MINOR THESIS: FROM FIBERED CATEGORIES TO ALGEBRAIC STACKS

THOMAS BARNET-LAMB

Abstract. It is well known that it is not possible to give a moduli scheme for elliptic curves in the classical sense, due to the existence of twists. However, there is an generalisation of the concept of a scheme, that of an algebraic stack, such that one can give an algebraic stack which is such a moduli space. We develop the theory of these objects to the point where we can see how they help shed light on moduli problems. We—very briefly—sketch applications to elliptic curves, following [8]. Along the way, we construct corresponding extensions of the notion of a presheaf and a sheaf, called a fibered category and a stack respectively.

The idea of algebraic stacks is due to Mumford, in [8], and Deligne and Mumford gave the definition in [4]. The canonical technical reference is [7], which is our main source on algebraic stacks. For the material on fibered categories and stacks, we follow the splendid article [10] by Vistoli. There is an extension of the notion of algebraic stack due to Artin which we do not consider.

Contents

1. Introduction

Classically, a moduli problem on the category **Sch** of schemes is a functor $F : \mathbf{Sch}^{op} \to \mathbf{Set}$, and to *solve* the moduli problem is to represent this functor. Explicitly, this means to give a scheme X and a 'universal' element ϕ in $F(X)$, such that for all other schemes T, the map:

$$\begin{aligned} \{\text{morphisms } f : T \to S\} &\to F(T) \\ f &\mapsto (Ff)(\phi) \end{aligned}$$

is a bijection. Since we think of morphisms $T \to S$ as 'T valued points of S', this says that the T valued points of S correspond naturally to the elements of $F(T)$, or that to give an element of $F(T)$ is simply to give a T valued point of S. We can similarly think about moduli problems on $\mathbf{Sch}/S$ for any scheme S.

One moduli problem of considerable algebraic and number theoretic interest is $F_e : (\mathbf{Sch}/\mathbb{Q})^{op} \to \mathbf{Set}$ given on objects $T \in \mathbf{Sch}/S$ by

$$T \mapsto \{\text{elliptic curves over T modulo isomorphism}\}$$

and on morphisms $\phi : T \to T'$ by

$$\begin{aligned} (F_e\phi) : \{\text{elliptic curves over T mod iso}\} &\to \{\text{elliptic curves over T mod iso}\} \\ (\mathcal{E} \to T) &\mapsto \text{the pullback of } \mathcal{E} \to T \text{ along } \phi \text{ to } T' \end{aligned}$$

(One verifies that the isomorphism class of the pullback is unaffected by the choice of the representative $\mathcal{E} \to T$ of the isomorphism class on the LHS.)

Unfortunately, this moduli problem is not solvable. To see this, consider the following elliptic curves $/\mathbb{Q}$

$$\{x^3 - x = 2y^2\} \qquad \text{and} \qquad \{x^3 - x = y^2\}$$

One easily verifies these are not isomorphic. However, after base change to $\bar{\mathbb{Q}}$, they are isomorphic, via $(x, y) \mapsto (x, \sqrt{2}y)$. We conclude that the map $F_e(\mathbb{Q}) \to F_e(\bar{\mathbb{Q}})$ is not injective. On the other hand, for any scheme X, the map $X(\mathbb{Q}) \to X(\bar{\mathbb{Q}})$ from $\mathbb{Q}$ valued points of X to $\bar{\mathbb{Q}}$ valued points *is* injective. If X represents F_e we would have a naturality diagram

$$\begin{tikzcd} F_e(\mathbb{Q}) \arrow[r] \arrow[d, "\cong"] & F(\bar{\mathbb{Q}}) \arrow[d, "\cong"] \\ X(\mathbb{Q}) \arrow[r, hook] & X(\bar{\mathbb{Q}}) \end{tikzcd}$$

telling us that $F_e(\mathbb{Q}) \to F_e(\bar{\mathbb{Q}})$ is injective, a contradiction.

This is a shame, as we would really like for the moduli problem to be soluable. There are two classical workarounds to this problem, which are in some sense dual to each other. An example of the first approach is to replace the functor F_e with the functor which attaches to each T not just the set of elliptic curves over T modulo isomorphism, but the set of elliptic curves with a distinguished torsion point of degree N, for some fixed $N \geqslant 4$. This functor *is* representable, and obviously quite closely related to the original functor: one might hope that in many applications, one can use it instead.

The other is to define a so-called *coarse moduli space*, which is an object X which 'comes as close as possible to representing F_e', in the sense that there is a natural transformation $\alpha : F_e \to h_X$ such that *a)* for any other space T and natural transformation $\beta : F_e \to h_T$ we have a factorisation of β as $F_e \to h_X \to h_T$ for some map $h_X \to h_T$ and *b)* the component of α at each algebriaically closed field is a bijection.

Unfortunately, each of these approaches lose information, and even in combination they are not ideal. Luckily, there is another way forward. The point is that moduli problems do not really live in a category (the category of functors $\mathbf{Sch}^{op} \to \mathbf{Set}$), but rather in a *2-category* of *fibered categories over* **Sch**. When we understand the situation in terms of fibered categories, we gain both a clear perspective on the problem, and an elegant solution; it turns out that most of the problems in dealing with these moduli problems have arisen from our shoehorning the problem from the world of 2-categories into the world of categories. (Of course, this is a natural enough thing to attempt, since categories are a familiar language, whereas 2-categories are less familiar.) While we cannot find a scheme which represents the functor, there is a natural generalisation of the concept of scheme, that of an *algebraic stack*; and there *are* algebraic stacks which 'represent' the functor.

To get to the point where we can see all this, we will need to work to reformulate various familiar concepts into this new language of fibered categories. Here is a basic 'dictionary' between analogous concepts in the two languages:

Old	**New**
functor $\mathcal{C}^{op} \to \mathbf{Set}$ =presheaf	*either* fibered category over **Sch** *or* category fibered in groupoids
separated presheaf	prestack
sheaf	stack
scheme	algebraic stack

It is worth pointing out here that the terminology is quite misleading. Despite the fact that 'stacks' and 'algebraic stacks' sound very similar, they are in fact quite distinct notions, playing very different roles in the theory: as different as the roles sheaves and schemes play.

In the next section, we will introduce fibered categories, and translate into this new language all the sheaf theory we need. We will also see that descent theory, in all its many forms, can be very naturally expressed in this language. We will be following the third and fourth chapters of [10]. In the subsequent section, we will move on to use the clarity that this language brings to diagnose what was going wrong in trying to represent F_e. We will develop the concepts of algebraic spaces and algebraic stacks, successive generalisations of the concept of a scheme. We will then proceed to see that there is a moduli algebraic stack $\mathcal{M}$ of elliptic curves, thus solving the problem we started with.

We assume familiarity with elementary algebraic geometry (e.g. Hartshorne's book), Grothendieck topologies (including the étale, fppf and fpqc topologies, and the fact that a representable functor on $\mathbf{Sch}/S$ is an fpqc sheaf), and basic category theory.

2. The language of fibered categories and stacks

2.1. **Categorification, decategorification, and 2-categories.** We will begin by considering the process of decategorification, which is a process which passes from a category to a set. Much of what will be said is obvious, but it is worth considering because a category is to a set as a 2-category is to a category, and the corresponding process of de-2-categorification, which passes from a 2-category to a category, is more mysterious and of considerable importance for thinking about moduli problems. It will therefore pay briefly to consider its little brother.

As has been said, decategorification is a process that turns a category into a set. In particular, we get the set whose elements are the isomorphism classes of objects in the category. This is a common and very useful process in mathematics (although the name is much less common). For instance, if we decategorify the category of finite sets, we get the set of natural numbers. Often extra operations on the category we start with will descend to give extra structure on the decategorified set: for instance, the operations of cartesian product and disjoint union on the category of finite sets descend to give addition and multiplication on $\mathbb{N}$, turning it into a ring.

The process of decategorification is very useful. First, it is very tiresome keeping track of explicit isomorphisms. Second, the decategorification of a set can often be significantly simpler in structure than the set you start with, which makes calculations easier. (For instance, elements of $\mathbb{N}$ are easily written decimally for calculation.) Third, once we have decategorified, we can apply some processes which can be very useful, such as adjoining formal additive or multiplicative inverses, which it is hard to make sense of without decategorifying.

Many objects of great interest are decategorifications (like $\mathbb{N}$) or decategorifications with inverses adjoined (like $\mathbb{Z}$). For example, if we take the fundamental groupoid of a topological space (whose objects are points of the space and morphisms are homotopy classes of maps), we get the set of its path components. If we decategorify the theory of line bundles on a curve, we get the Picard group. If we decategorify the category of representations of a finite group G, and adjoin additive inverses, we get G's representation ring. If we decategorify the theory of vector bundles on a fixed topological space, and adjoin additive inverses, we get the K-theory. As a final example, we remark that topologists, who spend a lot of their time dealing with isomorphism classes in the category **TopHtpy** of topological spaces with homotopy classes of maps between them, often work in more-or-less decategorified terms, saying 'X is an S^2' to express the homotopy equivalence between X and S^2, and trying to suppress explicit mention of the maps inducing homotopy equivalences where possible.

Nevertheless, information is lost when we decategorify, and sometimes working with the original categorical formulation can provide more insight. Combinatorialists, having proved two sets have the same size by decategorified means, will often look for a so-called *bijective proof* of the result—an explicit isomorphism of sets between the two sides—for precisely this reason. When working with K-theory, we know we must sometimes 'get our hands dirty' and think of actual (stable classes of) vector bundles.

We will now turn to the more tricky matter of de-2-categorification.[1]

There are several equivalent ways of thinking about 2-categories. Morally, a 2 category $\mathcal{C}$ is like a category, except as well as objects and morphisms (which we now call 1-morphisms), we also have 2-morphisms; that is morphisms between morphisms. Thus given objects A and B, rather than a Hom set $\mathrm{Hom}(A, B)$, we have a Hom *category* (which has objects the 1-morphisms from A to B and arrows the 2-morphisms between the 1-morphisms), and the assosciativity and identity rules are replaced by functorial analogues. Formally,

Definition 2.1.1. A (strict[2]) 2-category consists of a set O of objects, equipped with categories $\mathcal{HOM}(A, B)$ for each $A, B \in O$. For each $A \in O$, $\mathcal{HOM}(A, A)$ has a distinguished *identity* object id_A, and for all A, B, C we have a composition functor

$$F_{A,B,C} : \mathcal{HOM}(A, B)\times\mathcal{HOM}(B, C)\rightarrow\mathcal{HOM}(A, C)$$

satisfying:

- For all A, B, C, D we have
$$F_{A,C,D} \circ (F_{A,B,C}\times\mathrm{id}_{\mathcal{HOM}(C,D)}) = F_{A,B,D} \circ (\mathrm{id}_{\mathcal{HOM}(A,B)}\times F_{B,C,D})$$
- We have
$$F_{A,A,B}(\mathrm{id}_A, x) = x = F_{A,B,B}(x, \mathrm{id}_B)$$
for all $x \in \mathrm{Ob}\ \mathcal{HOM}(A, B)$, and
$$F_{A,A,B}(\mathrm{id}_{\mathrm{id}_A}, f) = f = F_{A,B,B}(f, \mathrm{id}_{\mathrm{id}_B})$$
for all $f \in \mathrm{Mor}\ \mathcal{HOM}(A, B)$.

[1]By the way, the reader interested in decategorification might wish to read [2], where the authors argue that many results, even in theories that are not in any obvious way a decategorification of something, can have light shed on them by *finding* some category which decategorifies to give the theory in question, then finding a result which decategorifies to give the result under study. The paper sketches, with varying degrees of completeness, this process for $\mathbb{Q}$, $\mathbb{Z}$, before moving on to Quantum field theory and Feynmann diagrams...

[2]All 2-categories considered in this essay will be strict, apart from a few remarks where we explicitly say otherwise.

Finally, we require the middle 4 interchange condition; if $A, B, C \in O$ and we have arrows $f : X \to Y, g : Y \to Z$ of $\mathcal{HOM}(A, B)$ and arrows $w : Q \to R, v : R \to S$ of $\mathcal{HOM}(A, B)$ (so A, B, C are objects, X, Y, Z, Q, R, S are 1-morphisms, and f, g, w, v are 2-morphisms), then we have

$$F_{A,B,C}(f \circ g, v \circ w) = F_{A,B,C}(f, v) \circ F_{A,B,C}(g, w)$$

The canonical example of a 2-category is the 2-category **Cat** of categories. Here the objects are categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations. Another example would be the 2-category $\mathbf{Top_2}$ whose objects are topological spaces, whose 1-morphisms are maps between them, and whose 2-morphisms are homotopies between these maps, modulo reparametrisation.

There is a process of de-2-categorification analogous to the decategorification process we have already considered. Here, we take in a 2-category $\mathcal{C}$, and the result is an ordinary category $\mathcal{C}'$. This category $\mathcal{C}'$ has the same objects as the original 2-category $\mathcal{C}$, but we replace $\mathcal{C}$'s category of morphisms $\mathcal{HOM}(A, B)$ with a set of morphisms $\mathrm{Hom}(A, B)$ got by decategorifying $\mathcal{HOM}(A, B)$. It is easy to see that we get a bona fide category (in particular, that the composition functor from $\mathcal{C}$ descends to give a composition map as we decategorify).

It is easy to see that, in fact, the category **TopHtpy** mentioned above as being of interest to topologists is in fact none other than the de-2-categorification of $\mathbf{Top}_2$.

Remark 2.1.2. We can also *almost* see the fundamental groupoid of a space in these terms. If we try to form a 2-category with objects points of the space, 1-morphisms paths between points, and 2-morphisms endpoint preserving homotopies between paths, we don't form a strict 2-category. The problem is that composition of 1-morphisms is not assosciative in the sense we demand (that $a(bc) = (ab)c$) but rather in the weaker sense that there is a 2-isomorphism between $a(bc)$ and $(ab)c$—similar remarks hold for identities. There is, in fact, a generalisation of the notion of a strict 2-category to a notion of a *weak 2-category* or *bicategory*, of which this is an example. We do not give the rather technical and at-first counterintuitive details, but remark that the de-2-categorification of this weak 2 category is indeed the fundamental groupoid.

Just as with decategorification, de-2-categorification is a mixed blessing, and sometimes getting a proper insight on a problem will require looking at the original 2-category. But there is a difference. When considering a question about a set F that is the decategorification of some category $\mathcal{F}$, the idea that we should be considering $\mathcal{F}$ instead (to get a better idea of what is going on) is natural and easy, since $\mathcal{F}$ is a category, and categories are relatively familiar and friendly. But when considering a question about a category $\mathcal{F}$ which is the de-2-categorification of a 2-category $\mathcal{F}'$, it feels like rather a big step to decide to consider $\mathcal{F}'$ instead, since 2-categories are unfamiliar and seem difficult to work with. Thus while the advice 'sometimes, it's better not to decategorify', while true, is normally superfluous (because in the cases where the advice applies, it's quite a natural thing to think of doing anyway), the advice 'sometimes, it's better not to de-2-categorify' is genuinely useful, because the idea of thinking in 2-categorical terms is not the first thing one would think of doing.

This is essentially the situation with moduli problems. We will see that the truly natural setting to think of moduli problems is in the *2-category* of categories fibered over **Sch** (or $\mathbf{Sch}/S$). By considering them as functors $\mathbf{Sch}^{op} \to \mathbf{Set}$, we have de-2-categorified, losing valuable information, which is the key to unravelling our difficulties. Our first step, then, is to investigate this 2-category of fibered categories, and see how moduli problems naturally live there.

2.2. **Fibered categories: Definition and first properties.** Let us fix a base category $\mathcal{C}$ (this will usually be **Sch** or $\mathbf{Sch}/S$, but for now we can work in generality. By a *category over* $\mathcal{C}$, we mean a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$. (This is precisely analogous to a scheme over S being a scheme T with a map $T \to S$.) A convenient notation for dealing with this kind of situation is to use diagrams where we mix objects from $\mathcal{C}$ and $\mathcal{F}$. Arrows between objects in $\mathcal{C}$ will simply refer to arrows in $\mathcal{C}$. Similarly for arrows between objects of $\mathcal{F}$. The only other arrows will be arrows $\xi \mapsto U$ *from* an object ξ of $\mathcal{F}$ *to* an object U of $\mathcal{C}$; this means $p\xi = u$. (Such arrows will be written using the $\mapsto$ style to distinguish them from ordinary $\mathcal{C}$ or $\mathcal{F}$ arrows.) Moreover, if we say

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

commutes, we mean that $p\phi = f$.

Definition 2.2.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\phi : \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h : p\zeta \to p\xi$ in $\mathcal{C}$ with $p(\phi) \circ h = p(\psi)$, there exists a unique arrow $\theta : \zeta \to \xi$ with $p(\theta) = h$ and $\phi \circ \theta = \psi$, as in the diagram:

$$\begin{tikzcd} \zeta \arrow[dr, "\theta"] \arrow[drr, "\psi"] \arrow[dd, mapsto] & & \\ & \xi \arrow[r, "\phi"] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\ p(\zeta) \arrow[dr, "h"] \arrow[drr] & & \\ & p(\xi) \arrow[r] & p(\eta) \end{tikzcd}$$

We say that ξ is a *pullback of* η *along* $p(\phi)$.

Remark 2.2.2. This is the definition used in [10], where it is pointed out that this is more restrictive than the definition given in SGA1. Nevertheless, the notions of fibered category for the two definitions coincide. (Some authors have called arrows satisfying our condition *strongly cartesian*)

Remark 2.2.3. It is possible to 'take literally' the notation of our diagrams mixing objects of $\mathcal{C}$ and $\mathcal{F}$, and consider a category with objects the disjoint union of the objects of $\mathcal{C}$ and $\mathcal{F}$, and morphisms freely generated by a) the morphisms of $\mathcal{C}$, b) the morphisms of $\mathcal{F}$, and c) morphisms $p_\xi : \xi \to U$ where $p(\xi) = U$, quotiented by the relation that if $\phi : \xi \to \eta$ and $f : U \to V$ have $p(\phi) = f$, $f \circ p_\xi = p_\eta \circ \phi$. (In other words, if the 'mixed' diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

is one of the ones we have called 'commutative', we decree that the diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d] & \eta \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

in our new category should commute.) It should be easy to see that for $\phi : \xi \to \eta$ to be cartesian is for us to have a cartesian square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d] & \eta \arrow[d] \\ p(\xi) \arrow[r, "p(\phi)"] & p(\eta) \end{tikzcd}$$

in this 'disjoint-ish union' category.

The previous remark immediately suggests that cartesian arrows should satisfy analogs of the properties of cartesian (or 'pull-back') squares in general categories. Indeed, we have the following facts, whose proofs are all trivial analogs of the proofs in general categories, and are left to the reader.

Proposition 2.2.4. *We have that:*

- *Suppose $\phi : \xi \to \eta$ and $\tilde{\phi} : \tilde{\xi} \to \eta$ are cartesian arrows with the same target, which map under p to the same arrow. Then there is a unique isomorphism $i : \xi \to \tilde{\xi}$ such that $i\phi = \tilde{\phi}$. In other words, a pullback is unique up to unique isomorphism.*
- *The composite of cartesian arrows is cartesian.*
- *An arrow in $\mathcal{F}$ whose image is an isomorphism is cartesian iff it is itself an isomorphism.*
- *If $\xi \to \eta$ and $\eta \to \zeta$ are arrows in $\mathcal{F}$ and $\eta \to \zeta$ and the composite $\xi \to \zeta$ are cartesian, then $\xi \to \eta$ is cartesian.*

It is also easy to show

Proposition 2.2.5. *If $\mathcal{F}$ is a category over $\mathcal{C}$ and $\mathcal{G}$ a category over $\mathcal{F}$, and we have a morphism ϕ in $\mathcal{G}$ which is cartesian over its image in $\mathcal{F}$, the image in $\mathcal{F}$ in turn being cartesian over* its *image in $\mathcal{C}$, then if we consider $\mathcal{G}$ as a fibered category over $\mathcal{C}$ using the functor composite $\mathcal{G} \to \mathcal{F} \to \mathcal{C}$, then ϕ is cartesian.*

Definition 2.2.6. A *fibered category over $\mathcal{C}$* is a category $\mathcal{F}$ over $\mathcal{C}$ such that given:

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

we can complete to a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

with ϕ cartesian. (In other words, every object has a pullback along every morphism that it could.)

Definition 2.2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories $\mathcal{C}$. A *(1-)morphism of fibered categories* $\mathcal{F} \to \mathcal{G}$ is a functor $\mathcal{F} \to \mathcal{G}$ which is base preserving ($p_{\mathcal{F}} = p_{\mathcal{G}} \circ F$) and sends cartesian arrows to cartesian arrows. A *2-morphism* of fibered categories is a natural transformation a of functors $F, G : \mathcal{F} \to \mathcal{G}$, such that for each object ξ of $\mathcal{F}$, the component a_ξ is in $\mathcal{G}(p\xi)$. Thus fibered categories form a 2-category, as promised.

(We note that we need an actual *equality* of functors $p_{\mathcal{F}} = p_{\mathcal{G}} \circ F$; a natural isomorphism is not enough.)

Proposition 2.2.8. *If $\mathcal{G}$ is a fibered category over $\mathcal{F}$ and $\mathcal{F}$ is a fibered category over $\mathcal{C}$, then $\mathcal{G}$ is a fibered category over $\mathcal{C}$.*

Proof. This follows from prop 2.2.5. □

2.3. **Pseudo-functors.** We have seen that fibered categories form a two-category. But they are meant to correspond, under our grand analogy between 2-categorical and 1-categorical things, to functors $\mathcal{C}^{op} \to \mathbf{Set}$; it is less clear how they do this. We now turn to the task of understanding this.

Definition 2.3.1. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$, and U be an object of $\mathcal{C}$. Then the *fiber* $\mathcal{F}(U)$ *of* $\mathcal{F}$ *over* U is the subcategory of $\mathcal{F}$ determined by the objects that p maps to U and the morphisms which p maps to id_U. (Note well the restriction on the morphisms: this is *not* the full subcategory 'lying over' U.)

It is easy to see that given a morphism $F : \mathcal{F} \to \mathcal{G}$ of fibered categories over $\mathcal{C}$ and a $U \in \mathcal{C}$ sends $\mathcal{F}(U)$ to $\mathcal{G}(U)$; thus we get a restriction functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

Note that the same definition of the fiber over U could be given for any category over $\mathcal{C}$, and not just for a fibered category. However, the resulting notion is not well behaved; for instance, isomorphic objects in $\mathcal{C}$ could have unrelated fibers, in the strong sense that we could arrange for the fiber over one to be anything we like, and the fiber over the other to be anything else. Let us see how the fibered category axiom rules out this kind of pathology.

Suppose $\mathcal{F} \to \mathcal{C}$ is a fibered category, and $f : U \to V$ an arrow in $\mathcal{C}$. What does the condition that $\mathcal{F} \to \mathcal{C}$ is fibered give us? Well, for each $\xi \in \mathcal{F}(V)$, we know there exists a pullback along f to an element in $\mathcal{F}(U)$. Let us choose a pullback $f^*\eta$ for each $\eta \in \mathcal{F}(V)$. Then

Claim 2.3.2. *We can extend the map* $f^* : \mathrm{Ob}\ \mathcal{F}(V) \to \mathrm{Ob}\ \mathcal{F}(U)$ *to a functor in a natural way.*

We achieve this by sending a morphism $\beta : \eta \to \eta'$ into the unique arrow $f^*\beta$ making

$$\begin{tikzcd} f^*\eta \arrow[r] \arrow[d, "\beta"] & \eta \arrow[d, "\beta"] \\ f^*\eta' \arrow[r] & \eta' \end{tikzcd}$$

commute. (It is easy to see this is indeed a functor.) Thus we get from $f : U \to V$ in $\mathcal{C}$ a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$, but only once we choose pullbacks of each object in $\mathcal{F}(V)$. A choice of these pullbacks for all morphisms f in $\mathcal{C}$ is called a *cleavage*.

Definition 2.3.3. A *cleavage* of a fibered category $\mathcal{F} \to \mathcal{C}$ consists of a class K of cartesian arrows in $\mathcal{F}$ such that for each diagram

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

there is a unique completion to a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

with ϕ in K.

Thus a cleavage gives a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ for each $f : U \to V$. It would be tempting to think that the assignation $U \mapsto \mathcal{F}(U), f \mapsto f^*$ gives us a functor from $\mathcal{C}^{op}$ to the category $\mathbf{Cat}$

of categories[3], but it would be wrong. Suppose, for instance, that we have morphisms $f : U \to V$ and $g : V \to W$ in $\mathcal{C}$, and suppose ξ is an object of $\mathcal{F}$ over W. Then $f^*g^*\xi$ is a pullback of ξ to U; but pullbacks are not unique, and there is no reason that it should be the *same* pullback that we chose for ξ along gf. In other words f^*g^* need not equal $(gf)^*$. Similarly, the identity on U need not give the identity functor on $\mathcal{F}(U)$.

We might think that we were somewhat lax in choosing pullbacks arbitrarily: maybe, we might think, a more cunning choice would give us a functor. Unfortunately, this is not always possible. It is easy to see that a cleavage gives rise to a true functor iff it is closed under composition and contains all the identities (where we think of the cleavage as simply being a class of arrows). (In this case, we say it is a *splitting*.) Here is an example showing we cannot always choose our cleavage to be a splitting. We may consider a group G as a one-object category, and a homomorphism of groups $G \to H$ gives a functor $G \to H$ of the corresponding categories. Arrows of G are always cartesian, so this exhibits G as a fibered category as long as G surjects on H.

Then, a cleavage is a subset of G mapping bijectively onto H; it is a splitting iff it is a subgroup. Thus the fibered category G over H admits a splitting iff our surjection $G \mapsto H$ has a splitting in the usual sense (i.e. a subgroup of G such that our surjection restricts to give an isomorphism between the subgroup and H). It is well known such a splitting need not exist.

We might now be tempted to give up; but all is not lost. We recall that **Cat** is a 2-category, not merely a category, in which it makes sense to say that two 1-arrows are isomorphic even if not equal, and we might try to ask not for identities $f^*g^* = (gf)^*$, but isomorphisms $\alpha_{f,g} : f^*g^* \cong (gf)^*$. Similarly, rather than asking that $\mathrm{id}_U^* = \mathrm{id}_{\mathcal{F}(U)}$ we ask for an isomorphism $\epsilon_U : \mathrm{id}_U^* \cong \mathrm{id}_{\mathcal{F}(U)}$.

The mere existence of such isomorphisms turns out to be not quite enough, for the following reason. Suppose we have $f : U \to V, g : V \to W, h : W \to T$ and $\theta \in \mathcal{F}(T)$. Then, of course, we'd like $f^*g^*h^*\theta \cong (hgf)^*\theta$. Fortunately, what we have already asked for guarantees the existence of such an isomorphism. Unfortunately, what we already have actually furnishes us with *two* such isomorphisms. Since we would like a *canonical* isomorphism $f^*g^*h^*\theta \cong (hgf)^*\theta$, we should insist these isomorphisms are the same[4]. We can impose similar conditions to ensure we get a *canonical* isomorphism $f^*\mathrm{id}^* \cong f^* \cong \mathrm{id}^*f^*$.

This is the last condition we must impose. Having done so, we have no further work to do. These constraints now ensure that for any pair of things that we can manufacture an isomorphism between using the isomorphisms we have been given, all the different ways of constructing the isomorphism give the same answer.

(Readers familiar with the process of defining a weak 2-category, or a weak monoidal category, or a weak symmetry on a strict monoidal category, will notice the similarity. In each of these cases, we take the definition of some object, and generalise by replacing the requirement that some identities hold with a stipulation that for each there is a prescribed isomorphism between the two sides of the would-have-been identity. Then, just as above, one must impose so-called *coherence* constraints. These ensure that for every pair of things that we can manufacture an isomorphism between, all possible ways of manufacturing the isomorphism give the same answer.)

Let us now introduce some formal language for the above ideas.

Definition 2.3.4. A *pseudo-functor* or *lax 2-functor* Φ from a category $\mathcal{C}$ to a 2-category $\mathcal{D}$ consists of the following data:

- For each object U of $\mathcal{C}$, an object ΦU.

[3]We will not consider size issues in this essay—the interested reader is welcome to formalize our argument using any of the many standard approaches

[4]Carried away by the spirit of the moment, we might think to ask that the isomorphisms were *isomorphic* rather than equal might be a better idea. Luckily, such confusions are not needed, and in fact don't make sense. Since **Cat** is a 2-category, not a 3-category, there is no notion of natural transformations being isomorphic, as opposed to equal.

- For each arrow $f : U \to V$ of $\mathcal{C}$, a 1-morphism $\Phi f : \Phi U \to \Phi V$.
- For each object U of $\mathcal{C}$, a 2-isomorphism $\epsilon_U : \Phi(\mathrm{id}_U) \cong \mathrm{id}_{\Phi U}$ of 1-morphisms $\Phi U \to \Phi U$. (A 2-isomorphism is a 2-morphism with an inverse).
- For each pair of arrows $f : U \to V, g : V \to W$ of $\mathcal{C}$, a 2-isomorphism
$$\alpha_{g,f} : \Phi(g)\Phi(f) \cong \Phi(gf)$$
of 1-morphisms $\Phi U \to \Phi W$.

Satisfying the conditions

(1) If $f : U \to V$ is an arrow of $\mathcal{C}$, we have equalities of 2-morphisms:
$$\alpha_{f,\mathrm{id}_U} = \epsilon_U \Phi(f) \quad \text{and} \quad \alpha_{\mathrm{id}_V,f} = \Phi(f)\epsilon_U$$
($\epsilon_U \Phi(f)$ is the conventional notation for what is more properly written $\epsilon_U \mathrm{id}_{\Phi(f)}$.)

(2) Whenever we have arrows $f : U \to V, g : V \to W, h : W \to T$, we have the following commutative diagram in $\mathcal{HOM}(\Phi U, \Phi T)$:
$$\begin{tikzcd}
\Phi(h)\Phi(g)\Phi(f) \arrow[r, "\alpha_{h,g}\Phi(f)"] \arrow[d, "\Phi(h)\alpha_{g,f}"] & \Phi(hg)\Phi(f) \arrow[d, "\alpha_{hg,f}"] \\
\Phi(h)\Phi(gf) \arrow[r, "\alpha_{h,gf}"] & \Phi(hgf)
\end{tikzcd}$$

Note that a functor may always be thought of as a pseudo-functor, where all the αs and ϵs are identities. Our discussion amounts to saying that even though $f \mapsto f^*$ might not be a functor into **Cat**, we might hope it is a pseudo-functor $\mathcal{C}^{op} \to \mathbf{Cat}$, which it is.

Theorem 2.3.5. *A fibered category with a cleavage defines a pseudo-functor* $\mathcal{C}^{op} \to \mathbf{Cat}$.

Proof. First we construct the isomorphisms $\alpha_{f,g}$. Given $f : U \to V$ and $g : V \to W$ and ζ above W, we know that both $f^* g^* \zeta$ and $(gf)^*\zeta$ are pullbacks of ζ to W, since the composition of cartesian morphisms is cartesian. Thus they are canonically isomorphic. It is easy to see these isomorphisms piece together to give an isomorphism of functors $\alpha_{f,g}$. Similarly, given $\zeta \in \mathcal{F}(W)$, both $\mathrm{id}_W^* \zeta$ and ζ are pullbacks of ζ along id_U; we get isomorphisms which piece together to give the isomorphism of functors we need.

Now we need the compatibility conditions. We'll show this for a), as b) is similar. Given $f : U \to V, g : V \to W, h : W \to T$ and ζ above T, then since $f^* g^* h^* \zeta$ and $(hgf)^*\zeta$ are pullbacks of ζ, so they is a *unique* ismorphism between them in $\mathcal{F}(U)$. But clearly both $\alpha_{gh,f} \circ \alpha_{f,g}(h^*\zeta)$ and $\alpha_{g,hf} \circ (f^* \alpha_{h,f}(\zeta))$ are such isomorphisms. Thus they are equal, and since they are equal for all ζ
$$\alpha_{gh,f} \circ (\alpha_{f,g} h^*) = \alpha_{g,hf} \circ (f^* \alpha_{h,f})$$
which is as required. □

In the sequel, we shall always use 'pseudo-functor on $\mathcal{C}$' to refer to a contravariant pseudo-functor from $\mathcal{C}$ to **Cat**; that is, a pseudo-functor $\mathcal{C}^{op} \to \mathbf{Cat}$.

In particular, if U and V are isomorphic objects (via, say, $\iota : U \to V$) in the underlying category $\mathcal{C}$, then the fact that we have a pseudo-functor $\mathcal{C}^{op} \to \mathbf{Cat}$ tells us that the functor $\iota^* : \mathcal{F}(V) \to \mathcal{F}(U)$ induced by our isomorphism is an equivalence of categories. Thus, in a fibered category, the fibers above isomorphic objects are related in a very satisfactory way.

Having seen a way of constructing a pseudo-functor from a fibered category with a cleavage, it is natural to ask whether one can go the other way. The answer is 'yes', but the details are rather fiddly. We shall give the construction, and leave the details of verifying the result is a fibered category for the reader to check. (Some—but not all—can be found in [10]; and in any case, it's one of those things that it is easier to work through oneself.)

Consider a pseudo-functor Φ on $\mathcal{C}$. We wish to define a fibered category $\mathcal{F}$ over $\mathcal{C}$ which corresponds to Φ. We take as objects pairs (ξ, U), where U is an object of $\mathcal{C}$, and ξ is an object of $\Phi(U)$. An arrow $(a, f) : (\xi, U) \to (\eta, V)$ will be a pair consisting of an arrow $f : U \to V$ in $\mathcal{C}$ together with an arrow $a : \xi \to \Phi(f)(\eta)$ in $\Phi(U)$. Given two arrows $(a, f) : (\xi, U) \to (\eta, V)$ and $(b, g) : (\eta, V) \to (\zeta, W)$, their composite is defined to be $(\alpha_{f,g}(\zeta) \circ \Phi(b) \circ a, gf)$; note that $\alpha_{f,g}(\zeta) \circ \Phi(b) \circ a$ is indeed an arrow from ξ to $\Phi(gf)(\zeta)$ in $\Phi(U)$, since we have

$$\xi \xrightarrow{a} \Phi(f)\zeta \xrightarrow{\Phi(f)b} \Phi(f)\Phi(g)\zeta \xrightarrow{\alpha_{f,g}} \Phi(gf)\zeta$$

We can now make explicit how the notion of a fibered category is related to the notion of a presheaf, and in some sense its 2-categorical analog. Starting with the notion of a presheaf (a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$) we replace **Set** in this construction with **Cat** (and thus go 'one stage more categorical'). We also replace the notion of a functor (which is the 'right' notion when going to a category but not when going to a 2-category) with a pseudo-functor. We get the notion of a pseudofunctor $\mathcal{C}^{op} \to \mathbf{Cat}$, which we have seen is morally the same as a fibered category. (Morally, in the sense that the only difference is a choice of cleavage, and these are equivalent in some sense.) And since a pseudofunctor has this rather arbitrary cleavage attached, whereas the fibered category doesn't, the fibered category is the more natural object to work with. So we see fibered categories really are a natural 2-categorical analog of presheaves.

Moreover, just as the structure of **Set** as a category gives a structure turning the set $[\mathcal{C}^{op}, \mathbf{Set}]$ of functors to **Set** form $\mathcal{C}^{op}$ into a category (this just natural transformations!), so the structure of **Cat** as a 2-category makes the set of pseudo-functors into a 2-category. The details are rather fiddly, and never used in this essay, so we will not give them[5]; but the reader should be able to work them out, and see that this gives the same 2-categorical structure as the structure we have put on the 2-category of fibered categories.

2.4. **Examples.** Having seen a lot or rather abstract definitions, it is useful to see some examples to get a feel for the definitions.

Example 2.4.1. Maybe the clearest example of a fibered category is the category of vector spaces with attached vector bundles. Indeed, [5] goes as far as to motivate the definition of a fibered category as 'something that pulls back like bundles'. So let $\mathcal{C} = \mathbf{Top}$, and let $\mathcal{F}$ be the category whose objects are vector bundles $V \to B$ (that is, maps equipped with linear structures on the fibers and local trivialisations), and whose morphisms are commutative squares

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d, "\phi"] & V_2 \arrow[d, "g"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

with the map

$$\phi|_{V_1 \times_{B_1} \{b\}} : V_1 \times_{B_1} \{b\} \to V_2 \times_{B_2} \{f(b)\}$$

linear for each $b \in B_1$. This is a category over **Top** by forgetting about the vector bundles; it is clear that the cartesian morphisms in this category are the cartesian squares.

[5]Roughly, a 1-morphisms of pseudo-functors is a 1-morphism of **Cat** for each object of $\mathcal{C}^{op}$, satisfying a naturality condition; rather than insisting that the analogs of naturality squares commute, we only ask for each square that there is a prescribed isomorphism ι between the maps going round the square the two possible ways. We then need a compatibility criterion for these isomorphisms, which says that given composable morphisms f, g the prescribed isomorphisms for the naturality squares for f, g and fg are related properly with the $\alpha_{f,g}$ for the pseudofunctor, and a similar condition for identities.

A 2-morphism of pseudo-functors is a 2-morphism of **Cat** for each object of $\mathcal{C}$, which together satisfiy a naturality condition coming from the prescribed isomorphisms ι_f from the domain and the codomain.

Now, we know that given a bundle $V \to B$ and a map of spaces $A \to B$, we can form a *pullback bundle* $V' \to A$, unique up to unique isomorphism, which makes

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d, "g"] \\ A \arrow[r] & B \end{tikzcd}$$

cartesian. (Essentially, we take a fibered product of V and A and show it's possible to put a nice linear structure on it.) One can easily see this is exactly what we need to make the category a fibered category.

Note that this example is particularly instructive in that there is no canonical cleavage. While we know we always can construct pullback bundles, there are different concrete ways of doing it, which result in different pullbacks (all canonically isomorphic, of course), which is why we talk of 'a' pullback, not 'the' pullback.

Example 2.4.2. The simplest example of a fibered category is probably as follows. Let $\mathcal{C}$ be a category with fibered products. Then we can consider the category **Arr** $\mathcal{C}$, whose objects are the arrows of $\mathcal{C}$, and where a morphism from an arrow $f : X \to U$ to an arrow $g : Y \to V$ is a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ U \arrow[r] & V \end{tikzcd}$$

which we turn into a category over $\mathcal{C}$ by the functor sending each arrow to its target and each diagram to its bottom row. We see that the cartesian arrows are precisely those for which the square above is cartesian; thus we have enough to make **Arr** $\mathcal{C}$ into a fibered category, since $\mathcal{C}$ has fibered products. (Note also that picking some particular construction of fibered products gives this category a cleavage, but since there is no canonical construction for the fibered product, there is no canonical cleavage; and since $A \times_B B \times_X C$ and $A \times_X C$ are merely isomorphic, and not equal, the cleavage isn't a splitting.)

Definition 2.4.3. A class $\mathcal{P}$ of arrows n a category $\mathcal{C}$ is *stable* iff we have

(1) If $f : X \to U$ is in $\mathcal{P}$ and we are given isomorphisms $\phi : X \cong X'$ and $\psi : Y \cong Y'$, then $\psi \circ f \circ \phi$ is in $\mathcal{P}$.
(2) Given an arrow $Y \to V$ in $\mathcal{P}$, and any other arrow $U \to V$, then we can form the fibered product $U \times_V Y$, and the projection $U \times_V Y$ is in $\mathcal{P}$. (So it's always possible to base change a $\mathcal{P}$-morphism by any other morphism, and being-in-$\mathcal{P}$ is stable under base-change.)

Example 2.4.4. We can extend the previous example as follows. Let $\mathcal{P}$ be a stable class of arrows in $\mathcal{C}$. Then we can form a fibered category over $\mathcal{C}$ with objects the arrows of $\mathcal{P}$, and morphisms commutative squares as in the previous example. We use the same functor to $\mathcal{C}$ as in the previous example; then it is clear we have a fibered category, with cartesian morphisms again being cartesian squares.

Example 2.4.5. Consider the forgetful functor U : **Top**$\to$**Set**. Let Y be a topological space, and suppose we are given a set X and a map of sets $X \to UY$. We can give X the (analyst's) weak topology; that is, the coarsest topology s.t. the map to Y is continuous; then a function $T \to X$ is continuous iff the map $T \to X \to Y$ is; this ensures that the map $X \to Y$ of spaces is cartesian. (The fiber over a particular set is equivalent to the category with objects topologies on the set, and morphisms given by the usual partial order on the topologies.) In this example, there is a canonical splitting.

Having constructed some examples directly as fibered categories, let us construct a few via the point of view of pseudo-functors.

Example 2.4.6. We consider a site $\mathcal{C}$. As is normal, for X an object in $\mathcal{C}$, a 'sheaf on X' shall mean a sheaf on $\mathcal{C}/X$. Let us write $\mathrm{Sh}\ X$ for the category of sheaves on X. Thus $\mathrm{Sh}\ -$ associates to each element of $\mathcal{C}$ a category. I claim that $\mathrm{Sh}\ -$ extends to a strict functor from $\mathcal{C}$ to **Cat**. Since strict functors are a particular kind of pseudo-functors, and pseudo-functors give rise to fibered categories, this will define a fibered category.

To a morphism $f : X \to Y$ of $\mathcal{C}$ we must associate a functor f^* from $\mathrm{Sh}\ Y$ to $\mathrm{Sh}\ X$. An object of $\mathrm{Sh}\ Y$ is a functor G from $\mathcal{C}/Y$ to **Set**.

To give f^* on objects, we want to define the functor $f^*G : \mathcal{C}/X \to \mathbf{Set}$. Let f^*G on an object $U \to X$ be $G(U \to Y)$, where $U \to Y$ is the composite of $U \to X$ with f. A morphism $\phi : U \to V$ from $U \to X$ to $V \to X$ defines a morphism from $U \to X \to Y$ to $V \to X \to Y$, to which G associates a map of sets from $G(U \to Y)$ to $G(V \to Y)$; that is to say, a map from $f^*G(U \to X)$ to $f^*G(V \to X)$. We define $f^*G(\phi)$ to be this morphism. Functoriality is obvious. It is then easy to see that f^*G satisfies the sheaf condition.

Given a natural transformation of sheaves $\alpha : G \to G'$, there is an obvious induced natural transformation from $f^*G \to f^*G'$ (essentially, the restriction of α). This makes f^* into a functor, as may easily be verified.

It is easy to check that id_X^* is the identity functor and that $f^*g^* = (gf)^*$, so we have indeed defined a functor from $\mathcal{C}$ to **Cat**.

The previous example has a splitting; as a final example, we define a fibered category via a pseudo-functor which is a 'genuinely pseudo'; that is, not a strict functor.

Example 2.4.7. We work over the category $\mathbf{Sch}/S$, for S our favourite base scheme. For each scheme X/S, we have the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves on X; moreover, morphisms $X \to Y$ of base schemes do indeed give us pullback functors $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(Y)$. But this definitely does not give us a functor from $\mathbf{Sch}/S$ to **Cat**, since $f^*g^* \neq (gf)^*$; rather, there is a canonical isomorphism of functors between the two.

This probably leads one to suspect that **QCoh** is a pseudofunctor, and this is indeed the case. Checking the axioms (1) and (2) is rather unenlightening—see [10], pp57–59.[6]

2.5. **Categories fibered in groupoids, sets and equivalence relations.** We saw in §2.3 that fibered categories are equivalent (modulo choice of cleavage) to pseudo-functors $\mathcal{C}^{op} \to \mathbf{Cat}$, and thus closely related to presheaves (functors $\mathcal{C}^{op} \to \mathbf{Set}$). But we also know that we can view sets as categories if we like, by equipping each element with an identity morphism (and having

[6]Vistoli's exposition is almost always exemplary, but at this point I found there were moments in the proof where I required a little head-scratching before I could see what had been done. In particular, at the end of the proof, he says 'similar arguments work for the second part of the first condition and for the second condition'. Each of these cases requires the use of a manipulation which the first case didn't. We just outline how the second part of the first condition goes, since the second condition is then more of the same...

We need to show that $f^*\epsilon_V(\mathcal{N}) \cong \alpha_{f,\mathrm{id}_V}(\mathcal{N})$, where $f : U \to V$ is a map of schemes and $\mathcal{N}$ a sheaf on V. By definition, the natural transformation associated with $\alpha_{f,\mathrm{id}_V}(\mathcal{N})$ is

$$\Theta_f(\mathrm{id}_V^*\mathcal{N}, -) \circ \Theta_{\mathrm{id}_V}(\mathcal{N}, f_*-) \circ \Theta_f(\mathcal{N}, -)^{-1}$$

and that associated with $f^*\epsilon_V(\mathcal{N})$ is $- \mapsto - \circ f^*\epsilon_V(\mathcal{N})$; so we want to know

$$\begin{tikzcd}
 & \mathrm{Hom}(f^*\mathrm{id}_V^*\mathcal{N}, -) \arrow[dr, "- \mapsto - \circ f^*\epsilon_V(\mathcal{N})"] & \\
\mathrm{Hom}(\mathrm{id}_V^*\mathcal{N}, f_*-) \arrow[ur, "\Theta_f(\mathrm{id}_V^*\mathcal{N}, -)"] & & \mathrm{Hom}(f^*\mathcal{N}, -) \\
 & \mathrm{Hom}(\mathcal{N}, (\mathrm{id}_V f)_*-) \arrow[ul, "\Theta_{\mathrm{id}_V}(\mathcal{N}, f_*-)"] \arrow[ur, "\Theta_f(\mathcal{N}, -)"'] &
\end{tikzcd}$$

no other morphisms). This suggests that we might be able to consider functors $\mathcal{C}^{op} \to \mathbf{Set}$ as pseudo-functors $\mathcal{C}^{op} \to \mathbf{Cat}$, all of whose image categories have only identity morphisms; that is, view presheaves as fibered categories all of whose fibers have only identity morphisms.

This also suggests thinking that the real difference between fibered categories and presheaves is the possibility of having morphisms in the fibers $\mathcal{F}(U)$ other than just the identity morphisms. We might hope that the cases where one has no such nonidentity morphisms (we call these *categories fibered in sets*) correspond exactly to presheaves. This is indeed the case, as we will see in this section.

Insofar as work with moduli problems is concerned, the only morphisms in the fibers which we are interested in are the isomorphisms. This suggests that we might as well discard all the non-isomorphisms, to simplify the situation. This is indeed possible; the resulting kind of fibered category, where all morphisms in the fibers are isomorphisms, is called a *category fibered in groupoids*, and we shall study them in this section also.

We have said that we can view a set as a category with only the identity morphisms. Let us make a definition:

Definition 2.5.1. A category is a *set* if all morphisms are identity morphisms (this is sometimes called being a *discrete* category).

We can also see how a function gives rise to a functor on the corresponding category, so we have an inclusion of categories $\mathbf{Set} \to \mathbf{Cat}$. In fact, this is an inclusion of 2-categories, as long as we consider $\mathbf{Set}$ as a trivial 2-category, with the only 2-morphisms the identity morphisms. Thus we have a strict map of 2-categories $\mathbf{Set} \to \mathbf{Cat}$ (by strict, we mean that all the things like $FfFg = F(fg)$ are equalities, rather than isomorphisms), and any strict functor whose image objects are all discrete categories factors through this inclusion (this is because, trivially, the only functors between discrete categories are those that come form functions on their underlying sets).

This notion of a category which is a set is somewhat unsatisfactory since 'being a set' is not preserved under equivalence of categories. Natural properties of categories should normally be preserved in this way, which suggests we might also want to investigate the related property of 'being equivalent to a set', which is preserved. Luckily, there is a nice characterization of such categories.

Definition 2.5.2. A category $\mathcal{C}$ is an *equivalence relation* iff for any $x, y \in \operatorname{Ob} \mathcal{C}$, there is at most one morphism from x to y, and all morphisms are invertible.

It is easy to see where the terminology comes from: the morphisms of the category determine the structure of an equivalence relation on the objects (two objects are equivalent iff there is a map from the first to the second). Given an equivalence relation on some objects, there is obviously a unique way of making them into a category of the above kind giving rise to that equivalence relation. We leave the details for the reader to check.

commutes. But then it suffices to show

$$\begin{tikzcd}
 & \operatorname{Hom}(f^*\mathrm{id}_V^*\mathcal{N}, -) \arrow[dr, "- \mapsto - \circ f^*\epsilon_V(\mathcal{N})"] & \\
\operatorname{Hom}(\mathrm{id}_V^*\mathcal{N}, f_*-) \arrow[ur, "\Theta_f(\mathrm{id}_V^*\mathcal{N}, -)"] & \operatorname{Hom}(\mathcal{N}, f_*-) \arrow[l, "- \mapsto - \circ \epsilon_V(\mathcal{N})"'] \arrow[r, "\Theta_f(\mathcal{N}, -)"] & \operatorname{Hom}(f^*\mathcal{N}, -) \\
 & \operatorname{Hom}(\mathcal{N}, (\mathrm{id}_V f)_*-) \arrow[ul, "\Theta_{\mathrm{id}_V}(\mathcal{N}, f_*-)"] \arrow[ur, "\Theta_f(\mathcal{N}, -)"'] &
\end{tikzcd}$$

commutes. Which it does; the top diamond is an example of the square which we see is commutative at the top of Vistoli's page 58, while for the bottom diamond manifestly commutes, since $- \mapsto - \circ \epsilon_V(\mathcal{N})$ is, by definition, the same as $\Theta_{\mathrm{id}_V}(\mathcal{N}, f_*-)$.

Proposition 2.5.3. *A category $\mathcal{C}$ is equivalent to a set iff it is an equivalence relation.*

Proof. Suppose $\mathcal{C}$ is an equivalence relation. Let S be the set of connected components of $\mathcal{C}$; we consider it a category with only identity morphisms. Pick a representative $O_s \in \mathrm{Ob}\,\mathcal{C}$ for each $s \in S$. Define a functor $F : S \to \mathcal{C}$ as follows: send each $s \in S$ to O_s, and send the identity on s to id_{O_s}. (Since S only has identities, that's all the morphisms.) This is clearly faithful. We claim it is also full. Given a, b in S, we have two cases. Case 1 is that $a \neq b$, in which case O_a and O_b are in different connected components, and so there is no morphism from a to b, so $\mathcal{C}(Fa, Fb) = \mathcal{C}(O_a, O_b) = \oslash$, so we definitely and trivially surject $S(a,b) \to \mathcal{C}(Fa, Fb)$. Case 2 is that $a = b$; then $\mathcal{C}(Fa, Fb) = \mathcal{C}(Fa, Fa) = \mathcal{C}(O_a, O_a)$. Now $\mathcal{C}(O_a, O_a)$ certainly contains id_{O_a}, which we hit (from id_a); then since there is at most one morphism from O_a to O_a (by definition of equivalence relation), this is the only element of $\mathcal{C}(O_a, O_a)$, so we hit every element, as required.

We claim it is also essentially surjective. Given $X \in \mathrm{Ob}\,\mathcal{C}$, we know X lies in some connected component s. Then O_s and X lie in the same connected component; thus (since all morphisms are invertible, and a trivial induction) there is an isomorphism form X to O_s; but O_s is in the image of F, so we are done. □

Definition 2.5.4. A category is a *groupoid* if every morphism is invertible.

Definition 2.5.5. A *category fibered in sets* is a fibered category $\mathcal{F}(U)$ all of whose fibers $\mathcal{F}(U)$ are sets. A *category fibered in equivalence relations* is a fibered category $\mathcal{F}(U)$ all of whose fibers $\mathcal{F}(U)$ are sets. A *category fibered in groupoids* is a fibered category $\mathcal{F}(U)$ all of whose fibers $\mathcal{F}(U)$ are groupoids.

Now, given a presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$, we may map it through the inclusion $\iota : \mathbf{Set} \to \mathbf{Cat}$, and we get a strict functor $F' = \iota F : \mathcal{C}^{op} \to \mathbf{Cat}$. All strict functors may be considered to be pseudo-functors, so we have a pseudo-functor $F' : \mathcal{C}^{op} \to \mathbf{Cat}$, which gives us a fibered category $\mathcal{F}$. The fiber $\mathcal{F}(U)$ is $F'(U) = \iota(F(U))$, so lies in the image of ι, so is a set. Thus every presheaf is a category fibered in sets. Can we go backwards?

Given a category fibered in sets, we have a pseudo-functor $\mathcal{C}^{op} \to \mathbf{Cat}$ whose image lies in $\mathbf{Set} \subset \mathbf{Cat}$. Then we have

Claim 2.5.6. *A pseudo-functor Φ whose image lies in $\mathbf{Set} \subset \mathbf{Cat}$ is a strict functor.*

Proof. We need to show that the 2-isomorphisms $\alpha_{g,f}$ and ϵ_U are identities. (Seeing as we work in $\mathbf{Cat}$, '2-isomorphism' means the same as 'natural isomorphism'.) But α is a natural transformation between functors Φf and Φg in the image of Φ; in particular, this means that their common codomain S lies in $\mathbf{Set} \subset \mathbf{Cat}$; i.e. it is a discrete category. Since every component of $\alpha_{g,f}$ must be a morphism in S, which has only identity morphisms, $\alpha_{f,g}$ has all components the identity. So $\alpha_{g,f}$ is indeed the identity.

A similar argument shows the ϵ_U are identities. □

So this is in fact a strict functor mapping to $\mathbf{Set} \subset \mathbf{Cat}$; this then factors through the inclusion ι to give a presheaf $\mathcal{C}^{op} \to \mathbf{Set}$. It is easy to see these operations are equivalent. Thus we have a chain of equivalent objects

$$\begin{aligned}
\text{category fibered in sets} \quad &\sim \quad \text{fibered category with every } \mathcal{F}(U) \text{ a set} \\
&\sim \quad \text{pseudo-functor } \mathcal{C}^{op} \to \mathbf{Cat} \text{ whose image lies in } \mathbf{Set} \subset \mathbf{Cat} \\
&\sim \quad \text{strict functor } \mathcal{C}^{op} \to \mathbf{Cat} \text{ whose image lies in } \mathbf{Set} \subset \mathbf{Cat} \\
&\sim \quad \text{strict functor } \mathcal{C}^{op} \to \mathbf{Set} \\
&\sim \quad \text{presheaf}
\end{aligned}$$

(It is worth remarking that, as one can easily see, especially given the equivalent definition of a category fibered in sets that we give below, that there is only one cleavage in a category fibered in sets. Thus there are no issues here with choice of a cleavage.)

What of categories fibered in equivalence relations? Since fibered categories live in a 2-category, there is a notion of equivalence of fibered categories[7]. We would hope that just as the categories which are equivalent to sets are precisely the equivalence relations, the fibered categories equivalent (as fibered categories) to categories fibered in sets are the categories fibered in equivalence relations. This is indeed the case; before we can prove it, though, we need a useful criterion for when two fibered categories are equivalent.

Lemma 2.5.7. *Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then F is an equivalence if and only if the restriction $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence for each object U of $\mathcal{C}$.*

Proof. We shall first show that F is full and faithful. Let ξ', η' be two objects of $\mathcal{F}$, lying over U and V respectively; we wish to show that F gives a bijection between arrows $\xi' \to \eta'$ and arrows $F\xi' \to F\eta'$. We shall do this by showing separately for each $f : U \to V$ that F induces a bijection between the arrows $\xi' \to \eta'$ over f and the arrows $F\xi' \to F\eta'$ over f; since every map $\xi' \to \eta'$ or $F\xi' \to F\eta'$ must be over some such f, this will suffice.

Pick a pullback $\phi' : \eta'_1 \to \eta'$ of η' along f. Then let $\xi = F\xi', \eta = F\eta'$ and $\phi = F\phi'$. Note that ϕ is also cartesian. By the cartesian property, each map $\psi : \xi \to \eta$ factors uniquely as $\psi_1 \circ \phi$ for some map ψ_1 over id_U. Analogously, each map $\psi' : \xi' \to \eta'$ factors uniquely as $\psi'_1 \circ \phi'$ for some ψ'_1 over id_U. Thus since each map over id_U in $\mathcal{G}$ comes from a unique map over id_U in $\mathcal{G}$, we're done. Thus *the map is full and faithful.*

Now, for each object of $\mathcal{G}$, pick an object $G\xi$ of $\mathcal{F}(U)$, where $U = p\xi$, whose F-image in $\mathcal{G}(U)$ is isomorphic in $\mathcal{G}(U)$ to ξ; we can do this since F_U is essentially surjective. Let $\alpha_\xi : \xi \cong F(G\xi)$ be an isomorphism between them. Now, for every arrow $\phi : \xi \to \eta$ in $\mathcal{G}$, there is, since F is full and faithful, a unique arrow $G\phi : G\xi \to G\eta$ such that $F(G\phi)$ is $\alpha_\eta \circ \phi \circ \alpha_\xi^{-1}$; that is, making the following diagram commute:

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, "\alpha_\xi"] & \eta \arrow[d, "\alpha_\eta"] \\ F(G\xi) \arrow[r, "F(G\phi)"] & F(G\eta) \end{tikzcd}$$

It is easy to see that G is a functor $\mathcal{G} \to \mathcal{F}$; and the commutativity of the above diagram means that the α_ξ are the components of a natural transformation $\mathrm{id}_{\mathcal{G}} \to F \circ G$.

All that remains is to show $G \circ F$ is isomorphic to the identity $\mathrm{id}_{\mathcal{F}}$. For each object ξ' of $\mathcal{F}$, we have an isomorphism

$$\alpha_{F\xi'} : F\xi' \to FGF\xi'$$

which, since F is full and faithful, is $F\beta_{\xi'}$ for a unique $\beta_{\xi'} : \xi' \to GF\xi'$. A trivial check using the naturality of α and the fact F is full and faithful shows the $\beta_{\xi'}$ to be the components of a natural transformation. □

Theorem 2.5.8. *A fibered category $\mathcal{F}$ is fibered in equivalence relations if and only if it is equivalent to a category fibered in sets.*

Proof. First suppose $\mathcal{F}$ is equivalent to a category fibered in sets $\mathcal{F}'$; let $F : \mathcal{F} \to \mathcal{F}'$ be an equivalence. Let $U \in \mathrm{Ob}\, \mathcal{C}$, we want to show $\mathcal{F}(U)$ is an equivalence relation. Now, by the lemma, F_U gives an equivalence $\mathcal{F}(U) \to \mathcal{F}'(U)$; and $\mathcal{F}'(U)$ is a set, as $\mathcal{F}'$ is fibered in sets. Thus, $\mathcal{F}(U)$, being equivalent to a set, is an equivalence relation.

[7]This is the obvious thing; two fibered categories $\mathcal{F}, \mathcal{G}$ are equivalent iff there are functors $a : \mathcal{F} \to \mathcal{G}, b : \mathcal{G} \to \mathcal{F}$ such that both ab and ba are 2-isomorphic to the identity.

Conversely, suppose that $\mathcal{F}$ is fibered in equivalence relations. For each object of U, let ΦU be the set of equivalence classes in $\mathcal{F}(U)$. Given an arrow $F : U \to V$ in $\mathcal{C}$, can construct a function Ob $\mathcal{F}(V) \to \Phi U$ sending an object to the isomorphism class of a pullback to U; since pullbacks are isomorphic, this is well defined. In fact, isomorphic objects of $\mathcal{F}(V)$ pull back to give isomorphic objects in $\mathcal{F}(U)$, so we get a function $\Phi f : \Phi V \to \Phi U$; it is easy to see Φ is a functor. If we consider the associated category fibered in sets, we clearly have a morphism $cF \to \Phi$, which is an equivalence on each fiber (it's the map going the other way to the map we constructed in prop 2.5). So $\mathcal{F}$ is equivalent to a category fibered in sets. □

There are equivalent definitions for categories fibered in sets, groupoids and equivalence relations in common use in the literature:

Proposition 2.5.9. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then*

(1) *$\mathcal{F}$ is a category fibered in sets if and only if for every diagram*

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

there is a unique completion to a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

(2) *$\mathcal{F}$ is a category fibered in groupoids if and only if every arrow in $\mathcal{F}$ is cartesian, and for every diagram*

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

there is a completion to a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

(which need not be unique).

(3) *$\mathcal{F}$ is a category fibered in equivalence relations if and only if for ξ, η objects of $\mathcal{F}$, each arrow $f : p\xi \to p\eta$ has at most one arrow $\xi \to \eta$ over it and for every diagram*

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

there is a completion to a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

(which need not be unique).

Proof. We take each part in turn

(1) First suppose we have the condition given. We claim the category is fibered. Given a diagram like the one on the left

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

we can find, via the existence part of the condition, a ϕ as in the right-hand diagram. We just then need ϕ cartesian. Given a diagram

$$\begin{tikzcd} \zeta \arrow[dr, dashed, "\theta"] \arrow[drr, "\psi"] \arrow[dd, mapsto] & & \\ & \xi \arrow[r, "\phi"] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\ p(\zeta) \arrow[dr, "h"'] \arrow[drr] & & \\ & p(\xi) \arrow[r] & p(\eta) \end{tikzcd}$$

the existence part of the condition tells us that we can find a θ as marked, and the uniqueness tells us that $\phi\theta = \psi$, since they both lie above the same arrow and have the same target, so the diagram commutes. Thus ϕ is cartesian.

Now let $\xi \in \mathcal{F}(U)$—then the condition tells us that there is a unique arrow above id_U ending in ξ; that is, a unique arrow in $\mathcal{F}(U)$ ending in ξ, which must be the identity. Thus there are only identity arrows in the fiber $\mathcal{F}(U)$, and the category is fibered in sets.

Conversely, suppose $\mathcal{F}$ is fibered in sets. Given the lefthand diagram below

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

we know we can find an arrow (indeed a cartesian arrow) ϕ as in the righthand picture, via the fibered category condition. We just need to show this is unique. But any other arrow $\phi' = \xi' \to \eta$ over f factors (using the cartesian condition) as the top line of

$$\begin{tikzcd} \xi' \arrow[r, "\psi"] \arrow[d, mapsto] & \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "="] & U \arrow[r, "f"] & V \end{tikzcd}$$

and the map ξ', being in the fiber $\mathcal{F}(U)$, which is a set, must be an identity. Thus $\phi' = \phi\psi = \phi\mathrm{id}_\xi$, so we have uniqueness.

(2) First suppose the condition holds. We first claim we have a fibered category. This is trivial; consider the diagrams:

$$\begin{tikzcd} & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

given the lefthand diagram we need to complete to a square as in the righthand diagram, with ϕ cartesian. But the second part of the condition tells us that we can find a ϕ giving a square as in the righthand diagram; then the first part tells us this ϕ is cartesian. Now, if $\phi : \xi \to \eta$ is an arrow of $\mathcal{F}(U)$, then since it is cartesian, we can find an arrow $\psi : \eta \to \xi$ over id_U s.t. $\phi\psi = \mathrm{id}_\eta$. Doing the same thing to ψ, there is a $\phi' : \xi \to \eta$ s.t. $\psi\phi' = \mathrm{id}_\xi$. Then $\phi = \phi\psi\phi' = \phi'$, and ϕ is an inverse to ψ in $\mathcal{F}(U)$

Conversely, suppose $\mathcal{F}$ is fibered in groupoids. Since $\mathcal{F}$ is fibered, the second part of the condition is clear. For the first part, let $\phi : \xi \to \eta$ be an arrow in $\mathcal{F}$, mapping to $f : U \to V$ in $\mathcal{C}$. Choose a pullback η' of η along f, with map $\alpha : \eta' \to \eta$. Then we have a map $\iota : \xi \to \eta$ in $\mathcal{F}(U)$ s.t. $\iota\alpha = \phi$. Now ι is an isomorphism, so ϕ, being isomorphic to a pullback of η, is a pullback of η, ensuring that ϕ is cartesian.

(3) First suppose the condition holds. We have that the category is fibered using almost precisely the same argument as the first part (just change 'the same target' to 'the same source and target'). It is also clear that the fibers are equivalence relations.

Now suppose that $\mathcal{F}$ is fibered in equivalence relations. Since $\mathcal{F}$ is fibered, the second part of the condition is clear. For the first part, suppose we have two morphisms $\phi, \phi' : \xi \to \eta$ lying over the same arrow $f : U \to V$. Since our category is fibered in equivalence relations, it is in particular fibered in groupoids, so every arrow is cartesian, so ϕ is, so ϕ' factors as the top line of

$$\begin{tikzcd} \xi \arrow[r, "\iota"] \arrow[d, mapsto] & \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "="] & U \arrow[r, "f"] & V \end{tikzcd}$$

Now, ι and lies in $\mathcal{F}(U)$, where is an equivalence relation, so there is at most one morphism between any two objects. But id_ξ is also in $\mathcal{F}(U)$ and maps from and to the same objects as iota, so $\iota = \mathrm{id}_\xi$. Then we have that $\phi' = \phi\iota = \phi\mathrm{id} = \phi$, which was as required.

□

We have said that it is possible to take a fibered category $\mathcal{F}$ and carry out an operation which removes all the non-isomorphisms form the fibers $\mathcal{F}(U)$, but leaves all the isomorphisms, which is useful since only the isomorphisms are of relevance to moduli problems, and the resulting category is much simpler.

Definition 2.5.10. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$; then the *associated category fibered in groupoids* $\mathcal{F}$ is the category obtained by discarding all morphisms in $\mathcal{F}$ which are not cartesian. (We know that the composite of cartesian morphisms is cartesian; and identities are cartesian since they are isomorphisms and they map down to identities, which are are isomorphisms—and a map over an isomorphism is cartesian iff an isomorphism).

It is clear (using the second and fourth parts of 2.2.4) that every morphism of $\mathcal{F}'$ is cartesian, so $\mathcal{F}'$ is a fibered category over $\mathcal{C}$—indeed, using prop 2.5.9, we can see $\mathcal{F}'$ is fibered in groupoids. There is an obvious map of categories fibered over $\mathcal{C}$ from $\mathcal{F}'$ to $\mathcal{F}$. Moreover, it is clear that every map from a category fibered in groupoids to $\mathcal{F}$ factors through $\mathcal{F}'$. Also, since we know a map above an isomorphism is cartesian iff an isomorphism, and since the morphisms in the fibers $\mathcal{F}(U)$ all lie above the identity of U, which is an isomorphism, we end up having discarded from the fiber precisely the maps which are not isomorphisms.

We know that the Yoneda lemma embeds $\mathcal{C}$ in $[\mathcal{C}^{op}, \mathbf{Set}]$, and we have just embedded $[\mathcal{C}^{op}, \mathbf{Set}]$ into the category of fibered categories over $\mathcal{C}$, as the categories fibered in sets. Thus we have an embedding of $\mathcal{C}$ into the 2-category of fibered categories over $\mathcal{C}$, which we call the 2-Yoneda embedding. We might ask what fibered category an object $X \in \mathrm{Ob}\,\mathcal{C}$ maps to. A very brief reflection reveals that U maps to the slice category $\mathcal{C}/X$, which is clearly a fibered category over $\mathcal{C}$. A morphism $f : X \to Y$ goes to the morphism of fibered categories $\mathcal{C}/f : (\mathcal{C}/X) \to (\mathcal{C}/Y)$ sending an object $U \to X$ of $\mathcal{C}/X$ to the composite $U \to X \to$, and does the corresponding thing on arrows. Now, the Yoneda embedding is full and faithful, and so is the embedding of presheaves in fibered categories (this is trivial to check); so

Proposition 2.5.11. *The 2-Yoneda embedding is full and faithful.*

The usual way of proving that the Yoneda embedding is full and faithful is to prove the Yoneda lemma. We might ask if we could also have proved the previous proposition as a deduction from some kind of 2-Yoneda lemma. We can; it is easy to see that it follows from

Theorem 2.5.12. *There is an equivalence of categories*

$$\mathcal{HOM}((\mathcal{C}/X), \mathcal{F}) \to \mathcal{F}(X)$$

Proof. This is one of those almost-tautologies it is best to prove oneself. (See [10], p68, for slightly more explanation; but not much more!) □

We say a fibered category is if it is equivalent to a comma category $\mathcal{C}/X$.

2.6. **Stacks.** We now need to proceed to construct an analogue for the sheaf condition for functors to **Set** in the world of fibered categories. Fibered categories which satisfy this analogous condition are *stacks*.

Before we proceed to put together a technical definition for being a stack, and seeing why this is a natural extension of the sheaf condition for functors $\mathcal{C}^{op} \to \mathbf{Set}$, let us pause for a moment. We remember the machinery of sheaves can seem slightly formidable at first sight in all its abstract glory, but is much more naturally understood if one tempers the abstraction by having in mind at all times the canonical example of a sheaf: continuous functions from open subsets of a topological space X to some fixed space (say $\mathbb{R}$). In many ways, it is easiest to understand the notion of a sheaf as a abstraction of the key properties of this situation (just as one might think of the notion of a ring as an abstraction of the key properties of $\mathbb{Z}$).

In the same way, the abstraction inherent in the machinery of stacks is better tempered, I feel, by keeping in mind a motivating example; so before we move to the definitions, let us consider (following [5]) what is perhaps the canonical example of a stack, the category of vector bundles on a topological space, fibered over **Top** as in example 2.4.1.

The key point about the continuous functions, which means that they are a sheaf, is that given a covering $U_i \to U$, and functions on each of the U_i, we can glue the functions together to get a unique function on U, so long as they agree on the 'overlaps' (that is, the fibered products). It will be the corresponding gluing construction for vector bundles that makes *them* a stack. (And a stack is, in [5]'s words, therefore 'something which glues like bundles'.) So let us ask what the gluing property for vector bundles is.

Well, the basic story is the same: if we have vector bundles $V_i \to U_i$, where $U_i \to U$ is a covering, and they agree on the overlaps, then we can glue them together to make a vector bundle on the whole space U. The subtlety comes when we try and be specific about what we mean by 'agreeing on the overlaps'. Suppose we have an overlap $U_i \times_U U_j$, with projections

$$\mathrm{pr}_1 : U_i \times_U U_j \to U_i, \mathrm{pr}_2 : U_i \times_U U_j \to U_j$$

we do not want to insist on the *equality* of the two pullbacks $\mathrm{pr}_1^* V_i$ and $\mathrm{pr}_2^* V_j$; vector bundles are (basically) never equal; the best one hopes for is isomorphism. But just asking for an isomorphism is too weak. Rather, we require that *particular* isomorphisms be specified

$$\alpha_{ij} : \mathrm{pr}_1^* V_i \cong \mathrm{pr}_2^* V_j$$

and we then require that these satisfy the *consistency condition* $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ wherever this makes sense (i.e. on $U_i \times_U U_j \times_U U_k$). This will then ensure that if we join the collections of trivialisations on the different parts using the isomorphisms α_{ij}, the resulting collection of trivialisations will have changeover isomorphisms which are transitive on their overlaps, as is required for the definition of a vector bundle.

It should be clear, in some sense, how this is a 2-categorification, or at least how we made everything 'one level more categorical'; since bundles on U form a category (whereas functions on U form a set, which is 'one less categorical') we didn't want to insist on equality (which is the natural thing to ask for for elements of a set, but not for elements of a category) but rather prescribed isomorphisms α_{ij}; we then demand a consistency critereon[8].

In the case of continuous functions, the other key thing was a uniqueness criterion: if we had two functions that agreed when restricted to every set on an open covering, they agree. What corresponds to this in the case of vector bundles? Well, we would like that if two vector bundles V_1, V_2 on a common base B are 'the same', when pulled back to each space in an open covering of B, they are the same. In the spirit of what has gone before, we of course don't require the pullbacks to be the same in the sense of equality, but rather isomorphism; and then we should only ask for an isomorphism between V_1 and V_2. Thus the uniqueness condition boils down to an ability to stitch together isomorphisms. In fact, it turns out to be slightly better to ask to be able to stitch together all morphisms, not just isomorphisms (but see the end of this section).

Now, let us turn to the task of putting together formal definitions. We will give two; the first uses the language of pseudofunctors, the second the language of fibered categories directly. Let us consider for a moment the sheaf condition on a functor F. It starts with a given space $B \in \mathcal{C}$ and covering $B_i \to B$. We have a map $FB \to FB_i$ for all i, giving a map $f : FB \to \prod FB_i$. We then ask that f is injective, and 'as surjective as possible'. In particular, we know that the image of this map lies in the subset where the two obvious maps $\prod FB_i \to \prod F(B_i \times_B B_j)$ agree; we then require that f surjects onto that subset.

We can rephrase this as follows. We have maps $\mathrm{pr}_1 : B_i \times_B B_j \to B_i, \mathrm{pr}_2 : B_i \times_B B_j \to B_j$ Define a set

$$F\{B_i \to B\} = \{x_i \in \prod_i FB_i | \mathrm{pr}_1^* x_i = \mathrm{pr}_2^* x_j \in F(B_i \times_B B_j)\}$$

(We write f^* for Ff). We then note that the natural map $f : FB \to \prod FB_i$ factors through this subset, giving a natural map

$$\bar{f} : FB \to F\{B_i \to B\}$$

which we require to be bijective.

Now, to extend this to the case of pseudofunctors $\mathcal{F}$, we would expect everything to go 'one stage up' in categorification. Thus we should try and define a *category* $\mathcal{F}\{B_i \to B\}$, and a functor from $\mathcal{F}(B)$ (which is definitely a category!) to $\mathcal{F}\{B_i \to B\}$. Instead of asking this to be a bijection, we should ask for the corresponding thing for categories: an equivalence. (We might for a moment think about asking for an isomorphism of categories; but this is almost never the right thing to look for...)

[8]The reader should compare with the definition of a pseudo-functor; again, in response to going to **Cat** rather than **Set**, we no longer wanted an equality between f^*g^* and $(gf)^*$, but a prescribed isomorphism; these then needed to satisfy a consistency critereon.

So what should the category $\mathcal{F}\{B_i \to B\}$ (called the *category of objects with descent data*) be? Well, we'll want objects to be elements of $\mathcal{F}(B_i)$ for each i, satisfying some 'consistency critereon' akin to '$\mathrm{pr}_1^* x_i = \mathrm{pr}_2^* x_j \in F(B_i \times_B B_j)$'. Guided by the vector bundle example, we know we don't want an equality, but rather a prescribed isomorphism, satisfying a compatibility condition. A little thought, and the following seems right

Definition 2.6.1. The objects of the category of objects with descent data $\mathcal{F}\{B_i \to B\}$ are tuples $x_i \in \prod \mathrm{Ob}\ \mathcal{F}(B_i)$, together with isomorphisms

$$\alpha_{ij} : \mathrm{pr}_1^* x_i \cong \mathrm{pr}_2^* x_j \in \mathcal{F}(B_i \times_B B_j)$$

which are required to be compatible in the sense that

$$\mathrm{pr}_{23}^* \alpha_{jk} \circ \mathrm{pr}_{12}^* \alpha_{ij} = \mathrm{pr}_{13}^* \alpha_{ik}$$

where $\mathrm{pr}_{12} : B_i \times_B B_j \times_B B_k \to B_i \times_B B_j$, $\mathrm{pr}_{13} : B_i \times_B B_j \times_B B_k \to B_i \times_B B_k$, and $\mathrm{pr}_{23} : B_i \times_B B_j \times_B B_k \to B_j \times_B B_k$

Morphisms are then the only thing that makes sense; again, the reader can easily see that this corresponds nicely to our example of vector bundles.

Definition 2.6.2. The morphisms of the category of objects with descent data $\mathcal{F}\{B_i \to B\}$ from an object (x_i) to an object (y_i) are tuples $\phi_i \in \prod \mathrm{Hom}_{\mathcal{F}(B_i)}(x_i, y_i)$, such that the diagram

$$\begin{tikzcd}
\mathrm{pr}_1^* x_i \arrow[r, "\alpha_{ij}"] \arrow[d, "\mathrm{pr}_1^* \phi_i"] & \mathrm{pr}_2^* x_j \arrow[d, "\mathrm{pr}_2^* \phi_j"] \\
\mathrm{pr}_1^* y_j \arrow[r, "\alpha_{ij}"] & \mathrm{pr}_2^* y_j
\end{tikzcd}$$

commutes for all i, j. Morphisms compose in the obvious way.

Now, there is a natural functor $\mathcal{F}(B) \to \mathcal{F}\{B_i \to B\}$; we send an object to the tuple of its pullbacks under the maps $B_i \to B$; the isomorphisms α_{ij} come from the α maps of the pseudo-functor; and the compatibility is inherited from that of the α for the pseudofunctor. Morphisms are similarly pulled back; the commutativity of the square in definition 2.6.2 comes from the naaturality of the α for the pseudo-functor.

Then the condition that we have a stack is that this functor is an equivalence. It is easy to see this corresponds to what we expect in the vector bundle case. In fact we can say more; it is the essential surjectivity that corresponds to the statement 'we can glue bundles', and the full and faithful property corresponds to the 'uniquely'. In the case of sheaves, there is a name for a functor where we have the uniqueness even if we can't always glue (that is, when $\bar{f} : FB \to F\{B_i \to B\}$ injects): a *separated functor*. Thus we correspondingly invent a term for a fibered category where the functor $\mathcal{F}(B) \to \mathcal{F}\{B_i \to B\}$ is full and faithful (even if not essentially surjective): a *prestack*[9].

Remark 2.6.3. The interested reader will have noticed the strong familiarity between these definitions and those of descent theory. In fact, finding a clean systematic framework for descent theory was one of the motivations for the invention of the language of stacks. We shall see how a few of the results of descent theory might be written out in this framework in the next section.

Given that stacks are meant to be an extension of the concept of sheaf, we should obviously prove a result like:

[9]The terminology is due to Grothendeick, and is slightly odd, since a 'presheaf' is just a functor, so we'd expect a prestack to be simply a fibered category; and what we call a prestack should be called a separated prestack. But the old terminology is established, and anyhow there is a notion, in the theory of algebraic stacks, of a *separated stack*—as we shall see—so introducing a different notion of a separated prestack would be very confusing!

Theorem 2.6.4. *Let F be a presheaf; then F is a sheaf iff the corresponding fibered category $\mathcal{F}$ is a stack. F is a separated functor iff $\mathcal{F}$ is a prestack.*

Proof. Let us take a covering $U_i \to U$. The fiber $\mathcal{F}(U)$ is just the set $F(U)$, while $\mathcal{F}\{U_i \to U\}$ is nothing more than the set $F\{U_i \to U\}$. (The morphisms in the category of objects with descent data are just isomorphisms in this case, so $\mathcal{F}\{U_i \to U\}$ is a set. Also, since the only isomorphisms in the $\mathcal{F}(V)$ are the identities, saying there exist specified isomorphisms $\mathrm{pr}_1^* x_i \cong \mathrm{pr}_2^* x_j \in \mathcal{F}(U_i \times_U U_j)$ actually guarantees $\mathrm{pr}_1^* x_i = \mathrm{pr}_2^* x_j$ (and we are forced to specify the identity morphisms—which always satisfy the consistency critereon). Now, to say that a function, thought of as a functor between discrete categories, is full and faithful is just to say it injects; while to say it is an equivalence is to say it's a bijection.

$$\begin{aligned} \mathcal{F} \text{ is a stack (resp prestack)} \iff & \mathcal{F}(U) \to \mathcal{F}\{U_i \to U\} \text{ is an equivalence (resp full and faithful)} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall\, U_i \to U \\ \iff & F(U) \to F\{U_i \to U\} \text{ bijects (resp injects)} \quad \forall\, U_i \to U \\ \iff & F \text{ is a stack (resp separated functor)} \end{aligned}$$

as required. □

The above definition appeared to depend rather heavily on the choice of a cleavage, which we know to be somewhat unnatural. We sketch a definition which works without a choice of cleavage.

Let $\{U_i \to U\}$ be a covering. We shall refer to $U_i \times_U U_j$ as U_{ij}, and similarly define U_{ijk}. The projections $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13}$ will be as normal. By 'the ijk cube', we shall mean the commutative diagram

$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\mathrm{pr}_{23}"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dd, "\mathrm{pr}_{13}"] & & U_{jk} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dd, "\mathrm{pr}_2"] \\
U_{ij} \arrow[rr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"] & & U_j \arrow[dd] & \\
& U_{ik} \arrow[rr, "\mathrm{pr}_2"] \arrow[dl, "\mathrm{pr}_1"] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U &
\end{tikzcd}$$

We define an object with descent data to be be a triple of collections

$$(\{\xi_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}, \{\xi_{ijk}\}_{i,j,k \in I})$$

where each ξ_α is an object of $\mathcal{F}(U_\alpha)$, together with, for each ijk, a commutative diagram of cartesian arrows in $\mathcal{F}$:

$$\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_j & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_k \\
\xi_i & & &
\end{tikzcd}$$

which projects down to the corresponding part of the ijk cube. An arrow

$$(\{\xi_i\}_{i\in I}, \{\xi_{ij}\}_{i,j\in I}, \{\xi_{ijk}\}_{i,j,k\in I}) \to (\{\eta_i\}_{i\in I}, \{\eta_{ij}\}_{i,j\in I}, \{\eta_{ijk}\}_{i,j,k\in I})$$

is a collection of arrows $\phi_\alpha : \xi_\alpha \to \eta_\alpha$ for $\alpha \in I, I \times I$, or $I \times I \times I$, satisfying the obvious compatibility conditions (for example, the pullback of the arrow ϕ_i to U_{ij} should be ϕ_{ij}). Let us call this category $\mathcal{F}_1\{U_i \to U\}$. We cannot define a functor $\mathcal{F}(U) \to \mathcal{F}_1\{U_i \to U\}$ per se, but we can do something basically as good. We define another category $\mathcal{F}_2\{U_i \to U\}$. This has objects quadruples

$$(\xi, \{\xi_i\}_{i\in I}, \{\xi_{ij}\}_{i,j\in I}, \{\xi_{ijk}\}_{i,j,k\in I})$$

equipped with, for each ijk, a commutative diagrams of cartesian arrows

$$\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_j \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_k \arrow[dl] \\
\xi_i \arrow[rr] & & \xi &
\end{tikzcd}$$

mapping down to the ijk cube. Arrows are defined analogously to $\mathcal{F}_1\{U_i \to U\}$; we then easily check that $\mathcal{F}_2\{U_i \to U\}$ is equivalent to $\mathcal{F}(U)$, via the map where we forget all but the first element of the quadruple representing an object (and similarly for morphisms). There is now an obvious functor $\mathcal{F}_2\{U_i \to U\} \to \mathcal{F}_1\{U_i \to U\}$. We leave it to the reader to check that functor is an equivalence iff $\mathcal{F}$ is a stack, and it is fully faithful iff $\mathcal{F}$ is a prestack. (The basic idea is to construct a functor $\mathcal{F}_1\{U_i \to U\} \to \mathcal{F}\{U_i \to U\}$, and show it is an equivalence. This functor goes basically as follows. For an object $(\{\xi_i\}_{i\in I}, \{\xi_{ij}\}_{i,j\in I}, \{\xi_{ijk}\}_{i,j,k\in I})$ of $\mathcal{F}_1\{U_i \to U\}$, the arrows $\xi_{ij} \to \xi_i, \xi_{ij} \to \xi_j$ induce isomorphisms $\xi_{ij} \cong \mathrm{pr}_1^* \xi_i$ and $\xi_{ij} \cong \mathrm{pr}_2^* \xi_j$, which gives us an isomorphism $\mathrm{pr}_1^* \xi_i \cong \mathrm{pr}_2^* \xi_j$, which one may easily check satisfies the condition. A morphism in $\mathcal{F}_1\{U_i \to U\}$ determines a morphism in $\mathcal{F}\{U_i \to U\}$ by discarding all but the ϕ_i components.)

There is another definition, which is possibly the most common in the literature. We sketch the details

Definition 2.6.5. An object of $\mathcal{F}\{U_i \to U\}$ is called a *descent datum*. It is *effective* if it is isomorphic to an object in the image of $\mathcal{F}(U) \to \mathcal{F}\{U_i \to U\}$.

If every descent datum is effective, then that just says the functor $\mathcal{F}(U) \to \mathcal{F}\{U_i \to U\}$ is essentially surjective (and conversely). Now $\mathcal{F}(U) \to \mathcal{F}\{U_i \to U\}$ being an equivalence ($\mathcal{F}$ being a stack) is the same as it being full and faithful (i.e. $\mathcal{F}$ being a prestack) and essentially surjective (i.e. every descent datum being effective), so we have

Proposition 2.6.6. *$\mathcal{F}$ is a stack iff it is a prestack and every descent datum is effective.*

We now define the so-called functor of arrows. Pick a cleavage of $\mathcal{F}$, a U in Ob $\mathcal{C}$ and objects $\xi, \eta \in \mathcal{F}(U)$. Given the cleavage, we may consider $\mathcal{F}$ as a pseudofunctor, when useful. We define a presheaf H on $\mathcal{C}/U$ as follows. For an object $f : W \to U$ of $\mathcal{C}/U$, we define

$$H(f) = \mathrm{Hom}_{\mathcal{F}(W)}(f^*\xi, f^*\eta)$$

For a map $g : W_1 \to W_2$ between objects $f_1 : W_1 \to U$ and $f_2 : W_2 \to U$, we define a map $H(g) : H(f_2) \to H(f_1)$: given $\phi \in H(f_2)$, so $\phi : f_2^*\xi \to f_2^*\eta$, we have $g^*\phi : g^* f_2^* \xi \to g^* f_2^* \eta$, and we define

$H(g)(\phi) \in H(f_1) = \mathrm{Hom}_{\mathcal{F}(W)}(f_1^*\xi, f_1^*\eta)$ to be the composite:

$$f_1^*\xi = (f_2 g)^*\xi \xrightarrow{(\alpha_{g,f_2})_\xi^{-1}} g^* f_2^*\xi \xrightarrow{g^*\phi} g^* f_2^*\eta \xrightarrow{(\alpha_{g,f_2})_\eta} (f_2 g)^*\eta = f_1^*\eta$$

It is easy to see that H is a functor. An exercise which we leave to the reader is to show that this functor is independent of the choice of cleavage, up to isomorphism of functors.

Definition 2.6.7. We say *Homs form a sheaf* iff for all U and $\xi, \eta \in \mathcal{F}(U)$ as above, the presheaf just constructed is a sheaf. (In the case of a category fibered in groupoids, we might also say *isomorphisms form a sheaf.*)

Proposition 2.6.8. *$\mathcal{F}$ is a prestack iff Homs form a sheaf.*

Proof. Take an object U of $\mathcal{C}$, a covering $\{U_i \to U\}$, and two objects ξ, η of $\mathcal{F}(U)$. Let us denote $(\{\xi_i\}, (\alpha_{ij})$ and $(\{\eta_i\}, (\beta_{ij})$ the descent data associated with ξ and η respectively; then the arrows from the former to the latter are collections of arrows $(\phi_i : \xi_i \to \eta_i)$ such that the pullbacks of ϕ_i, ϕ_j to $U_i j$ coincide. The sheaf condition for H for this covering tells us precisely that each such comes from a unique map $\xi \to \eta$, which tells us precisely that the functor $\mathcal{F}(U) \to \mathcal{F}\{U_i \to U\}$ is full and faithful. ☐

Putting together the two propositions, we get the equivalent definition in common use:

Theorem 2.6.9. *$\mathcal{F}$ is a stack iff Homs form a sheaf and every descent datum is effective.*

There is yet another definition, based on sieves, in [10]; while it is the most elegant, it is the least practical, and since we will not be using it, we will not give the details.

Before we close this section, there is a difference in terminology which will be worth remarking on. As we have constructed it, the concept of being a stack (or not) can be applied to any fibered category. This general concept of stack finds considerable application in descent theory where we wish to be able to construct via descent morphisms which are not isomorphisms. (It is, for example, the one used in [10].) But in a large class of cases, including those of constructing moduli spaces as algebraic objects, the maps in the $\mathcal{F}(U)$ which are not isomorphisms are of no interest. For these uses, it is better to restrict the notion of 'stack' only to cases where the underlying category is fibered in groupoids.[10] All of [5], [9], [4] and [7] make this definition.

We take the a middle ground. We have given the definition of a stack in the general case, with a general fibered category rather than a category fibered in groupoids, both because the other definition is a special case of this one, and because we will want to state some descent theory results which are best stated in this language. But, starting in the next section, when we wish to analyse algebraic stacks, we will adopt the convention that all stacks are fibered in groupoids.

As a final point, it is worth noting that there is a connection between the two worlds, which will prove useful. Recall that for any fibered category $\mathcal{F}$, we can form an associated category fibered in groupoids $\mathcal{F}'$ by forgetting about all morphisms in the fibers that aren't *iso*morphisms.

Proposition 2.6.10. *Let $\mathcal{C}$ be a site, $\mathcal{F}$ a category fibered over $\mathcal{C}$, and $\mathcal{F}'$ the associated category fibered in groupoids. Then*

(1) *If $\mathcal{F}$ is a stack, so is $\mathcal{F}'$.*
(2) *If $\mathcal{F}$ is a prestack and $\mathcal{F}'$ is a stack, then $\mathcal{F}$ is in fact a stack.*

Proof. We know that $\mathcal{F}'$ has the same objects as $\mathcal{F}$; also, since the morphisms in a fiber $\mathcal{F}'(U)$ are just the isomorphisms in $\mathcal{F}(U)$, we know that the fibers $\mathcal{F}(U)$ and $\mathcal{F}'(U)$ have the *same* isomorphisms. Since the objects of the category $\mathcal{F}\{U_i \to U\}$ only depend on the objects of $\mathcal{F}$ and the isomorphisms in the fibers of $\mathcal{F}$, we have that the objects of $\mathcal{F}\{U_i \to U\}$ and $\mathcal{F}'\{U_i \to U\}$ are

[10] I believe that this is also the natural formulation to make for many of the geometric applications of the theory of stacks.

the same. Moreover, whether descent data are effective also only depends on what the objects and isomorphisms in fibers are; so the same descent data are effective for $\mathcal{F}$ and $\mathcal{F}'$. Thus it suffices to show that if $\mathcal{F}$ is a prestack then so is $\mathcal{F}'$.

So let $U_i \to U$ be a covering and write U_{ij} for $U_i \times_U U_j$. ξ and η be objects of $\mathcal{F}(U)$, and let ξ_i, ξ_{ij} be pullbacks of ξ to $\mathcal{F}(U_i)$, $\mathcal{F}(U_{ij})$ respectively. (Similarly η_i, η_{ij}.) Now, suppose we have $\alpha_i \in \mathrm{Hom}_{\mathcal{F}'(U_i)}(\xi_i, \eta_i)$, s.t. $\mathrm{pr}_1^* \alpha_i = \mathrm{pr}_2^* \alpha_j : \xi_{ij} \to \eta_{ij}$; we want these to glue to a morphism in $\mathrm{Hom}_{\mathcal{F}'(U)}(\xi, \eta)$.

Since every morphism in $\mathcal{F}'$ is a morphism in $\mathcal{F}$, we can think of the α_i is being arrows in $\mathrm{Hom}_{\mathcal{F}(U_i)}(\xi_i, \eta_i)$, which then by assumption glue to give an arrow α in $\mathrm{Hom}_{\mathcal{F}(U)}(\xi, \eta)$; we want to show this morphism in $\mathcal{F}(U)$ gives us a morphism in $\mathcal{F}'(U)$, which is true precisely if it is an isomorphism (since the morphisms in the fibers of $\mathcal{F}'$ are precisely the isomorphisms in $\mathcal{F}'$). So let us find an inverse.

Now, the α_i are morphisms in $\mathcal{F}'(U_i)$, hence isomorphisms; their inverses α_i^{-1} can be thought of as in $\mathcal{F}(U_i)$ and patch to give a morphism β in $\mathcal{F}(U)$ (we get this from the patching condition for the α_i). Then $\alpha \circ \beta$ and $\beta \circ \alpha$ pull back to give the identity on each U_i (as $\alpha_i^{-1} \alpha_i = \alpha_i \alpha_i^{-1} = \mathrm{id}$); so $\alpha \circ \beta = \beta \circ \alpha = \mathrm{id}$. □

2.7. **Descent theory.** At this point, the reader should be able to see that the standard results of algebraic-geometric descent theory are simply assertions that particular fibered categories are stacks. This is in some sense unsurprising, since part of the motivation for the definition of stacks was that they provide a natural framework in which to think about descent theory. Nevertheless, the framework is *so* natural that I cannot resist stating a couple of the results of the theory here. (One of these will be used in the next section, so this is not pure indulgence!) I will not give proofs, since they are rather technical and involved; the details are in [10].

Recall that we defined, in example 2.4.2, a fibered category $\mathbf{QCoh}/S$ over $\mathbf{Sch}/S$, whose fiber over X/S is the category of quasi-coherent schemes on X. Then

Theorem 2.7.1. *This fibered category* $\mathbf{QCoh}/S$ *is a stack in the fpqc topology (and hence, furthermore, in the fppf and étale topologies).*

The class of affine arrows $A \to B$ in $\mathbf{Sch}/S$ is stable in the sense of definition 2.4.3, so as in example 2.4.4 we can form a fibered category $\mathbf{AffM}/S$ with objects affine arrows in $\mathbf{Sch}/S$ and morphisms commutative squares.

Theorem 2.7.2. *This fibered category* $\mathbf{AffM}/S$ *is a stack.*

This is useful, but obviously limited in scope; the morphisms of greatest importance in algebraic geometry are not the affine ones, but the projective. Projective morphisms, alas, do not form a stack, but there is a standard way around this fact. If we restrict our attention to a subclass of projective morphisms for which there is a canonically defined ample invertible sheaf, then everything works. Before we state the precise theorem, we need to define a local class of arrows.

Definition 2.7.3. Fix a topology on $\mathcal{C}$. A class of arrows $\mathcal{P}$ in $\mathcal{C}$ is *local* iff it is stable and whenever one has an arrow $X \to U$ in $\mathcal{C}$ and a covering $U_i \to U$ such that $X \times_U U_i \to U_i$ is in $\mathcal{P}$ for all i, we have that $X \to U$ is in $\mathcal{P}$ also.

Theorem 2.7.4. *Suppose we have a local class of arrows all of which are flat, proper and of finite presentation. Let $\mathcal{F}$ be the associated fibered category. Suppose that for each object $\xi : X \to U$ of $\mathcal{F}$ one is given an invertible sheaf $\mathcal{L}_\xi$ on X which is ample with respect to the morphism $X \to U$,* *and that for each cartesian morphism in $\mathcal{F}$*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"] & Y \arrow[d, "\eta"] \\ U \arrow[r, "\phi"] & V \end{tikzcd}$$

one is given an isomorphism $\rho_{f,\phi} : f^*\mathcal{L}_\eta \cong \mathcal{L}_\xi$ of invertible sheaves. Suppose further that these isomorphisms are compatible in the following manner: whenever we have a composable pair of cartesian morphisms in $\mathcal{F}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"] & Y \arrow[r, "g"] \arrow[d, "\eta"] & Z \arrow[d, "\zeta"] \\ U \arrow[r, "\phi"] & V \arrow[r, "\psi"] & W \end{tikzcd}$$

then the diagram

$$\begin{tikzcd} f^*g^*\mathcal{L}_\zeta \arrow[r, "\alpha_{f,g}(\mathcal{L}_\zeta)"] \arrow[d, "f^*\rho_{g,\psi}"] & (gf)^*\mathcal{L}_\zeta \arrow[d, "\rho_{gf,\psi\phi}"] \\ f^*\mathcal{L}_\eta \arrow[r, "\rho_{f,\phi}"] & \mathcal{L}_\xi \end{tikzcd}$$

of quasi-coherent sheaves on X commutes.

Then we may conclude that $\mathcal{F}$ is a stack.

Example 2.7.5. For any fixed base scheme S, and any non-negative integer g we can consider the class $\mathcal{P}_{g,S}$ of proper smooth morphisms, whose geometric fibers are connected curves of genus g. These are a local class in $\mathbf{Sch}/S$.

If $g \neq 1$, then the theorem applies. For $g \geqslant 2$ we can take $\mathcal{L}_{X \to U}$ to be a suitable power of the cotangent sheaf $\Omega^1_{X/U}$, while for $g = 0$ we can take the dual of the cotangent sheaf. Thus we get a stack. The assosciated category fibered in groupoids, usually denoted $\mathcal{M}_{g,S}$, is then also a stack, and is of considerable importance.

For curves of genus 1, there is no naturally defined ample sheaf, and the theorem doesn't apply.

Fact 2.7.6. *In fact, $\mathcal{P}_{1,S}$ does not give rise to a stack: see the references in* [10].

Now, we know that while a curve of genus 1 does not have a canonical ample invertible sheaf, once we distinguish some point on the curve to form an elliptic curve, there is a canonical projective embedding and hence a canonical ample invertible sheaf. Thus we might hope that while families of curves of genus 1 do not form a stack, families of elliptic curves do; this is indeed the case. We limit ourselves to the case of schemes over a field k.

Theorem 2.7.7. [11]*Consider the category $\mathcal{E}$ whose objects are smooth morphisms $p : E \to B$ in $\mathbf{Sch}/k$, whose geometric fibers are connected curves of genus 1, equipped with a section $s : B \to E$.*

[11]In fact, [10] doesn't quite give enough prove this fact, since the theorem he proves—theorem 2.7.4 above—only applies when the fibered category is a category of arrows in the base category. Nevertheless, it is easy to extend the methods in his article; we briefly outline what's necessary. All we need is to extend theorem 2.7.4 (his theorem 4.38) to cover the case where rather than just having a category of arrows in the base, we have arrows with a section.

To do this, one first extends his proposition 4.31, which shows that a stable class of arrows is always a prestack if the site is canonical, to the case of arrows with sections—this just necessitates us showing that if we have two elliptic curves (E_1, p_1, s_1) and (E_2, p_2, s_2) over a common base B, and a map from E_1 to E_2 in the category $\mathbf{Sch}/B$, then this map behaves properly with respect to the section (and so gives rise to a map of elliptic curves) iff it does so locally with respect to a covering in the topology; this follows from representable functors being seperable, which follows in turn from the fact the site is canonical.

Morphisms are maps $e : E_1 \to E_2$ and $b : B_1 \to B_2$ s.t. both the diagrams

$$\begin{tikzcd} E_1 \arrow[r, "e"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "b"] & B_2 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} E_1 \arrow[r, "e"] & E_2 \\ B_1 \arrow[r, "b"] \arrow[u, "s_1"] & B_2 \arrow[u, "s_2"'] \end{tikzcd}$$

commute, and we consider this as a category over $\mathbf{Sch}/k$ by the functor which forgets about E, p and s from objects and e from morphisms. It is easy to see this is a fibered category (this is the fact one can pull back elliptic curves), whose cartesian morphisms are those where the first of the squares above are cartesian. This is a stack.

We will call this stack $\mathcal{M}_{\mathrm{ell}}{}'$, and shall denote the associated category fibered in groupoids, which is also a stack $\mathcal{M}_{\mathrm{ell}}$.

3. Algebraic spaces and algebraic stacks

From now on, when we say 'stack' we shall mean 'category fibered in groupoids which is a stack'.

3.1. **A diagnosis.** Having added lots of 2-categorical strings to our bow, let us now return to the moduli problems which first motivated our entire discursion into matters 2-categorical; our aim will be to see how these new tools can shed light on what was going wrong before, and might suggest a way forward.

Our first port of call is to understand how to view moduli problems as fibered categories, rather than functors. For concreteness, let us consider in particular the moduli problem of elliptic curves over a field. We constructed a category fibered in groupoids $\mathcal{M}_{\mathrm{ell}}$ in the previous section. Upon brief reflection, it is easy to see that this category encodes all the information we need about the moduli problem.

First consider the fiber $\mathcal{M}_{\mathrm{ell}}{}'(U)$ above any given object U of $\mathbf{Sch}/k$, we see the category of families of elliptic curves over U; the objects of this give us the families of elliptic curves over U. But we need to know more than just what the elliptic curves over U are. There are way too many of these, and what we are really interested in is knowing what they are *up to isomorphism.* Luckily, since we have $\mathcal{M}_{\mathrm{ell}}{}'(U)$, the *category* of families of elliptic curves, we can just read off the isomorphisms. In fact, since we're only interested in the *isomorphisms* in $\mathcal{M}_{\mathrm{ell}}{}'(U)$, we might as well simplify things by passing to $\mathcal{M}_{\mathrm{ell}}(U)$, which has all the same objects and isomorphisms, but nothing else to confuse matters.

That gives us everything we need over U. The only other thing we need at all, when you think about it, is knowledge of how these things pull back; and the fibered category structure on $\mathcal{F}$ gives us precisely this. These are the key ingredients of a moduli problem: understanding the families in question, the isomorphisms between them, and the notions of pulling back.

How do these things relate to the classical picture involving a functor $F : (\mathbf{Sch}/k)^{op} \to \mathbf{Set}$? When we were showing that a category fibered in equivalence relations $\mathcal{G}$ over $\mathcal{C}$ was equivalent to a category fibered in sets, we constructed a category fibered in sets $\mathcal{G}'$ (essentially, for an object U of $\mathcal{C}$, $\mathcal{G}'(U)$ was the set of isomorphism classes in $\mathcal{G}(U)$), and an equivalence $\mathcal{G} \to \mathcal{G}'$. We can apply this to our moduli problem $\mathcal{F}$, and we'll get a category fibered in sets $\mathcal{F}'$ (which we can think of as a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$); this will be equivalent to $\mathcal{F}$ if $\mathcal{F}$ was fibered in equivalence relations (and, as one can easily see, conversely). It is easy to see that this functor is the classical moduli problem. Essentially, since we cannot keep track of isomorphisms using the classical framework,

Then, we modify the very end of the proof of Vistoli's theorem 4.38. His proof constructs a family of curves of genus 1 by gluing together our elliptic curves, using the canonical ample line bundle. We then simply add the observation that we can also glue together the sections of these elliptic curves to give a section of the curve of genus 1, again using the fact that the topology is subcanonical. (But now using the full sheaf condition, not just separatedness.)

we have to declare isomorphic objects equal (and then we forget about the isomorphisms); this gives us the classical moduli problem. Critically, if there were non-identity automorphisms for any object in any fiber of $\mathcal{F}$ (i.e. if $\mathcal{F}$ was a genuine category fibered in groupoids, and *not* a category fibered in equivalence relations), forgetting about the isomorphisms also forgets about these extra automorphisms, meaning that the collapse process is not an equivalence.

We are now very close to seeing how the stacky approach may well succeed where the previous approach failed. We just need to take another look at why twists caused a problem in solving our moduli problem. We had two curves E_1 and E_2 over $\mathbb{Q}$; if the elliptic curve moduli problem $M_{\text{ell}} : \mathbf{Sch}{\rightarrow}\mathbf{Set}$ were representable by X, these would give be points of $X(\mathbb{Q})$. They became isomorphic over $\bar{\mathbb{Q}}$, so should pull back to the same point of $X(\bar{\mathbb{Q}})$. So the map $X(\mathbb{Q}){\rightarrow}X(\bar{\mathbb{Q}})$ would have to be non-injective. Now, before we just observed that this was impossible using our general knowledge about field extensions: under an extension of fields, the old points always inject into the new points. But there is a less ad-hoc way of expressing why the map $X(\mathbb{Q}){\rightarrow}X(\bar{\mathbb{Q}})$ must be injective. The key point is that $\bar{\mathbb{Q}}{\rightarrow}\mathbb{Q}$ is an étale covering, and so the map $G(\mathbb{Q}){\rightarrow}G(\bar{\mathbb{Q}})$ is injective for all étale sheaves. And we know representable functors are étale sheaves. Since X's functor of points is representable (d'uh!), it must be a sheaf, so $X(\mathbb{Q}){\rightarrow}X(\bar{\mathbb{Q}})$ is injective.

Putting this the other way, we observe that since M_{ell} has two points over $\mathbb{Q}$ (E_1 and E_2), which map to the same point over $\bar{\mathbb{Q}}$, we know $M_{\text{ell}}(\mathbb{Q}){\rightarrow}M_{\text{ell}}(\bar{\mathbb{Q}})$ *isn't* injective, so M_{ell} *isn't* a sheaf in the étale topology, so it *cannot* be representable.

Now, the key point is this. While M_{ell} doesn't satisfy the sheaf condition, $\mathcal{M}_{\text{ell}}$ *does* satisfy the stack condition, as we saw at the end of the last section. And whatever the 2-categorical analogue of a scheme is, the stacks that are representable by such a thing (we'll use the phrase '2-representable' for such stacks, as the word 'representable' already means 'representable by a scheme') ought to satisfy not the sheaf condition, but the stack condition. Thus, since $\mathcal{M}_{\text{ell}}$ satisfies the stack condition, it looks like it has a chance of being 2-representable.

We could stop there, having seen that the fibered category approach has avoided the obstacle which prevented the classical moduli problems being solvable. But at the moment it is quite unclear precisely what 'special something' is making the fibered category approach succeed where the other failed! The fact that $\mathcal{M}_{\text{ell}}$ is a stack emerged from considerations of rather great abstraction, and it is hard to *see* what is making it tick. The rest of this section is intended to give some example-based (and maybe less rigorous) explication to try and make it clear what that 'special something' is, and hopefully linking these ideas with some more classical thoughts about the representability or otherwise of moduli problems. This should compliment the rather more formal considerations of descent theory which tell us rigorously that $\mathcal{M}_{\text{ell}}$ does succeed where M_{ell} failed, while giving us less of an idea *why*.

As a starting point, we return to our discussion about when the process of collapse from $\mathcal{M}_{\text{ell}}$ to M_{ell} was an equivalence. We might hope that the obstructions to this being an equivalence are the things that make $\mathcal{M}_{\text{ell}}$ succeed; and we remember that the obstructions were essentially the non-identity automorphisms in the fibers $\mathcal{M}_{\text{ell}}(U)$. This is promising, since there is plenty of anecdotal evidence that the existence of extra automorphisms of the objects of the fibers $\mathcal{M}_{\text{ell}}(U)$ are the key to our difficulties. (And not only the large number of authors who gnostically mention that the extra automorphisms are the nub of the problem without mentioning quite why—see [9] and many others!) Let us briefly rehearse at a few, following [3], moving gradually from rather hand-wavy motivation to something quite precise.

Example 3.1.1. As a first clue, we can consider the moduli problem of elliptic curves over $\mathbb{C}$. This functor is not representable, but only just. Although we can form the j-line, which 'ought' to be the moduli space, it doesn't quite manage it because we cannot give a universal family of elliptic curves parametrised by j. The best we can do is the family

$$y^2 = 4x^3 - \frac{27j}{j-1728}x - \frac{27j}{j-1728}$$

which works everywhere except $j = 0$ and $j = 1728$, *which correspond to the elliptic curves with extra automorphisms*. And there is also the suggestion that the automorphism (inversion) present in all the elliptic curves in the family has something to do with the problem: we know well that if we modify the moduli problem and ask for an elliptic curve with a specified 3-torsion point, so that the automorphisms go away (this is called *rigidifying*, and a moduli problem 'without extra automorphisms' is *rigid*), then the resulting moduli problem is solvable: $Y_1(4)$ has a bona fide universal family.

Example 3.1.2. For something more concrete, we can turn to our original pair

$$E = \{x^3 - x = 2y^2\} \quad E' = \{x^3 - x = y^2\}$$

of elliptic curves which are nonisomorphic over $\mathbb{Q}$ but which become isomorphic over $\bar{\mathbb{Q}}$; say by an isomorphism $\phi : E \to E'$. Then, for any element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have that ϕ^σ is also an isomorphism $E \to E'$, since E, E' are defined over $\mathbb{Q}$. So $\phi(\phi^\sigma)^{-1}$ is an automorphism of E_1 defined over $\mathbb{Q}$. The fact that the isomorphism ϕ is not defined over $\mathbb{Q}$ tells us that for some σ, $\phi^\sigma \neq \phi$, which implies that $\phi(\phi^\sigma)^{-1} \neq \mathrm{id}_E$. Thus twists allow us to construct nontrivial automorphisms.

In fact, as is well known, the connection between twists and automorphisms goes well further than this.

Theorem 3.1.3. *There is a bijection*

$$\left\{\begin{matrix} \mathbb{Q}\text{-}\textit{isomorphism classes of elliptic curves} \\ \textit{which become isomorphic to } E \textit{ over } \bar{\mathbb{Q}} \end{matrix}\right\} \cong H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Aut}(E))$$

Proof. See Silverman, AEC □

Thus, if there are no nontrivial automorphisms, there are no twists; but if $\mathrm{Aut}(E) \neq 0$, then we can expect $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Aut}(E)) \neq 0$, which would mean we will get twists, and not get a representable functor.

(Of course, this only gives us insight in cases where a field is not algebraically closed, which doesn't cover everything; for instance, in example 3.1.1 we have automorphisms, and failure to be representable, but everything takes place over $\mathbb{C}$, so there is no Galois cohomology to take. We shall eventually see–in the theorem at the end of this section–that we can in fact fit example 3.1.1 into the framework of this one, by replacing Galois cohomology with étale cohomology, of which it is a special case. But we will not do this for now, as we will find it easier once we have first considered some other matters.)

Let us now proceed head on to try and understand why $\mathcal{M}_{\text{ell}}$ is a stack while M_{ell} is not a sheaf, still focusing on the pair of elliptic curves we had before. We could draw the following picture:

$$\begin{tikzcd}
E_1 \times_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] \arrow[rr] \arrow[dr, leftrightarrow, "\cong"] \arrow[dd] & & E_1 = \{x^3 - x = 2y^2\}/\mathbb{Q} \arrow[dd] & \\
& E_2 \times_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] \arrow[rr] \arrow[dd] & & E_2 = \{x^3 - x = y^2\}/\mathbb{Q} \arrow[dd] \\
\text{Spec } \mathbb{Q}[\sqrt{2}] \arrow[rr, twoheadrightarrow] \arrow[dr, equal] & & \text{Spec } \mathbb{Q} \arrow[dr, equal] & \\
& \text{Spec } \mathbb{Q}[\sqrt{2}] \arrow[rr, twoheadrightarrow] & & \text{Spec } \mathbb{Q}
\end{tikzcd}$$

The stack condition failed because we didn't have the injectivity condition: the map from curves over $\mathbb{Q}$ to curves on the one-piece cover $\bar{\mathbb{Q}}$ which 'agree on overlaps' (i.e. behave correctly pulled back to $\mathbb{Q}[\sqrt{2}] \times_Q \mathbb{Q}[\sqrt{2}]$) isn't injective. The analogical condition for stacks is the fact that functor from the fiber $\mathcal{M}_{\text{ell}}(\text{Spec } \mathbb{Q})$ to the category of descent data $\mathcal{M}_{\text{ell}}\{\text{Spec } \mathbb{Q}[\sqrt{2}] \to \text{Spec } \mathbb{Q}\}$ is full and faithful; so if a problem arose, it would arise with the f+f condition. Let us ask why it doesn't.

In particular, the fullness condition tells us we can take an isomorphism in the category of objects with descent data, and use it to construct an isomorphism over $\mathbb{Q}$. So we ask ourselves why we cannot construct an isomorphism between the E_i over $\mathbb{Q}$ by using the isomorphism between them over $\mathbb{Q}[\sqrt{2}]$ to construct an isomorphism in the category of objects with descent data (which would then give an isomorphism over $\mathbb{Q}$). The answer must be that the isomorphism between the $E_i \times_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}]$ does not satisfy the compatibility criterion we need for it to give rise to a morphism between the objects with descent data corresponding to the $E_i \times_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}]$. Let us consider this compatibility criterion. As usual for étale coverings like $\mathbb{Q}[\sqrt{2}] \to \mathbb{Q}$ which are Galois, the compatibility criterion on an isomorphism ϕ boils down to a Galois condition:

$$\text{For all } \sigma \text{ in } \text{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}),\ \phi^\sigma \phi^{-1} = \text{id}$$

and in this case, for σ the nontrivial element of $\text{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q})$, we have $\phi^\sigma \phi^{-1} \neq \text{id}$; rather, $\phi^\sigma \phi^{-1}$ is a nontrivial automorphism of E_1 (inversion). This automorphism provides the obstruction to ϕ descending to give a morphism over $\mathbb{Q}$.

Thus the key point is that *automorphisms provide us with excuses for isomorphisms over* $\bar{\mathbb{Q}}$ *not to descend to give isomorphisms over* $\mathbb{Q}$, allowing $\mathcal{M}_{\text{ell}}$ to be a stack even though there are isomorphisms over $\bar{\mathbb{Q}}$ which do not come from isomomrphisms over $\mathbb{Q}$. When we collapse from the fibered category point of view to the functorial one, we get rid of these automorphisms, and the stack condition (which is then a sheaf condition) then requires that *every* isomorphism over $\bar{\mathbb{Q}}$ comes from one over $\mathbb{Q}$; since this is not the case, the resulting functor is not a sheaf, and so not representable.

We have seen that automorphisms can play an important role in giving an obstruction to descending an isomorphism from (say) over $\bar{\mathbb{Q}}$ to over $\mathbb{Q}$; if they are doing so, then collapsing to transform into a classical (functorial) moduli problem will result in a functor which is not a sheaf. From this vantage, we can see that theorem 3.1.3 gives a precise condition for when automorphisms are playing this role (in this Galois case): when they give rise to some H^1. If they do, then we will have twists, and those automorphisms will be busy providing the reason that their isomorphism over the extension of $\mathbb{Q}$ does not give an isomorphism over $\mathbb{Q}$ itself.

In the non-Galois case, things become slightly more tricky. Again, the automorphisms provide an obstruction to giving, but it is less easy to give an intuitive picture of precisely how: we will not go into the details, since we're only trying to give an intuitive picture anyway[12]. We have a nice generalisation of theorem 3.1.3, working in a completely arbitrary stack over a completely general site (not just Galois-type examples): we simply replace Galois cohomology with arbitrary sheaf cohomology

Theorem 3.1.4. *Let ξ be an object of a stack $\mathcal{F}$ over a site $\mathcal{C}$, lying above an object U of $\mathcal{C}$. We say an object $\xi' \in \operatorname{Ob} \mathcal{F}(U)$ is a twist of ξ if there is some one-piece covering $V \to U$ s.t. the pullbacks of ξ and ξ' to $\mathcal{F}(V)$ are isomorphic. Then there is a natural bijection between $\mathcal{F}(U)$-isomorphism classes of twists of ξ and the group $H^1(U, \operatorname{Aut}(\xi))$, where we take sheaf cohomology with respect to the given site $\mathcal{C}$. (That is, $H^1(U, \operatorname{Aut}(\xi))$ categorises objects which become isomorphic to ξ over some V which covers U, up to isomorphism over U.)*

Proof. This is an long, but easy, application of the stack conditions, and the proof is unenlightening, so I don't want to give it here. Since I couldn't find a reference, however, I give the proof in appendix to this essay. □

Thus even in non-Galois cases, if $\operatorname{Aut}(\xi) \neq 0$, we will 'usually' have $H^1(U, \operatorname{Aut}(\xi)) \neq 0$; this will give rise to twists, which will mean that after collapsing our fibered category to a functor, it will certainly not be representable.

Remark 3.1.5. It is worth mentioning a promising fact before we close. We might fear that, even though replacing functors with fibered categories gets rid one obstacle to representability: in the form of the moduli functor being not-a-sheaf (since the moduli fibered category *is* a stack), that there might be other things that get in our way. (Not every sheaf is representable, and we shouldn't expect every stack to be 2-representable either.)

The fact is that for classical moduli problems that are 'elliptic curves with extra structure', it follows from the work of Igusa that having extra automorphisms is one of only a few obstacles to representability[13]. But having extra automorphisms, we have seen, is really only a problem in the classical approach. (This is basically because we forget about them as we construct the moduli problem as a functor; this means that even though the original fibered category moduli problem was a stack—since the automorphisms gave us an 'excuse' not to provide descent isomorphisms—the functorised version will not be a sheaf.) Thus we have hope that the stacky approach will get us round this obstacle; and so then the fact that this is one of few obstacles gives us good confidence that we will then be close to having a representable functor.

3.2. **Algebraic spaces.** We are now nearly ready to see how to construct algebraic stacks, which are our fibered-category generalisations of schemes. However, it is easier to do this in two stages. There is an intermediate generalisation of the scheme concept, called an algebraic space. In this section, we shall see how algebraic spaces emerge as a generalisation of a scheme, and in the next section, we will see how to extend the category of algebraic spaces to get the category of algebraic stacks.

To understand where the idea of algebraic spaces comes from, we actually need to go right back to the definition of the concept of a scheme. The point is that a scheme is not the most

[12] In some sense, the best way of getting such an intuitive picture is to just meditate on the proof of 3.1.4 below. The proof shows exactly how automorphisms, if they give rise to H^1, give the stack axioms an 'excuse' not to have isomorphisms descend.

[13] The precise result says that if a moduli problem which adds extra structure to the moduli problem for elliptic curves is relatively representable, rigid and affine, then it is representable. But we will not go into the definitions!

fundamental object in algebraic geometry: that honour falls to the affine scheme. The category of affine schemes is really nothing other than the category of rings reversed (although an affine scheme carries around an extra geometric interpretation). We then define schemes to be 'things we get by piecing together affine schemes', or more formally, objects in a suitable category which are locally isomorphic to affine schemes. This raises two questions: firstly: what 'suitable category'? And secondly: locally in what topology?

The answers, of course, are well known. The 'suitable category', whose objects form (in some sense) the 'substrate' upon which we piece together the affine schemes to get schemes, is nothing other than the category of locally ringed spaces. (The 'extra geometric interpretation' that an affine scheme has compared to its underlying ring is basically its structure as a locally ringed space.) And 'locally' here means locally in the Zariski topology. (What else could it be, since the Zariski topoloogy is the only bona-fide *geometric* topology we have, and locally ringed spaces are very geometric things.) Thus a scheme is a locally ringed space X such that there is a surjective morphism $U \to X$ for U a disjoint union of affine schemes.

We know, however, that the Zariski topology has serious shortcomings for many things in algebraic geometry, and we 'grow up' we learn to generalise many techniques to allow us to replace it with the étale topology, which is much more satisfactory. So the question naturally arises: can we find some generalisation of the concept of a scheme in which we piece together affine schemes in the étale topology rather than the Zariski one.

The answer is yes, but it is not easy to see precisely how to do this. The problem is that the 'substrates', locally ringed spaces, on which we glue together the affine schemes to form schemes are rather heavily geometric objects, and so really require we use a bona-fide geometric topology like the Zariski topology. We need to find some other substrates which interact better with the categorical language of Grothendeick topologies. It turns out that the answer is to use sheaves $\mathbf{Aff}^{op} \to \mathbf{Set}$, where $\mathbf{Aff}$ is the category of affine schemes (equivalently, we could think of these as functors from rings to $\mathbf{Set}$). (We use sheaves because they are, somehow, determined by what they do locally. As an example of this, we have the fact that we could also use sheaves $\mathbf{Sch}^{op} \to \mathbf{Set}$, since a sheaf is determined by what it does on schemes; we will use the two points of view interchangably.)

We first give some rather technical definitions for what an algebraic space is. It will not be that clear how what we are doing is an extension of the process of gluing together affine schemes to get schemes. In fact, we will follow the standard practice in the literature, and get our algebraic spaces by gluing together schemes, not affine schemes (not that it makes much difference). But we will then consider an alternative perspective on the definitions, coming from the notion of an equivalence relation. This perspective will hopefully give more of a feel for what's going on, and how we are indeed generalizing the concept of an affine scheme.

First, the technical definition! We recall that sheaf categories have fibered products. This allows us to make the following definition for a morphism of sheaves; in some sense, it demands that the fibers above schemes are schemes.

Definition 3.2.1. Let $X \to Y$ be a morphism of sheaves $\mathbf{Aff}^{op} \to \mathbf{Set}$. It is *schematic* iff whenever $S \to Y$ is a morphism for S a scheme, the fibered product $S \times_Y X$ is isomorphic to a scheme.

Proposition 3.2.2. *The diagonal map $X \times X \to X$ is schematic iff every map from a scheme to X is schematic.*

Proof. Omitted. This is implied by the corresponding 2-theorem in the next section. □

For any property P of maps of schemes which is stable under base change, we can say that a schematic map $X \to Y$ has that property iff for every scheme S, the map $S \times_Y X \to S$ has that property. Now we can define an algebraic space.

Definition 3.2.3. An *algebraic space* is a sheaf whose diagonal is schematic, quasi-compact and separated, such that there exists an étale map $U \to X$ from a scheme (the map is schematic by the proposition, so it makes sense to assert that it is étale): we call this an *atlas*.

Basically, we ask that the diagonal has certain 'nice' properties, and then, critically, assert that the functor looks 'locally' like a scheme (since an étale map is a local isomorphism).

Remark 3.2.4. If we'd wanted to glue together affine schemes, we'd have insisted that U was a disjoint union of affine schemes. As you can see, this makes almost no difference; given an atlas $U \to X$ which is not affine, pick a collection U_i of affines covering U then use $\coprod U_i \to U \to X$.

Now, on to the promised intuitive second look. The basic idea (following [1]) is that if we want to glue together schemes in the étale topology, we do not actually need to fund some 'substrate' to do the gluing on. Instead, we might simply be able find an *equivalence relations* on our schemes, and then simply decree that we are thinking of equivalent points as the same. In order to carry this out, we need to have some idea what an equivalence relation in the category of schemes should be. We shall doing this by expressing the usual definition of an equivalence relation of sets in 'as categorical terms as possible'; this will make it easy to generalise.

An equivalence relation on a set A is a subset R of $A \times A$ which is reflexive, symmetric and transitive. It is easy to see how we express the reflexivity condition; we just ask that the map $A \to A \times A : a \mapsto (a, a)$ factors through the inclusion of R. Similarly, for symmetry we can just ask that the map $\sigma : (x, y) \mapsto (y, x) : A \times A \to A \times A$ descends to give a map $R \to R$. Transitivity takes a bit more work. We need that if there is are points (a, b) and (c, d) in R with $c = b$, then (a, d) is in R. Now 'pairs of points (a, b) and (c, d) in R with $c = b$' can be expressed categorically as 'points of $R \times_{p_2,U,p_1} R$'. Given such a point of $R \times_{p_2,U,p_1} R$, (a, d) is just the image of $p_1 \times p_2$ (so p_1 on the first R and p_2 on the second) in $A \times A$. So we just ask that this map factors through the inclusion of $R \to A \times A$.

Definition 3.2.5. An *equivalence relation* on a scheme U is a closed subscheme $\iota : R \to U \times U$, such that a) the map

$$x \mapsto (x, x) : U \to U \times U$$

factors through $U \to R \to U \times U$, b) if we define the switch map $\sigma : (x, y) \mapsto (y, x) : U \times U \to U \times U$ then we have a morphism $R \to R$ making

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R \arrow[d] \\ U \times U \arrow[r, "\sigma"] & U \times U \end{tikzcd}$$

commute, and c) the map $p_1 \times p_2 : R \times_{p_2,U,p_1} R \to U \times U : (r_1, r_2) \mapsto (p_1 r_1, p_2 r_2)$ factors through ι.

Then given an algebraic space X, we can construct an étale equivalence relation as follows. We have an atlas $U \to X$. Define $R = U \times_X U$; this is a scheme since it is the pullback to U of $U \to X$, which is a schematic map (since it is a map from a scheme to X, so schematic, since X has schematic diagonal). The two projections $R \to U$ are étale, being pullbacks of the map $U \to X$, which is étale. We have a map '$x \mapsto (x, x) : U \to U \times_X U = R$'; the map $U \to R$ induced from the square

$$\begin{tikzcd} U \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

so we can factor $x \mapsto (x, x) : U \to U \times U$ as $U \to R = U \times_X U \to U \times U$. Similarly, map '$(x, y) \mapsto (y, x)$' can be defined as the map $p_2 \times p_1 : R \to R$, which gives the symmetry condition. We have

$$R \times_U R = (U \times_X U) \times_U (U \times_X U) = U \times_X U \times_X U \to U \times_X U$$

via projection onthe the first and third factor; one checks that this provides a factorisation of $p_1 \times p_2 : R \times_{p_2,U,p_1} R \to U \times U : (r_1, r_2) \mapsto (p_1 r_1, p_2 r_2)$ through ι.

Theorem 3.2.6. *Every étale equivalence relation comes from an algebraic space in this manner; and it is possible to reconstruct the algebraic space from the equivalence relation up to isomorphism.*

Proof. This follows from the stack version of this theorem which we give in the next section. □

Now, we can always arrange for our étale cover $U \to X$ to have U a disjoint union of affines; then R will be a disjoint union of quasi-affines. Thus algebraic spaces can be thought of not just as the quotients of schemes by étale equivalence relations, but also as the quotient of disjoint unions of affine schemes by étale equivalence relations which are disjoint unions of quasi-affines.

This gives us our way of seeing algebraic spaces as a generalisation of schemes. For, given a scheme S, we have our map $U \to S$ from disjoint union of affines to S; then we can define $R = U \times_S U$, and we have that R is an equivalence relation (exactly as above), which will again be a disjoint union of quasi-affines. But in this case, the projections $R \to S$ are not merely étale, but rather each projection maps each component of R isomorphically onto an open subset of U. Thus we can think of schemes as quotients of disjoint unions U of affine schemes by equivalence relations R which are disjoint unions of quasi-affines, and whose structural maps send each component of the relation R isomorphically onto an open subset of U. This is actually quite a natural way of thinking about schemes, and even calculating with them.

For instance, suppose we have a scheme S, with its atlas (collection of affines) U and its equivalence relation R. What is a map from an affine scheme A into S in terms of U and R? Well, we get a pullback $U \times_S A$. This is a collection of open quasi-affines, covering A. These are the inverse images of coordinate patches on U. We have a map from each of these inverse images to the corresponding coordinate patch. These are then compatible, in that if two points of $U \times_S A$ lie above the same point of A, then their images in U will be related by one of the equivalences in R. This is pretty much the usual way we think of a morphism to a scheme.

So we can see that we get algebraic spaces from schemes simply by no longer insisting the structural maps are isomorphisms of each component to an open subset of U (which is maybe some kind of notion of 'being a local isomorphism'), but rather are étale (which we know to be a much more satisfactory notion of 'local isomorphism').

Now, one can in fact transfer much of the theory of schemes to the world of algebraic spaces; it's just that rather than having affine covers that are local isomorphisms in the sense of isomorphisms on open sets, they are local isomorphisms in the sense of being étale; this doesn't matter for many arguments. This is done at length in [6]. Problems do arise for things like Chow's lemma and the Serre criterion, whose traditional proofs rely on having a true local isomorphism; but other proofs have been found for the algebraic space case.

One might ask whether algebraic spaces are, in fact, the same thing as schemes. They are not, but it is actually slightly tricky to construct such a thing. They have the flavour of the Hironaka examples of non-projective complete schemes, or compact complex surfaces which are not schemes. In particular, Hartshorne (in appendix A) gives a compact complex surface which is not a scheme, by quotienting a non-projective complete scheme by a group action (which you can do in the analytic world), and proving the result is not a scheme. One can in fact see quite easily that this quotient is an algebraic space, so there are examples of algebraic spaces which are not schemes.

3.3. **Algebraic stacks.** Now, the extension of this to the 2-categorical case is pretty clear. We have defined an algebraic space to be a functor $(\mathbf{Sch}/S)^{op} \rightarrow \mathbf{Set}$ which is étale-locally a scheme, in some sense. An algebraic stack is simply a fibered category which is étale locally a scheme. Again, we need to make the technical conditions on the diagonal. This section is dedicated to giving analogues of most of the things we proved in the previous case.

Definition 3.3.1. We fix a base category $\mathcal{C}$, and suppose we have fibered categories $\mathcal{B}, \mathcal{E}\, cF$ with morphisms $\mathrm{pr}_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{B}, \mathrm{pr}_\mathcal{E} : \mathcal{F} \rightarrow \mathcal{B}$. A *fibered product* is a fibered category $\mathcal{X}$ equipped with morphisms $\mathrm{pr}_1 : \mathcal{X} \rightarrow \mathcal{E}, \mathrm{pr}_2 : \mathcal{X} \rightarrow \mathcal{F}$ such that the composites $\mathrm{pr}_\mathcal{E}\mathrm{pr}_1$ and $\mathrm{pr}_\mathcal{F}\mathrm{pr}_2$ are isomorphic via some natural transformation α (we do *not* require equality), such that for any $\mathcal{Z}$ with projections $\mathrm{pr}_1', \mathrm{pr}_2'$ which satisfies the same condition via some natural transformation α', we have a projection $\pi : \mathcal{Z} \rightarrow \mathcal{X}$ (unique up to natural isomorphism) and natural isomorphisms $\gamma_i : \mathrm{pr}_i \pi \cong \mathrm{pr}_i'$ such that

$$(\gamma_2 \mathrm{pr}_c F)\alpha(\gamma_1 \mathrm{pr}_c E)^{-1} = \alpha' \text{ as natural transformations } \mathrm{pr}_\mathcal{E}\mathrm{pr}_1 \rightarrow \mathrm{pr}_\mathcal{F}\mathrm{pr}_2$$

This is rather confusing, so let us draw some pictures:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathcal{E} \arrow[d, "\mathrm{pr}_\mathcal{E}"] \\
\mathcal{F} \arrow[r, "\mathrm{pr}_\mathcal{F}"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{B}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{Z} \arrow[r, "\mathrm{pr}_1'"] \arrow[d, "\mathrm{pr}_2'"] & \mathcal{E} \arrow[d, "\mathrm{pr}_\mathcal{E}"] \\
\mathcal{F} \arrow[r, "\mathrm{pr}_\mathcal{F}"] \arrow[ur, Rightarrow, "\beta"] & \mathcal{B}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{Z} \arrow[dr, "\pi"] \arrow[drr, "\mathrm{pr}_1'"] \arrow[ddr, "\mathrm{pr}_2'"'] & & \\
& \mathcal{X} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \arrow[u, phantom, "\Uparrow\gamma_1"] \arrow[l, phantom, "\Leftarrow\gamma_2"] & \mathcal{E} \arrow[d, "\mathrm{pr}_\mathcal{E}"] \\
& \mathcal{F} \arrow[r, "\mathrm{pr}_\mathcal{F}"] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{B}
\end{tikzcd}$$

(we draw natural transformations as double arrows; all natural transformations are isomorphisms). We are given a picture like the left one, such that given any other picture like the middle one, we can draw a picture like the right-hand one. Then we can go across from $\mathcal{Z} \rightarrow \mathcal{F} \rightarrow \mathcal{B}$ to $\mathcal{Z} \rightarrow \mathcal{E} \rightarrow \mathcal{B}$ using either the natural transformation in the middle diagram or the composite in the right-hand diagram; we require these are the same.

One can see that a fibered product is unique up to equivalence, if it exists. And it always does. Define a category with objects triples (ξ_1, ξ_2, ι), where $\xi_1 \in \mathcal{E}$, $\xi_2 \in \mathcal{F}$ are objects over the same element of U $\mathcal{C}$, and where α is an isomorphism in $\mathcal{B}(U)$ between $\mathrm{pr}_\mathcal{E}(\xi_1)$ and $\mathrm{pr}_\mathcal{F}(\xi_2)$, and an arrow $(\xi_1, \xi_2, \iota) \rightarrow (\xi_1', \xi_2', \iota')$ being arrows $\phi_1 : \xi_1 \rightarrow \xi_1'$ in $\mathcal{E}$ and $\phi_2 : \xi_2 \rightarrow \xi_2'$ in $\mathcal{F}$ lying above the same arrow of $\mathcal{C}$, with $\iota' \mathrm{pr}_\mathcal{E} \phi_1 = \iota \mathrm{pr}_\mathcal{F} \phi_2$; it is easy to see this is a fiber product.

The discussion of fibered products above, where we explicitly kept track of the various natural isomorphisms, is rather tedious. Therefore, from now on, we will suppress them where possible, drawing a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

and calling it '2-commutative' if we can find a natural isomorphism to 'fill in' the picture; these isomorphisms will have to satisfy compatibility properties, which we leave unstated, but should be clear from context; they ensure that any two possible isomoprhisms between 1-morphisms in the diagram are the same.

For a scheme S, a 'fibered category over S' shall mean a fibered category over **Sch** equipped with a map to $\mathbf{Sch}/S$, or (which is the same thing) a fibered category over $\mathbf{Sch}/S$ (which we may think of as coming with a map to $\mathbf{Sch}/S$ in the 2-category of fibered categories over $\mathbf{Sch}/S$, since $\mathbf{Sch}/S$ is terminal in the 2-category of fibered categories over $\mathbf{Sch}/S$). We will write the absolute product of stacks over $\mathbf{Sch}/S$ as $\times_S$, since it is also the fibered product over the terminal object $S = (\mathbf{Sch}/S)$.

Definition 3.3.2. A morphism of fibered categories $\mathcal{F}\to\mathcal{G}$ is *schematic* iff, whenever X is a scheme mapping into $\mathcal{G}$, the pullback $\mathcal{G} \times_{\mathcal{F}} X\to X$ of $\mathcal{F}\to\mathcal{G}$ along $X\to\mathcal{G}$ has $\mathcal{G} \times_{\mathcal{F}} X$ a scheme. It is *representable* if the pullback is an algebraic space.

Proposition 3.3.3. *Let $\mathcal{F}$ be a fibered category equipped with a morphism to a scheme S. Then the diagonal $\mathcal{F} \times_S \mathcal{F}\to\mathcal{F}$ is schematic iff every map from a scheme to $\mathcal{F}$ is.*

Proof. First assume the diagonal map $\Delta_{\mathcal{F}} : \mathcal{F}\to\mathcal{F} \times_S \mathcal{F}$ is schematic. If $f : X\to\mathcal{F}, g : Y\to\mathcal{F}$ are arrows from schemes, we want to show the fibered product $X \times_{\mathcal{F}} Y$ is a scheme. We have a diagram

$$\begin{tikzcd} X \times_{\mathcal{F}} Y \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ X \times_S Y \arrow[r, "f \times g"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

which 2-commutes; moreover (one can check) these projections make $X \times_{\mathcal{F}} Y$ into a fibered product of $\mathcal{F}$ and $X \times_S Y$ over $\mathcal{F} \times_S \mathcal{F}$ (we say the square is cartesian, as usual). Thus, since $\mathcal{F}\to\mathcal{F} \times_S \mathcal{F}$ is schematic and $X \times_S Y$ is a scheme, the pullback $X \times_{\mathcal{F}} Y\to X \times_S Y$ along $X \times_S Y\to\mathcal{F} \times_S \mathcal{F}$ has $X \times_{\mathcal{F}} Y$ a scheme.

Conversely, suppose that every morphism from a scheme to F is schematic. Let $h : X\to\mathcal{F}\times_S\mathcal{F}$ be a morphism from a scheme; we want to know that the pullback of $\Delta_{\mathcal{F}}$ along this map has source a scheme. h corresponds to two morphisms $f : X\to\mathcal{F}$ and $g : X\to\mathcal{F}$, and we can write $h = (f \times g) \circ \Delta_X$. We have a two-commutative diagram.

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{F}\times_S\mathcal{F}} X \arrow[r] \arrow[d] & X \times_{f,\mathcal{F},g} X \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ X \arrow[r, "\Delta_X"] & X \times_S Y \arrow[r, "f \times g"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

We can check each square is cartesian, so the rectangle is, so left column is the pullback of $\Delta_{\mathcal{F}}$ along $h = (f \times g) \circ \Delta_X$, so its source, $\mathcal{F} \times_{\mathcal{F}\times_S\mathcal{F}} X$, is what we wish to show is a scheme. If we can see $X \times_{\mathcal{F}} X$ is a scheme, then the left square will exhibit $\mathcal{F} \times_{\mathcal{F}\times_S\mathcal{F}} X$ as a fibered product of two schemes ($X \times_{\mathcal{F}} X$ and X) over a scheme ($X \times_S Y$), so it will be a scheme and we will be done.

But $X \times_{\mathcal{F}} X$ is the source of the pullback of $f : X\to\mathcal{F}$ (a morphism from a scheme, so schematic by assumption) along $g : X\to\mathcal{F}$ (a morphism from a scheme), so by definition of schematic it is a scheme. □

Definition 3.3.4. An *algebraic stack* is a stack X with schematic, quasicompact and separated diagonal, and such that there exists some scheme U with an étale, surjective map $U\to X$. This U is called an atlas.

Remark 3.3.5. There is a more general definition, due to Artin, which we will not deal with in detail in this essay. We outline the main points of departure. Artin's definition is

> An *algebraic stack* is a stack X with representable, quasicompact and separated diagonal, and such that there exists some algebraic space U with a smooth surjective map $U\to X$. This U is called an atlas.

The really important point is that we ask that the atlas only be smooth. In fact, if we change 'smooth' back to 'étale', giving

> An *algebraic stack* is a stack X with representable, quasicompact and separated diagonal, and such that there exists some algebraic space U with an étale surjective map $U \to X$. This U is called an atlas.

then we are back to the same notion as our 'official' definition above. In particular, there will be a scheme S with an étale map $S \to U$ and we can take $S \to U \to X$ to get an atlas which is a scheme, and there is a theorem that says that we can deduce that the diagonal is schematic (rather than just representable) in this case—see [7], lemma 4.2.

Just as with algebraic spaces, many results carry over directly to the world of algebraic spaces, since they only ever work with the space in ways that are local in the étale topology. Other results take more work. The reference for all these things is the encyclopedic [7].

Just as we could see algebraic spaces from the alternative perspective of étale equivalence relations, so we can see algebraic stacks from the alternative perspective of étale groupoids. Just as before, we first have to take the definition of a groupoid and 'categorify' it so we can ask what it means to be a groupoid in categories other than set.

Claim 3.3.6. *A groupoid is determined by the following data: sets M of morphisms and O of objects, and maps of sets $s, t : M \to O$, $e : M \to O$ and $i : M \to M$, and a map $\mu : M \times_{s,O,t} M \to M$ such that*

$$si = t, ti = s, se = te = \mathrm{id}_O, \mu \circ (i \times \mathrm{id}_M) = e, \mu \circ (\mathrm{id}_M \times i) = e, s\mu = s\mathrm{pr}_2, t\mu = t\mathrm{pr}_1$$

and such that we have the associativity condition that the two maps $\mu \circ (\mu \times \mathrm{id}_M)$ and $\mu \circ (\mu \times \mathrm{id}_M)$ from $M \times_{s,O,t} M \times_{s,O,t} M$ (defined in the obvious way) to M agree.

Proof. This is all relatively easy to check. We take the set of morphisms as M, and objects as O. s, t give the source and target of a morphism, e gives the identity on each element, and i sends a morphism to its inverse. Two morphisms can be composed iff the source of the first is the target of the second, so the set of composable pairs of morphisms is precisely $M \times_{s,O,t} M$, so μ is the multiplication map. $\square$

Definition 3.3.7. A *groupoid object in the category of schemes/S* consists of schemes O and M over S ('objects' and 'morphisms' respectively), with morphisms $s, t : M \to O$, an inversion map $i : M \to M$, an identity map $e : O \to M$, and a multiplication map $\mu : M \times_{s,O,t} M \to M$ s.t.

$$si = t, ti = s, se = te = \mathrm{id}_O, \mu \circ (i \times \mathrm{id}_M) = e, \mu \circ (\mathrm{id}_M \times i) = e, s\mu = s\mathrm{pr}_2, t\mu = t\mathrm{pr}_1$$

and such that we have the associativity condition that the two maps $\mu \circ (\mu \times \mathrm{id}_M)$ and $\mu \circ (\mu \times \mathrm{id}_M)$ from $M \times_{s,O,t} M \times_{s,O,t} M$ (defined in the obvious way) to M agree. It is an *étale groupoid* iff the maps s and t are étale.

Given an algebraic stack $\mathcal{F}$ with atlas $U \to \mathcal{F}$, we can construct a groupoid as follows. We take $O = U$, and we let $M = U \times_{\mathcal{F}} U$, which is a scheme since U is and $\mathcal{F}$ has schematic diagonal; Note that this is *not* a subscheme of $U \times_S U$; each (u_1, u_2) point of $U \times_S U$ has a point above it for each isomorphism in $\mathcal{F}$ from the object of $\mathcal{F}$ hit by u_1 to that hit by u_2,. $M = U \times_{\mathcal{F}} U$ has two natural projections $U \to M$ which are both étale, being pullbacks of $U \to \mathcal{F}$, which is étale. These are the source and target maps. We have a diagonal map $U \to U \times_{\mathcal{F}} U = M$, an inverse map which switches the two factors, and a composition map

$$M \times_U M = (U \times_{\mathcal{F}} U) \times_U (U \times_{\mathcal{F}} U) = U \times_{\mathcal{F}} U \times_{\mathcal{F}} U \to U \times_{\mathcal{F}} U$$

which is just the projection onto the first and third factors. It is easy to check that this is a groupoid.

We call this étale groupoid a *presentation* of $\mathcal{F}$. Different atlases will give different groupoids.

Theorem 3.3.8. *Every étale groupoid object in the category of schemes (up to isomorphism) arises from an algebraic stack in this manner, and it is possible to reconstruct the stack from the étale groupoid.*

Proof. Given an étale groupoid G with morphisms M and objects U, we will construct an algebraic stack $\mathcal{F}$ with atlas $U \to \mathcal{F}$.

(1) First, given an étale surjection of schemes $T' \to T$, we construct a category $\mathcal{X}(T' \to T)$. First note that $(T', T' \times_T T')$ will be an étale groupoid H, for the same reasons as $(U, U \times_{\mathcal{F}} U)$ was in the discussion preceding this theorem. We let the objects of $\mathcal{X}(T' \to T)$ be morphisms in the category of étale groupoids of schemes from the groupoid $(T', T' \times_T T')$ to (U, M); such a morphism is a pair (ψ, Ψ) where $\psi : T' \to U$, $\Psi : T' \times_T T' \to M$ and these are compatible with the structure maps of the groupoids. We can think of these objects of $\mathcal{X}(T' \to T)$ as, in some sense, 'functor objects in the category of schemes' between the groupoids G and H, which are certainly category objects in the category of schemes. The morphisms of $\mathcal{X}(T' \to T)$ will be 'natural transformation objects in the category of schemes'.

    Precisely, the morphisms from (ψ_1, Ψ_1) to (ψ_2, Ψ_2) are morphisms $\alpha : T' \to M$ such that

$$s_G \circ \alpha = \Phi_1 \quad t_G \circ \alpha = \Phi_2$$

    (here s_G, t_G are the source and target maps of G, and similarly for H). The reader should be able to see how this is indeed a natural transformation object. Thus $\mathcal{X}(T' \to T)$ is in some sense 'the analog of $[G, H]$, with schemes replacing sets'.

    Also, just as ordinary natural transformations into an ordinary groupoid are always invertible, so it is that $\mathcal{X}(T' \to T)$ has all its morphisms invertible.

(2) If we have two étale surjective morphisms $T' \to T$ and $V' \to V$ and compatible maps $T \to V$ and $T' \to V'$, then we get an induced morphism of groupoids from $H_{T' \to T}$ to $H_{V' \to V}$ and hence, by composition, a functor from $\mathcal{X}(V' \to V)$ to $\mathcal{X}(T' \to T)$.

(3) Let $\mathcal{X}(T)$ be the direct limit of the $\mathcal{X}(T' \to T)$ over the category of étale maps $T' \to T$. For $f : T \to V$ a morphism of $\mathbf{Sch}/S$, we have for each étale $V' \to V$ that $V' \times_V T \to T$ is étale and there is a functor

$$\mathcal{X}(V' \to V) \to \mathcal{X}(V' \times_V T \to T)$$

    Now, $\mathcal{X}(V' \times_V T \to T)$ maps into the limit $\mathcal{X}(T)$, and by composition there is a functor $\mathcal{X}(V' \to V) \to \mathcal{X}(T)$. These are compatible between the V's, and there is an induced functor from $\mathcal{X}(V)$ to $\mathcal{X}(T)$.

(4) One checks that is a pseudo-functor, and that the associated fibered category is a stack.

(5) We have a trivial groupoid (U, U) with each projection the identity. (This is the groupoid with underlying scheme of objects U, and only identity morphisms). There is a morphism $(U, U) \to G$, sending each object to itself, and each identity to the identity on that object. (So formally, $\psi = \mathrm{id}_U$ and $\Psi = e_G$.) This gives an object in $\mathcal{X}(U)$ and hence a morphism $U \to \mathcal{X}$. One checks this is étale and surjective.

(6) One can check that the diagonal is schematic, quasi-compact and separated.

Now, the reader can check that if we take the algebraic stack $\mathcal{X}$, with atlas $U \to \mathcal{X}$, and construct the groupoid $(U, U \times_{\mathcal{F}} U)$, it is canonically isomorphic back to our original groupoid (U, M). One can also check that if we start with a stack $\mathcal{G}$, form a presentation, then form $\mathcal{X}$ as above, $\mathcal{G}$ and $\mathcal{X}$ are equivalent (use our criterion for equivalence). □

3.4. **The moduli stack of elliptic curves.** In order to give at least some impression that these rather abstract objects can be constructed, used, and calculated with, we will give a (very vague!) sketch of how we can construct a moduli stack of elliptic curves over $\mathbb{C}$, and actually consider it concretely enough that we can calculate something about it.

We have seen that a moduli stack of elliptic curves exists; we want to know it is an algebraic stack. We will not go into too much detail on how the technical conditions on the diagonal can be put together. Given two schemes A, B with morphisms to $\mathcal{M}_{\mathrm{ell}}$, we want to find a fibered product $A \times_{\mathcal{M}_{\mathrm{ell}}} B$ as a scheme (this will then tell us the diagonal is schematic). The morphisms $A \to \mathcal{M}_{\mathrm{ell}}, B \to \mathcal{M}_{\mathrm{ell}}$ give us families of elliptic curves over A, B respectively, and it is easy to see that if we can find an absolute product in the category schemes with families of elliptic curves, that will correspond to the fibered product we seek. And we can; this is a consequence of rather general results of Grothendeick—see the reference on p50 of [8]. We also have that the map from this absolute product to $A \times_S B$ (the usual product of A and B as schemes) is quasi-compact and separated. An argument akin to that of 3.3.3 (applying this result where $A = B$) will then establish the diagonal of $\mathcal{M}_{\mathrm{ell}}$ is quasi-compact and separated.

The curve $Y_1(4)$, with its universal family of elliptic-curves-with-4-torsion-points, gives in particular a family of elliptic curves over $Y_1(4)$, i.e. a map $Y_1(4) \to \mathcal{M}_{\mathrm{ell}}$. We claim this map is étale, so we have an atlas. We need to show that for every family $E \to X$ of elliptic curves (with corresponding map $X \to \mathcal{M}_{\mathrm{ell}}$) the projection $X \times_{\mathcal{M}_{\mathrm{ell}}} Y_1(4) \to X$ is étale. But the points of $X \times_{\mathcal{M}_{\mathrm{ell}}} Y_1(4)$ are just points of X with a choice of a 4-torsion point in the curve above X; since every curve has 12 choices of a 4-torsion point, every point has 12 preimages under $X \times_{\mathcal{M}_{\mathrm{ell}}} Y_1(4)$; so the cover is twelvefold with no ramification[14]. Thus it is easy to see the map is indeed étale.

We can see the elements of $\mathcal{M}_{\mathrm{ell}}$ above a space X more concretely. Given a family of elliptic curves on S ($\pi : \mathcal{X} \to E$, say), the j invariant gives us a map $S \to \mathbb{C}$, doubly ramified at each point where $j = 0$, triply where $j = 1728$ and unramified everywhere else. This does not quite determine the family, however; but (perhaps rather suprisingly) it is very easy to get something which does. Consider the standard family lying above the j line minus $(0, 1728)$; let's call this $\pi_0 : \mathcal{E} \to \mathbb{C} - \{0, 1728\}$. We can form the product $I = X \times_{\mathcal{M}_{\mathrm{ell}}} \mathbb{C} - \{0, 1728\}$. This is a double cover of X above every point where $j \neq 0, 1728$, and has no points above the other points of X, since every elliptic curve with $j \neq 0, 1728$ has only 2 automorphisms. Thus we have a double étale cover of $j^{-1}(\mathbb{C} - \{0, 1728\}) \subset X$; this extends to give a covering I of all of X (now no longer étale). Then this, with $j : X \to \mathbb{C}$, determines the family uniquely.

To see this, one first checks that there is at most one family $\mathcal{X} \to S$ extending the restriction $\mathcal{X} \times_S j^{-1}(\mathbb{C} - \{0, 1728\}) \to j^{-1}(\mathbb{C} - \{0, 1728\})$ of the family to $j^{-1}(\mathbb{C} - \{0, 1728\}) \subset S$, so we may assume $j(S) \subset (\mathbb{C} - \{0, 1728\})$ ($= S_0$, say). We have our standard family $\pi_0 : \mathcal{E} \to S_0$ over most of the j line, our product $I = X \times_{\mathcal{M}_{\mathrm{ell}}} S_0$ and its universal family $\mathcal{Y} \to I$, giving a diagram

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathcal{X} \arrow[dd, "\pi"] & & \mathcal{E} \arrow[dd, "\pi_0"] \\
 & I \arrow[dl] \arrow[dr] & \\
S & & S_0
\end{tikzcd}$$

Now, the family $\mathcal{Y} \to I$ is determined by j and I/S, since it is just the family induced over I by base extension

$$I \longrightarrow S \xrightarrow{j} S_0$$

[14] An equivalent argument here is as follows. The number of points above a point of X with attached curve $\mathcal{E}$ is nm where n is the number of points of $Y_0(N)$ whose attached curve is isomorphic to $\mathcal{E}$, and m is the number of automorphisms of $\mathcal{E}$. For $j(\mathcal{E}) = 0$, these are 3 and 4 respectively, for $j = 1728$ they are 2 and 6, and otherwise they are 6 and 2.

from the family $\mathcal{E}$. But $\mathcal{Y}$ is also pulled back from $\mathcal{X}$ along $I \to S$, which is a double étale cover; so $\mathcal{Y} \to \mathcal{X}$ is a double étale cover. We could reconstruct $\mathcal{X}$ from this cover if we had the involution ι of the cover interchanging the sheets. But we have an involution of I over S swapping the sheets, which gives an involution i_1 of I over S_0 too; and inversion gives an involution i_1 of $\mathcal{E}$ over S_0; together these determine an involution $i_0 \times i_1$ of $I \times_{S_0} \mathcal{E}$, which is just $\mathcal{Y}$; one checks this involution is ι.

Now, let us briefly see how we can set up the Picard group of a moduli problem. We have a category **QCoh** of quasi-coherent sheaves, which is fibered over **Sch** and so over $\mathbf{Sch}/\mathbb{C}$. We can restrict to the invertible sheaves (since invertible sheaves pull back to invertible sheaves) and we get a category **InSh** fibered over $\mathbf{Sch}/\mathbb{C}$. An invertible sheaf on an object X is then just an element of $\mathbf{InSh}(U)$, which by Yoneda is the same (up to equivalence) as an element of $\mathcal{HOM}(((\mathbf{Sch}/\mathbb{C})/X, \mathbf{InSh})$. Now $((\mathbf{Sch}/\mathbb{C})/X$ is the stack assosciated to X, so we are motivated to define the category of invertible sheaves on an arbitrary stack $\mathcal{F}$ to be $\mathcal{HOM}(\mathcal{F}, \mathbf{InSh})$.

Concretely, an invertible sheaf L on a stack $\mathcal{M}$ is, for each object $\eta \in \mathcal{F}$, an invertible sheaf $L(\eta)$ on $p\eta \in \mathcal{C}$; and for each morphism $\phi : \eta \to \xi$ in $\mathcal{F}$, lying over $f : U \to V$, an isomorphism between $L(\eta)$ and $f^*(L(\xi))$, where f^* pulls back the sheaf $L(\xi)$ on V to U, with the isomorphisms satisfying a compatibility condition for composites of morphisms (we've seen many conditions like this now!)

Tensor products and 'inverses' work as normal, and we get an abelian group of equivalence classes of these things: the *Picard group* of the stack.

Now, let us try and understand the Picard group of $\mathcal{M}_{\text{ell}}$. Let us first try and extract some invariants from an invertible sheaf on L on $\mathcal{M}_{\text{ell}}$. We start as follows. Every elliptic curve has the 'multiplication by -1 automorphism' ρ; given a family of curves $\mathcal{X} \to S$, this gives an automorphism of the family

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\rho"] \arrow[d] & \mathcal{X} \arrow[d] \\ S \arrow[r, "\text{id}"] & S \end{tikzcd}$$

which in turn gives an automorphism of the sheaf $L(\mathcal{X} \to S)$. This must be of order 2 (as ρ is), and given by multiplication of an element α of $\Gamma(S, O_S^*)$ (standard fact about automorphisms of invertible sheaves); thus we have an element α with $\alpha^2 = 1$, so on each connected component, $\alpha \in \{1, -1\}$. In particular, for each curve C, applying this to the one-curve family $C \to \text{Spec } \mathbb{C}$ gives a number $\alpha(C) \in \{1, -1\}$. And we know that whenever two curves appear above points in a family with connected base, they have the same α (which is the α on the family). But $Y_1(4)$ gives us a connected family containing all curves. Thus we have extracted a number α from L, and indeed have a homomorphism from Pic $\mathcal{M}_{\text{ell}}$ to $\mathbb{Z}/2$.

We can do more however. We have two special curves with more automorphisms: C_A say (with 4) and C_B (with 6). Then we have two vector spaces $L(\mathbb{C}_A \to \text{Spec } \mathbb{C})$, $L(\mathbb{C}_B \to \text{Spec } \mathbb{C})$ with an action of $\text{Aut}(C_A) = \mathbb{Z}/4$ on the former and $\text{Aut}(C_B) = \mathbb{Z}/6$ on the latter. IF we pick generators of $\text{Aut}(C_A), \text{Aut}(C_B)$ (say σ, τ respectively), then σ acts on $L(\mathbb{C}_A \to \text{Spec } \mathbb{C})$ by multiplication by a 4th root of 1, $L(\sigma)$ say. Similarly we get a 6th root of 1, $L(\tau)$. Now, clearly $\sigma^2 = \tau^3 = \rho$, so $L(\sigma)^2 = L(\tau)^3 = \alpha(L)$. If we pick a 12th root of 1 ζ, we can find an integer β mod 12 s.t. $\zeta^{6\beta} = \alpha(L); \zeta^{3\beta} = L(\sigma); \zeta^{2\beta} = L(\tau)$.

We thus have a homomorphism β : Pic $\mathcal{M}_{\text{ell}} \to \mathbb{Z}/12$. As it is, it depends on the arbitrary choices ζ, σ, τ (unlike α). We shall show that it is surjective by finding a (canonical, as it happens) invertible sheaf Λ such that $\beta(\Lambda)$ is a generator. This has the pleasant side effect that we can normalise β by insisting $\beta(\Lambda) = 1$, so we no longer depend on those arbitrary choices.

We take $\Lambda(\pi : \mathcal{X} \to S) = R^1\pi_*(O_{\mathcal{X}})$. This is known to be a locally free sheaf of rank $g = 1$. To check $\beta(\Lambda)$ generates, we just need that σ acts faithfully on $\Lambda(C_A \to \text{Spec } \mathbb{C})$, and τ on

$\Lambda(C_A \to \text{Spec } \mathbb{C})$. Now $\Lambda(C_A \to \text{Spec } \mathbb{C})$ is $H^1(C_A, O_{C_A})$, which by Serre duality is dual to the space of differentials. We can take C_A as $y^2 = x^3 - x$; a basis for regular differentials is given by dx/y, and σ can be taken to be $x \mapsto -x, y \mapsto iy$, which acts on dx/y by multiplication by i, which has order 4, the same as σ, so we are faithful in this case. The other case is similar.

Thus the Picard group has $\mathbb{Z}/12$ as a quotient. In fact, the map above is surjective, and Pic $\mathcal{M}_{\text{ell}}$ is just $\mathbb{Z}/12$—see p74 of [8].

4. Appendix—proof of theorem 3.1.4

We shall actually give a slightly simplified proof, which works in the case that our topology is such that given a cover $\{U_i \to U\}$ we can take the disjoint union V of the U_i and get a one-piece cover $V \to U$—this is certainly true for all the usual topologies. The general case is only harder from a standpoint of notation, and it is much easier to see what is going on in this case. The sheaf cohomology $H^1(U, \text{Aut}(\xi))$ is just the limit of the Čech groups $H^1(\{U_i \to U\}, \text{Aut}(\xi))$; given the above condition, this will be the same as the Čech group $H^1(V \to U, \text{Aut}(\xi))$, so it suffices to show that there is a bijection

$$\left\{ \begin{matrix} \mathcal{F}(U)\text{-isomorphism classes of objects } \eta \in \text{Ob } \mathcal{F}(U) \\ \text{s.t. } \xi, \eta \text{ pull back to isomorphic objects over } V \end{matrix} \right\} \cong H^1(V \to U, \text{Aut}(\xi))$$

and we can then take limits.

Before we do anything else, let's fix some notation. We will care about the following objects of $\mathcal{C}$: $U, U' = V, U'' = V \times_U V$ and $U''' = V \times_U V \times_U V$. We have two projection morphisms from $U'' = V \times_U V$ to V, which we shall call pr_a and pr_b. We have three projection morphisms $U''' \to U''$, which we call $\text{pr}_{12}, \text{pr}_{23}, \text{pr}_{13}$ in the obvious fashion. Finally, we have projections $V'' \to V$, which we call $\text{pr}_1, \text{pr}_2, \text{pr}_3$. Then there are obvious identities

$$\text{pr}_a\text{pr}_{12} = \text{pr}_1 \quad \text{pr}_b\text{pr}_{12} = \text{pr}_2 \quad \text{pr}_a\text{pr}_{23} = \text{pr}_2 \quad \text{pr}_b\text{pr}_{23} = \text{pr}_3 \quad \text{pr}_a\text{pr}_{13} = \text{pr}_1 \quad \text{pr}_b\text{pr}_{13} = \text{pr}_3$$

For each object ζ of $\mathcal{F}(U)$, we pick pullbacks $\zeta', \zeta'', \zeta'''$ to U', U'', U''' respectively, along these projection maps. (One might worry, for example, about whether U'' is the pullback of U' along pr_a or pr_b. It does not matter. Since there is an isomorphism $\lambda : V \times_U V \to V \times_U V$ s.t. $\text{pr}_a \lambda = \text{pr}_b$, there is an isomorphism between a pullback along pr_a and a pullback along pr_b; so a pullback along pr_a is also pullback along pr_b.) We will use $\text{pr}_a : \zeta'' \to \zeta'$ for the map in $\mathcal{F}$ above pr_a, and similarly for the other projections. It is clear that we have arranged for the identities

$$\text{pr}_a\text{pr}_{12} = \text{pr}_1 \quad \text{pr}_b\text{pr}_{12} = \text{pr}_2 \quad \text{pr}_a\text{pr}_{23} = \text{pr}_2 \quad \text{pr}_b\text{pr}_{23} = \text{pr}_3 \quad \text{pr}_a\text{pr}_{13} = \text{pr}_1 \quad \text{pr}_b\text{pr}_{13} = \text{pr}_3$$

to hold for these maps in $\mathcal{F}$ too.

This, which amounts to a partial choice of cleavage, gives us functors $\text{pr}_a^* : \mathcal{F}(U') \to \mathcal{F}(U'')$ etc. It is fairly easy to see that the identities $\text{pr}_{12}^*\text{pr}_a^* = \text{pr}_1^*$ etc. hold exactly (rather than up to isomorphism—i.e we have cunningly arranged for our partial cleavage to be a partial splitting), because of the way we have chosen pullbacks. For a map $\psi : \zeta_1' \to \zeta_2'$, we will write ψ_a as a shorthand for $\text{pr}_a^*(\psi)$ and so on. We have

$$(\psi_a)_{12} = \psi_1 \quad (\psi_b)_{12} = \psi_2 \quad (\psi_a)_{23} = \psi_2 \quad (\psi_b)_{23} = \psi_3 \quad (\psi_a)_{13} = \psi_1 \quad (\psi_b)_{12} = \psi_3$$

As a final piece of notation, for a map $\psi : \zeta_1 \to \zeta_2$, we'll write $\psi_* : \zeta_1' \to \zeta_2'$ for the pullback to U'; then $(\phi_*)_a = (\phi_*)_b$.

We will construct, given a η s.t. ξ, η pull back to isomorphic objects over V, a 1-cochain ϕ; that is, an element of $\text{Aut}(\xi'')$. Given our η, we know there is an isomorphism $\iota : \eta' \to \xi'$. We define $\phi = \iota_a \iota_b^{-1}$; we have

$$\xi'' \xrightarrow[b]{\iota^{-1}} \eta'' \xrightarrow[a]{\iota} \xi''$$

Claim 4.0.1. *ϕ is a 1-cocycle.*

Proof. We need to verify that $\phi_{23}\phi_{13}^{-1}\phi_{12} = \mathrm{id}$. And

$$\begin{aligned}
\phi_{23}\phi 13^{-1}\phi_{12} &= (\iota_a\iota_b^{-1})_{23}(\iota_a\iota_b^{-1})_{13}^{-1}(\iota_a\iota_b^{-1})_{12} \\
&= (\iota_a)_{23}(\iota_b)_{23}^{-1}((\iota_a)_{13}(\iota_b)_{13}^{-1})^{-1}(\iota_a)_{12}(\iota_b)_{12}^{-1} \\
&= \iota_2\iota_3^{-1}(\iota_1\iota_3^{-1})^{-1}\iota_1\iota_2^{-1} \\
&= \iota_2\iota_3^{-1}\iota_3\iota_1^{-1}\iota_1\iota_2^{-1} \\
&= \iota_2\iota_2^{-1} = \mathrm{id}
\end{aligned}$$

so we are done. □

Thus we get a map

$$\left\{\begin{matrix}\text{objects } \eta \in \mathrm{Ob}\ \mathcal{F}(U) \text{ with a specified isomorphism} \\ \iota \text{ between the pullbacks } \eta' \text{ and } \xi'\end{matrix}\right\} \to Z^1(V \to U, \mathrm{Aut}(\xi))$$

and we can apply projection to get a map

$$\left\{\begin{matrix}\text{objects } \eta \in \mathrm{Ob}\ \mathcal{F}(U) \text{ with a specified isomorphism} \\ \iota \text{ between the pullbacks } \eta' \text{ and } \xi'\end{matrix}\right\} \to H^1(V \to U, \mathrm{Aut}(\xi))$$

Claim 4.0.2. *ϕ is well defined up to a coboundary, independent of the choice of ι. Indeed, it is also independent of the choice of η within an isomorphism class in $\mathcal{F}(U)$.*

Proof. Suppose that instead of η we had used ζ, which is isomorphic to η via $\theta : \zeta \to \eta$, and we choose an arbitrary isomorphism $\kappa : \zeta' \to \xi'$. We want to show that

$$\kappa_a\kappa_b^{-1} \quad \text{and} \quad \iota_a\iota_b^{-1}$$

are cohomologous, that is, there is some $\alpha \in \mathrm{Aut}(\xi')$ s.t.

$$\alpha_a\kappa_a\kappa_b^{-1} = \iota_a\iota_b^{-1}\alpha_b$$

Consider $\alpha = \iota\theta_*\kappa^{-1}$; we have

$$\begin{aligned}
\alpha_a\kappa_a\kappa_b^{-1} &= \iota_a(\theta_*)_a\kappa_a^{-1}\kappa_a\kappa_b^{-1} \\
&= \iota_a(\theta_*)_a\kappa_b^{-1} \\
&= \iota_a(\theta_*)_b\kappa_b^{-1} \\
&= \iota_a\iota_b^{-1}\iota_b(\theta_*)_b\kappa_b^{-1} \\
&= \iota_a\iota_b^{-1}\alpha_b
\end{aligned}$$

and we are done. □

Thus we have a map

$$\left\{\begin{matrix}\mathcal{F}(U)\text{-isomorphism classes of objects } \eta \in \mathrm{Ob}\ \mathcal{F}(U) \\ \text{s.t. } \xi, \eta \text{ pull back to isomorphic objects over } V\end{matrix}\right\} \to H^1(V \to U, \mathrm{Aut}(\xi))$$

Claim 4.0.3. *This map is injective.*

Proof. Suppose we have two objects η, ζ in $\mathrm{Ob}\ \mathcal{F}(U)$, which map to the same element of $H^1(V \to U, \mathrm{Aut}(\xi))$; we want to construct an isomorphism between them in $\mathcal{F}(U)$. We give an isomorphism θ between their pullbacks η', ζ' to U', and show that this satisfies the condition to descend to an isomorphism of η with ζ.

We know η' and ζ' are isomorphic to ξ'; let us choose isomorphisms

$$\iota : \eta' \to \xi' \quad \kappa : \zeta' \to \xi'$$

Since η, ζ map to the same element of $H^1(V \to U, \mathrm{Aut}(\xi))$, the 1-cocyles they give rise to, $\iota_a\iota_b^{-1}$ and $\kappa_a\kappa_b^{-1}$, must be cohomologous, so there is an $\alpha \in \mathrm{Aut}(\xi')$ such that $\alpha_a\kappa_a\kappa_b^{-1} = \iota_a\iota_b^{-1}\alpha_b$.

We let $\theta = \iota^{-1}\alpha\kappa : \zeta' \to \eta'$. To show this descends to give a morphism $\zeta \to \eta$ we need only show that $\theta_a = \theta_b$. Now

$$\begin{aligned}\theta_a &= \iota_a^{-1}\alpha_a\kappa_a = \iota_a^{-1}\alpha_a\kappa_a\kappa_b^{-1}\kappa_b \\ &= \iota_a^{-1}\iota_a\iota_b^{-1}\alpha_b\kappa_b \\ &= \iota_b^{-1}\alpha_b\kappa_b = \theta_b\end{aligned}$$

and we are done. □

Claim 4.0.4. *This map is surjective.*

Proof. Suppose we have a 1-cocycle ϕ in $Z^1(V \to U, \mathrm{Aut}(\xi))$. This is an automorphism of ξ''. The consider the pair $(\{\xi'\}, \{\phi^{-1}\})$; I claim it is a well defined object in the category of objects with descent data. The condition for this to be the case is

$$\phi_{23}\phi_{13}^{-1}\phi_{12} = 1$$

which is precisely the 1-cocycle condition. Thus we can descend to get an object $\eta \in \mathrm{Ob}\ \mathcal{F}(U)$. By definition, ξ' is a pullback of η, and so there is an isomorphism between ξ' and η', the pullback of η we already chose. Thus the class of η lies in the left hand side of

$$\left\{\begin{matrix}\mathcal{F}(U)\text{-isomorphism classes of objects } \eta \in \mathrm{Ob}\ \mathcal{F}(U) \\ \text{s.t. } \xi, \eta \text{ pull back to isomorphic objects over } V\end{matrix}\right\} \to H^1(V \to U, \mathrm{Aut}(\xi))$$

Next, consider the pair $(\{\eta'\}, \{\mathrm{id}_{\eta''}\})$; this is also manifestly an object with descent data, and (equally obviously), the object they η maps to them. Then the isomorphism $\mathrm{id}_\eta : \eta \to \eta$ in $\mathcal{F}(U)$ gives rise to an isomorphism

$$(\{\xi'\}, \{\phi^{-1}\}) \to (\{\eta'\}, \{\mathrm{id}_{\eta''}\})$$

in $\mathcal{F}\{V \to U\}$. That is, we have an isomorphism $\iota : \eta' \to \xi'$, such that we have a commutative square

$$\begin{tikzcd} \xi'' \arrow[r, "\phi^{-1}"] \arrow[d, "\iota_a"] & \xi'' \arrow[d, "\iota_b"] \\ \eta'' \arrow[r, "="] & \eta'' \end{tikzcd}$$

And we can read off that $\phi = \iota_a\iota_b^{-1}$, which means the class of ϕ is indeed the image of the class of η under the map. □

References

[1] M Artin, *Algebraic spaces*, New Haven: Yale University Press, 1971

[2] J Baez, J Dolan, *From finite sets to Feynman diagrams* in "Mathematics Unlimited — 2001 and Beyond", vol. 1 (B Engquist, W Schmid, eds) (2001), 29–50, New York: Springer-Verlag

[3] S Dasgupta, *Complex multiplication*, course given at Harvard University, spring 2005

[4] P Deligne, D Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES, **36** (1969), 75–110

[5] B. Fantechi, *Stacks for everyone*, online notes, `www.cgtp.duke.edu/~drm/PCMI2001/fantechi-stacks.pdf`

[6] DM Knutson, *Algebraic spaces*, New York: Springer Verlag, 1971

[7] G. Laumon and L. Moret-Bailly, *Champs algébriques*, New York: Springer-Verlag, 2000

[8] D Mumford, *Picard groups of moduli problems*, in "Arithmetical Algebraic Geometry" (OFG Schilling, ed) (1965), 33–81, New York: Harper & Row

[9] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Inventiones mathematicae, **97** (1989), 613–670

[10] A. Vistoli, *Notes on Grothendieck topologies, fibered categories, and descent theory*, online notes, `www.dm.unibo.it/~vistoli/descent.pdf`