular βH) on S: projection from a codimension 2 plane through a point of C under the embedding given by P realizes a map to $\mathbb{P}^1$ of degree equal to $C \cdot P - 1$. So any divisor Γ on C realizing the gonality satisfies $\deg \Gamma < \deg_{\beta H} C$, and therefore has degree at least $(\alpha - \beta)H^2$. □

Specializing to $\beta = 1$ gives an alternative proof, which does not use the Cayley-Bacharach condition, of [9, Lemma 4.4], used to understand the gonality of complete intersection curves.

2.2. Examples: curves on $E \times \mathbb{P}^1$.

As a first example, let us now see how these techniques apply when C is a smooth curve on $S = E \times \mathbb{P}^1$. We denote the projection maps

$$E \xleftarrow{\ \pi_1\ } E \times \mathbb{P}^1 \xrightarrow{\ \pi_2\ } \mathbb{P}^1$$

to E and $\mathbb{P}^1$, respectively. We then have that

$$\operatorname{Pic} S \simeq \pi_1^* \operatorname{Pic} E \oplus \pi_2^* \operatorname{Pic} \mathbb{P}^1.$$

As is standard, if $\mathcal{L}_1$ is a line bundle on E and $\mathcal{L}_2$ is a line bundle on $\mathbb{P}^1$, we write $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ for $\pi_1^*\mathcal{L}_1 \otimes \pi_2^*\mathcal{L}_2$. Furthermore, the Neron-Severi group is $\mathrm{NS}(S) = \mathbb{Z} \oplus \mathbb{Z}$, spanned by the classes F_1 and F_2 of fibers of the first and second projections, respectively. These satisfy the familiar intersection relations

$$F_1^2 = 0, \qquad F_2^2 = 0, \qquad F_1 \cdot F_2 = 1.$$

We will denote the numerical class $xF_1 + yF_2$ of a divisor by (x, y). The effective cone of S is then all classes with $x, y \geqslant 0$, and the ample cone is all classes with $x, y \geqslant 1$.

The following geometric result is the main ingredient in the proof of Theorem 1.

Theorem 2.7. *Let C be a smooth curve on $S = E \times \mathbb{P}^1$ in numerical class (γ, α) for*

$$2 \leqslant \gamma/2 \leqslant \alpha \leqslant \gamma.$$

Then $\operatorname{gon}(C) = \gamma$, *and W_eC does not contain any positive-dimensional abelian varieties for $e < \alpha$.*

Remark 2.8. Note that Bertini's theorem guarantees that there exist smooth curves in numerical class (γ, α) once $\gamma \geqslant 2$ and $\alpha \geqslant 1$, as the linear equivalence class is necessarily basepoint free.

Proof. For any such curve, the linear equivalence class of C is $\mathcal{O}_E(\gamma e) \boxtimes \mathcal{O}_{\mathbb{P}^1}(\alpha)$ for some point $e \in E$. The two projection maps exhibit C as a γ-sheeted cover of $\mathbb{P}^1$, and an α-sheeted cover of E. Therefore $\operatorname{gon}(C) \leqslant \gamma$ and $E \hookrightarrow W_\alpha C$.

(1) **The gonality of C is γ:** Suppose to the contrary that Γ is a divisor on C of degree at most $\gamma - 1$ that moves in a basepoint free pencil. Then

$$\deg \Gamma \leqslant \gamma - 1 < C^2/4 = \alpha\gamma/2,$$

as $\alpha \geqslant 2$. So by Proposition 2.3, there exists an effective divisor D on S with at least 2 sections, in numerical class $xF_1 + yF_2$ for some $x \geqslant 0$ and $y \geqslant 0$ such that

2.3(2): $C \cdot D < C^2/2 \quad \Rightarrow \quad x\alpha + y\gamma < \alpha\gamma$;

2.3(3): $D \cdot (C - D) \leqslant \deg \Gamma \quad \Rightarrow \quad x\alpha + y\gamma - 2xy < \gamma \ (\leqslant 2\alpha)$.

Upon rearrangement we have

$$\alpha(\gamma/2 - x) + \gamma(\alpha/2 - y) > 0, \qquad (\gamma/2 - x)(\alpha/2 - y) > (\gamma/2)(\alpha/2 - 1) \geqslant 0$$

as $\alpha \geqslant 2$. Therefore both $(\gamma/2 - x)$ and $(\alpha/2 - y)$ have to be nonnegative. Together with the fact that $x, y \geqslant 0$, we have

$$(\gamma/2)(\alpha/2 - y) > (\gamma/2)(\alpha/2 - 1), \qquad \Rightarrow \ y = 0.$$

Plugging that back in, we see

$$(\alpha/2)(\gamma/2 - x) > (\gamma/2)(\alpha/2 - 1) \qquad \Rightarrow \ x < \gamma/\alpha \leqslant 2.$$

So $x = 0, 1$. But every divisor of numerical class 0 or F_1 has at most 1 section, which is a contradiction.

(2) **There do not exist positive-dimensional abelian varieties in W_eC for $e < \alpha$:** Suppose to the contrary that there exists a positive dimensional abelian variety $A \hookrightarrow W_eC$ for $e \leqslant \alpha - 1$; and further that e is minimal for this property. Then the points $p \in A_2$ parameterize basepoint free linear systems Γ_p of degree at most $2e \leqslant 2\alpha - 2 < \alpha\gamma/2$, since $\gamma \geqslant 4$. By part (1), the degree is at least γ. Another application of Proposition 2.3 produces a divisor D_p on $E \times \mathbb{P}^1$ with $h^0(D_p) \geqslant 2$ in numerical class $xF_1 + yF_2$. The necessary intersection theory properties are:

2.3(2): $\alpha x + \gamma y < \alpha\gamma$;

2.3(3): $\alpha x + \gamma y - 2xy \leqslant 2\alpha - 2 \leqslant 2\gamma - 2$.

Furthermore, we also have $h^0(D_p|_C - \Gamma_p)$ is effective. We may write these two inequalities as

$$\alpha(\gamma/2 - x) + \gamma(\alpha/2 - y) > 0, \qquad (\gamma/2 - x)(\alpha/2 - y) > \alpha(\gamma/4 - 1) \geqslant 0,$$

with the right-most inequality coming from our assumption that $\gamma \geqslant 4$. Therefore both $(\gamma/2 - x)$ and $(\alpha/2 - y)$ must be positive. As $-y$ is nonpositive and $(\gamma - 2x)$ is nonnegative, the second of these inequalities implies

$$\alpha(\gamma/2 - x) > \alpha(\gamma/4 - 1) \qquad \Rightarrow x < 2.$$

Similarly for y we obtain $y < 2\alpha/\gamma \leqslant 2$. Combining this with the requirement that D_p move in a pencil on $E \times \mathbb{P}^1$, we see that it must be in numerical class F_2 or $F_1 + F_2$.

By possibly enlarging D_p, there always exists a point $q \in E$ such that

$$(D|_C := \mathcal{O}_E(q) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|_C) - \Gamma_p$$

is effective. By the Kunneth formula

$$H^0(E \times \mathbb{P}^1, \mathcal{O}_E(q) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(E, \mathcal{O}_E(q)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

and so all divisors in $|D|$ are reducible: the union of the fiber $\pi_1^{-1}(q)$ and some fiber of π_2. Furthermore, as

$$\begin{aligned} H^1(S, D - C) &= H^1(E \times \mathbb{P}^1, \mathcal{O}_E(q - \gamma e) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1 - \alpha)) \\ &\simeq H^1(E, \mathcal{O}_E(q - \gamma e)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1 - \alpha)) \oplus H^0(E, \mathcal{O}_E(q - \gamma e)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1 - \alpha)) \\ &= 0, \end{aligned}$$

the map $H^0(E \times \mathbb{P}^1, D) \to H^0(C, D|_C)$ is surjective. We therefore have $h^0(C, D|_C) = 2$: every divisor linearly equivalent to $D|_C$ is the union of $\pi_1^{-1}(q) \cap C$ and $\pi_2^{-1}(z) \cap C$ for some $z \in \mathbb{P}^1$. The linear system $\left|D|_C\right|$ has baselocus exactly $\pi_1^{-1}(q) \cap C$.

By assumption $|\Gamma_p|$ is a basepoint free sub-linear series of $|D|_C|$. As such, it must not pass through the basepoints of $|D|_C|$, and so Γ_p must be supported in a fiber of the second projection $\pi_2 \colon C \to \mathbb{P}^1$. Therefore $\mathcal{O}_{\mathbb{P}^1}(1)|_C - \Gamma_p$ is effective. The fact that

$$\deg \Gamma_p \geqslant \gamma = \deg \mathcal{O}_{\mathbb{P}^1}(1)|_C$$

forces $\Gamma_p = \mathcal{O}_{\mathbb{P}^1}(1)$ for all p. This contracts the fact that A has positive dimension. □

2.3. **Cones in $N^1(S)_{\mathbb{R}}$.** We now show how Lemma 2.1 in combination with Proposition 2.3, can prove the nonexistence of positive-dimensional abelian subvarieties in W_eC when C lies on an arbitrary smooth surface with $h^1(S, \mathcal{O}_S) = 0$.

Theorem 2.9. *Let $S/\mathbb{C}$ be a smooth projective surface with $h^1(S, \mathcal{O}_S) = 0$, and let P be any very ample polarization. For any closed subcone $N \subseteq \mathrm{Amp}(S)$, there exists a finite set $\mathrm{Exc}(N)$ of integral classes in N such that if C is a smooth curve on S with $[C] \in N \smallsetminus \mathrm{Exc}(N)$, then W_eC contains no positive-dimensional abelian varieties for $e \leqslant P \cdot C$.*

We also prove an effective result for surfaces of Picard rank 1. Here we allow that the class H generating $\mathrm{Pic}\, S = \mathbb{Z} \cdot H$ may not itself be very ample, but some multiple βH is very ample; specializing $\beta = 1$ will imply the result in the introduction.

Theorem 2.10. *Let $S/\mathbb{C}$ be a smooth surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$, H ample and effective, and let $C \in |\alpha H|$ be a smooth curve with $\alpha \geqslant 8\beta^2 + \beta + 1$. Then for $e \leqslant \alpha\beta H^2 = \deg_{\beta H} C$, $W_e(C)$ does not contain any abelian varieties of dimension at least 1.*

The linear series that arise from Lemma 2.1 have degree at most twice the original degree e for which we assumed $A \subset W_eC$. Therefore to prove Theorem 2.9, we will need to use Proposition 2.3 for divisors Γ of degree up to $2P \cdot C$. The hypotheses of Proposition 2.3 already restricts us to classes $H \in N$ for which

$$2P \cdot H < H^2/4.$$

The first thing we do is show that this restriction only removed finitely many classes of curves in N. In all finiteness theorems here, it is important to keep in mind that our essential assumption $h^1(S, \mathcal{O}_S) = 0$ allows us to translate finiteness results for *numerical classes* of curves, into finiteness results for *linear equivalence classes* of curves.

Lemma 2.11. *Let $N \subseteq \operatorname{Amp}(S)$ be a closed subcone of the ample cone, and let P be a fixed very ample polarization on S. Then the set of integral classes $H \in N$ for which*

$$2P \cdot H \geqslant H^2/4,$$

is finite. Call this set $\operatorname{Exc}_1(N) := \operatorname{Exc}_1(N, P)$.

We will deduce this from the following elementary result.

Lemma 2.12. *Suppose that $N \subset \mathbb{R}^n$ is a closed cone and let $f : N \to \mathbb{R}$ be a continuous function taking positive values away from 0. Let Λ be any lattice in $\mathbb{R}^n$. If for all $H \in N$ and all $\lambda \geqslant 0$, we have*

$$f(\lambda H) = \lambda f(H),$$

then for any $c \in \mathbb{R}$, the set

$$\{H \in N : f(H) \leqslant c\} \cap \Lambda$$

is finite.

Proof. Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^n$. Set $c_{min} = \inf\{f(H) | H \in \mathbb{S} \cap N\}$. Since $\mathbb{S}$ is compact and N is closed, the intersection $\mathbb{S} \cap N$ is compact. So this minimum is achieved by f on $\mathbb{S} \cap N$, and in particular $c_{min} > 0$. By the hypothesis $f(\lambda H) = \lambda f(H)$, we then have that $f(H) > rc_{min}$ for all $H \in N \backslash B_r$, where B_r is the closed ball of radius r. Then for any $c > 0$, the set

$$\{H \in N | f(H) \leqslant c\}$$

is a closed set contained in the compact set $B_{c/c_{min}}$, and is hence compact. So its intersection with the discrete set Λ is finite. □

Proof of Lemma 2.11. Let $f : N_{\mathbb{R}} \to \mathbb{R}$ be the continuous function

$$H \mapsto f(H) = \frac{H^2}{8P \cdot H}.$$

As H and P are both ample, the function f is positive and clearly satisfies $f(\lambda H) = \lambda f(H)$. Therefore by Lemma 2.12 there are only finitely many integral classes H for which

$$f(H) = \frac{H^2}{8P \cdot H} \leqslant 1.$$

□

By an abuse of notation, if N is cone (containing the origin), we will use the notation N° for its interior **union the origin**.

Given any closed subcone $M \subseteq N^1(S)_{\mathbb{R}} = N_1(S)_{\mathbb{R}}$, we naturally have the dual cone $M^\vee \subseteq N_1(S)_{\mathbb{R}} = N^1(S)_{\mathbb{R}}$ defined by

$$M^\vee := \{D \in N_1(S)_{\mathbb{R}} : H \cdot D \geqslant 0 \text{ for all } H \in M\}.$$

This satisfies $(M^\vee)^\vee = M$. For any ample class H, write $H^\vee$ for the dual to the ray generated by H; this is a half space. For any cone $M \subset N_1(S)_{\mathbb{R}}$, write $M_{P=1}$ for the convex set

$$M_{P=1} := \{D \in M : P \cdot D = 1\}.$$

For any nonzero divisor class $R \in N^1(S)$, and any closed cone $\Lambda \subset ((R \cup P)^\vee)^\circ \subset N_1(S)_{\mathbb{R}} = N^1(S)_{\mathbb{R}}$, define the function

$$m(R, \Lambda) := m(R, \Lambda, P) := \min\left(X \cdot R : X \in \Lambda_{P=1}\right).$$

In the course of the proof of Theorem 2.9 we will also need the following two (dual) finiteness results, whose proofs follows a similar strategy.

Lemma 2.13. *Let $M \subset ((N \cup P)^\vee)^\circ$ be any closed subcone; in particular, by Kleiman's criterion for ampleness, we could take $M = \overline{\operatorname{Eff}}(S)$.*

(1) There are finitely many integral classes $H \in N$ for which $m(H, M) \leqslant c$, for any $c \in \mathbb{R}$.

(2) There are finitely many integral classes $D \in M$ at which $m(D, N) \leqslant c$ for any $c \in \mathbb{R}$.

Proof. The function m is continuous, positive (as $M \subseteq (N^\vee)^\circ$), and scales linearly by $\lambda \in \mathbb{R}$. The result therefore follows from from Lemma 2.12. □

2.4. **Gonality.** The contradiction in the proof of Theorem 2.9 will come from a lower bound on the gonality of the curve in question.

Definition 2.14. We say that the gonality of a class $H \in \mathrm{Pic}(S)$, denoted $\mathrm{gon}(H)$, is the minimal gonality of a smooth projective curve $C \in |H|$.

For any ample class H, define the set

$$\Sigma(H, n) := \{D \in \mathrm{Eff}(S) : h^0(S, D) \geqslant 2, H \cdot D < H^2/2, D \cdot (H - D) \leqslant n\}.$$

Given an ample curve in class H having a basepoint free pencil of degree $n < H^2/4$, Proposition 2.3 then guarantees, in particular, the existence of a divisor $D \in \Sigma(H, n)$.

Lemma 2.15. *If $D \in \Sigma(H, n)$, then $H \cdot D < 2n$.*

Proof. If $D \in \Sigma(H, n)$, we have in particular that

$$H \cdot D < H^2/2, \qquad \text{and} \qquad D \cdot (H - D) \leqslant n. \tag{10}$$

Furthermore, as H is ample, the Hodge Index Theorem implies that

$$H^2 D^2 \leqslant (H \cdot D)^2.$$

Combining this with the left inequality of (10), we have

$$D^2 < H \cdot D/2.$$

We combine this with the right inequality of (10), and so

$$H \cdot D - H \cdot D/2 < D \cdot (H - D) \leqslant n \qquad \Rightarrow \qquad H \cdot D < 2n.$$ □

Lemma 2.16. *For fixed H, the set $\Sigma(H, n)$ is finite.*

Proof. The function $f(D) = H \cdot D$ on $\overline{\mathrm{Eff}}(S)$ satisfies the hypotheses of Lemma 2.12, and therefore takes only finitely many values less than $2n$ for integral D. □

Lemma 2.17. *For any closed cone $M \subseteq ((N \cup P)^\vee)^\circ$ containing the pseudoeffective cone $\overline{\mathrm{Eff}}(S)$, we have that*

$$\mathrm{gon}(H) \geqslant \min(H^2/4, m(H, M)/2).$$

Proof. Let C be a smooth curve in class H. Suppose that Γ is an effective divisor moving in a basepoint-free pencil of degree $\mathrm{gon}(C)$. Then either $\mathrm{gon}(C) \geqslant C^2/4$, and we are done, or $\mathrm{gon}(C) < C^2/4$. Then by Proposition 2.3, there exists an effective divisor $D \in \Sigma(H, \mathrm{gon}(C))$ such that $h^0(C, D|_C - \Gamma) > 0$. Since D is integral and effective, we have that $D \in M$ and $D \cdot P \geqslant 1$. Therefore, using Lemma 2.15, we have

$$m(H, M) \leqslant D \cdot H < 2\,\mathrm{gon}(C),$$

as desired. □

Lemma 2.18. *For any constant $c \in \mathbb{R}$, there are finitely many integral classes $H \in N$ for which $\mathrm{gon}(H) \leqslant c$.*

Proof. Using Lemma 2.17, for any closed cone M such that $\overline{\mathrm{Eff}}(S) \subset M \subset ((N \cup P)^\vee)^\circ$, we have that

$$\min(H^2/4, m(H, M)/2) \leqslant \mathrm{gon}(H) \leqslant c.$$

There are only finitely many integral classes $H \in N$ with bounded self-intersection H^2 and with bounded $m(H, M)$ by Lemma 2.13. □

2.5. **Proof of Theorem 2.9.** By Lemma 2.11, there are only finitely many integral classes $H \in N$ for which $2P \cdot H \geqslant H^2/4$. We may therefore discard such classes into our exceptional set $\mathrm{Exc}_1(N) \subseteq \mathrm{Exc}(N)$ and assume that we are dealing exclusively with curves C such that $2P \cdot C < C^2/4$. We next prove a sufficient condition for the nonexistence of abelian varieties in W_eC.

Lemma 2.19. *Let $C \subset S$ be a smooth curve on a smooth projective surface with $h^1(S, \mathcal{O}_S) = 0$, such that $2P \cdot C < C^2/4$. If for all divisors $D \in \Sigma(C, 2P \cdot C)$, we have that*

$$D^2 < \mathrm{gon}(C)/2,$$

then W_eC contains no positive-dimensional abelian varieties for $e \leqslant P \cdot H$.

Proof. Suppose to the contrary that there existed a positive-dimensional abelian variety $A \subset W_eC$ for some $e \leqslant P \cdot H$; we further assume that e is minimal with respect to this property. Then by Lemma 2.1, the points $p \in A_2$ parameterize effective divisor classes Γ_p moving in basepoint-free pencils of degree $2e \leqslant 2P \cdot H$. Therefore, by Proposition 2.3, for each $p \in A_2$, there exists a divisor $D_p \in \Sigma(C, 2P \cdot C)$ such that

$$h^0(C, D_p|_C - \Gamma_p) \geqslant 0.$$

By the upper semicontinuity of $\dim H^0$, given any particular $D = D_p$, the locus of $q \in A_2$ such that $\mathcal{O}_C(D|_C - \Gamma_q)$ is effective is closed in A_2. Moreover, the union of all these loci is A_2. Since $\Sigma(C, 2P \cdot C)$ is finite by Lemma 2.16, there hence must be some single $D \in \Sigma(C, 2P \cdot C)$ such that $\mathcal{O}_C(D|_C - \Gamma_q)$ is effective for all $q \in A_2$.

Let $f := \deg(D|_C - \Gamma_q)$. Residuation in D therefore gives a map

$$A_2 \hookrightarrow W_fC \subset \mathrm{Pic}^f C, \qquad q \mapsto \mathcal{O}_C(D|_C - \Gamma_q).$$

By part (3) of Proposition 2.3 we have

$$f = D \cdot C - \deg \Gamma_q \leqslant D^2.$$

Again by Lemma 2.1 we have that the points of $(A_2)_2 \subseteq W_{2f}C$ parameterize effective divisor classes moving in basepoint-free pencils of degree at most $2D^2$. Since we assumed that all $D \in \Sigma(C, 2P \cdot C)$ satisfy

$$D^2 < \mathrm{gon}(C)/2,$$

this is a contradiction. ☐

We finally have that the conditions of Lemma 2.19 are satisfied for all but finitely many classes of curves H in N.

Lemma 2.20. *For any closed cone $M \subset (N^\vee)^\circ$, there are finitely many integral divisors $D \in M$ such that there exists an ample class $H \in N$ with*

$$D \cdot H < 4P \cdot H. \tag{11}$$

There is therefore a well-defined maximum value of D^2, as D ranges over integral divisors that satisfy (11) *for some H.*

Proof. Since inequality (11) is invariant under scaling H, we may assume that $H \cdot P = 1$. The result now follows from Lemma 2.13 for $m(D, N) \leqslant 4$. ☐

Lemma 2.21. *The set of all integral classes $H \in N$ satisfying $2P \cdot H < H^2/4$, for which there exists a divisor $D \in \Sigma(H, 2P \cdot H)$ with*

$$D^2 \geqslant \mathrm{gon}(H)/2,$$

is finite. Call this set $\mathrm{Exc}_2(N) := \mathrm{Exc}_2(N, P)$.

Proof. By Lemma 2.15, any divisor $D \in \Sigma(H, 2P \cdot H) \subset \overline{\mathrm{Eff}}(S)$ satisfies (11) and therefore the quantity $2D^2$ is absolutely bounded independent of H by Lemma 2.20. The result now follows from Lemma 2.18. ☐

Proof of Theorem 2.9. The set $\mathrm{Exc}(N) = \mathrm{Exc}_1(N) \cup \mathrm{Exc}_2(N)$ will consist of all integral classes $H \in N$ such that either $2P \cdot H \geqslant H^2/4$ or $\mathrm{gon}(H)$ is bounded above by the $2D^2$ for some $D \in \Sigma(H, 2P \cdot H)$. By Lemmas 2.11 and 2.21, this is a finite set.

If C is a smooth curve on S with $[C] \in N \smallsetminus \mathrm{Exc}(N)$, then $2P \cdot C < C^2/4$ and for all $D \in \Sigma(C, 2P \cdot C)$, we have that $D^2 < \mathrm{gon}(C)/2$. Therefore C satisfies the hypotheses of Lemma 2.19, and so W_eC contains no positive-dimensional abelian varieties for $e \leqslant P \cdot C$. □

2.6. **Effective results for Picard rank 1.** In this section we give effective bounds on the set $\mathrm{Exc}(\mathrm{Amp}(S))$ in the case that $\mathrm{Pic}\, S \simeq \mathbb{Z} \cdot H$.

Proposition 2.22. *Let S be a smooth projective surface such that $\mathrm{Pic}\, S \simeq \mathbb{Z} \cdot H$ with effective H. For any smooth curve $C \in |\alpha H|$ with $\alpha > 8\beta^2 + \beta$, we have that W_eC contains no positive-dimensional abelian varieties for $e \leqslant \alpha\beta H^2$.*

Proof. Suppose for contradiction that for some $e \leqslant \beta\alpha H^2$, we have an inclusion $A \subset W_eC$ of a positive-dimensional abelian variety. We will assume e is minimal with respect to this. Then by Lemma 2.1, points $p \in A_2$ parameterize basepoint-free line bundles $\mathcal{O}_C(\Gamma_p)$ of degree at most $2\beta\alpha H^2$.

Since $\alpha > 8\beta^2 + \beta \geqslant (2\beta+1)^2$, for $\beta \geqslant 1$, we have

$$2\alpha\beta < (2\beta+1)(\alpha - (2\beta+1)),$$

and so Γ_p satisfies the degree requirements of Proposition 2.5 with $k = 2\beta+1$. Therefore $\mathcal{O}_C(2\beta H - \Gamma_p)$ is effective.

First assume that $\mathcal{O}_C((2\beta-1)H - \Gamma_p)$ is not effective. Then by Proposition 2.5,

$$\deg \Gamma_p \geqslant 2\beta(\alpha - 2\beta)H^2.$$

Therefore

$$\deg \mathcal{O}_C(2\beta H - \Gamma_p) \leqslant 2\alpha\beta H^2 - 2\beta(\alpha - 2\beta)H^2 = 4\beta^2 H^2.$$

Residuation in the effective divisor $2\beta H$ therefore gives an embedding

$$A_2 \hookrightarrow W_f(C), \qquad f \leqslant 4\beta^2 H^2.$$

An application of Lemma 2.1 again shows that a general point q of $(A_2)_2$ corresponds to an effective divisor Λ_q of degree at most $8\beta^2 H^2$ moving in a basepoint-free pencil. By our requirements on α, we have that $\alpha > 8\beta$, and so

$$\deg \Lambda_q \leqslant 8\beta^2 H^2 < \beta H \cdot C.$$

Applying Corollary 2.6, we have

$$8\beta^2 H^2 \geqslant \deg \Gamma \geqslant (\alpha - \beta)H^2,$$

which is a contradiction, as $\alpha > 8\beta^2 + \beta$. Similarly, if $\mathcal{O}_C((2\beta-1)H - \Gamma_p)$ were effective, we would obtain a contradiction with a smaller f. □

Note that in the case $\beta = 1$, we see that once $\alpha \geqslant 10$, W_eC contains no positive-dimensional abelian varieties for $e \leqslant C \cdot H$.

2.7. **Example: curves on $\mathbb{P}^1 \times \mathbb{P}^1$.** While the set $\mathrm{Exc}(N)$ is not specifed explicitly in the statement of Theorem 2.9, the same method of proof can be used to compute the exceptional set explicitly when the divisor structure on S is sufficiently well understood. We present one example here.

Proposition 2.23. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let C be a smooth curve of any bidegree (d_1, d_2) with $d_1 \leqslant d_2$ and $(d_1, d_2) \neq (3,3)$ or $(2,2)$. Then, for $e < d_1$, $W_e(C)$ contains no positive-dimensional abelian varieties.*

Remark 2.24. The assumption that $(d_1, d_2) \neq (3,3)$ in the proposition is necessary, as we now explain. Such curves (the complete intersection of a quadric and a cubic surface) are canonical curves of genus 4, and there exist bielliptic genus 4 curves. Explicitly, if the cubic surface is the cone over a smooth plane cubic and the quadric is general, then projection from the cone point gives a degree 2 map from the curve to the cubic plane curve.

Likewise, if $(d_1, d_2) = (2,2)$, then the curve is elliptic and $W_1(C)$ contains the elliptic curve C.

Proof. The cases of curves with $d_1 \leqslant 1$ are trivial. If $d_1 = 2$, by assumption $d_2 \geqslant 3$, so the genus of the curve is $d_2 - 1 > 1$, and $W_1(C)$ contains no positive-dimensional abelian varieties. So we may assume $d_1 \geqslant 3$ and $d_2 \geqslant 4$. Let C be a smooth curve of bidegree (d_1, d_2) with $d_1 \geqslant 3$ and $d_2 \geqslant 4$, and suppose the conclusion of the proposition fails for C. Let e be minimal such that $W_e(C)$ contains a positive dimensional abelian variety A. By Lemma 2.1, the points p of A_2 give rise to basepoint free pencils Γ_p of degree $2e < 2d_1$. In particular they satisfy the key hypothesis

$$\deg \Gamma_p < C^2/4$$

since $C^2 = 2d_1 d_2 \geqslant 8d_1$.

So we apply Proposition 2.3 to guarantee the existence of an effective divisor D. The divisor D is necessarily in class (x, y) for some $x, y \geqslant 0$. The requirements are:

2.3(2): $d_1 y + d_2 x = C \cdot D < C^2/2 = d_1 d_2$;

2.3(3): $d_1 y + d_2 x - 2xy = D \cdot (C - D) \leqslant 2e < 2d_1 \leqslant 2d_2$.

In exactly the same way as in the proof of Theorem 2.7, this forces $x, y \leqslant 1$. This leaves at most 3 options for (x, y): it can be of class $(0,1)$, $(1,0)$, or $(1,1)$. In the first two cases, sending Γ_p to the effective divisor of class $D|_C - \Gamma_p$, whose degree is at most D^2 by 2.3(3), induces an isomorphism between A_2 and $W_0(C) = \mathrm{pt}$, which is absurd.

Now we consider the case $D = (1,1)$. The degree restriction of Proposition 2.3(3) gives us the inequality

$$2e \geqslant d_1 + d_2 - 2.$$

If $d_2 > d_1$, this inequality contradicts $e < d_1$. So we have that $d_1 = d_2$ and $e = d_1 - 1$. Then the map sending Γ_p for $p \in A_2$ to the effective divisor of class $D|_C - \Gamma_p$ induces a inclusion $A_2 \to W_2(C)$, so $W_2(C)$ contains a positive-dimensional abelian variety. This is a contradiction since we have $e = d_2 - 1 \geqslant 3$ and assumed e was minimal such that $W_e(C)$ contains an abelian variety. ☐

3. Number-Theoretic Consequences

Recall the statement of Faltings' Theorem:

Theorem 3.1 (Faltings [6], [5])**.** *Let k be a number field. Let $X \subset J$ be a subvariety of an abelian variety J over k. If $X(k)$ is infinite then X contains a translate of a positive-dimensional abelian subvariety of J. And furthermore if X contains a translate of a positive-dimensional abelian subvariety of J, then there exists a finite extension K/k such that $X(K)$.*

We will apply this to the image map from the eth symmetric power of C to $\mathrm{Pic}^e(C)$. Note that if e is strictly less than the k-gonality of C, then the natural map $C^{(e)} \to W_e(C) \subseteq \mathrm{Pic}^e C$ is one-to-one on k-points:

$$C^{(e)}(k) \hookrightarrow W_e(C)(k), \qquad \text{when } e < \mathrm{gon}_k(C).$$

Therefore if e is strictly less than the k-gonality of C, then $C^{(e)}(k)$ is finite if and only if $W_e(C)(k)$ is finite.

Remark 3.2. From this we see that $\mathrm{a.irr}_{\overline{k}}(C) \leqslant e$ if and only if $\mathrm{gon}(C_{\overline{k}}) \leqslant e$ or $W_f(C_{\overline{k}})$ contains a positive-dimensional abelian subvariety for $f \leqslant e$. As both of these conditions depend only upon the $\overline{k}$-isomorphism class of C, this is indeed a geometric property.

Proof of Theorem 1. We will show that for every possible γ, α such that $0 < \gamma/2 \leqslant \alpha \leqslant \gamma$, there is a nice curve C over a number field k such that $\operatorname{a.irr}_k(C) = \operatorname{a.irr}_{\overline{k}}(C) = \alpha$ and $\operatorname{gon}_k(C) = \operatorname{gon}_{\overline{k}}(C) = \gamma$. In fact, as the proof will show, we can take $k = \mathbb{Q}$.

For $\gamma = 1$, the only allowed α is 1 (which is achieved for $\mathbb{P}^1_{\mathbb{Q}}$). For $\gamma = 2$, $\alpha = 1$ is achieved for a positive rank elliptic curve (which exists over $\mathbb{Q}$), and $\alpha = 2$ is achieved for any hyperelliptic curve of genus at least 2. For $\gamma = 3$, $\alpha = 2$ is achieved for a bielliptic, non-hyperelliptic curve (see Remark 2.24 for a construction in genus 4), and $\alpha = 3$ is achieved for all non-hyperelliptic, non-bielliptic trigonal curves.

Therefore, we may assume that $\gamma \geqslant 4$ (and so $\alpha \geqslant 2$). Let E be a positive rank elliptic curve over $\mathbb{Q}$. By Theorem 2.7 a smooth curve in numerical class (γ, α) has $\operatorname{gon}_{\overline{\mathbb{Q}}}(C) = \gamma$ and $\operatorname{a.irr}_{\overline{\mathbb{Q}}}(C) \geqslant \alpha$. As C has a map π_1 of degree α to E, we further have $\pi_1^{-1}(E(\mathbb{Q})) \subseteq C_\alpha$, and so $\operatorname{a.irr}_{\overline{\mathbb{Q}}}(C) \leqslant \alpha$; therefore equality holds. □

Proof of Theorems 2 and 4. Let P_C be some very ample divisor on a curve C over k. Then the complete linear series $|P_C|$ gives rise to an embedding

$$C \hookrightarrow \mathbb{P}H^0(C, P_C)^\vee := \mathbb{P}^r_k.$$

Projecting from a k-rational codimension 2 plane of $\mathbb{P}^r_k$ exhibits C as a $(\deg P_C)$-cover of $\mathbb{P}^1_k$; therefore

$$\operatorname{gon}_k(C) \leqslant \deg P_C,$$

for any very ample divisor P_C.

If, further, $W_e(C_{\overline{k}}) \subset \operatorname{Pic}^e(C_{\overline{k}})$ contains no positive-dimensional abelian varieties for $e \leqslant \deg P_C$, then by Faltings' Theorem (Theorem 3.1 above), $W_e(C)$ has only finitely many k-points for $e \leqslant \operatorname{gon}_k(C) \leqslant \deg P_C$. Therefore if $W_e(C_{\mathbb{C}})$ contains no positive-dimensional abelian varieties for $e \leqslant \operatorname{gon}_k(C)$, then the arithmetic degree of irrationality $\operatorname{a.irr}_k(C) \geqslant \operatorname{gon}_k(C)$, and therefore is equal to $\operatorname{gon}_k(C)$.

Suppose that $C \hookrightarrow S/k$ with very ample polarization P. Then $P|_C$ is very ample and so $\deg P|_C = P \cdot C$ gives an upper bound on $\operatorname{gon}_k(C)$. Theorems 2.9 and 2.10 prove the nonexistence of positive-dimensional abelian varieties in $W_e(C_{\mathbb{C}})$ for $e \leqslant P \cdot C$, provided that $[C]$ is "sufficiently ample": it lies outside an exceptional set in any closed cone N in Theorem 2.9, respectively is a large enough multiple of the class generating $\operatorname{Pic} S_{\overline{k}}$ in Theorem 2.10. □

Proof of Corollary 3. The gonality of any complete intersection curve $C \subset \mathbb{P}^r_k$ of type $(d_1, d_2, \ldots, d_{r-1})$ is at most $\deg C = d_1 d_2 \cdots d_{r-1}$. It therefore suffices to show the nonexistence of abelian subvarieties in $W_e(C_{\mathbb{C}})$ for $e \leqslant d_1 d_2 \cdots d_{r-1}$. By [9, Theorem 3.1], every smooth complete intersection curve over $\mathbb{C}$ of type $d_1 < d_2 \leqslant \cdots \leqslant d_{n-1}$ with $4 \leqslant d_1$ lies on a smooth complete intersection surface S of type $(d_2, \cdots, d_{n-1})$ with $\operatorname{Pic} S \simeq \mathbb{Z} \cdot [\mathcal{O}_S(1)]$ and the class of C equal to $\mathcal{O}_S(d_1)$. Therefore, the result follows from Theorem 2.10 with $P = \mathcal{O}_S(1)$, as $P \cdot C = d_1 d_2 \cdot d_{r-1}$. □

Proof of Theorem 6. Let C be a nice curve of type (d_1, d_2) on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with $1 \leqslant d_1 \leqslant d_2$. Then we claim that $\operatorname{gon}_k(C) = \operatorname{gon}_{\overline{k}}(C) = d_1$. The upper bound is provided by the projection

$$C \hookrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$$

onto the first factor. The lower bound $\operatorname{gon}_{\overline{k}}(C) \geqslant d_1$ comes from the fact that $d_1 - 1$ points on C always impose independent conditions on sections of $K_C = \mathcal{O}_C(d_1 - 2, d_2 - 2)$. By Proposition 2.23, $W_e(C)$ contains no positive-dimensional abelian subvarieties for $e < d_1$ as long as $(d_1, d_2) \neq (2, 2)$ or $(3, 3)$. Therefore $C_{d_1 - 1}$ is finite for all such (d_1, d_2) and $\operatorname{a.irr}_k(C) = \operatorname{gon}_k(C) = d_1$. If $(d_1, d_2) = (2, 2)$, then $C_{\overline{k}}$ is an elliptic curve and so $\operatorname{a.irr}_{\overline{k}}(C) = 1$. If $(d_1, d_2) = (3, 3)$, then C is a canonical curve of genus 4. By the work of Harris–Silverman [8], there exists a finite extension K/k such that $(C_K)_2$ is infinite if and only if C is bielliptic (as it is never hyperelliptic); in this case $\operatorname{a.irr}_{\overline{k}}(C) = 2$. Otherwise $\operatorname{a.irr}_{\overline{k}}(C) = 3$. □

Appendix A. The exceptional set is effectively computable

Assume that the closed cone $N \subseteq \mathrm{Amp}(S)$ is finite rational polyhedral, that is the convex cone spanned by finitely many vectors in $N^1(S)_{\mathbb{R}}$ with rational coordinates. Under this assumption, we sketch here that the proof of Theorem 2.9 gives an effective method for computing a finite set of integral classes in N containing the exceptional set $\mathrm{Exc}(N)$, and highlight upon what data of the surface this depends. This is a harmless assumption in practice: for any closed subcone N of $\mathrm{Amp}(S)$, there exists a finite rational polyhedral subcone N' such that $N \subseteq N' \subset \mathrm{Amp}(S)$.

Inputs:

- The lattice $N^1(S)$ with its intersection product
- Integral generators $H_1, \ldots, H_k$ for the rational polyhedral subcone N,
- A rational number ϵ witnessing the fact that $N \subset \mathrm{Amp}(S)$: such that the ball of radius ϵ about each H_i is contained in $\mathrm{Amp}(S)$.
- A very ample polarization $P \in N$ (or, by Reider's Theorem [13], an identification of the class of $K_S \in N^1(S)$, since for any integral ample class $L \in N$ and $m > 3$, $K_S + mL$ is very ample)

In what follows we will use that we can compute the maximum and minimum of a continuous function on a compact set specified by finitely many linear inequalities.

Steps of computation:

1. Using the explicit lower bound ϵ for the distance to the boundary of the ample cone, compute generators $A_1, \ldots, A_{k'}$ for a closed subcone M contained in $\mathrm{Amp}(N)$ and containing N in its interior.
2. Compute the dual cone $M^\vee$ as the intersection of the half-spaces $A_1^\vee \cap \cdots \cap A_{k'}^\vee$. Write $E_1, \ldots E_r$ for the integral generators of $M^\vee$.
3. Compute the minimum m_1 of the continuous function $H \mapsto H^2$ on the compact convex set $N_{P=1}$.
4. For each $i \leqslant k$, compute the compact convex set Σ_i of $D \in M^\vee$ such that $D \cdot H_i \leqslant 4P \cdot H_i$ as the intersection of $M^\vee$ with a translate of the half-space $-H_i^\vee$. Compute the maximum $m_{2,i}$ of the continuous function $D \mapsto D^2$ on Σ_i. Let m_2 be the max of these $m_{2,i}$. (Then m_2 is the maximum of D^2 such that there exists $H \in N$ with $D \cdot H \leqslant 4P \cdot H$).
5. Enumerate integral classes $H = a_1 H_1 + \cdots + a_k H_k$ such that $P \cdot H \leqslant 8/m_1$, or $H^2 \leqslant 8m_2$, or there exists j such that $H \cdot E_j \leqslant 4m_2(P \cdot E_j)$.
   (a) Add them to the set $\mathrm{Exc}_1(N, P)$ if $H^2 \leqslant 8P \cdot H$, otherwise
   (b) Add them to the set $\mathrm{Exc}_2(N, P)$ if $H^2 \leqslant 8m_2$ or if there exists $j \leqslant r$ such that $(H \cdot E_j)/(P \cdot E_j) \leqslant 4m_2$
6. Return the set $\mathrm{Exc}(N) = \mathrm{Exc}(N, P) = \mathrm{Exc}_1(N, P) \cup \mathrm{Exc}_2(N, P)$.

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