CATEGORY $\mathcal{O}$

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Abstract. Notes from a talk on Category $\mathcal{O}$ given at David Nadler's seminar. My notation, phrasing, and choice of topics are for the most part taken directly from a course on this subject I took from Victor Ginzburg at the University of Chicago. Obviously, any mistakes I've introduced in typing this up are my own.

1. Introduction and Motivation

We are interested in the unitary representation theory of some Lie groups G which contains some maximal compact subgroup K. This is a very difficult analytic problem, and we'd like to reduce it to something we can handle. So let's take our G-module V and look at the space of vectors V_{fin} which transform finitely under the action of K. Luckily for us, we know all about the representation theory of *compact* groups, and so we find:

Proposition 1.1. *$V_{fin} = \bigoplus_\rho V_\rho$, and V_{fin} is dense in V.*

Proof. We can construct a projector onto the ρ-isotypic component to get V_ρ, and we can construct delta-shaped sequences to approximate vectors. □

If all the ρ-isotypic components are finite-dimensional, we say that V is *admissible.*

Let's restrict ourselves further to $V_{\text{fin}} \cap V^\infty$, where V^∞, the set of *smooth* vectors, is defined to be $\{v \in V \mid [v \mapsto gv] \in C^\infty(G)\}$. This at least is a space which is pretty manageable. The problem of course is that by restricting ourselves to the action of K we've lost a lot of information about how G acts, but at the same time, the space we have now isn't a G-module. Solution: look at *infinitesimal* G-actions, *i.e.*, the action of $\mathfrak{g}$.

We can abstract the situation we find ourselves in with a new definition:

Definition 1.2. A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module with compactible K-action (*i.e.,* we have $\mathrm{Ad}k(x)m = kxk^{-1}m$).

Let's make one more reduction:

Definition 1.3. A $(\mathfrak{g}, K)$-module is a *Harish-Chandra module* if:

(1) The ρ-isotypic component of V is finite-dimensional for all ρ.
(2) The $Z\mathfrak{g}$ action on M is locally finite (*i.e.*, $Z\mathfrak{g} \cdot m$ is finite dimensional for all $m \in M$).

The key theorem is due to Harish-Chandra:

Theorem 1.4. *(1) Let V be a unitary irrep of G. Then $V^\infty \cap V_{fin}$ is a Harish-Chandra module.*
(2) Let V_1, V_2 be two unitary irreps whose corresponding Harish-Chandramodules are isomorphic. Then $V_1 \cong V_2$ as G-reps.

This result, which involves some difficult analysis, shows that we haven't lost anything by the reductions we've made so far. This has essentially reduced the problem of studying unitary irreps of G to algebra. So from now on we can forget about G and restrict our attention to $\mathfrak{g}$.

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2. Chevalley and Harish-Chandra isomorphisms

Before we define Category $\mathcal{O}$, we'll need to know a bit more about how the Lie algebra works. There are two theorems in particular which are very important:

Theorem 2.1 (Chevalley restriction). *The restriction map* $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ *induces an isomorphism of algebras* $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \cong \mathbb{C}[\mathfrak{h}]^W$.

Proof. Injectivity: if a function is zero on $\mathfrak{h}$, it's zero on all semisimple elements, which are Zariski-dense in $\mathfrak{g}$. For surjectivity: note that $\mathbb{C}[\mathfrak{h}]^W$ is spanned by elements of the form $b_{n,\lambda}(x) := \sum_w \lambda(x)^n$. Let $a_{n,\lambda}(x) = Tr(x^n|_{V_\lambda})$ and note that $a_{n,\lambda}|_{\mathfrak{h}}$ is $b_{n,\lambda}$ plus some contributions from $\mu < \lambda$, so we can induct on $|\lambda|$ and we're done. □

From results in invariant theory, we know that actually $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ on $n = \mathrm{rk}(\mathfrak{g})$ generators.

Now we've got to talk about the Harish-Chandra isomorphism, which is a little bit trickier. First we've got to introduce the notion of a Verma module:

$$\Delta(\lambda) : Ind_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda.$$

In other words, we declare that $\mathfrak{b}$ acts by the character λ, and then we induce to $\gg$. In the second description, we see that $1 \otimes 1$ is a cyclic vector for $\Delta(\lambda)$. By Frobenius reciprocity, $\Delta(\lambda)$satisfies the following universal property: for any $\gg$-module M, $\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), M) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}^\lambda, M)$.

Let's do the case of $\mathfrak{sl}_2$: Here λ is totally described by the complex number $l := \langle \lambda, \alpha^\vee \rangle$. The weight of $\mathfrak{h}$ on $F^k \cdot 1$ is $l - 2k$. We see that if $l \notin \mathbb{Z}_+$, then $\Delta(\lambda)$ is irreducible; otherwise, it fits into an exact sequence

$$0 \to \Delta(-\lambda - 2) \to \Delta(\lambda) \to V_\lambda \to 0.$$

We'll see that this is basically indicative of the general situation with Verma modules. We also know:

Proposition 2.2. *Let α be a simple positive root, $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha^\vee \rangle =: n \in \mathbb{Z}_{>0}$. Then there is a nonzero map $\Delta(\lambda - (n+1)\alpha) \to \Delta(\lambda)$.*

Proof. Pick Chevalley generators $e_1, \dots, e_r, f_1, \dots, f_r, h_1, \dots, h_r$. In other words f_i, e_i, h_i generate a copy of $\mathfrak{sl}_2$, and all other commutators vanish. Also, number the generators so that the first copy of $\mathfrak{sl}_2$ is the one corresponding to α. Now let $u = f_1^{n+1} 1_\lambda$. Then $e_1 u = 0$ because every commutator subtracts 1, and $e_i u = 0$ for $i \neq 1$ because e_i commutes with f_i. And we can see that $h_i u = h\lambda(h_i) - (n+1)\alpha(h_i)$. So $1_\lambda - (n+1)\alpha \mapsto u$ extends to a map of Vermas. □

Now let's define the *universal Verma module* $\Delta := U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}$. In other words, with this Verma module we're killing any element of $\mathfrak{n}$ on the right, but we're not specifying the $\mathfrak{h}$ action. Thus we have $\Delta(\lambda) = \Delta \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$ for any Verma module $\Delta(\lambda)$.

If we write $\Delta = U(\mathfrak{g}) \otimes_U (\mathfrak{b}) U(\mathfrak{h})$, it becomes clear that Δ is a $(\mathfrak{g}, \mathfrak{h})$-bimodule, and a little bit of thinking should convince us that $\mathrm{End}_{(\mathfrak{g},\mathfrak{h})} \Delta \cong U(\mathfrak{h})$. This means in particular that any element $z \in Z\mathfrak{g}$ acts on $v \in \Delta$ by

$$z \cdot v = v \cdot \Theta(v).$$

In other words, we have constructed a map $Z\mathfrak{g} \to \mathrm{Sym}(\mathfrak{h})$.

Before we can state the Harish-Chandra theorem, we need one last bit of notation. We are going to define a new action of W on $\mathfrak{h}^*$, which we'll call the *dot action* and denote by $w\dot{\lambda}$, by: $w\dot{\lambda} := w(\lambda + \rho) - \rho$. Now we can state Harish-Chandra's theorem.

Theorem 2.3. *The map $\Theta : Z\mathfrak{g} \to Sym(\mathfrak{h})$ is injective, with image $Sym(\mathfrak{h})^{W\cdot}$.*

We won't go into detail about the proof of this result; the idea is to show that Θ respects the filtration on both sides, then apply Chevalley restriction to conclude that $\mathrm{gr}\Theta$ is an isomorphism of the associated graded algebras, hence Θ is an isomorphism.

Combining this with a result stated above, we see that $Z\mathfrak{g} \cong \mathbb{C}[x_1, \dots, x_n]$.

3. Category $\mathcal{O}$

Now we are finally in a position to learn why Category $\mathcal{O}$ is so good.

Definition 3.1. The BGG Category $\mathcal{O}$ is the category of $U(\mathfrak{g})$-modules M such that:

(i.) M is finitely generated over $U\mathfrak{g}$
(ii.) The $\mathfrak{h}$-action on M is diagonal
(iii.) The $U\mathfrak{n}$-action on M is locally finite.

Now let's list some of the nice properties of $\mathcal{O}$ which explain why we like this category so much.

(1) $\mathcal{O}$ is an abelian category
(2) $\Delta(\lambda) \in \mathcal{O}$
(3) Any object $M \in \mathcal{O}$ is a quotient of $\Delta(E) := U\mathfrak{g} \otimes_{U\mathfrak{b}} E$ for some finite-dimensional $\mathfrak{h}$-semisimple $U\mathfrak{b}$-module E (Proof: let E be free on the generators of M)
(4) For M be a finitely generated $U\mathfrak{g}$-module which is $\mathfrak{h}$-semisimple, we have: $M \in \mathcal{O} \iff \operatorname{Spec} M$ is bounded, *i.e.*, there exists a finite subset $S \subset \mathfrak{h}^*$ such that $\operatorname{Spec} M \subset S - Q^+$
(5) If $M \in \mathcal{O}$, then M_μ is finite-dimensional for all $\mu \in \mathfrak{h}^*$ (Proof: this is true of $\Delta(E)$)
(6) M is finitely generated over $U(\mathfrak{n}^-)$
(7) $\dim \operatorname{Hom}_{\mathcal{O}}(M,N) < \infty$. (Proof: $\operatorname{Hom}(M,N) \hookrightarrow \operatorname{Hom}_{U\mathfrak{g}}(\Delta(E),N) = \operatorname{Hom}_{U\mathfrak{b}}(E,N) \cong \bigoplus_{\mu\in\operatorname{Spec}(E)} \operatorname{Hom}(E_\mu, N_\mu)$)
(8) Any $M \in \mathcal{O}$ is annihilated by an ideal $I \subset Z(\mathfrak{g})$ of finite codimension (Proof: The map $Z(\mathfrak{g}) \to \operatorname{End}_{\mathcal{O}} M$ is a map to a finite-dimensional algebra, hence its kernel has finite codimension)
(9) $\mathcal{O} = \bigoplus_{\chi\in\mathfrak{h}^*/W} \mathcal{O}_\chi$.
(10) Any $M \in \mathcal{O}_\chi$ has a simple subquotient, and the simple objects of $\mathcal{O}_\chi$ are $\{L(w(\chi+\rho)-\rho) \mid w \in W\}$. (Proof: Spec M is bounded from above, so we can pick a maximal weight and map to it from $\Delta(\lambda)$)
(11) Any object of category $\mathcal{O}$ has finite length (Proof: Assume $M \in \mathcal{O}_\chi$. Suppose M has an infinite composition series, each containing a simple subquotient $L(\lambda)$. Finitely many λ are in the $W\cdot$ orbit of χ, so this would mean M has some infinite-dimensional isotypic component)
(12) If λ is dominant, $\Delta(\lambda)$ is a projective in $\mathcal{O}$; if $-\lambda$ is dominant, then $\Delta(\lambda)$ is simple.

#11 is really important: The Grothendieck group $K(\mathcal{O})$ is free abelian on simples $L(\lambda)$. We'll write the multiplicity of $L(\mu)$ in M as $[M : L(\mu)]$, so $[M] = \sum_\mu [M : L(\mu)][L(\mu)]$.

Anyway: at this point David is probably starting to get mad at me because I haven't been drawing enough pictures. So I'll say some words of homological algebra which will allow me to draw a picture which we like.

Definition 3.2. A *projective generator* in a category $\mathbb{C}$ is a projective object P such that $\operatorname{Hom}(P,M) \neq 0$ for any $M \in \mathcal{C}$.

Theorem 3.3. *Suppose $\mathcal{C}$ is a k-linear abelian category suc that all Hom-spaces are finite-dimensional over k, all objects have finite length, and $\mathcal{C}$ admits a projective generator P. Then $\mathcal{C}$ is equivalent to End(P)-mod by $M \mapsto$ Hom(P,M).*

Proof. Elementary homological algebra, so I leave it as an exercise and encourage you to do it yourself. Alternatively, this is a corollary of the Barr-Beck theorem. ☐

Proposition 3.4. *$\mathcal{O}_\chi \cong$ mod–End(P) for some P.*

We can actually say exactly what this P is: $\mathcal{O}_\chi$ contains finitely many simples $L(\lambda)$, and $\mathcal{O}_\chi$ has enough projectives, so take a projective P_λ covering each simple λ. Then we can take $P := P_\lambda$.

Now let's work things out in detail for $\mathfrak{g} = \mathfrak{sl}_2$. Here we have $\mathfrak{h} \cong \mathbb{C}$, $W \cong \mathbb{Z}/2$, the root lattice is $2\mathbb{Z}$ and the weight lattice is $\mathbb{Z}$. Take a dominant weight $\lambda = m \in \mathbb{Z}_{\geq 0}$. Let $L(m)$ be the finite dimensional irrep with highest weight m. $\Delta(m) \twoheadrightarrow L(m)$, and the diagram is as follows: [diagram].

Note that $\rho = 1$, so W acts as $\lambda \to -\lambda - 2$. Because $-m-2$ is antidominant, we have $\Delta(-m-2) = L(-m-2)$.

Let $P = P(m) \oplus P(-m-2)$. This is a projective generator for $\mathcal{O}_\lambda$. The first summand has $[P(m)] = [L(m)] + [L(-m-2)]$, and the second has $[P(-m-2)] = [\Delta(-m-2)] + [\Delta(m)] = [L(-m-2)] + [L(m)] + [L(-m-2)]$. We have a map $u : P(-m-2) \to P(m)$ and $v : P(m) \to P(-m-2)$ which satisfy $v \circ u = 0$. We see that End(P) is the quotient of the path algebra of the quiver $\bullet \leftrightarrow \bullet$ by this relation: $\operatorname{End}(P) = kQ/(vu)$. By the result above, we see that $\mathcal{O}_\lambda$ is the category of representations of that quiver satisfying $v \circ u = 0$.

We've seen this quiver before, in Zorn's second talk. Now things are starting to fit together, and I can explain why this is really the same as what we've been studying. First let me draw the quiver for $\mathfrak{sl}_3$: [diagram]. As you can see, things start to get more complicated pretty quickly: there's a lot of calculation required here, which I why I'm sticking to $\mathfrak{sl}_2$ as much as I can in this talk.

Anyway, the connection comes from the following theorem:

Theorem 3.5 (Beilinson-Bernstein)**.** *There is an equivalence of categories between the category of $U\mathfrak{g}$-modules with central character χ and the category of $\mathcal{D}_{\Theta(\chi)}(G/B)$-modules, i.e., the category of sheaves (on the flag variety G/B) of modules for the ring of differential operators on the line bundle corresponding to the character which is the image of χ under the Harish-Chandra isomorphism.*

Obviously, I'm not going to prove this now.

Of course, in category $\mathcal{O}$ we require something extra: we have the condition of local $\mathfrak{n}$-finiteness, which on the geometric side translates into an N-equivariance condition on our $\mathcal{D}$-module. (Likewise, the K-equivariance condition for Harish-Chandra modules translates into a K-equivariance condition.)

Theorem 3.6. *Suppose M is an N- or K-equivariant $\mathcal{D}_{\Theta(\chi)}(G/B)$-module. Then M is regular holonomic.*

Now: look at how N acts. Two orbits: 0, and everything else. (Note: at this point I've run out of time to write everything out. I recommend the end of Chapter 9 of [4], but also working all this out yourself is a really good exercise. The thing to remember is that SL_2 acts on $\mathbb{P}^1$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)[x : y] = [ax + by : cx + dy]$, and the point $[1 : 0]$ has stabilizer the upper-triangular unipotent matrices B, so that $G/B \cong \mathbb{P}^1$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [1 : 0] = [a : c]$.) So we're talking about perverse sheaves on the stratification $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{pt\}$. Also interesting to look at how K acts: it fixes $[1 : i]$ maybe?

[One more fun thing: Any point of G/B has tangent space $\mathfrak{g}/\mathfrak{b} \cong \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$, hence the top exterior power is $\bigwedge_\alpha \mathfrak{g}_\alpha$, and so we see that the canonical bundle on the flag variety corresponds, in a certain sense, to the sum of all negative roots. Thus ρ represents a sort of square root of the canonical bundle. In general this leads to questions about spin structures &c., but in the simple case of SL_2 this is an easy way of seeing that $\mathcal{L}(\rho)$ is the sheaf $\mathcal{O}(-1)$ on $G/B \cong \mathbb{P}^1$.]

I probably have time now to say some more things about things that are "BGG." First let me say one related thing: we might note that $[\Delta(\lambda)] = \sum_{\lambda' \leq \lambda} [L(\lambda')] = [L(\lambda)] + \sum_{\lambda' < \lambda} [L(\lambda')]$. Since the $[L(\lambda)]$ form a basis for $K(\mathcal{O})$, we can see that the $[\Delta(\lambda)]$ do too. One natural question to ask is: what is the change-of-basis matrix? This is the *Kazhdan-Lusztig* problem, and it turns out to be very hard. Hodge structures and ℓ-adic cohomology is involved. This is all I have to say right now. I encourage anyone who is interested to read the excellent sketch of ideas involved in the proof at the end of [1].

Anyway, moving on: let's state the BGG reciprocity theorem.

Theorem 3.7. $[P(\lambda) : \Delta(\mu)] = [\Delta(\mu) : L(\lambda)]$.

There's more that I would like to say about this, and about the BGG resolution, but I'm certainly out of time.

References

[1] Joseph Bernstein. "Algebraic Theory of D-modules."
[2] Dennis Gaitsgory. "Geometric Representation Theory, Fall 2005."
[3] Benjamin Gammage. Course notes from "Geometry of Semisimple Groups," course taught by Victor Ginzburg, Spring 2014. [Unpublished]
[4] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, Perverse Sheaves, and Representation Theory.*