Mirror symmetry for very affine hypersurfaces

Benjamin Gammage (joint work with Vivek Shende)

UC Berkeley

A new approach to Fukaya categories

(Ganatra-Pardon-Shende, following conjectures of Kontsevich, Nadler):

Fukaya category is covariantly functorial and localizes over skeleton:

- V Weinstein manifold $\rightsquigarrow$ *skeleton* $\mathbb{L}_V$
- $(V, W : V \to \mathbb{C})$ LG model $\rightsquigarrow$ *relative skeleton* $\mathbb{L}_W$

Associated Fukaya category is global sections of cosheaf of categories $\mathcal{F}$ on $\mathbb{L}$.

Locally, this cosheaf is a category of microlocal sheaves $\rightsquigarrow$ *easy to compute.*

Functoriality

Functor $\mathrm{Coh}(-)$ satisfies proper descent $\implies$ $\mathrm{Fuk}(-)$ should satisfy descent along the mirror to proper morphisms.

Example

$V = T^*M$. Precosheaf on M given by $U \mapsto \mathrm{Fuk}(T^*U)$ is actually a *cosheaf.*

For $B \subset M$ a ball, the Fukaya category is $\mathrm{Fuk}(T^*B) \cong \mathrm{Perf}_k$. Hence

$$\mathrm{Fuk}(T^*M) \cong \mathrm{colim}_M \mathrm{Fuk}(T^*B) \cong \mathrm{colim}_M \mathrm{Perf}_k \cong C_*(\Omega M) - \mathrm{Perf}.$$

For general V: using Nadler's theory of arboreal singularities, we can write the Fukaya category $\mathrm{Fuk}(V)$ as a colimit of categories $\mathrm{Rep}(Q)$ for Q an acyclic quiver.

Skeleta

Every Weinstein manifold V has an associated Liouville vector field X. Its skeleton $\mathbb{L}$ is the stable set of X.

In case $V = T^*X$ and $z_0 \in \mathbb{C}$ is a regular value of $W : V \to \mathbb{C}$, the relative skeleton of the LG model (V, W) is $\mathbb{L}_W := \mathrm{Cone}(\mathbb{L}_{W^{-1}(z_0)}) \cup X$.

Example

$V = T^*S^1 \cong \mathbb{C}^\times$, $W : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ given by $W(z) = z + z^{-1}$.

The skeleton $\mathbb{L}_W$ is the union of S^1 with a cotangent fiber.

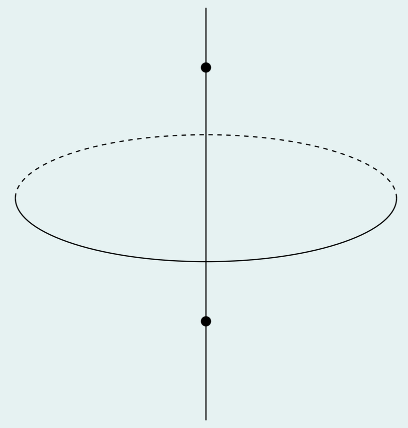

Figure 1: The union of S^1 with the cone on $W^{-1}(\frac{5}{2}) = \{\frac{1}{2}, 2\}$.

This is the mirror to $\mathbb{P}^1$. An easy check: $\mu\mathrm{Sh}(\mathbb{L}_W) \cong \mathrm{Rep}(\bullet \rightrightarrows \bullet) \cong \mathrm{Coh}(\mathbb{P}^1)$.

Better: This is two copies of the mirror to $\mathbb{A}^1$, glued along the mirror to $\mathbb{G}_m$. We see the *same colimit* on both sides!

The equivalence constructed by matching colimits is part of a more general story.

Toric mirrors

Let Σ be a fan with primitive vectors $v_1, \dots, v_n$.

Mirror to the toric variety $\mathbf{T}_\Sigma$ is LG model $((\mathbb{C}^\times)^n, W_\Sigma = \sum_j z^{v_j})$.

Following Bondal, [FLTZ] conjecture: Relative skeleton $\mathbb{L}_{W_\Sigma}$ is equal to

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} \sigma^\perp \times \sigma \subset T^*T^n = (\mathbb{C}^\times)^n$$

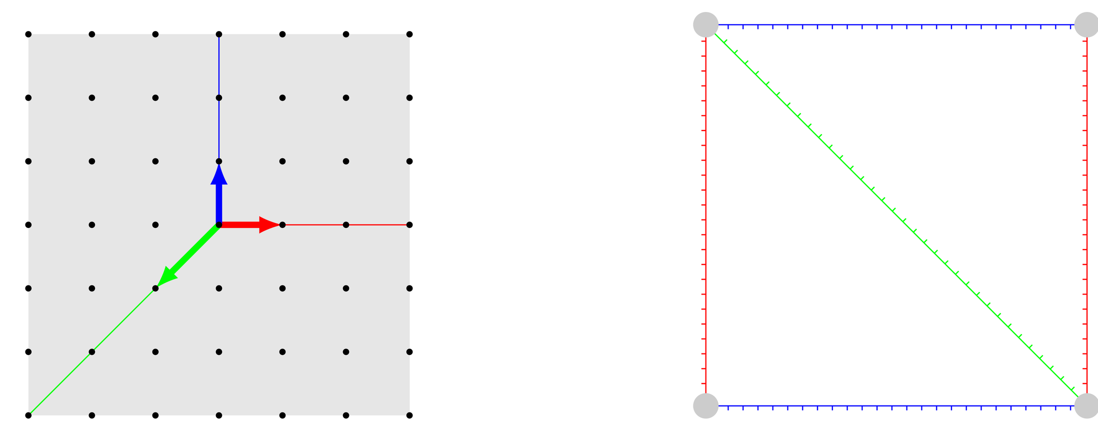

Figure 2: The fan and FLTZ skeleton for $\mathbb{P}^2$.

Theorem [Ku]: $\mathrm{Sh}_\Lambda(T^n) \cong \mathrm{Coh}(\mathbf{T}_\Sigma)$.

We prove the conjecture:

Theorem [G.-Shende]: Λ_Σ is a relative skeleton for W_Σ.

Corollary: Kuwagaki's theorem is a mirror symmetry equivalence.

Computing the skeleton

We need to compute the skeleton of the hypersurface $W_\Sigma^{-1}(0) \subset (\mathbb{C}^\times)^n$. Following Nadler, we use Mikhalkin's tropical pants decomposition. Pants

$$\mathcal{P}_{n-1} := \{z_1 + \cdots + z_n + 1 = 0\} \subset (\mathbb{C}^\times)^n$$

are bulding block of toric hypersurfaces.

[M] isotopes pants to "tailored pants." Now the Morse function

$$\mathrm{Log}^\ell : z \mapsto \|(\log|z_1| - \ell, \dots, \log|z_n| - \ell)\|^2, \qquad \ell \gg 0,$$

gives "nice" computable skeleton $\mathbb{L}_{\mathcal{P}_{n-1}}$, confirms [FLTZ] conjecture for $\mathbb{A}^n$.

Patchworking allows us to globalize this construction.

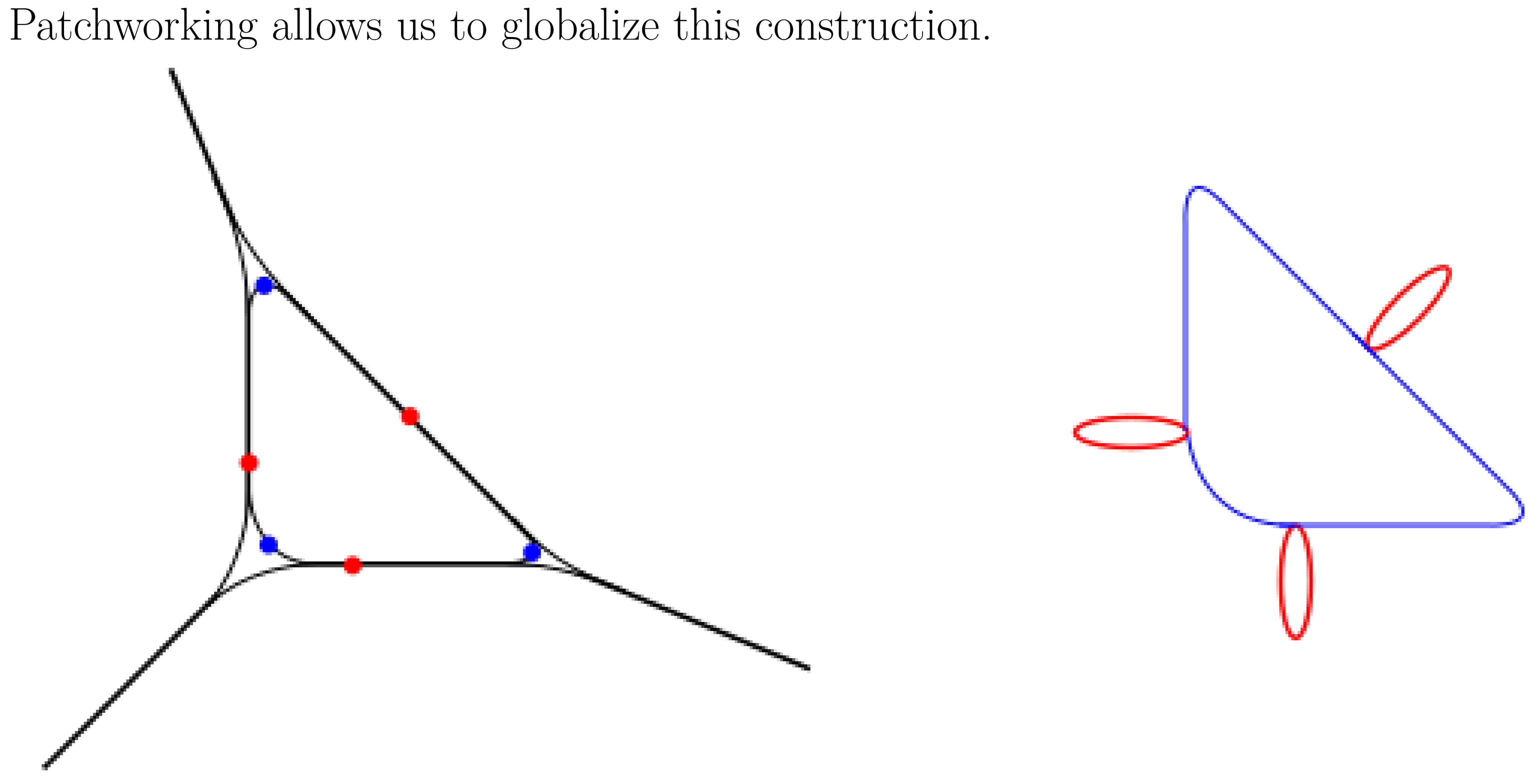

Figure 3: The amoeba of a tailored hypersurface, with images of index 0 and index 1 critical points, and the resulting skeleton.

The resulting skeleton is a union of Legendrian lifts of tori; its cone is precisely the [FLTZ] Lagrangian.

Mirror symmetry for hypersurfaces

A generic hypersurface $H_\Delta \subset (\mathbb{C}^\times)^n$ is determined by its Newton polytope

$$\Delta = \mathrm{Conv}(v_1, \dots, v_n) \subset \mathbb{R}^n.$$

This is a "large-volume limit." Its mirror is the large-complex-structure limit of the toric stack $\mathbf{T}_{\Sigma_\Delta}$, where the fan Σ_Δ has primitive rays $\{v_1, \dots, v_n\}$; in other words,

$$H_\Delta \subset (\mathbb{C}^\times)^n \text{ is mirror to the toric boundary } \partial\mathbf{T}_{\Sigma_\Delta}.$$

We prove this homological mirror symmetry equivalence.

Theorem (Mirror symmetry for hypersurfaces)

With notation as above, there is an equivalence of categories

$$\mathrm{Fuk}(H_\Delta) \cong \mathrm{Coh}(\mathbf{T}_{\Sigma_\Delta}).$$

Since we allow $\mathbf{T}_\Sigma$ to be a (Deligne-Mumford) stack, the theorem applies to *all hypersurfaces* (including general type)!

The proof of mirror symmetry

Toric boundary $\partial\mathbf{T}_\Sigma$ = (toric varieties, glued along toric varieties).

The construction from [M] allows us to match this on the mirror: The skeleton $\mathbb{L}$ = (skeleta of mirrors to toric varieties, glued along skeleta of mirrors to toric varieties).

We can apply descent as soon as we know what the mirror to pushforward is.

Lemma [G.-Shende]: Pushforward of orbit closures is mirror to *microlocalization.*

Now we apply Kuwagaki's theorem to match up pieces and observe that both sides are the same colimit.

$$\mathrm{Coh}(\mathbf{T}_\Sigma) = \mathrm{colim}_{\sigma \in \Sigma} \mathrm{Coh}(\mathbf{T}_\sigma) \cong \mathrm{colim}_{\sigma \in \Sigma} \mathrm{Fuk}(W_\sigma) = \mathrm{Fuk}(\mathbb{L}).$$

Moral

If you can produce a nice skeleton, mirror symetry becomes easy.

Further directions

- Toric degenerations: toric varieties glued together along toric varieties should have nice mirror skeleta.
- New functorialities, expected from mirror-symmetry structure: cf. [Au].
- New calculations for symplectic resolutions.

References

[Au] Speculations on homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$.

[FLTZ] B. Fang, C.-C. M. Liu, D. Treumann, and E. Zaslow, T-duality and homological mirror symmetry of toric varieties.

[GS] B. Gammage and V. Shende, Mirror symmetry for very affine hypersurfaces.

[GPS] S. Ganatra, J. Pardon, and V. Shende, Covariantly functorial Floer theory on Liouville sectors.

[Ku] T. Kuwagaki, The nonequivariant coherent-constructible correspondence for toric stacks.

[M] G. Mikhalkin, Decomposition into pairs of pants for complex algebraic hypersurfaces.

[N] D. Nadler, Wrapped microlocal sheaves on pairs of pants.