ON THE SATAKE ISOMORPHISM

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In this paper, we present an expository treatment of the Satake transform. This gives an isomorphism between the spherical Hecke algebra of a split reductive group G over a local field and the representation ring of the dual group $\hat{G}$.

If one wants to use the Satake isomorphism to convert information on eigenvalues for the Hecke algebra to local L-functions, it has to be made quite explicit. This was done for $G = GL_n$ by Tamagawa, but the results in this case are deceptively simple, as all of the fundamental representations of the dual group are minuscule. Lusztig discovered that, in the general case, certain Kazhdan-Lusztig polynomials for the affine Weyl group appear naturally as matrix coefficients of the transform. His results were extended by S. Kato.

We will explain some of these results below, with several examples.

Contents

1. **The algebraic group $\underline{G}$**

Throughout the paper, we let F be a local field with ring of integers A. We fix a uniformizing parameter π for A, and let q be the cardinality of the residue field $A/\pi A$.

Let $\underline{G}$ be a connected, reductive algebraic group over F. We will assume throughout that $\underline{G}$ is split over F. Then $\underline{G}$ is the general fibre of a group scheme (also denoted $\underline{G}$) over A with reductive special fibre; in the semi-simple case $\underline{G}$ is a Chevalley group scheme over A. We fix a maximal torus contained in a Borel subgroup $\underline{T} \subset \underline{B} \subset \underline{G}$, all defined over A, and define the Weyl group of $\underline{T}$ by $W = N_G(\underline{T})/\underline{T}$.

Define the characters and co-characters of $\underline{T}$ by

$$X^\bullet = X^\bullet(\underline{T}) = \mathrm{Hom}(\underline{T}, \mathbb{G}_m)$$
$$X_\bullet = X_\bullet(\underline{T}) = \mathrm{Hom}(\mathbb{G}_m, \underline{T})$$

These are free abelian group of rank $l = \dim(\underline{T})$, which are paired into $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. The first contains the roots Φ of $\underline{G}$: the characters of $\underline{T}$ occuring in the adjoint representation on Lie $(\underline{G})$, and the second contains the coroots [1, Ch.6,§1].

The subset Φ^+ of positive roots which occur in the representation on Lie $(\underline{B})$ satisfies: $\Phi = \Phi^+ \cup -\Phi^+$. It determines a root basis $\Delta \subset \Phi^+$ of positive, indecomposible roots. When $\underline{G}$ is of adjoint type, the elements of Δ give a $\mathbb{Z}$-basis of $X^\bullet$.

The root basis determines a positive Weyl chamber P^+ in $X_\bullet(\underline{T})$, defined by

$$P^+ = \{\lambda \in X_\bullet : \langle \lambda, \alpha \rangle \geq 0 \text{ all } \alpha \in \Phi^+\} \quad (1.1)$$
$$= \{\lambda \in X_\bullet : \langle \lambda, \alpha \rangle \geq 0 \text{ all } \alpha \in \Delta\}$$

Let

$$2\rho = \sum_{\Phi^+} \alpha \quad \text{in } X^\bullet(T) \quad (1.2)$$

Then, for all λ in P^+, the half-integer $\langle \lambda, \rho \rangle$ is non-negative.

There is a partial ordering on P^+, written $\lambda \geq \mu$, if the difference $\lambda - \mu$ can be written as the sum of positive coroots. If $\check{\alpha}$ is a basic coroot in $\check{\Delta}$, then [1, Ch.6,§1.10]

$$\langle \check{\alpha}, \rho \rangle = 1 \quad (1.3)$$

Hence $\lambda \geq \mu$ implies that $\langle \lambda - \mu, \rho \rangle$ is a non-negative integer.

Let $\hat{G}$ be the complex dual group of $\underline{G}$. This is a connected, reductive group over $\mathbb{C}$ whose root datum is dual to $\underline{G}$. If we fix a maximal torus in a Borel subgroup $\hat{T} \subset \hat{B} \subset \hat{G}$, there is an isomorphism

$$X^\bullet(\hat{T}) \simeq X_\bullet(\underline{T}) \tag{1.4}$$

which takes the positive roots corresponding to $\hat{B}$ to the positive co-roots corresponding to B. The elements λ in $P^+ \subset X^\bullet(\hat{T})$ index the irreducible representations V_λ of the group $\hat{G} : \lambda$ is the highest weight for $\hat{B}$ on V_λ. Let $\chi_\lambda = \mathrm{Trace}(V_\lambda)$ be the character of V_λ, viewed as an element of $\mathbb{Z}[X^\bullet(\hat{T})]$. Then χ_λ lies in the subring

$$R(\hat{G}) = \mathbb{Z}[X^\bullet(\hat{T})]^W$$

fixed by the Weyl group. If we write

$$\chi_\lambda = \sum m_\lambda(\mu) \cdot [\mu],$$

then $m_\lambda(\mu) = m_\lambda(w\mu)$. Hence it suffices to determine the integers $m_\lambda(\mu)$ for μ in P^+, as these weights represent the orbits of the Weyl group on $X^\bullet(\hat{T})$. A simple result is that the integer $m_\lambda(\mu) = \dim V_\lambda(\mu)$ is non-zero if and only if $\lambda \geq \mu$ in P^+ [5, pg.202-203].

2. The Gelfand pair (G, K)

We define compact and locally compact topological groups by taking the A- and F-rational points of the group scheme $\underline{G}$:

$$K = \underline{G}(A) \subset G = \underline{G}(F) \tag{2.1}$$

Then K is a hyperspecial maximal compact subgroup of G [17, 3.8.1]. Similarly, we have the locally compact, closed subgroups

$$T = \underline{T}(F) \subset B = \underline{B}(F) \subset G = \underline{G}(F). \tag{2.2}$$

We let $N = \underline{N}(F)$, where $\underline{N}$ is the unipotent radical of $\underline{B}$. Then

$$B = T \ltimes N, \quad \text{and} \tag{2.3}$$

$$\det(\mathrm{ad}(t) | \mathrm{Lie}(N)) = 2\rho(t). \tag{2.4}$$

The Hecke ring $\mathcal{H} = \mathcal{H}(G, K)$ is by definition the ring of all locally constant, compactly supported functions $f : G \to \mathbb{Z}$ which are K-bi-invariant: $f(kx) = f(xk') = f(x)$ for all k, k' in K. The multiplication in $\mathcal{H}$ is by convolution

$$f \cdot g\,(z) = \int_G f(x) \cdot g(x^{-1}z)dx \tag{2.5}$$

where dx is the unique Haar measure on G giving K volume 1. We will see below that the product function $f \cdot g$ also takes values in $\mathbb{Z}$. The characteristic function of K is the unit element of $\mathcal{H}$.

Each function f in $\mathcal{H}$ is constant on double cosets KxK; since it is also compactly supported it is a finite linear combination of the characteristic functions $\mathrm{char}(KxK)$ of double cosets. Hence these characteristic functions give a $\mathbb{Z}$-basis for $\mathcal{H}$.

For any $\lambda \in X_\bullet(\underline{T}) = \mathrm{Hom}(\mathbb{G}_m, \underline{T})$ we have the element $\lambda(\pi)$ in $T(F)$. Since $\lambda(A^*) \subset T(A) \subset K$, the double coset $K\lambda(\pi)K$ depends only on λ, not on the choice of uniformizing element. Here we view λ multiplicatively, so $(\lambda+\mu)(\pi) = \lambda(\pi)\cdot\mu(\pi)$.

Proposition 2.6. (cf. [17, p.51]) *The group G is the disjoint union of the double cosets $K\lambda(\pi)K$, where λ runs through the co-characters in P^+.*

This is a refinement of the Cartan decomposition: $G = KTK$; for $\underline{G} = GL_n$ it is proved by the theory of elementary divisors [13, pg.56-57]. It follows that the elements

$$c_\lambda = \mathrm{char}(K\lambda(\pi)K) \qquad \lambda \in P^+ \tag{2.7}$$

give a $\mathbb{Z}$-basis for $\mathcal{H}$, and multiplication is determined by the products

$$c_\lambda \cdot c_\mu = \sum n_{\lambda,\mu}(\nu)\cdot c_\nu, \quad \text{with } n_{\lambda,\mu}(\nu) \in \mathbb{Z}. \tag{2.8}$$

To obtain an explicit formula for the integers $n_{\lambda,\mu}(\nu)$, we write

$$\begin{aligned} \nu(\pi) &= t \\ K\lambda(\pi)K &= \coprod x_i K \\ K\mu(\pi)K &= \coprod y_j K \end{aligned}$$

Then

$$\begin{aligned} n_{\lambda,\mu}(\nu) &= (c_\lambda \cdot c_\mu)(t) \\ &= \int_G c_\lambda(x) c_\mu(x^{-1}t)dx \\ &= \sum_i \int_{x_iK} c_\mu(x^{-1}t)dx \\ &= \sum_i \int_K c_\mu(kx_i^{-1}t)dk \\ &= \sum_i c_\mu(x_i^{-1}t) \\ &= \#\{(i,j) : \nu(\pi) \in x_i y_j K\} \end{aligned}$$

Since we can take $x_i = \lambda(\pi)$ and $y_j = \mu(\pi)$, this shows that $n_{\lambda,\mu}(\lambda+\mu) \geq 1$. In fact, we will see later that $n_{\lambda,\mu}(\lambda+\mu) = 1$ and that $n_{\lambda,\mu}(\nu) \neq 0$ implies that $\nu \leq (\lambda+\mu)$. Therefore

$$c_\lambda \cdot c_\mu = c_{\lambda+\mu} + \sum_{\nu<(\lambda+\mu)} n_{\lambda,\mu}(\nu) \cdot c_\nu \tag{2.9}$$

The most important property of $\mathcal{H}$ is not evident from this calculation. It is the fact that $n_{\lambda,\mu}(\nu) = n_{\mu,\lambda}(\nu)$, or in other words:

Proposition 2.10. *The Hecke ring $\mathcal{H}$ is commutative.*

This is equivalent to the statment that (G, K) is a Gelfand pair. It is usually proved via Gelfand's Lemma (cf. [6, pg.279]), which requires the existence of an anti-involution of G which fixes each double coset. For $\underline{G} = GL_n$ one takes $g \mapsto {}^t g$, which fixes the torus $\underline{T}$ of diagonal matrices. In general, there is an involution σ of $\underline{G}$ over A which acts as -1 on $\underline{T}$ (cf. [1, Ch.8,§2]), and one takes $g \mapsto \sigma(g^{-1})$.

A more involved proof of commutativity is via the Satake transform, which injects $\mathcal{H}$ into the commutative ring $R(\hat{G}) \otimes \mathbb{Z}[q^{-1/2}, q^{1/2}]$. We will discuss this transform in the next section.

One case when $\mathcal{H}$ is obviously commutative is when $\underline{G} = \underline{T}$ is a torus! We then have an exact sequence of locally compact groups

$$o \to \underline{T}(A) \to \underline{T}(F) \underset{\gamma}{\to} X_\bullet(\underline{T}) \to 0 \tag{2.11}$$

with $\gamma(t)$ the cocharacter satisfying

$$\langle \gamma(t), \chi \rangle = \mathrm{ord}(\chi(t)) \tag{2.12}$$

for all characters χ in $X^\bullet(T) = \mathrm{Hom}(\underline{T}, \mathbb{G}_m)$. The choice of uniformizing parameter π gives a splitting of this sequence: map λ in $X_\bullet(\underline{T})$ to $\lambda(\pi)$ in T.

Since each $\underline{T}(A)$ double coset is a single right coset, we have

$$c_\lambda \cdot c_\mu = c_{\lambda+\mu}$$

in $\mathcal{H}$. This agrees with (2.9), as there are no elements ν in $P^+ = X_\bullet(\underline{T})$ with $\nu < \lambda + \mu$. In other words, we have an isomorphism of rings

$$\begin{aligned} \mathcal{H}_T &\simeq \mathbb{Z}[X_\bullet(\underline{T})] \\ c_\lambda &\longleftrightarrow [\lambda] \end{aligned} \tag{2.13}$$

3. The Satake transform

The Satake transform gives a ring homomorphism

$$\mathcal{S} : \mathcal{H}_G \longrightarrow \mathcal{H}_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}],$$

with image in the invariants for the Weyl group. It is defined by an integral of a type much studied by Harish-Chandra.

Fix $\underline{T} \subset \underline{B} = \underline{T} \cdot \underline{N} \subset \underline{G}$ over A, and let dn be the unique Haar measure on the unipotent group $N = \underline{N}(F)$ which gives $\underline{N}(A) = N \cap K$ volume 1. Let

$$\delta : B \longrightarrow \mathbb{R}_+^* \tag{3.1}$$

be the modular function of B, defined by the formula

$$d(bnb^{-1}) = \delta(b) \cdot dn \tag{3.2}$$

Then δ is trivial on N, so defines a character $\delta : T \longrightarrow \mathbb{R}_+^*$. Let $\delta^{1/2}$ be the positive square-root of this character; if $t = \mu(\pi)$ with μ in $X_\bullet(\underline{T})$ we have

$$\begin{aligned} \delta^{1/2}(t) &= |\det(\operatorname{ad} t | \operatorname{Lie}(N))|^{1/2} \\ &= |\pi^{\langle \mu, 2\rho \rangle}|^{1/2} \\ &= q^{-\langle \mu, \rho \rangle}. \end{aligned} \tag{3.3}$$

In particular, $\delta^{1/2}$ takes values in the subgroup $q^{(1/2)\mathbb{Z}}$. If $\rho \in X^\bullet(\underline{T})$ then $\delta^{1/2}$ takes values in the subgroup $q^{\mathbb{Z}}$.

For f in $\mathcal{H}_G$, we define $\mathcal{S}f$, the Satake transform, as a function on T by the integral

$$\mathcal{S}f(t) = \delta(t)^{1/2} \cdot \int_N f(tn) dn \tag{3.4}$$

Then $\mathcal{S}f$ is a function on $T/T \cap K = X_\bullet(T)$ with values in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. The main result is that the image lies in the subring

$$\left(\mathcal{H}_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]\right)^W = R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \tag{3.5}$$

of W-invariants, and that furthermore (cf. [11],[3, p.147]),

Proposition 3.6. *The Satake transform gives a ring isomorphism* $\mathcal{S} : \mathcal{H}_G \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \simeq R(\hat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$. *If* ρ *is an element of* $X^\bullet(\underline{T})$, *then the Satake transform gives a ring isomorphism*

$$\mathcal{H}_G \otimes \mathbb{Z}[q^{-1}] \simeq R(\hat{G}) \otimes \mathbb{Z}[q^{-1}].$$

Without giving the full proof, we give a sample calculation that illustrates the main idea. Assume that λ and μ lie in P^+; we wish to evaluate $\mathcal{S}(c_\lambda)$ on the element $t = \mu(\pi)$ of T.

Recall that c_λ is the characteristic function of the double coset $K\lambda(\pi)K = \coprod x_i K$. Since $G = BK$, we may assume each $x_i = t(x_i)n(x_i)$ lies in $B = TN$. Then

$$\begin{aligned}\mathcal{S}(c_\lambda)(t) =&\delta(t)^{1/2}\int_N c_\lambda(tn)dn \\ =&q^{-\langle\mu,\rho\rangle}\sum_i \int_{N\cap t^{-1}x_i K} dn.\end{aligned} \tag{3.7}$$

The intersection is empty unless $t^{-1}\cdot t(x_i)$ lies in $\underline{T}(A)$, when the integral is equal to 1. Since $t = \mu(\pi)$,

$$\mathcal{S}(c_\lambda)(t) = q^{-\langle\mu,\rho\rangle}\#\{i : t(x_i) \equiv \mu(\pi) \pmod{\underline{T}(A)}\}$$

Hence the transform counts the number of single cosets where the "diagonal entries" $t(x_i)$ have the same valuation as $\mu(\pi)$. In particular, as we can take $x_i = \lambda(\pi)$, we have

$$\mathcal{S}(c_\lambda)(\lambda(\pi)) \neq 0. \tag{3.8}$$

In fact, one has

$$\mathcal{S}(c_\lambda)(\lambda(\pi)) = q^{\langle\lambda,\rho\rangle}.$$

Moreover, if μ is in P^+ and $\mathcal{S}(c_\lambda)(\mu(\pi)) \neq 0$, then $\mu \leq \lambda$ [3, p.148]. Therefore we have

$$\mathcal{S}(c_\lambda) = q^{\langle\lambda,\rho\rangle}\chi_\lambda + \sum_{\mu<\lambda} a_\lambda(\mu)\chi_\mu \tag{3.9}$$

in $R(\hat{G}) \otimes \mathbb{Z}[q^{-1/2}, q^{1/2}]$. From this the isomorphism of rings follows easily.

If one takes the scaled basis

$$\varphi_\lambda = q^{\langle\lambda,\rho\rangle}\cdot\chi_\lambda \tag{3.10}$$

of $R(\hat{G}) \otimes \mathbb{Z}[q^{-1/2}, q^{1/2}]$, one finds that

$$\mathcal{S}(c_\lambda) = \varphi_\lambda + \sum_{\mu<\lambda} b_\lambda(\mu)\varphi_\mu \tag{3.11}$$

with coefficients $b_\lambda(\mu)$ in $\mathbb{Z}$. Inversely, for all λ in P^+ we have the identity:

$$q^{\langle\lambda,\rho\rangle}\,\mathrm{Trace}(V_\lambda) = \varphi_\lambda = \mathcal{S}(c_\lambda) + \sum_{\mu<\lambda} d_\lambda(\mu)\mathcal{S}(c_\mu) \tag{3.12}$$

with integers $d_\lambda(\mu)$.

If λ is a minuscule weight for $\hat{G}$, there are no elements μ in P^+ with $\mu < \lambda$. In this case we obtain

$$q^{\langle \lambda, \rho \rangle} \operatorname{Trace}(V_\lambda) = \mathcal{S}(c_\lambda) \tag{3.13}$$

This pleasant situation occurs for <u>all</u> fundamental representations $V_\lambda = \bigwedge^i \mathbb{C}^n$ of $\hat{G}(\mathbb{C}) = GL_n(\mathbb{C})$, in the case when $G = GL_n$. This gives the formula of Tamagawa [15], [13, pg.56-62]:

$$q^{i(n-i)/2} \operatorname{Trace}(\bigwedge^{i} \mathbb{C}^n) = \mathcal{S}\Big(\operatorname{char}(K \underset{n-i \text{ times}}{\overset{i \text{ times}}{\begin{pmatrix} \pi & & & & \\ & \ddots & & & \\ & & \pi & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}}} K)\Big) \tag{3.14}$$

Similarly, one obtains the transform of minuscule λ corresponding to real forms $G_{\mathbb{R}}$ of G with Hermitian symmetric spaces. The term $\langle \lambda, \rho \rangle$ is half of the complex dimension of $G_{\mathbb{R}}/K_{\mathbb{R}}$. For example, when $\underline{G} = GSp_{2n}$ and $\hat{G}(\mathbb{C}) = \mathrm{CSpin}_{2n+1}(\mathbb{C})$ one has a minuscule weight λ corresponding to the spin representation V_λ of dimension 2^n. We find [8, 2.1.3]

$$q^{\frac{n(n+1)}{4}} (\operatorname{Trace} V_\lambda) = \mathcal{S}\Big(\operatorname{char}(K \underset{n \text{ times}}{\overset{n \text{ times}}{\begin{pmatrix} \pi & & & & \\ & \ddots & & & \\ & & \pi & & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}}} K)\Big) \tag{3.15}$$

4. **Kazhdan-Lusztig polynomials**

If one wishes to determine $\mathcal{S}(c_\lambda)$ for non-minuscule λ, one has to calculate the integers $b_\lambda(\mu)$ in (3.11), or equivalently the integers $d_\lambda(\mu)$ of the inverse matrix in (3.12). These depend on λ, μ, and the cardinality q of the residue field of A; in fact, we will see that $d_\lambda(\mu) = P_{\mu,\lambda}(q)$ is a polynomial in q with non-negative integer coefficients. Lusztig realized that $P_{\mu,\lambda}(q)$ is a Kazhdan-Lusztig polynomial for the affine Weyl group of $\hat{G}$ [9]. His work was completed by S. Kato [7], and we review it below. We assume in this section that $\underline{G}$ is a group of adjoint type.

For any μ in $X_\bullet(T)$, we let

$$\hat{P}(\mu) = \sum_{\mu=\sum n(\alpha^\vee)\alpha^\vee} q^{-\sum n(\alpha^\vee)} \tag{4.1}$$

be the polynomial in q^{-1} which counts the number of expressions of μ as a non-negative sum of positive coroots. If μ cannot be expressed as such a sum, $\hat{P}(\mu) = 0$. Since we include the empty sum, when $\mu = 0$ we have $\hat{P}(0) = 1$. In all cases,

$$q^{\langle \mu,\rho\rangle} \cdot \hat{P}(\mu) \tag{4.2}$$

is a polynomial in q with integral coefficients. If μ is in P^+ and $\mu \geq 0$, the constant coefficient of $q^{\langle \mu,\rho\rangle}\hat{P}(\mu)$ is equal to 1. Let

$$2\rho^\vee = \sum_{(\Phi)^+} \alpha^\vee \qquad \text{be the sum of all positive coroots.} \tag{4.3}$$

Proposition 4.4. *The coefficient $d_\lambda(\mu)$ appearing in the Satake isomorphism is given by the formula*

$$d_\lambda(\mu) = P_{\mu,\lambda}(q) = q^{\langle \lambda-\mu,\rho\rangle} \sum_W \varepsilon(\sigma)\hat{P}\big(\sigma(\lambda+\rho^\vee) - (\mu+\rho^\vee)\big),$$

where $\varepsilon(\sigma) = \det(\sigma|X_\bullet(T))$ is the sign character on the Weyl group W.

Kato shows that the polynomial $P_{\mu,\lambda}(q)$ defined by the alternating sum in Proposition 4.4 is a Kazhdan-Lusztig polynomial for the affine Weyl group W_a of $\hat{G}$. It is associated to the pair of elements $w_\mu \leq w_\lambda$ in the extended affine Weyl group $\tilde{W}_a = X^\bullet(\hat{T}) \rtimes W$ of maximal length in the double cosets $W\mu W$ and $W\lambda W$ respectively. These elements have lengths: $\ell(w_\mu) = \langle \mu, 2\rho\rangle + \dim(G/B)$ and $\ell(w_\lambda) = \langle \lambda, 2\rho\rangle + \dim(G/B)$. The general theory of Kazhdan-Lusztig polynomials then implies that $P_{\mu,\lambda}(q)$ has non-negative integer coefficients, and has degree strictly less than $\langle \lambda-\mu,\rho\rangle$ in q [9, pg.215].

If we set $q = 1, \hat{P}(\mu)$ becomes the partition function, and $P_{\mu,\lambda}(1) = \dim V_\lambda(\mu)$ by a formula of Kostant (cf. [5, pg.421]). More generally, R. Brylinski [2] has shown how to obtain $P_{\mu,\lambda}(q)$ from the action of a principal SL_2 in $\hat{G}$ on the space $V_\lambda(\mu)$.

Assume $\mu \leq \lambda$ in P^+. Then $P_{\mu,\lambda}(q)$ has constant coefficient $= 1$. In particular,

$$\dim V_\lambda(\mu) = 1 \Rightarrow P_{\mu,\lambda}(q) = 1. \tag{4.5}$$

A non-trivial case is due to Lusztig [9, p.226]. Assume G is simple and λ is the highest coroot (= the highest weight of the adjoint representation of $\hat{G}$). Then $0 \leq \lambda$ in P^+ and

$$P_{0,\lambda}(q) = \sum_{i=1}^{l} q^{m_i - 1} \tag{4.6}$$

where $m_1, m_2, , \ldots , m_l$ are the exponents of G [1, ch.5,§6].

5. Examples

We first treat the case $\underline{G} = PGL_2$. Then $\hat{G} = \mathcal{S}L_2(\mathbb{C})$ and $X_\bullet(T) = X^\bullet(\hat{T}) = \mathbb{Z} \cdot \chi$, where χ is the character of the standard representation on $\mathbb{C}^2$. If $\lambda = n\chi$ and $\mu = m\chi$ are elements of $P^+(m, n \geq 0)$, then $\lambda \geq \mu$ if and only if

$$\begin{cases} n \geq m \\ n \equiv m \pmod 2. \end{cases} \tag{5.1}$$

Since $\dim V_\lambda(\mu) = 1$, we have $d_\lambda(\mu) = 1$ in this case. If we use the traditional notation

$$T_{\pi^m} = \mathrm{char}(K(\begin{smallmatrix} \pi^m & \\ & 1 \end{smallmatrix})K) \tag{5.2}$$

for c_μ in $\mathcal{H}$ we obtain the well-known formula [12, p.73]

$$q^{\frac{n}{2}}\big(\operatorname{Trace}\operatorname{Sym}^n(\mathbb{C}^2)\big) = \mathcal{S}\Big(\sum_{\substack{m \leq n \\ m \equiv n \pmod 2}} T_{\pi^m}\Big). \tag{5.3}$$

The simplest adjoint group with a fundamental co-weight λ which is not minuscule is $G = PSp_4 = SO_5$. Then $\hat{G} = Sp_4(\mathbb{C})$ and V_λ is the 5-dimensional orthogonal representation. We have $0 \leq \lambda$ in P^+, and $P_{0,\lambda}(q) = 1$. Hence

$$\begin{aligned} q^2 \cdot \mathrm{Trace}(V_\lambda) &= \mathcal{S}(c_\lambda) + \mathcal{S}(c_0) \\ &= \mathcal{S}(c_\lambda) + 1. \end{aligned}$$

When $G = G_2$, so $\hat{G} = G_2(\mathbb{C})$, neither of the fundamental co-weights λ_1, λ_2 are minuscule. If $V_1 = V_{\lambda_1}$ is the seven-dimensional representation and $V_2 = V_{\lambda_1}$ is the fourteen-dimensional adjoint representation, we have

$$0 \leq \lambda_1 \leq \lambda_2 \qquad \text{in } P^+. \tag{5.5}$$

The polynomials $P_{\mu,\lambda}(q)$ are given by

$$\begin{cases} P_{0,\lambda_1}(q) = 1 \\ P_{\lambda_1,\lambda_2}(q) = 1 \\ P_{0,\lambda_2}(q) = 1 + q^4 \end{cases} \tag{5.6}$$

as the exponents for G_2 are $m_1 = 1, m_2 = 5$. Hence we find

$$\begin{cases} q^3 \cdot \mathrm{Trace}(V_1) = S(c_{\lambda_1}) + 1 \\ q^5 \cdot \mathrm{Trace}(V_2) = S(c_{\lambda_2}) + S(c_{\lambda_1}) + (1 + q^4). \end{cases} \tag{5.7}$$

6. L-functions

One application of the Satake isomorphism is in the calculation of L-functions. This is based on the fact that the characters

$$\omega : R(\hat{G}) \otimes \mathbb{C} \longrightarrow \mathbb{C} \tag{6.1}$$

of the representation ring are indexed by the semi-simple conjugacy classes s in the dual group $\hat{G}(\mathbb{C})$. The value of the character ω_s on χ_λ in $R(\hat{G})$ is given by

$$\omega_s(\chi_\lambda) = \chi_\lambda(s) = \mathrm{Trace}(s|V_\lambda) \tag{6.2}$$

Since we have a fixed isomorphism

$$\mathcal{S} : \mathcal{H}_G \otimes \mathbb{C} \simeq R(\hat{G}) \otimes \mathbb{C}$$

we see that to any complex character of the Hecke algebra $\mathcal{H}_G$, we can associate a semi-simple class s in $\hat{G}(\mathbb{C})$, its Satake parameter.

Let π be an irreducible, smooth complex representation of G. The space π^K of K-fixed vectors has dimension ≤ 1. When $\dim(\pi^K) = 1$, we say π is unramified; in this case π gives a character of $\mathcal{H}_K$:

$$\mathcal{H}_K \longrightarrow \mathrm{End}(\pi^K) = \mathbb{C}. \tag{6.3}$$

We let $s = s(\pi)$ be the Satake parameter of this character.

Proposition 6.4. (cf. [3, ch. III]) *The map $\pi \longrightarrow s(\pi)$ gives a bijection between the set of isomorphism classes of unramified irreducible representations of G and the set of semi-simple conjugacy classes in $\hat{G}(\mathbb{C})$.*

We write $\pi(s)$ for the unramified representation with Satake parameter s in $\hat{G}(\mathbb{C})$. It is known that $\pi(s)$ is tempered, so lies in the support of the Plancherel measure, if and only if s lies in a compact subgroup of $\hat{G}(\mathbb{C})$. Macdonald has determined the Plancherel measure on the unramified unitary dual [10, Ch.V].

If $\pi = \pi(s)$ is an unramified representation of G, and V is a complex, finite-dimensional representation of $\hat{G}(\mathbb{C})$, we define the local L-function $L(\pi, V, X)$ in $\mathbb{C}[[X]]$ by the formula.

$$(6.5) \qquad L(\pi, V, X) = \det(1 - sX|V)^{-1}.$$

When π is tempered, the eigenvalues of s on V have absolute value 1, so $L(\pi, V, X)$ has no poles in the disc $|X| < 1$. In general, we have

$$(6.6) \qquad \det(1 - sX|V) = \sum_{k=0}^{\dim V} (-1)^k \operatorname{Tr}(s|\bigwedge^k V)X^k.$$

Hence, if we can write the elements $\operatorname{Tr}(\bigwedge^k V)$ in $R(\hat{G})$ as polynomials in the elements $S(c_\lambda)$, we can calculate the local L-function from the eigenvalues of c_λ in $\mathcal{H}$ acting on π^K.

For example, let $\underline{G} = GL_n$, so $\hat{G} = GL_n(\mathbb{C})$. We take $V = \mathbb{C}^n$, the standard representation. Let α_i in $\mathbb{C}$ be the eigenvalue of the elements c_{λ_i} on π^K. Then by Tamagawa's formula (3.14):

$$(6.7) \qquad L(\pi, V, X) = \Big(\sum_{k=0}^{n} (-1)^k q^{-k(n-k)/2} \cdot \alpha_k \cdot X^k\Big)^{-1}$$

Equivalently, one has [13, pg.61]

$$L(\pi, V, q^{\frac{n-1}{2}} \cdot X) = \Big(\sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \cdot \alpha_k \cdot X^k\Big)^{-1}.$$

For $n = 2$, this gives

$$L(\pi, V, q^{\frac{1}{2}} \cdot X) = (1 - \alpha_1 X + q\alpha_2 X^2)^{-1}$$

Next, consider the case of $G = SO_5$, so $\hat{G} = Sp_4(\mathbb{C})$. First we want to consider the L-function of the standard representation $V_{\lambda_1} = \mathbb{C}^4$, with minuscule weight. We have $\bigwedge^2 V_{\lambda_1} = V_{\lambda_2} \oplus \mathbb{C}$, where V_{λ_2} is the other fundamental representation (of dimension 5). Let α_i be the eigenvalues of c_{λ_i} on π^K.

Then by our previous calculation of the Satake transform:

$$q^{3/2} \operatorname{Tr}(s|V_{\lambda_1}) = \alpha_1$$
$$q^2 \operatorname{Tr}(s|V_{\lambda_2}) = \alpha_2 + 1.$$

Hence

$$q^2 \operatorname{Tr}(s|\bigwedge^2 V_{\lambda_1}) = \alpha_2 + 1 + q^2,$$

and we find (cf.[14], [16]):

$$(6.8) \quad L(\pi, V_{\lambda_1}, q^{\frac{3}{2}} X) = (1 - \alpha_1 X + (q\alpha_2 + q + q^3)X^2 - q^3\alpha_1 X^3 + q^6 X^4)^{-1}$$

Now consider the L-function of the representation V_{λ_2}. We have $\bigwedge^2 V_{\lambda_2} \simeq \bigwedge^3 V_{\lambda_2} = V_{2\lambda_1}$, the adjoint representation of dimension 10. Using the fact that $V_{\lambda_1}^{\otimes 2} = V_{2\lambda_1} + \bigwedge^2 V_{\lambda_1}$, we find the relation

$$\chi_{2\lambda_1} = (\chi_{\lambda_1})^2 - \chi_{\lambda_2} - 1 \qquad \text{in } R(\hat{G}).$$

Hence

$$\begin{aligned} \mathrm{Tr}(s|V_{2\lambda_1}) &= \mathrm{Tr}(s|V_{\lambda_1})^2 - \mathrm{Tr}(s|V_{\lambda_2}) - 1 \\ &= \frac{\alpha_1^2}{q^3} - \frac{(\alpha_2+1)}{q^2} - 1 \end{aligned}$$

and the L-function of V_{λ_2} is given by

$$\begin{aligned} L(\pi, V_{\lambda_2}, q^2X) =& (1 - (\alpha_2+1)X + (q\alpha_1^2 - q^2\alpha_2 - q^2 - q^4)X^2 \\ & - (q^3\alpha_1^2 - q^4\alpha_2 - q^4 - q^6)X^3 \\ & + q^6(\alpha_2+1)X^4 - q^{10}X^5)^{-1} \end{aligned} \tag{6.9}$$

Finally, consider the case when $G = G_2$, so $\hat{G} = G_2(\mathbb{C})$. Let V be the 7-dimensional representation of $\hat{G}$, with weight λ_1, and let λ_2 be the weight of the 14-dimensional representation. We have

$$\begin{aligned} q^3\chi_1(s) &= \alpha_1 + 1 \\ q^5\chi_2(s) &= \alpha_2 + \alpha_1 + 1 + q^4 \end{aligned}$$

where α_i is the eigenvalue of c_{λ_i}. On the other hand,

$$\begin{aligned} \mathrm{Tr}(V) &= \chi_1 \\ \mathrm{Tr}(\bigwedge^2 V) &= \chi_2 + \chi_1 \\ \mathrm{Tr}(\bigwedge^3 V) &= \chi_1^2 - \chi_2 \end{aligned}$$

in $R(\hat{G})$, and $\bigwedge^k V \simeq \bigwedge^{7-k} V$ for all k. Hence we find

$$\begin{aligned} L(\pi, V, q^3X) =& \Big(1 - (\alpha_1+1)X + (q\alpha_2 + (q+q^3)\alpha_1 + (q+q^3+q^5))X^2 \\ & - (q^3\alpha_1^2 + (2q^3 - q^4)\alpha_1 - q^4\alpha_2 + (q^3 - q^4 - q^8))X^3 \\ & + \ldots - q^{21}X^7\Big)^{-1} \end{aligned} \tag{6.10}$$

7. The trivial representation

One interesting unramified representation π of G is the trivial representation. Then $\pi = \pi^K$ affords a representation of $\mathcal{H}$, and c_λ acts by multiplication by

$$\deg(c_\lambda) = \#\text{ of single } K\text{-cosets in } K\lambda(\pi)K. \tag{7.1}$$

This integer is given by a polynomial in q, with leading term $q^{\langle\lambda,2\rho\rangle}$. Let $P_\lambda \subset \underline{G}$ be the standard parabolic subgroup defined by the co-character λ. We have

$$\mathrm{Lie}(P_\lambda) = \mathrm{Lie}(T) + \bigoplus_{\langle\lambda,\alpha\rangle\geq 0} \mathrm{Lie}(G)_\alpha \tag{7.2}$$

and

$$\dim(G/P_\lambda) = \#\{\alpha \in \Phi : \langle\lambda,\alpha\rangle < 0\} \tag{7.3}$$

If $\lambda = 0$ we find $P_\lambda = G$; if λ is regular we find $P_\lambda = B$. Let

$$\ell : \tilde{W}_a = X^\bullet(\hat{T}) \rtimes W \longrightarrow \mathbb{Z}$$

be the length function on the extended affine Weyl group, defined in [9, pg.209]. The following is a simple consequence of the Bruhat-Tits decomposition of G [17, 3.3.1].

Proposition 7.4. *For all λ in P^+, we have:*

$$\deg(c_\lambda) = \sum_{W\lambda W} q^{\ell(y)} \Big/ \sum_{W} q^{\ell(w)} = \frac{\#(G/P_\lambda)(q)}{q^{\dim(G/P_\lambda)}} \cdot q^{\langle\lambda,2\rho\rangle}.$$

Moreover, λ is a minuscule co-weight if and only if

$$\deg(c_\lambda) = \#(G/P_\lambda)(q)$$

It is also known that the Satake parameter of the trivial representation is the conjugacy class $s = \rho(q) = 2\rho(q^{1/2})$ in $\hat{G}(\mathbb{C})$. Equivalently, if

$$s_0 = \begin{pmatrix} q^{1/2} & \\ & q^{-1/2} \end{pmatrix} \quad \text{in } SL_2(\mathbb{C}) \tag{7.5}$$

is the Satake parameter of the trivial representation of PGL_2, then s is the image in $\hat{G}(\mathbb{C})$ of s_0 in a principal SL_2.

This gives a check on our various formulas. For example, when $G = G_2$ we found

$$q^3\chi_1(s) = \alpha_1 + 1$$
$$q^5\chi_2(s) = \alpha_2 + \alpha_1 + 1 + q^4$$

On the trivial representation, we find

$$\alpha_1 = \deg(c_{\lambda_1}) = q^6 + q^5 + q^4 + q^3 + q^2 + q$$
$$\alpha_2 = \deg(c_{\lambda_2}) = q^{10} + q^9 + q^8 + q^7 + q^6 + q^5$$

Since

$$V_1 = S^6(\mathbb{C}^2)$$
$$V_2 = S^{10}(\mathbb{C}^2) + S^2(\mathbb{C}^2)$$

as representations of the principal SL_2 in G_2, we find

$$\begin{aligned} q^3\chi_1(s) &= q^3(q^3 + q^2 + q + 1 + q^{-1} + q^{-2} + q^{-3}) \\ &= q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 \\ q^5\chi_2(s) &= q^{10} + q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1 \end{aligned}$$

which checks!

One consequence of Proposition 7.4 is that the degrees of Hecke operators are quite large. For example, if $G = E_8$ and $\lambda \neq 0$, $\deg(c_\lambda) > q^{58}$.

8. Normalizing the Satake isomorphism

One unpleasant, but necessary, feature in the Satake isomorphism is the presence of the irrationalities $q^{m/2}$. As already noted, these are not needed in the case when $\rho \in X^\bullet(T)$. When the derived group G' of G is simply-connected, we will see how they can be removed by a choice of normalization.

Let Y be the quotient torus G/G'. The exact sequence:

$$1 \longrightarrow G' \longrightarrow G \longrightarrow Y \longrightarrow 1 \tag{8.1}$$

induces an exact sequence

$$0 \longrightarrow X^\bullet(Y) \longrightarrow X^\bullet(T) \longrightarrow \mathrm{Hom}(\mathbb{Z}\check{\Phi}, \mathbb{Z}) \longrightarrow 0 \tag{8.2}$$

Since $\langle \check{\alpha}, \rho \rangle$ is integral for all coroots, there is a class ρ_Y in $\frac{1}{2}X^\bullet(Y)$ with

$$\rho \equiv \rho_Y \qquad \bmod X^\bullet(T) \tag{8.3}$$

The class ρ_Y is well-determined in the quotient group $\frac{1}{2}X^\bullet(Y)/X^\bullet(Y)$. A choice of representative in $\frac{1}{2}X^\bullet(Y)$ will be called a normalization.

Since $X^\bullet(Y) = X_\bullet(\hat{Z})$, where $\hat{Z}$ is the connected center of $\hat{G}$, a normalization ρ_Y gives us a central element

$$z = \rho_Y(q) = 2\rho_Y(q^{1/2}) \quad \text{in } \hat{Z}(\mathbb{C}) \subset \hat{G}(\mathbb{C}). \tag{8.4}$$

We adopt the convention that the normalized Satake parameter of an unramified representation π is given by:

$$s' = z \cdot s(\pi) \quad \text{in } \hat{G}(\mathbb{C}). \tag{8.5}$$

Of course, this depends on the choice of ρ_Y. It has the advantage that the relation between eigenvalues of the Hecke operators c_λ on π^K and the traces $\chi_\lambda(s')$ is now algebraic, involving only integral powers of q. Indeed, $\chi_\lambda(s') = q^{\langle \lambda, \rho_Y \rangle}\chi_\lambda(s)$, so

$$q^{\langle \lambda, \rho - \rho_Y \rangle}\chi_\lambda(s') = (c_\lambda|\pi^K) + \sum_{\mu<\lambda} d_\lambda(\mu)(c_\mu|\pi^K). \tag{8.6}$$

Note that $\rho - \rho_Y$ is an element of $X^\bullet(T)$.

One example, discussed by Deligne [4, pg.99-101], is for $G = GL_2$. Then $\rho = \frac{1}{2}\alpha = \frac{1}{2}(e_1 - e_2)$ is not in $X^\bullet(T) = \mathbb{Z}e_1 + \mathbb{Z}e_2$. The normalization

$$\rho = \tfrac{1}{2}(e_1 + e_2) = (\det)^{1/2} \tag{8.7}$$

gives the Hecke parameter $s' = q^{1/2} \cdot s$, and the normalization

$$\rho_Y = -\tfrac{1}{2}(e_1 + e_2) = (\det)^{-1/2} \tag{8.8}$$

gives the Tate parameter $s' = q^{-1/2} \cdot s$.

For the trivial representation $\pi = \mathbb{C}$ of $GL_2(F)$, the Satake parameter is the element

$$s = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \qquad \text{in } SL_2(\mathbb{C}).$$

The normalized Hecke parameter is the element

$$s' = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{in } GL_2(\mathbb{C}),$$

and the normalized Tate parameter is the element

$$s' = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix} \qquad \text{in } GL_2(\mathbb{C}).$$

One may choose to take a normalization even when ρ lies in $X^\bullet(T)$. For example, when $G = GL_n$ the normalization $\rho_Y = (\det)^{\frac{n-1}{2}}$ is popular, as in the formula following (6.7). In the global situation, the choice of normalization often depends on the infinity type, and is chosen to render the parameters s'_p integral. The Hecke normalization does this for holomorphic forms of weight 2 on the upper half-plane.

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