ALGEBRAIC STACKS

ANATOLY PREYGEL
SEPTEMBER 27, 2009

Acknowledgements: The author warmly thanks Dennis Gaitsgory for valuable suggestions and proofreading!

1. Introduction

Our goal is roughly to explain the meaning of

$$\begin{array}{ccccc}
\left\{\begin{array}{c}\text{Sheaves of groupoids}\\ \text{on } \mathbf{Aff}\end{array}\right\} & \hookleftarrow & \{\text{Algebraic Stacks}\} & \ni & \left\{\begin{array}{c}BG, \operatorname{Pic} X, \operatorname{Bun}_G, \mathcal{M}_{g,n}\\ \text{reasonable moduli problems}\end{array}\right\} \\
\big\uparrow & & \big\uparrow & & \\
\left\{\begin{array}{c}\text{Sheaves of sets}\\ \text{on } \mathbf{Aff}\end{array}\right\} & \hookleftarrow & \{\text{Algebraic Spaces}\} & \hookleftarrow & \{\text{Schemes}\}
\end{array}$$

Recall that the Yoneda/functor of points formalism tells us that $X \mapsto \operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} -, X)$ defines a fully faithful embedding

$$\{\text{Schemes}\} \hookrightarrow \operatorname{Psh} \overset{\text{def}}{=} \operatorname{Fun}(\mathbf{Aff}^{\text{op}}, \mathbf{Set})$$

and Grothendieck's theory of descent says that schemes land in the full subcategory Shv of (flat) sheaves. Unfortunately, it's hard to tell when a sheaf is the functor represented by a scheme. Artin discovered a class of sheaves, the *algebraic spaces*, that miraculously admit both a "gluing" description and a more "functorial" description. They possess better formal properties than schemes, but are still geometric enough to allow us to import many definitions and properties from scheme theory.

Meanwhile, *moduli problems* of objects "with automorphisms," like *principal G-bundles* or *elliptic curves*, often have the property that the functor

$$T \mapsto \{\text{isomorphism classes of} \ldots \text{over } T\}$$

is not only not a scheme but not even a sheaf. Indeed, taking ... to be "line bundles" show that it is not even a Zariski sheaf: The stalks of the presheaf are all trivial, but there can exist non-trivial sections.

This problem goes away if we don't pass to isomorphism classes and instead work with the underlying *groupoids*. So, a natural home for moduli problems is inside $\operatorname{Psh}^{\text{Gpd}} \overset{\text{def}}{=} \operatorname{Fun}(\mathbf{Aff}^{\text{op}}, \text{Gpd})$. In the examples above, *faithfully flat descent* for affine schemes (in the case of G-bundles for G affine), projective schemes with ample line bundle (for elliptic curves using $\mathcal{O}_E(3 \cdot \text{id}_E)$), and for quasi-coherent sheaves (for line bundles) show that the previous moduli problems are indeed *sheaves* of groupoids. *Algebraic stacks* form a full subcategory of "geometric enough" sheaves of groupoids, for which one can again import many definitions and properties from scheme theory. Algebraic stacks include the algebraic spaces (which are precisely the algebraic stacks taking values sets), and like algebraic spaces can be characterized both via a "gluing" description and a more "functorial" criterion. (Though we won't talk about these "functorial" criterion at all.)

2. Algebraic Spaces

References.

- Knutson's book (Algebraic Spaces, Springer LNM 203). Chapter 2, with back-references to Chapter 1 as needed. ⋆
- Laumon–Moret-Bailly (Champs Algébriques). Chapter 1, Appendix A, also Corollaries 10.4.⋆.
- De Jong's "Stacks Project". Not exactly concise, but in English and incredibly encyclopedic on the scheme theory/algebraic space theory required.

We'll begin with algebraic spaces as a warm-up for the slightly more complicated situation of algebraic stacks.

2.1. Sheaves of Sets.

Definition 2.1.1. Let **Aff** be the category of affine schemes (possibly over some fixed base scheme S which will be hereafter omitted), and

$$\mathrm{Psh}(\mathbf{Aff}) \stackrel{\text{def}}{=} \mathrm{Fun}\left(\mathbf{Aff}^{\mathrm{op}}, \mathbf{Set}\right)$$

the category of presheaves of sets. It has all limits and colimits, computed pointwise

$$(\varprojlim_{\alpha} \mathscr{F}_\alpha)(T) = \varprojlim_{\alpha}\left(\mathscr{F}_\alpha(T)\right) \qquad \text{and} \qquad (\varinjlim_{\alpha} \mathscr{F}_\alpha)(T) = \varinjlim_{\alpha}\left(\mathscr{F}_\alpha(T)\right).$$

Notation 2.1.2. We think of elements of $\mathscr{F}(T)$ as "living over T", so that for $f : T \to T' \in \mathbf{Aff}$ we denote the induced functor $\mathscr{F}(T') \to \mathscr{F}(T)$ by "f^*". Also, motivated by the Yoneda embedding, for any $\mathscr{G} \in \mathrm{Psh}(\mathbf{Aff})$, we write $\mathscr{F}(\mathscr{G})$ for $\mathrm{Hom}_{\mathrm{Psh}(\mathbf{Aff})}(\mathscr{G})$.

Definition 2.1.3. A presheaf $\mathscr{F} \in \mathrm{Psh}(\mathbf{Aff})$ is an fppf *sheaf* if for all fppf coverings $\pi : U \to X$ in $\mathrm{Psh}(\mathbf{Aff})$ the natural map

$$\mathscr{F}(X) \to \mathrm{equaliz}\left\{ \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U) \right\}$$

is an isomorphism. Let $\mathrm{Shv}(\mathbf{Aff})$ denote the full subcategory of $\mathrm{Psh}(\mathbf{Aff})$ consisting of sheaves. Formal nonsense shows that it is closed under fiber products.

2.1.4. Grothendieck's theory of descent tells us that there is a fully faithful embedding $\mathbf{Sch} \hookrightarrow \mathrm{Shv}(\mathbf{Aff})$. For $X \in \mathbf{Sch}$, its image in $\mathrm{Shv}(\mathbf{Aff})$ is

$$X(T) = \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, T, X).$$

We will henceforth think of a scheme as a sheaf isomorphic to one of this form.

2.2. Base-change and properties.

2.2.1. We will now pick out certain objects (and arrows, as their relative versions) in $\mathrm{Shv}(\mathbf{Aff})$:

$$\left\{ \begin{matrix} \text{Sheaves of sets} \\ \text{on } \mathbf{Aff} \end{matrix} \right\} \supset \{\text{Algebraic Spaces}\} \supset \{\text{Schemes}\}$$

$$\left\{ \begin{matrix} \text{Morphisms of} \\ \text{sheaves} \end{matrix} \right\} \supset \underbrace{\{\text{Relative algebraic spaces}\}}_{\text{“Representable map”}} \supset \underbrace{\{\text{Relative Schemes}\}}_{\text{“Schematic map”}}$$

What we *do not* have is a particularly interesting example of an algebraic space that is not a scheme: the examples tend to be a bit exotic.

2.2.2. We know how to define geometric properties for schemes and morphisms of schemes. Most properties of morphisms of schemes are *closed under base-change*: If $f : X \to Y$ has the property, then the base-change $f_T : X \times_Y T \to T$ has the property for all $T \to Y$. This lets us extend the definition of these properties to the following class of morphisms of presheaves.

Definition 2.2.3. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathrm{Psh}(\mathbf{Aff})$ is *schematic* (or a *relative scheme*) if for every scheme T and $\alpha \in \mathscr{G}(T) = \mathrm{Hom}(T, \mathscr{G})$, the fiber product $\mathscr{F} \times_{\mathscr{G}} T$ is a scheme.

Definition 2.2.4. A schematic morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathrm{Psh}(\mathbf{Aff})$ is

an isomorphism/flat/faithfully flat/quasi-compact/separated/proper/finite-type/étale/smooth

if for every scheme T and $\alpha \in \mathscr{G}(T)$ the base-changed morphism $\phi_T : \mathscr{F} \times_{\mathscr{G}} T \to T$ is

an isomorphism/flat/faithfully flat/quasi-compact/separated/proper/finite-type/étale/smooth

This definition can be applied for any property of morphisms of schemes which is preserved under base-change.

2.2.5. Checking a property for *every* scheme T and $\alpha \in \mathscr{G}(T)$ sounds hard. Many properties of schemes are *flat local*: $f : X \to Y$ has the property iff for *some* faithfully flat and locally of finite-presentation(=fppf) $T \to Y$ the base-change $f_T : X \times_Y T \to T$ has the property.

Lemma 2.2.6. *Suppose $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathrm{Shv}(\mathbf{Aff})$ is a schematic morphism, and $\pi : U \to \mathscr{G}$ a schematic fppf morphism. Then, ϕ is affine/étale/smooth/etc. iff $\phi_U : \mathscr{F} \times_{\mathscr{G}} U \to U$ is affine/étale/smooth/etc. This works for any property that is closed under base-change and flat local.*

Proof. The forward direction is automatic. We handle the converse in two steps. If $T \to \mathscr{G}$ factors through U, then ϕ_T is a base-change of ϕ_U and thus has the property. For the general case, observe that $\pi_T : T' = T \times_{\mathscr{G}} U \to T$ is an fppf morphism and $\phi_{T'}$ has the property by the previous case; thus, ϕ_T has the property by descent. □

2.3. **Algebraic Spaces.**

2.3.1. The definition of an algebraic space can be motivated in a few ways:

- Algebraic spaces are precisely the (flat) sheaves that arise as quotients of (flat) equivalence relations of schemes.
- They are the sheaves that, by the previous Lemma, allow us to "easily check" properties of all maps from schemes.

Definition 2.3.2. An *algebraic space* is an fppf-sheaf $\mathscr{F} \in \mathrm{Shv}(\mathbf{Aff})$ satisfying

(i) The diagonal morphism $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(ii) There exists a scheme U and a (schematic) étale map $\pi : U \to \mathscr{F}$.

2.3.3. The map $\pi : U \to \mathscr{F}$ is called an *(étale) atlas* for $\mathscr{F}$. The following non-trivial theorem of Artin asserts that the existence of an *fppf* atlas is sufficient to guarantee the existence of an *étale* atlas!:

Theorem 2.3.4 (Artin)**.** *Suppose $\mathscr{F} \in \mathrm{Shv}(\mathbf{Aff})$ is an fppf-sheaf. Then, $\mathscr{F}$ is an algebraic space iff the diagonal $\mathscr{F} \to \mathscr{F}^2$ is schematic, and there exists a scheme U and a (schematic) faithfully flat, locally of finite-presentation, map $\pi : U \to \mathscr{F}$*

Proof. See LMB, §10, Corollaire 10.4.1. One should be able to ignore hypothesis (iii) there with our definition. □

2.3.5. See Appendix 6 for more on equivalent variants of the definition.

The relation of the second bit of 2.3.1 to the definition above is clarified by:

Lemma 2.3.6. *The following are equivalent*

(i_1) *The diagonal morphism $\mathscr{F} \to \mathscr{F}^2$ is schematic;*
(i_2) *For any scheme T and $f : T \to \mathscr{F}$, f is schematic;*
(i_3) *For any schemes T, T' and $\alpha \in \mathscr{F}(T), \alpha' \in \mathscr{F}(T')$, the fiber product $T \times_{\mathscr{F}} T'$ is a scheme;*
(i_4) *For any scheme T and $\alpha \in \mathscr{F}(T), \beta \in \mathscr{F}(T)$, the fiber product $T \times_{\mathscr{F}} T$ is a scheme.*

Proof. Easy definition/diagram chase using

$$\begin{array}{ccc} X \times_{\mathscr{F}} Y & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow{\scriptstyle \Delta_{\mathscr{F}}} \\ X \times Y & \xrightarrow[\alpha\times\beta]{} & \mathscr{F}^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} X \times_{\mathscr{F}^2} \mathscr{F} & \longrightarrow & X \times_{\mathscr{F}} X & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_F} \\ X & \xrightarrow[\Delta_T]{} & X^2 & \xrightarrow[\alpha\times\beta]{} & \mathscr{F}^2 \end{array}$$

□

2.3.7. Exercise. Use similar arguments to prove the following: Suppose $\mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, such that $\mathscr{G} \to \mathscr{G}^2$ is schematic. Then, the following are equivalent:

(a) $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(b) $\mathscr{F} \to \mathscr{F} \times_{\mathscr{G}} \mathscr{F}$ is schematic.

(Hint: Consider the factorization $\mathscr{F} \to \mathscr{F} \times_{\mathscr{G}} \mathscr{F} \to \mathscr{F}^2$, and prove that $\mathscr{F} \times_{\mathscr{G}} \mathscr{F} \to \mathscr{F}^2$ is schematic and that its diagonal is schematic (in fact an isomorphism).)

Thanks to Lemma 2.3.6 and Exercise 2.3.7 the following seemingly limited Lemma will become amazingly useful:

Lemma 2.3.8. *The class of schematic monomorphisms of sheaves satisfies fppf descent. That is, the assignment*

$$\mathrm{ob}(T) = \left\{ \textit{Diagrams of sheaves} \quad \begin{array}{ccc} \mathscr{F} & \hookrightarrow & \mathscr{G} \\ & \searrow \quad \swarrow & \\ & T & \end{array} \quad \textit{with } \phi \textit{ a schematic monomorphism} \right\}$$

$$\mathrm{mor}(T) = \{\textit{Sheaf morphisms over } T\}$$

defines an fppf-sheaf in the appropriate sense (i.e., a version of Definition 3.1.3 for sheaves of categories).

Proof. Only effectivity is an issue, and we will reduce it to the similar statement in Lemma 5.0.6. Suppose $\mathscr{F} \to \mathscr{G}$ is a sheaf map over T, and $T' \to T$ an fppf cover so that $\mathscr{F} \times_T T' = \mathscr{F}_{T'} \to \mathscr{G}_{T'} = G \times_T T'$ is a schematic monomorhpism. Then, for every $U \to \mathscr{G}$ we have a diagram

$$\begin{array}{ccc} U \times_{\mathscr{G}} \mathscr{F} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathscr{G} \end{array} \qquad \text{and its base changes } - \times_T T' \qquad \begin{array}{ccc} (U \times_{\mathscr{G}} \mathscr{F})_{T'} = (U \times_{\mathscr{G}} \mathscr{F}) \times_T T' & \longrightarrow & \mathscr{F}_{T'} \\ \downarrow & & \downarrow \\ U_{T'} = U \times_T T' & \longrightarrow & \mathscr{G}_{T'} \end{array}$$

Since $\mathscr{F}_{T'} \to \mathscr{G}_{T'}$ was a schematic monomorphism, $(U \times_{\mathscr{G}} \mathscr{F})_{T'}$ is a scheme and $(U \times_{\mathscr{G}} \mathscr{F})_{T'} \to U_{T'}$ is a monomorphism of schemes. Thus we can apply Lemma 5.0.6 to conclude that that $U \times_{\mathscr{G}} \mathscr{F} \to U$ is also a monomorphism of schemes. So, $\mathscr{F} \to \mathscr{G}$ is a schematic monomorphism as desired. ☐

2.4. **Proofs of Key Properties.** Here's a first application of the preceding Lemma:

Corollary 2.4.1. *Suppose $\mathscr{F}$ is an fppf-sheaf, U a scheme, and $\pi : U \to \mathscr{F}$ a surjection of fppf-sheaves such that $U \times_{\mathscr{F}} U$ is a scheme and $U \times_{\mathscr{F}} U \to U$ is fppf. (This is satisfied, for instance, if π is schematic and fppf.) Then, $\mathscr{F}$ is an algebraic space.*

Proof. First we prove that the diagonal $\mathscr{F} \to \mathscr{F}^2$ is schematic. Suppose T is a scheme and $T^2 \to \mathscr{F}^2$ a morphism. Since π is a surjection, there is an fppf cover $T' \to T$ such that $(T')^2 \to \mathscr{F}^2$ factors through U^2:

$$\begin{array}{ccccccc} T' \times_{\mathscr{F}^2} \mathscr{F} & \longrightarrow & T \times_{\mathscr{F}^2} \mathscr{F} & \longrightarrow & U \times_{\mathscr{F}} U & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (T')^2 & \longrightarrow & T & \longrightarrow & U^2 & \longrightarrow & \mathscr{F}^2 \end{array}$$

Then, $T' \times_{\mathscr{F}^2} \mathscr{F} \to T$ is a monomorphism of schemes, it is the pullback of $U \times_{\mathscr{F}} U$. By Lemma 2.3.8 (or the underlying Lemma 5.0.6), this shows that $T \times_{\mathscr{F}^2} \mathscr{F} \to T$ is a monomorphism of schemes. So, $\mathscr{F} \to \mathscr{F}^2$ is schematic.

Now, π is automatically schematic and we'd like to apply Lemma 2.2.6 to $\pi : U \to \mathscr{F}$ to conclude that π is fppf but we of course can't since the assumption there is that "$\pi : U \to \mathscr{G}$ is a schematic fppf morphism". We note that the only place in the proof of Lemma 2.2.6 where we use this is to obtain the existence of $T' \to T$ fppf such that $T' \to \mathscr{G}$ factors through U." For this, it suffices that π is a surjection of fppf-sheaves. This completes the proof. ☐

Corollary 2.4.2. *Algebraic spaces are closed under fiber products.*

Proof. Suppose $\mathscr{F}, \mathscr{F}' \to \mathscr{G}$ are maps of algebraic space, and $U \to \mathscr{F}$, $U' \to \mathscr{F}'$ schematic fppf morphisms from schemes. Then, $U \times_{\mathscr{G}} U' \to U \times_{\mathscr{G}} \mathscr{F}'$ and $U \times_{\mathscr{G}} \mathscr{F}' \to \mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$ are both base-changed from schematic fppf maps, so that their composite $U \times_{\mathscr{G}} U' \to \mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$ is schematic and fppf. Since $\mathscr{G}$ is an algebraic space, $U \times_{\mathscr{G}} U'$ is a scheme. Applying Corollary 2.4.1, we conclude that $\mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$ is an algebraic space. □

With Lemma 2.3.8 under our belt, the following key Lemma is (deceptively?) simple:

Lemma 2.4.3. *Suppose $T \to S$ is an fppf morphism of schemes, and $\mathscr{F} \to S$ a morphism of fppf-sheaves. Then, $\mathscr{F}_T = \mathscr{F} \times_S T$ is an algebraic space iff $\mathscr{F}$ is.*

Proof. Note that applying $- \times_S T$ to $\mathscr{F} \to \mathscr{F} \times_S \mathscr{F}$ yields $\mathscr{F}_T \to \mathscr{F}_T \times_T \mathscr{F}_T$. Each of these is a monomorphism. Thus, by Exercise 2.3.7 and Lemma 2.3.8, $\mathscr{F} \to \mathscr{F}^2$ is schematic iff $\mathscr{F}_T \to \mathscr{F}_T^2$ is.

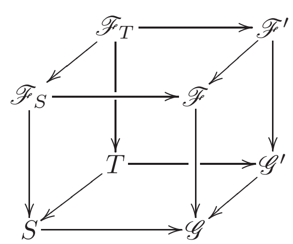
The property of being a schematic fppf morphism is preserved under base change and composition. If $\pi : U \to \mathscr{F}$ is an fppf morphism from a scheme, then the base-change $U \times_S T \to \mathscr{F}_T$ is as well; thus if $\mathscr{F}$ is an algebraic space, then $\mathscr{F}_T$ is as well. Conversely if $\pi' : V \to \mathscr{F}_T$ is an fppf morphism from a scheme, then the composite $V \to \mathscr{F}_T \to \mathscr{F}$ is as well; so, if $\mathscr{F}_T$ is an algebraic space, then $\mathscr{F}$ is as well. □

Definition 2.4.4. A morphism $\phi : \mathscr{F} \to \mathscr{G} \in \mathrm{Shv}(\mathbf{Aff})$ is *representable* or an *relative algebraic space* if for all schemes S and morphisms $S \to \mathscr{G}$ the fiber product $\mathscr{F} \times_{\mathscr{G}} S$ is an algebraic space.

2.4.5. Exercise. Suppose $\mathscr{G}$ is an algebraic space and $\mathscr{F} \to \mathscr{G}$ a morphism of sheaves. Then, $\mathscr{F} \to \mathscr{G}$ is a relative algebraic space iff $\mathscr{F}$ is an algebraic space. (Hint: One direction follows from Corollary 2.4.2. The other follows by taking an atlas $U \to \mathscr{G}$ and considering $\mathscr{F}_U \to \mathscr{F}$.)

Corollary 2.4.6. *Suppose $\mathscr{F} \to \mathscr{G}$ a map of sheaves and $\mathscr{G}' \to \mathscr{G}$ a schematic fppf morphism. Then, $\mathscr{F} \to \mathscr{G}$ is a relative algebraic space iff $\mathscr{F}' = \mathscr{F} \times_{\mathscr{G}} \mathscr{G}' \to \mathscr{G}'$ is a relative algebraic space.*

Proof. Here's a helpful commutative diagram with everything in sight cartesian



$\Rightarrow$ is a tautology. We prove $\Leftarrow$: Suppose $S \to \mathscr{G}$ is a morphism of schemes. Setting $T = S \times_{\mathscr{G}} \mathscr{G}'$, we have that $T \to S$ is an fppf morphism and $\mathscr{F}_T = \mathscr{F}' \times_{\mathscr{G}'} T$ is an algebraic space. So, by the previous Lemma $\mathscr{F}_S$ is an algebraic space. □

In light of descent for fppf-sheaves, this admits the following restatement:

Corollary 2.4.7. *The assignment*

$$T \mapsto \{\textit{Relative algebraic spaces over } T\}$$

satisfies fppf descent (i.e., it is an fppf-sheaf in the language of the next section). If we only consider it on algebraic spaces *(in particular, schemes) T, then the word "relative" may be dropped by Exercise 2.4.5.*

Finally, we hint at another way of thinking of the role of algebraic spaces:

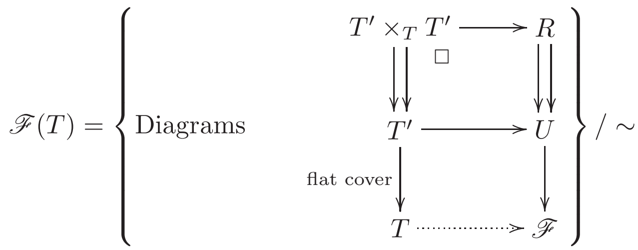
Corollary 2.4.8. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{Schematic surjections of fppf-sheaves} \\ U \twoheadrightarrow \mathscr{F}, \textit{ with } U \textit{ a scheme} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Equivalence relations in schemes} \\ R \rightrightarrows U \end{array} \right\}$$

Algebraic spaces can thus be described as precisely those fppf-sheaves arising as quotients of étale (or fppf) equivalence relations.

Proof. Given a surjection $\pi : U \to \mathscr{F}$, the corresponding equivalence relation has graph $R = U \times_{\mathscr{F}} U$. Conversely, given an equivalence relation we set $\mathscr{F}$ to be the quotient sheaf. It is an exercise to show that then $U \to \mathscr{F}$ is a surjection of fppf-sheaves and $U \times_{\mathscr{F}} U = R$, so that Corollary 2.4.1 implies that $\mathscr{F}$ is an algebraic space. It is an exercise to show that these assignments are inverse to one another. You may find the following Exercise helpful. □

2.4.9. Exercise. Suppose $\pi : U \to \mathscr{F}$ is an atlas of an algebraic space, $R = U \times_{\mathscr{F}} U$. Then, for a scheme T,



where $f_1 \sim f_2$ for $f_i : T_i \to U$ iff there is a common refinement $T' \to T_1, T_2$ such that $f \times g|_{T'} : (T')^2 \to U^2$ factors through R. This can be viewed as the sheafification of $T \mapsto U(T)/\sim_{R(T)}$, where we view $R(T) \subset U(T)^2$ as the graph of an equivalence relation on $U(T)$.

(Hints: With the dotted line, the square should be Cartesian. The universal property of fiber products implies that $\operatorname{coeq}\left\{ R(T) \rightrightarrows U(T) \right\} \longrightarrow \mathscr{F}(T)$ is injective. Now use that $\mathscr{F}$ is a sheaf and π a surjection of sheaves.)

3. Stacks

References.

- For background on classical faithfully flat descent, see e.g., Milne (Étale Cohomology).
- For details on the correct way to deal with "presheaves of groupoids" (i.e., categories fibered in groupoids), see something like Vistoli's notes.
- Laumon–Moret-Bailly (Champs Algébriques). Not exactly a concise reference, but reasonably comprehensive. Ch. 10 covers Artin's theorem that fppf atlases suffice.
- ⋆⋆⋆ Vistoli's Appendix to *Intersection theory on algebraic stacks and on their moduli spaces*. A wonderful intro/summary, but careful: for it an algebraic stack is a Deligne-Mumford stack, and representable = schematic.

Stacks address the following two issues:

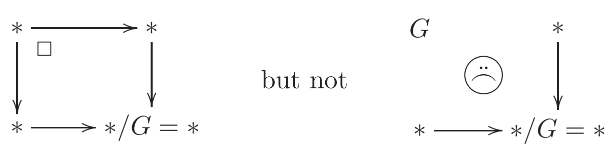
Example 3.0.10 (Non-example)**.** Taking an algebraic space with flat atlas $\pi : U \to \mathscr{F}$ to the flat equivalence relation $R = U \times_{\mathscr{F}} U \subset U^2$ defines (ignoring minor technicalities) a bijection

$$\left\{ \begin{array}{c} \text{Alg. space } \mathscr{F} \\ \text{w/(flat) atlas } \pi : U \to \mathscr{F} \end{array} \right\} = \left\{ \begin{array}{c} \text{Flat equiv. relation} \\ p_1, p_2 : R \to U \end{array} \right\}$$

But, this fact depends crucially on the fact that we are dealing with *equivalence relations* and not taking quotients with "stabilizers": There is no our fiber product construction can give us anything else. Indeed, if we try to carry out a more general type of quotient in sheaves of sets

$$*/G = \operatorname{colim}\left\{ * \leftleftarrows G \leftleftarrows G^2 \leftleftarrows \cdots \right\}^{\text{et-sh}} = *$$

One cannot recover the defining group(oid) from a presentation:



Example 3.0.11 (Non-example)**.** Plenty of moduli problems that involve classifying objects that naturally have automorphisms have the following issue: The presheaf of isomorphism classes of objects do not form sheaves (e.g., the example of line bundles from the introduction). Sheafifying will tend to lose some geometric meaning (for example, deformation theoretic computations), or at least not result in an algebraic space often enough. In other words, unlike algebraic spaces (or more general sheaves of sets), stacks do not merely have better formal properties but admit many *natural examples*.

Stacks resolve this by extending the above gluing context + "geometric object" picture:

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{Sheaves of groupoids}\\ \text{on } \mathbf{Aff}\end{array}\right\} & \hookleftarrow & \{\text{Algebraic Stacks}\} \\ \Big\uparrow & & \Big\uparrow \\ \left\{\begin{array}{c}\text{Sheaves of sets}\\ \text{on } \mathbf{Aff}\end{array}\right\} & \hookleftarrow & \{\text{Algebraic Spaces}\} \end{array}$$

3.1. **Sheaves of Groupoids.** For convenience and readability, I'm going to be intentionally sloppy about the correct notion of a sheaf of groupoids. In particular, I'll pretend that it is actually a functor $T \mapsto \mathscr{F}(T)$, etc. rather than a more correct variant like a "lax 2-functor" or a "category fibered in groupoids over **Aff**". The details can be found in excruciating detail in e.g., Vistoli's article.

3.1.1. A *groupoid* is a (small) category in which every morphism is invertible. Groupoids form a 2-category: In addition to functors, there are also natural transformations. One important consequence is that if $F : \mathcal{C} \to \mathcal{D}$ is a functor between groupoids, one almost never wants to ask that F literally have an inverse, instead one wants to ask that F be an *equivalence of categories*.

3.1.2. There is a fully faithful embedding of the category of sets (as a 2-category with no 2-morphisms) into the 2-category of groupoids: Every set may be regarded as a groupoid with no non-identity morphisms. Conversely, a groupoid is equivalent to a set (its set of components) iff it contains no non-identity automorphisms.

Definition 3.1.3. There is a 2-category of *presheaves of groupoids*, $\mathrm{Psh}^{\mathrm{Gpd}}(\mathbf{Aff}) \overset{\mathrm{def}}{=} \mathrm{Fun}(\mathbf{Aff}^{\mathrm{op}}, \mathrm{Gpd})$ (suitably interpreted–pullbacks are functorial only up to natural transformation), containing the category $\mathrm{Psh}(\mathbf{Aff})$ as a full subcategory. A (1-)morphism $\phi : \mathscr{F} \to \mathscr{G}$ is the data of, for each $T \in \mathbf{Aff}$, a functor $\phi(T) : \mathscr{F}(T) \to \mathscr{G}(T)$ satisfying the following functoriality: For any map $f : T \to T' \in \mathbf{Aff}$, there exists a natural isomorphism $\phi(T) \circ f^* \Rightarrow f^* \circ \phi(T')$ so that

$$\begin{array}{ccc} \mathscr{F}(T') & \longrightarrow & \mathscr{G}(T') \\ \Big\downarrow & \Rightarrow & \Big\downarrow \\ \mathscr{F}(T) & \longrightarrow & \mathscr{G}(T) \end{array}$$

commutes up to this natural isomorphism.
A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is said to be an *equivalence* if $\phi(T) : \mathscr{F}(T) \to \mathscr{G}(T)$ is an equivalence of categories for all $T \in \mathbf{Aff}$.

Example 3.1.4. A motivat(ing/ional) example is the following: Define the presheaf of groupoids QCoh by $X \mapsto \mathrm{QCoh}(X)^{iso}$, where the superscript denotes that we throw out all non-invertible morphisms (to get a groupoid). It is easy to see that we one can "glue" morphisms of sheaves, and even quasi-coherent sheaves themselves, from data defined on a Zariski open cover. The theory of *faithfully flat descent* extends this to fppf covers, and (in the language of the following definition) states that QCoh is an example of an fppf-sheaf of groupoids.

Definition 3.1.5. A presheaf $\mathscr{F} \in \mathrm{Psh}^{\mathrm{Gpd}}(\mathbf{Aff})$ is a (flat) sheaf if for every (flat) covering morphism $\pi : U \to X$ the natural map

$$\mathscr{F}(X) \to \mathrm{Hom}\left(\mathrm{hocolim}\left\{\cdots\right\}, \mathscr{F}\right) = \mathrm{holim}\left\{ \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U) \to \mathscr{F}(U \times_X U \times_X U) \right\}$$

is an *equivalence of categories*. We denote by $\mathrm{Shv}^{\mathrm{Gpd}}(\mathbf{Aff})$ the full subcategory spanned by the sheaves. Here, hocolim and holim are certain appropriate 2-categorical notions of colimits and limits. Rather than go into generalities, we will give explicit constructions.

Construction 3.1.6. With $\mathscr{F}$, $\pi : U \to X$ as above, the category $\mathrm{holim}\left\{\cdots\right\}$ admits the following description:

$$\mathrm{ob}\,\mathrm{holim}\left\{\cdots\right\} = \left\{ \begin{matrix} \alpha \in \mathrm{ob}\,\mathscr{F}(U), \text{ and an iso. } \phi : p_1^*\alpha \simeq p_2^*\alpha \\ \text{satisfying } p_{13}^*(\phi) = p_{23}^*(\phi) \circ p_{12}^*(\phi) \end{matrix} \right\}$$

$$\mathrm{Hom}_{\mathrm{holim}\{\cdots\}}\left((\alpha,\phi),(\alpha',\phi')\right) = \left\{ \begin{matrix} \text{(Iso)morphisms } \psi : \alpha \to \alpha', \text{ intertwining} \\ \phi, \phi', \text{ i.e., s.t. } p_2^*(\psi) \circ \phi = \phi' \circ p_1^*(\psi) \end{matrix} \right\}$$

where $p_i : U \times_X U \to U$, $i = 1, 2$, and $p_{ij} : U \times_X U \times_X U \to U \times_X U$, $(i,j) \in \{(1,2),(1,3),(2,3)\}$, are the projections. This is just the classical *category of descent data*. It is a groupoid since $\alpha \to \alpha'$ intertwines ϕ, ϕ' iff its inverse $\alpha' \to \alpha$ does.

3.1.7. Now, the sheaf condition can be described in two steps (as it usually is classically):

- The functor is *fully faithful*: This is equivalent to a sheaf condition on the presheaves of isomorphisms (i.e., that we can uniquely patch together isomorphisms from local data).

  Explicitly: For any $\alpha, \beta \in \mathscr{F}(X)$, fully faithfulness asks that the natural map

  $$\mathrm{Hom}_{\mathscr{F}(X)}(\alpha,\beta) \to \mathrm{Hom}_{\mathrm{holim}\{\cdots\}}((\pi^*\alpha, \mathrm{id}_{\pi^*_{U^2/X}\alpha}), (\pi^*\beta, \mathrm{id}_{\pi^*_{U^2/X}\alpha}))$$

  be a bijection. Writing out the definition we obtain

  $$\begin{aligned} &\mathrm{Hom}_{\mathrm{holim}\{\cdots\}}((\pi^*\alpha, \mathrm{id}_{\pi^*_{U^2/X}\alpha}), (\pi^*\beta, \mathrm{id}_{\pi^*_{U^2/X}\alpha})) \\ &= \mathrm{equaliz}\left\{ \mathrm{Hom}_{\mathscr{F}(U)}(\pi^*\alpha, \pi^*\beta) \underset{p_2^*(-)}{\overset{p_1^*(-)}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{F}(U\times_X U)}(\pi^*_{U^2/X}\alpha, \pi^*_{U^2/X}\beta) \right\} \end{aligned}$$

  Defining the presheaf $\mathrm{Isom}_X(\alpha,\beta)$ of sets on $\mathbf{Aff}/X$ by

  $$(f : T \to X) \mapsto \mathrm{Isom}_X(\alpha,\beta)(T \to X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathscr{F}(T)}(f^*\alpha, f^*\beta)$$

  we have thus shown that fully faithfulness is a sheaf condition on this presheaf.
- The functor is *essentially surjective*: This is what's usually known as "effectivity of descent."

Example 3.1.8. (Variants of) faithfully flat descent provides the following examples of fppf-sheaves of groupoids (in fact, all but the last two are fpqc-sheaves):

- Quasi-coherent sheaves: $\mathrm{QCoh}(T)$ is the groupoid of quasi-coherent sheaves on T and isomorphisms. This is the prototypical result. In applications to the below, it is useful to know that we also have descent for morphisms that are not necessarily isomorphisms. (Vistoli, Theorem 4.23)
- Affine morphisms: $\mathbf{Sch}^{\mathrm{Aff}}(T)$ is the groupoid of affine morphisms $T' \to T$ and T-isomorphisms. This follows from QCoh since an affine morphism is determined by a quasi-coherent sheaf of algberas. The underlying quasi-coherent sheaf and the extra structure maps all satisfy descent, and the algebra axioms may be checked after faithfully flat base change. (Vistoli, Theorem 4.33)
- Affine morphisms w/G-action: $G\mathbf{Sch}^{\mathrm{Aff}}(T)$ is the groupoid of affine morphisms $T' \to T$ together with G-action on T' as T-scheme, and G-equivariant isomorphisms over T. This is analogous to the above case.
- Polarized projective morphisms: $\mathbf{Sch}^{\mathrm{PProj}}(T)$ is the groupoid of pairs $(T' \to T, \mathscr{L})$ where $\mathscr{L}$ is a relatively ample line bundle over T'. This follows from QCoh since a polarized projective morphism is determined by a certain graded quasi-coherent sheaf of algebras generated in degree 1. (Vistoli, Theorem 4.38)
- Vector bundles of rank n: $\mathrm{Vect}_n(T)$ is the groupoid of vector bundles on T and isomorphims. This follows from QCoh since it is a full subcategory, and being a vector bundle of rank n is fppf local.

- (Non-Example) Arbitrary morphisms of schemes: $\mathbf{Sch}(T)$ is the groupoid of morphisms of schemes $T' \to T$ and T-isomorphisms. This is not a sheaf, since the natural functor is fully faithful but not an equivalence. However, only *half* the sheaf condition fails. The functor of Defn. 3.1.6 *is* fully faithful, it just need not be essentially surjective. Moreover, it remains fully faithful if we include non-invertible morphisms. (Vistoli, Prop. 4.31)
- Arbitrary morphisms of algebraic spaces: $\mathrm{AlgSp}(/T)$ is the groupoid of morphisms of algebraic spaces $T' \to T$ and T-isomorphisms. As mentioned in Corollary 2.4.7 this *is* a sheaf.

3.2. **Fiber product.**

Construction 3.2.1. Given $f : \mathscr{F} \to \mathscr{G}$ and $f' : \mathscr{F}' \to \mathscr{G}$, the (2-)fiber product $\mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$ may be described by:

$$\mathrm{ob}(\mathscr{F} \times_{\mathscr{G}} \mathscr{F}')(T) = \left\{ \begin{array}{c} \alpha \in \mathscr{F}(T), \alpha' \in \mathscr{F}'(T) \\ \text{and an iso. } \phi : f(a) \simeq f'(a') \in \mathscr{G}(T) \end{array} \right\}$$

$$\mathrm{mor}(\mathscr{F} \times_{\mathscr{G}} \mathscr{F}')(T)((\alpha, \alpha', \phi), (\beta, \beta', \varphi)) = \{\text{Maps } \alpha \to \beta,\ \alpha' \to \beta' \text{ intertwining } \phi \text{ and } \varphi\}$$

Remark 3.2.2. Suppose $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{G}$ are sheaves of groupoids. Unraveling the above definitions, one can show that $\mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$ is also a sheaf.

Remark 3.2.3. Suppose that $\mathscr{F}$ and $\mathscr{F}'$ are presheaves of *sets*. Then, there are no non-identity maps $\alpha \to \beta \in \mathscr{F}(T)$, $\alpha' \to \beta' \in \mathscr{F}'(T)$ and thus no non-identity maps in $(\mathscr{F} \times_{\mathscr{G}} \mathscr{F}')(T)$. So, if $\mathscr{F}$ and $\mathscr{F}'$ are sheaves of sets, then so is $\mathscr{F} \times_{\mathscr{G}} \mathscr{F}'$. If $\mathscr{G}$ is also a sheaf of sets, it is clear that the fiber product in $\mathrm{Psh}^{\mathrm{Gpd}}$ recovers the usual fiber product in Psh (of sets).

Example 3.2.4. Suppose X is a sheaf of sets and $\alpha, \beta \in \mathscr{F}(X)$. Consider first the fiber product[1]

$$\begin{array}{ccc} X \times_{\mathscr{F}} X & \longrightarrow & X \\ \downarrow & & \downarrow \beta \\ X & \xrightarrow{\alpha} & \mathscr{F} \end{array}$$

The above tells us $X \times_{\mathscr{F}} X$ is a sheaf of sets, and that

$$(X \times_{\mathscr{F}} X)(T) = \{f, g \in X(T), \text{and an iso. } \phi : f^*\alpha \simeq g^*\beta\}$$

Now, consider the fiber product

$$\begin{array}{ccc} X \times_{\mathscr{F}^2} \mathscr{F} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \Delta_{\mathscr{F}} \\ X & \xrightarrow{(\alpha,\beta)} & \mathscr{F}^2 \end{array}$$

By a 2-categorical version of Lemma 2.3.6, there is an equivalence $X^2 \times_{\mathscr{F}^2} \mathscr{F} \simeq X \times_{\mathscr{F}} X$ so that

$$X \times_{\mathscr{F}^2} \mathscr{F} \simeq X \times_{X^2} (X \times_{\mathscr{F}} X) \qquad \text{and} \qquad (X \times_{\mathscr{F}^2} \mathscr{F})(T) = \{f \in X(T), \text{and an iso } \phi : f^*\alpha \simeq f^*\beta\}.$$

so that $X \times_{\mathscr{F}^2} \mathscr{F}$ is precisely the presheaf in $\mathrm{Psh}(\mathbf{Aff})$ associated to $\mathrm{Isom}_X(a, b) \in \mathrm{Psh}(\mathbf{Aff}/X)$.

(Alternatively, the above construction tells us directly that

$$\mathrm{ob}(X \times_{\mathscr{F}^2} \mathscr{F})(T) = \left\{ \begin{array}{c} f : T \to X, \gamma \in \mathscr{F}(T) \\ \text{and an iso. } \phi : \gamma \times \gamma \simeq f^*\alpha \times f^*\beta \in \mathscr{F}(T) \times \mathscr{F}(T) \end{array} \right\}$$

with morphisms given by maps $\gamma \to \gamma'$ intertwining the isomorphisms. The inclusion of the full subcategory where $\gamma = f^*\alpha$ and ϕ is the identity on the first factor is an equivalence. This groupoid is equivalent to a set–there are no non-identity automorphisms, since the conditions force $\gamma \to \gamma'$ to be the identity map $f^*\alpha \to f^*\alpha$. Then,

$$(X \times_{\mathscr{F}^2} \mathscr{F})(T) = \{f : T \to X \text{ and an iso. } \phi : f^*\alpha \simeq f^*\beta\}$$

as before.)

[1] Convention/Technicality: A "commutative square" or "Cartesian square" in this context is only so up to certain natural transformations which I promised to ignore.↑

3.3. **Algebraic Stacks.** Once we have the fiber product we may import many definitions from the world of presheaves of sets into the world of presheaves of groupoids:

Definition 3.3.1. Suppose $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves of groupoids. We say that ϕ is *schematic*(*=relative scheme*), resp. *representable* (*=relative algebraic space*), if for all schemes X and morphisms $X \to \mathscr{G}$ the fiber product $X \times_{\mathscr{G}} \mathscr{F}$ is equivalent to a sheaf of sets represented by a scheme, resp. algebraic space. In this case, ϕ is quasi-compact/separated/étale/smooth/flat/faithfully flat/etc. iff $X \times_{\mathscr{G}} \mathscr{F} \to X$ is so (in the sense of scheme/algebraic space).

Definition 3.3.2. An *algebraic stack* is an fppf-sheaf of groupoids $\mathscr{F}$ satisfying

- The diagonal map $\Delta_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^2$ is representable; and,
- There exists a scheme U and a (representable) smooth surjection $\pi : U \to \mathscr{F}$.

As with algebraic spaces, there is the following non-trivial theorem of Artin, guaranteeing the existence of a *smooth* atlas given the existence of an *fppf* atlas:

Theorem 3.3.3 (Artin)**.** *Suppose $\mathscr{F}$ is an fppf-sheaf of groupoids. Then, $\mathscr{F}$ is an algebraic stack iff the diagonal $\mathscr{F} \to \mathscr{F}^2$ is representable, and there exists a scheme U and a (representable) faithfully flat, locally of finite-presentation, $\pi : U \to \mathscr{F}$.*

Proof. See LMB, §10, Théorème 10.1 One should be able to ignore the extra hypotheses in (ii) there with our definition. □

3.3.4. See Appendix 6 for more on equivalent variants of the definition.

3.3.5. As with algebraic spaces, there is another equivalent definition of an algebraic space, together with a generalization of the fact that schemes are fpqc-sheaves:

Proposition 3.3.6. *Suppose $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}^{\mathrm{Gpd}}(\mathbf{Aff})$ is an étale-sheaf. Then, $\mathscr{F}$ is an algebraic stack (in particular an fppf-sheaf) iff the diagonal $\mathscr{F} \to \mathscr{F}^2$ is representable, and there exists a scheme U and a (representable) smooth surjection $\pi : U \to \mathscr{F}$. If $\mathscr{F} \to \mathscr{F}^2$ is, in addition, quasi-affine, then $\mathscr{F}$ is in fact an fpqc-sheaf.*

Proof. See Appendix 6. □

3.3.7. The analog of Lemma 2.3.6 holds in this setting. That is, $\mathscr{F} \to \mathscr{F}^2$ representable(/schematic) $\Leftrightarrow$ $X \times_{\mathscr{F}} Y$ is an algebraic space(/scheme) for all schemes $X, Y \Leftrightarrow X \to \mathscr{F}$ is representable(/schematic) for all morphisms from a scheme. Using Example 3.2.4, it is not hard to show that this is also equivalent to $\mathrm{Isom}_X(\alpha, \beta)$ ($\approx X \times_{\mathscr{F}^2} \mathscr{F} \to X$) being an algebraic space(/scheme) over X for all schemes X and $\alpha, \beta \in \mathscr{F}(X)$.

3.3.8. Lemma 2.2.6 and Corollary 2.4.1 continue to hold for schematic morphisms in $\mathrm{Shv}^{\mathrm{Gpd}}(\mathbf{Aff})$. Any property which is étale-local on the source can be defined also for representable morphisms, in which case Lemma 2.2.6 extends also to that context. Similarly, Corollary 2.4.1 extends to the setting of representable morphisms. We'll leave the proper formulations and proofs to the reader, but special instances of the general argument will appear in the next section.

4. Examples

In this section, G will be an *affine algebraic group.*

4.1. *BG***.**

Definition 4.1.1. Define a presheaf of groupoids BG by

$$\mathrm{ob}\, BG(T) = \{\text{Principal } G\text{-bundles } \mathcal{P} \to T\}$$

$$\mathrm{Hom}_{BG(T)}(\mathcal{P}, \mathcal{P}') = \{G\text{-equivariant maps } \mathcal{P} \to \mathcal{P}' \text{ over } T\}$$

Lemma 4.1.2. *BG is an fppf (indeed fpqc) sheaf of groupoids.*

Proof. Using the interpretation of principal G-bundles as sheaf-torsors, this is easy. □

The rest of §4.1 will be devoted to proving that BG is an algebraic stack, and explicitly computing some pullbacks.

We begin with an important special case (recall the frowny face diagram from §3):

4.1.3. The trivial G-torsor $G \to *$ defines a map $* \to BG$. We claim that the following are Cartesian[2]

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \longrightarrow & BG \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG^2 \end{array}$$

Note that by Lemma 2.3.6 the two claims are equivalent, and by Example 3.2.4 equivalent to $\mathrm{Isom}_*(G,G) = G$. We can check this last claim on the functor of points, where it follows by noting that a G-equivariant map $G \to G$ is determined by where we send $\mathrm{id} \in G$ and so must be right translation by some $g \in G$.

More generally, the trivial G-torsor $G \times T \to T$ is always an object of $BG(T)$ and the same argument shows that $\mathrm{Isom}_T(G \times T, G \times T) \simeq G \times T$ (via the right action of $G \times T$). Note that this isomorphism depends on the choice of trivialization.

Now, we leverage this special case to the general case in a formulaic manner (compare 2.4.1).

4.1.4. Suppose T is a scheme and $\mathcal{P}, \mathcal{P}' \in BG(T)$. We will explicitly identify $T \times_{BG^2} BG$, equivalently $\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}')$, as a T-scheme.

First some notation: Set $\mathcal{E} = \mathcal{P} \times_T \mathcal{P}'$, so that $\mathcal{E}$ is a principal $G \times G$-bundle on T. The group scheme G^2 acts on the scheme G by left- and right-translation: At the functor-of-points level, $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Finally, let $(G)_\mathcal{E}$ denote the associated bundle of the principal G^2-bundle $\mathcal{E}$ via this action on G (i.e., $(G)_\mathcal{E} = (G \times \mathcal{E})/G^2$).

Claim.

$$\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}') \simeq (G)_\mathcal{E}$$

Proof Of Claim. Since both $\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}')$ and $(G)_\mathcal{E}$ are fppf-sheaves, we can construct the isomorphism flat locally. For this, consider the diagram

$$\begin{array}{ccccccc} G^2 \times \mathrm{Isom}_\mathcal{E}(\mathcal{P}|_\mathcal{E}, \mathcal{P}'|_\mathcal{E}) & \rightrightarrows & \mathrm{Isom}_\mathcal{E}(\mathcal{P}|_\mathcal{E}, \mathcal{P}'|_\mathcal{E}) & \longrightarrow & \mathrm{Isom}_T(\mathcal{P}, \mathcal{P}') & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^2 \times \mathcal{E} = \mathcal{E} \times_T \mathcal{E} & \rightrightarrows & \mathcal{E} & \longrightarrow & T & \xrightarrow[(\mathcal{P},\mathcal{P}')]{} & BG^2 \end{array}$$

Note that $\mathcal{P}|_\mathcal{E}$ and $\mathcal{P}'|_\mathcal{E}$ are trivial G-bundles, so that by 4.1.3 a choice of trivialization of each yields an isomorphism $\mathrm{Isom}_\mathcal{E}(\mathcal{P}|_\mathcal{E}, \mathcal{P}'|_\mathcal{E}) \simeq G \times \mathcal{E} (=: G_\mathcal{E})$. Thus, it suffices to compute the induced action of $G^2_\mathcal{E}$ on $\mathrm{Isom}_\mathcal{E}(\mathcal{P}|_\mathcal{E}, \mathcal{P}'|_\mathcal{E})$. This action is by changing the trivialization, and we can compute it at the level of functors-of-points: Now, G is just a group, $\mathcal{P}, \mathcal{P}'$ are two non-empty G-torsors in sets. Let $* \in \mathcal{P}$, $*' \in \mathcal{P}'$ be points (equiv. to trivializations). The identification $G = \mathrm{Isom}(\mathcal{P}, \mathcal{P}')$ is given by sending $g \in G$ to the unique G-equivariant map determined by $* \mapsto g*'$. If $(g_1, g_2) \in G^2$ acts on the trivializations by $* \mapsto g_1 *$, $*' \mapsto g_2 *$ then our isomorphism becomes $(g_1 *) \mapsto (g_1 g g_2^{-1})(g_2 *')$, as desired. □

Remark 4.1.5. If we had merely wanted to prove that $BG \to BG \times BG$ is representable, and not explicitly compute the relevant pullback, we could have avoided computing the G-action. Just the fact that there exists a flat cover $\mathcal{E} \to T$ over which $\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}')$ becomes representable is enough to give us descent data. Then, the fact that $\mathrm{Isom}_\mathcal{E}(\mathcal{P}, \mathcal{P}') \to \mathcal{E}$ was affine means that the descent data is effective, so that the sheaf $\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}')$ is itself representable (and affine over T). The extra work was in identifying precisely what that descent data is.

[2] i.e., the square commutes (up to natural transformation) and the resulting natural functor to the fiber product is an equivalence.↑

4.1.6. Suppose $\mathcal{P} \in BG(T)$. We claim that there is a Cartesian square

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & * \\ \downarrow & & \downarrow \\ T & \xrightarrow[\mathcal{P}]{} & BG \end{array}$$

We can give a few different arguments:

- We can prove this using the above

$$T \times_{BG} * = \mathrm{Isom}_T(G \times T, \mathcal{P}) = (G)_{\mathcal{P} \times_T (G \times T)} = (\mathcal{P} \times G^2)/(G^2) = \mathcal{P}.$$

- Thinking of principal bundles as sheaves, it is easy to see that $\mathrm{Isom}_T(G \times T, \mathcal{P}) = \mathrm{Sect}_T(\mathcal{P})$, the sheaf of sections of $\mathcal{P}$. But this is just $\mathcal{P}$ as a T-scheme.
- Or, we could do an argument analogous to that above by considering the diagram

$$\begin{array}{ccccccc} G^2 \times \mathcal{P} = G \times \mathcal{P} \times_T P & \rightrightarrows & G \times \mathcal{P} & \longrightarrow & T \times_{BG} * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G \times \mathcal{P} = \mathcal{P} \times_T \mathcal{P} & \rightrightarrows & \mathcal{P} & \longrightarrow & T & \xrightarrow[\mathcal{P}]{} & BG \end{array}$$

4.1.7. Since $\mathrm{Isom}_T(\mathcal{P}, \mathcal{P}') = (G)_{\mathcal{E}}$ is a scheme, $BG \to BG^2$ is schematic. Since $\mathcal{P} = T \times_{BG} * \to T$ is flat and surjective, $* \to BG$ is so. So, we've proved that BG is an algebraic stack. Moreover, we have proved that $* \to BG$ is "the universal G-torsor."

4.2. $X /\!\!/ G$.

In this section, X will be a scheme with G-action.

Definition 4.2.1. Define a presheaf of groupoids $X /\!\!/ G$ by

$$\operatorname{ob} X /\!\!/ G(T) = \left\{ \begin{array}{c} \text{Diagrams } T \leftarrow \mathcal{P} \to X \text{ where } T \leftarrow \mathcal{P} \text{ is a} \\ \text{principal } G\text{-bundle and } \mathcal{P} \to X \text{ is a } G\text{-equivariant map} \end{array} \right\}$$

$$\mathrm{Hom}_{X /\!\!/ G(T)}(\mathcal{P} \to X, \mathcal{P}' \to X) = G\text{-equivariant maps } \mathcal{P} \to \mathcal{P}' \text{ over } T \text{ and } X$$

Lemma 4.2.2. *$X /\!\!/ G$ is an fppf (indeed fpqc) sheaf of groupoids.*

Proof. Via the map $\mathcal{P} \to X \times T$, we may regard $X /\!\!/ G(T)$ as a full subcategory of the category $G\,\mathrm{Shv}_X(T)$ of G-equivariant sheaves (of sets) on $X \times T$. Since $G\,\mathrm{Shv}_X$ is a sheaf of groupoids, and the condition of being isomorphic to an object in $X /\!\!/ G$ is local, all that remains is to observe that G-bundles satisfy descent so that we don't lose effectivity of descent. □

4.2.3. Exercise. Suppose $X \to Y$ is a principal G-bundle. Show that there is a natural equivalence $X /\!\!/ G = Y$. (Hint: Apply descent to the G-equivariant map $\mathcal{P} \to X$.) So, if there is a scheme X/G such that the map $X \to X/G$ is a principal G-bundle, then $X/G = X /\!\!/ G$.

Here's a first way to prove that $X /\!\!/ G$ is representable:

4.2.4. There is a morphism $X /\!\!/ G \to BG$ determined on T-points by

$$\{T \leftarrow \mathcal{P} \to X\} \mapsto \{\mathcal{P} \to T\}$$

We claim that this map is representable (and schematic if X is quasi-projective with a G-equivariant ample line bundle), which will prove that $X /\!\!/ G$ is an algebraic stack by the following Lemma:

Lemma 4.2.5. *Suppose $\phi : \mathscr{F} \to \mathscr{G}$ is a representable map of sheaves of groupoids and that $\mathscr{G}$ is an algebraic stack. Then, $\mathscr{F}$ is an algebraic stack.*

To prove that $X/\!\!/G \to BG$ is representable, we will prove that for any $\mathcal{P} \in BG(T)$ there is a Cartesian square

$$\begin{array}{ccc} (X)_{\mathcal{P}} & \longrightarrow & X/\!\!/G \\ \downarrow & & \downarrow \\ T & \xrightarrow[\mathcal{P}]{} & BG \end{array}$$

This suffices since $(X)_{\mathcal{P}}$ is always an algebraic space, because algebraic spaces have effective fppf descent. If X is quasi-projective with a G-equivariant ample line bundle, then $(X)_{\mathcal{P}}$ will be a scheme.
Step 1: (The case of trivial bundles.) We claim that $\mathcal{P} \times_{BG} X/\!\!/G = X \times \mathcal{P}$. We may reduce this to showing $* \times_{BG} X/\!\!/G = X$. But,

$$(* \times_{BG} X/\!\!/G)(T) = \{G\text{-equivariant diagram } T \leftarrow \mathcal{P} \to X \text{ and a } G\text{-equivt. } T\text{-iso } \mathcal{P} \simeq G \times T\} = X(T)$$

where the last equality is by descent theory (identifying G-equivariant maps $G \times T \to X$ with maps $T \to X$.)
Step 2: (Gluing.) We combine step 1 with descent theory in the form of the diagram

$$\begin{array}{ccccccc} G \times (X \times \mathcal{P}) & \rightrightarrows & X \times \mathcal{P} & \longrightarrow & T \times_{BG} X/\!\!/G & \longrightarrow & X/\!\!/G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G \times \mathcal{P} = \mathcal{P} \times_T \mathcal{P} & \rightrightarrows & \mathcal{P} & \longrightarrow & T & \xrightarrow[\mathcal{P}]{} & BG \end{array}$$

We can compute that the G-action on $X \times \mathcal{P}$ is the expected one, so that $T \times_{BG} X/\!\!/G = (X \times \mathcal{P})/G = (X)_{\mathcal{P}}$.

Here's a sketch of a second (more direct) way to prove that $X/\!\!/G$ is representable:

4.2.6. Let $\alpha : G \times X \to X$ be the action map. Let G act on $G \times X$ by $g_1 \cdot (g, x) = (g_1 g, x)$. Then, $p_2 : G \times X \to X$ is a G-bundle while α is G-equivariant, so that

$$X \xleftarrow{\alpha} G \times X \xrightarrow{p_2} X$$

defines an element in $X/\!\!/G(X)$. One can check that there is a resulting Cartesian square

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ X & \longrightarrow & X/\!\!/G \end{array}$$

Then, a descent argument which the reader can imagine shows that given $(f : \mathcal{P} \to X) \in X/\!\!/G(T)$, there is a Cartesian square

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & X/\!\!/G \end{array}$$

This shows that $X \to X/\!\!/G$ is schematic and fppf. Finally, a second descent argument shows that there is a Cartesian square

$$\begin{array}{ccc} (\mathcal{P} \times_X \mathcal{P}')/G & \longrightarrow & T' \\ \downarrow & & \downarrow {\scriptstyle \mathcal{P}' \to X} \\ T & \xrightarrow[\mathcal{P} \to X]{} & X/\!\!/G \end{array}$$

where $(\mathcal{P} \times_X \mathcal{P}')/G$ is the (unique by descent) algebraic space over which $\mathcal{P} \times_X \mathcal{P}'$ is a principal G-bundle (definition left to the reader), which is a scheme if X is quasi-projective with a G-equivariant ample line bundle.

4.2.7. (Exercise) Suppose X, X' are schemes with G-actions, $f : X' \to X$ a G-equivariant morphism, and $(\mathcal{P} \to X) \in X/\!\!/G(T)$. Show that if $X' \to X$ is quasi-projective with a G-equivariant ample line bundle, then $(X' \times_X \mathcal{P})/\!\!/G$ is (equivalent to) a scheme. (Hint: Use Exercise 4.2.3.) Setting $X' = \mathcal{P}'$ (and noting that the hypothesis is satisfied, since this can be checked flat locally on T) we get the case of $(P' \times_X \mathcal{P})/G$ used above.

4.3. **Generalizations: Change of Group/Space.**

4.3.1. Suppose $G' \hookrightarrow G$ is a monomorphism of algebraic groups. There is a functor $G'(T) \to G(T)$ given by $G'(T) \ni \mathcal{P} \mapsto (G)_{\mathcal{P}} \in G(T)$.

(Exercise) There is a Cartesian square

$$\begin{array}{ccc} (G/G')_{\mathcal{P}} & \longrightarrow & BG' \\ \downarrow & & \downarrow {\scriptstyle (G)_{(-)}} \\ T & \xrightarrow[\mathcal{P}]{} & BG \end{array}$$

So $BG' \to BG$ is representable. If G/G' is quasi-projective with a G-equivariant ample line bundle, then it is schematic.

4.3.2. Suppose $f : X' \to X$ is a morphism of G-schemes. There is a natural functor $X/\!\!/G \to X'/\!\!/G$

$$X'/\!\!/G(T) \ni \{T \leftarrow \mathcal{P} \to X'\} \longmapsto \{T \leftarrow \mathcal{P} \to X' \xrightarrow{f} X\} \in X/\!\!/G(T)$$

Exercise. There are Cartesian squares

$$\begin{array}{ccc} X' & \longrightarrow & X'/\!\!/G \\ \downarrow & & \downarrow \\ X & \xrightarrow[\mathcal{P}\to X]{} & X/\!\!/G \end{array} \qquad \text{and more generally} \qquad \begin{array}{ccc} (X' \times_X \mathcal{P})/G & \longrightarrow & X'/\!\!/G \\ \downarrow & & \downarrow \\ T & \xrightarrow[\mathcal{P}\to X]{} & X/\!\!/G \end{array}$$

So, $X/\!\!/G \to X'/\!\!/G$ is representable and is schematic if $X' \to X$ is quasi-projective with a G-equivariant ample line bundle.

Example 4.3.3.

- This obviously includes the map $X/\!\!/G \to BG = */\!\!/G$ worked out above.
- Descent provides an equivalence $* = G/\!\!/G$, so that this result includes the map $* \to G$ worked out above.
- Descent provides an equivalence $X = (X \times G)/\!\!/G$, so that this result includes the map $X \to X/\!\!/G$ worked out above.
- Descent provides an equivalence $BG = G/\!\!/G^2$, so that this result includes the map $BG \to BG^2$ worked out above.
- Descent provides an equivalence $BG' = G/\!\!/(G' \times G) = (G/G')/\!\!/G$, so that this result includes the map $BG' \to BG = */\!\!/G$ mentioned above.

5. Appendix: Descent for monomorphisms

We begin by recalling certain descent results we're happy to take for granted:

Lemma 5.0.4. *Let S be a scheme, and X an S-scheme. Define presheaves of categories* $\mathrm{Shv}^{fppf}_{X/S}, \mathrm{Shv}^{fpqc}_{X/S} : \mathrm{Psh}(\mathbf{Sch}/S) \to \mathrm{Cat}$ *by*

$$\mathrm{Shv}^{fppf}_{X/S}(T) = \{\textit{The category of fppf-sheaves of sets on } X_T = X \times_S T\}$$

$$\mathrm{Shv}^{fpqc}_{X/S}(T) = \{\textit{The category of fpqc-sheaves of sets on } X_T = X \times_S T\}$$

Then, $\mathrm{Shv}^{fppf}_{X/S}$ *satisfies fppf descent. Analogously,* $\mathrm{Shv}^{fpqc}_{X/S}$ *satisfies fpqc descent.*

Proof. Somewhere. □

Lemma 5.0.5. *Let S be a scheme. Define presheaves of categories* $\mathrm{Open}_S, \mathrm{QAff}_S, \mathrm{QFin}_S^{qc}, \mathrm{Mono}_S^{qc} \subset \mathrm{Shv}_{S/S}$ *to be the full subcategories*

$$\mathrm{ob}\,\mathrm{Open}_S(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is an open immersion}\}.$$

$$\mathrm{ob}\,\mathrm{Mono}_S^{qc}(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is a quasi-compact monomorphism}\}.$$

$$\mathrm{ob}\,\mathrm{QFin}_S^{qc}(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is quasi-compact, separated, and locally quasi-finite}\}.$$

$$\mathrm{ob}\,\mathrm{QAff}_S(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is quasi-affine}\}.$$

Then, each of these satisfies fpqc descent.

Proof. (These should be in SGA I, Exp. VIII, but I've included a mini-sketch.) By Lemma 5.0.4, it suffices to prove effectivity of descent.

Any monomorphism is separated (the diagonal is an isomorphism, hence closed), and universally injective hence locally quasi-finite. Similarly, any étale morphism is separated and locally quasi-finite. So, descent for $\mathrm{Mono}_S^{\mathrm{qc}}$ follows from descent for QFin_S.

By Zariski's Main Theorem, any quasi-compact, separated, locally quasi-finite morphism is quasi-affine. So, descent for QAff_S implies descent for QFin_S. To prove descent for Open_S, one uses that an fpqc map is topologically a surjective quotient map.

To prove descent for QAff_S, one uses that the factorization $X_T \to \mathrm{Spec}_T\, f_*\mathcal{O}_{X_T} \to T$ of a (quasi-compact) quasi-affine map as a (quasi-compact) open immersion followed by a finite map commutes with fppf base-change (since formation of relative-Spec, and pushforward along a quasi-compact quasi-separated morphism do). This reduces one to the case of affine morphisms (handled using QCoh) and quasi-compact open immersions (handled as open immersions). □

Now, we carry out a trick which apparently comes from a 1971 paper of Raynaud & Gruson, and allows us to remove quasi-compactness hypotheses on the morphisms at the cost of passing to fppf (rather than fpqc) descent. This ingredient is missing in the treatment of algebraic spaces in Knutson's book, and LMB, and I learned of it from De Jong's online Stacks project.

Lemma 5.0.6. *Let S be a scheme. Define a presheaf of categories* $\mathrm{Mono}_S \subset \mathrm{Shv}_{S/S}$ *to be the full subcategory*

$$\mathrm{ob}\,\mathrm{Mono}_S(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is a monomorphism}\}.$$

Then, Mono_S *satisfies fppf descent.*

Proof. By the above lemma, all that we must show is effectivity of descent. By standard nonsense (being able to check monomorphism fppf locally) it suffices to show the following: Suppose $\mathscr{F} \in \mathrm{Shv}(S)$ is such that $\mathscr{F} \to S$ is a monomorphism and $\mathscr{F}_T$ is representable by a T-scheme; then $\mathscr{F}$ is representable by an S-scheme. For this, it suffices to write $\mathscr{F}$ as a union of subsheaves $\mathscr{V}_\alpha$ each of which is a scheme and such that $(V_\alpha)_T \to \mathscr{F}_T$ is an open immersion. Since schemes have effective Zariski descent, by a standard reduction we may assume S, T affine and $T \to S$ an fppf map.

We begin with the key claim:

Claim 5.0.7 (Raynaud-Gruson)**.** *Suppose $\mathscr{F} \in \mathrm{Shv}(S)$, that $T \to S$ is a quasi-compact and universally open morphism of schemes, and that $\mathscr{F}_T$ is a T-scheme. Suppose further that $U \subset \mathscr{F}_T$ is an open subscheme. Then, there exists a subsheaf $\mathscr{V} \subset \mathscr{F}$ such that $\mathscr{V}_T$ is a scheme and $U \subset \mathscr{V}_T \subset \mathscr{F}_T$. If U is quasi-compact, then $\mathscr{V}$ may be chosen so that $\mathscr{V}_T$ is quasi-compact.*

Supposing the claim for the moment, let us complete the proof: Write $\mathscr{F}_T = \bigcup_\alpha U_\alpha$ as a union of affine subschemes. Applying the first claim, we obtain subfunctors $\mathscr{V}_\alpha \subset \mathscr{F}$ such that $\mathscr{F}_T = \bigcup_\alpha (V_\alpha)_T$ and with $(\mathscr{V}_\alpha)_T \subset \mathscr{F}_T$ open quasi-compact sub-schemes. Surjectivity of sheaf maps is local, so that this implies $\mathscr{F} = \bigcup_\alpha V_\alpha$. Now, $(V_\alpha)_T$ is quasi-compact and T is separated (recall: assumed affine), so that $(V_\alpha)_T \to T$ is a quasi-compact. Quasi-compact monomorphisms have effective descent, by Lemma 5.0.5, so that $\mathscr{V}_\alpha$ is a scheme and $V_\alpha \to \mathscr{F}$ is an open subfunctor since $(V_\alpha)_T \to \mathscr{F}_T$ is an open immersion. □

Proof of Claim 5.0.7. Let $p_1, p_2 : \mathscr{F}_{T\times_S T} = \mathscr{F}_T \times_T (T \times_S T) \to \mathscr{F}_T$ be the two projections, which are quasi-compact open maps as they are base-changes of $T \to S$. The key observation is that $\mathscr{G} \subset \mathscr{F}_T$ has (automatically unique) descent data for $T \to S$ compatible with that on $\mathscr{F}_T$ iff $p_1^*\mathscr{G} = p_2^*\mathscr{G}$ as subsheaves of $\mathscr{F}_{T\times_S T}(\simeq p_1^*\mathscr{F}_T \simeq p_2^*\mathscr{F}_T)$. So by Lemma 5.0.4 it suffices to construct an open subscheme $V' \subset \mathscr{F}_T$ containing U and satisfying $p_1^{-1}V' = p_2^{-1}V' \subset \mathscr{F}_{T\times_S T}$, thus obtaining the desired sheaf $\mathscr{V}$ by descent.

If $U \subset \mathscr{F}_T$ is an open subscheme, then $V' = p_2(p_1^{-1}U)$ is an open subset of $\mathscr{F}_T$ (since p_2 is an open map). Then, $U \subset V'$ by virtue of the diagonal($/S$) morphism $U \to U \times_S T = p_1^{-1}U$, and $p_1^{-1}(V') = p_2^{-1}(V')$.

Moreover, since $p^{-1}U \to U$ is quasi-compact, we see that $p^{-1}U$ is quasi-compact when U is; then V' is quasi-compact as well, being its image. □

The same proof also shows the slightly stronger

Lemma 5.0.8 (De Jong's online Stacks project, More Morphisms, Lemma 11.1)**.** *Let S be a scheme. Define a presheaf of categories $\mathrm{QFin}_S^{loc,sep} \subset \mathrm{Shv}_{S/S}$ to be the full subcategory*

$$\mathrm{ob}\,\mathrm{QFin}_S^{loc,sep}(T) = \{T\text{-schemes } X_T \text{ for which } X_T \to T \text{ is separated and locally quasi-finite}\}.$$

Then, $\mathrm{QFin}_S^{loc,sep}$ satisfies fppf descent.

6. Appendix: Equivalent definitions of algebraic space/stack

6.1. Algebraic Spaces.

For algebraic spaces, we have the following four equivalent definitions:

Definition 6.1.1. An *algebraic space* is an étale-sheaf $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(b) There exists a scheme U and a (schematic) étale surjection $\pi : U \to \mathscr{F}$.

Definition 6.1.2. An *algebraic space* is an étale-sheaf $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(b) There exists a scheme U and a (schematic) smooth surjection $\pi : U \to \mathscr{F}$.

Definition 6.1.3 (Defn. 2.3.2)**.** An *algebraic space* is an fppf-sheaf $\mathscr{F} \in \mathrm{Shv}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(b) There exists a scheme U and a (schematic) étale surjection $\pi : U \to \mathscr{F}$.

Definition 6.1.4. An *algebraic space* is an fppf-sheaf $\mathscr{F} \in \mathrm{Shv}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is schematic;
(b) There exists a scheme U and a (schematic) fppf surjection $\pi : U \to \mathscr{F}$.

The equivalence of the last two was asserted earlier, as "Artin's Theorem" (2.3.4). This is slightly misleading, or at least at odds with standard naming of things. The result actually factors into the real Artin's Theorem(3.3.3) — telling us that an algebraic space in the fourth sense admits a *smooth* atlas — and the following fact (which those familiar with Deligne-Mumford stacks will appreciate as being useful):

Proposition 6.1.5. *Suppose $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}^{\mathrm{Gpd}}(\mathbf{Aff})$ is an étale-sheaf of groupoids satisfying the conditions of Defn. 6.2.1: i.e., $\mathscr{F} \to \mathscr{F}^2$ is representable and there exists a scheme U and a smooth surjection $\pi : U \to \mathscr{F}$. Suppose further that the diagonal $\mathscr{F} \to \mathscr{F}^2$ is* formally unramified*. Then, there exists a scheme V and an* étale *surjection $\pi' : V \to \mathscr{F}$.*

Proof. This is LMB, Théorème 8.1 (with slightly different hypotheses). The key idea is that the formally unramified diagonal hypothesis allows us to conclude that there is a surjection $\Omega_U \twoheadrightarrow \Omega_{U/\mathscr{F}}$, where the latter is locally free since π is smooth. This lets us immitate the usual proof that smooth morphisms (of schemes) have étale-local sections around any given point in the base to show that for any field point $y : \mathrm{Spec}\, k \to \mathscr{F}$ there is a scheme $V(y)$ and an étale map $V(y) \to \mathscr{F}$ such that $V(y) \times_{\mathscr{F}} (\mathrm{Spec}\, k) \neq \emptyset$. Taking a disjoint union over enough field points of $\mathscr{F}$, we can make the resulting map surjective. □

Now, the fact that the first two definitions become equivalent to the rest follows from the following Proposition, which also generalizes the statement that schemes are fpqc-sheaves:

Proposition 6.1.6. *Suppose $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}(\mathbf{Aff})$ is an étale-sheaf satisfying the conditions of Defn. 6.1.2: i.e., $\mathscr{F} \to \mathscr{F}^2$ is schematic, and there exists a scheme U and a smooth surjection $\pi : U \to \mathscr{F}$. Then, $\mathscr{F}$ is an fppf-sheaf. If $\mathscr{F} \to \mathscr{F}^2$ is quasi-compact, then it is in fact an fpqc-sheaf.*

Proof Sketch. Let $\mathscr{F}'$ be the fppf-sheafification of $\mathscr{F}$. The natural map $\mathscr{F} \to \mathscr{F}'$ is a monomorphism (Exercise!), and it suffices to show that it is an epimorphism of étale-sheaves. For this, it suffices to show the composite $U \to \mathscr{F}'$ is an epimorphism of étale-sheaves, and for this it suffices to show that it is schematic (Exercise: A schematic smooth surjection is an epimorphism of étale-sheaves.). In light of Lemma 2.3.6, it suffices to show that $\mathscr{F}'$ has representable diagonal. Since $\mathscr{F}' \to (\mathscr{F}')^2$ is a monomorphism, and $U \to \mathscr{F}'$ an epimorphism of fppf-sheaves, it suffices by Lemma 2.3.8 to consider the Cartesian diagram

$$\begin{array}{ccc} U \times_{\mathscr{F}'} U & \longrightarrow & \mathscr{F}' \\ \downarrow & & \downarrow \\ U^2 & \longrightarrow & (\mathscr{F}')^2 \end{array}$$

and to observe that since $\mathscr{F} \to \mathscr{F}'$ is a monomorphism $U \times_{\mathscr{F}'} U = U \times_{\mathscr{F}} U$, and the latter is known to be a scheme.

If $\mathscr{F} \to \mathscr{F}^2$ is quasi-compact, then an analogous argument with the fpqc-sheafification $\mathscr{F}''$ and fpqc descent for quasi-compact monomorphisms (Lemma 5.0.5) proves that $\mathscr{F} = \mathscr{F}''$. ☐

6.2. **Algebraic Stacks.** For algebraic stacks, we have the following three equivalent definitions:

Definition 6.2.1. An *algebraic stack* is an étale-sheaf of groupoids $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}^{\mathrm{Gpd}}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is representable;
(b) There exists a scheme U and a (representable) smooth surjection $\pi : U \to \mathscr{F}$.

Definition 6.2.2 (Defn. 3.3.2)**.** An *algebraic stack* is an fppf-sheaf of groupoids $\mathscr{F} \in \mathrm{Shv}^{\mathrm{Gpd}}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is representable;
(b) There exists a scheme U and a (representable) smooth surjection $\pi : U \to \mathscr{F}$.

Definition 6.2.3. An *algebraic stack* is an fppf-sheaf of groupoids $\mathscr{F} \in \mathrm{Shv}^{\mathrm{Gpd}}(\mathbf{Aff})$ such that

(a) $\mathscr{F} \to \mathscr{F}^2$ is representable;
(b) There exists a scheme U and a (representable) fppf surjection $\pi : U \to \mathscr{F}$.

The equivalence of the last two is Artin's Theorem (3.3.3). That the first definition is also equivalent to these follows from the following Proposition, generalizing the previous one:

Proposition 6.2.4. *Suppose $\mathscr{F} \in \mathrm{Shv}_{\mathrm{et}}^{\mathrm{Gpd}}(\mathbf{Aff})$ is an étale-sheaf of groupoids satisfying the conditions of the first definition above: i.e., $\mathscr{F} \to \mathscr{F}^2$ is representable, and there exists a scheme U and a smooth surjection $\pi : U \to \mathscr{F}$. Then, $\mathscr{F}$ is an fppf-sheaf. If $\mathscr{F} \to \mathscr{F}^2$ is quasi-affine, then it in fact an fpqc-sheaf.*

Proof Sketch. The proof is analogous to that of Prop. 6.1.6, using Corollary 2.4.7 in place of Lemma 2.3.8. ☐