SEMINAR NOTES: HIGGS BUNDLES, KOSTANT SECTION, AND LOCAL TRIVIALITY OF G-BUNDLES (OCT. 27, 2009)

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1. Local triviality (Steinberg's theorem)

1.1. Let k be an algebraically closed field of char. 0, and let G be a connected affine algebraic group over k.

The goal of this talk is to prove the following theorem:

Theorem 1.1.1. (Steinberg) *Let K be a field of rational functions of an algebraic curve over k. Then any G-bundle over K is trivial.*

In particular, this implies the following:

Corollary 1.1.2. *Let X be a smooth curve over k. Then any G-bundle on X admits a reduction to B (the Borel subgroup).*

Proof. Choose a reduction to B at the generic point of X, which is possible by Theorem 1.1.1.

Exercise 1.1.3. *Use the valuative criterion of properness to show that this reduction extends onto the entire curve.*

□

Corollary 1.1.4. *Let X be a smooth curve over k. Then any G-bundle on X is locally trivial in the Zariski topology.*

Exercise 1.1.5. *Deduce it from the previous corollary.*

1.2. Later in the seminar we'll see that Theorem 1.1.1 can be strengthened as follows:

Theorem 1.2.1. (Drinfeld-Simpson) *Let X be a complete curve over some field k. Let P_G be a G-bundle on $S \times X$, where S is a k-scheme. (For point* (3) *let $x \in X$ be a k-point.) Then there exists an etale base change $S' \to S$, such that the pull-back P'_G of P_G to $S' \times X$ satisfies:*
(1) *P'_G admits a reduction to B.*
(2) *The restriction of P'_G to $S' \times \mathrm{Spec}(K)$ is locally trivial in the Zariski topology.*
(3) *If the radical of G is unipotent, then the restriction of P'_G to $S' \times (X - x)$ is trivial.*

1.3. Let us briefly indicate the general strategy of the proof of Theorem 1.1.1. Evidently, we can (and from now on we will) assume that G is reductive (since a G-bundle with G unipotent is trivial on any affine scheme).

The main step, which is valid for any field K, is that given a G-bundle on $\mathrm{Spec}(K)$, we can always find its reduction to a certain group subscheme $J_K \subset G_K := G \times \mathrm{Spec}(K)$, such that J_K is a non-split torus, i.e., after an etale base change $K \to K'$, we have $J_K \underset{K}{\otimes} K' \simeq (\mathbb{G}_m)^{\times r}$ for some integer r.

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Having such a reduction, we'll show that when K is as in the theorem and J_K as above, any J_K-torsor over $\mathrm{Spec}(K)$ is trivial. This would be an easy consequence of Tsen's theorem.

Thus, we have to find J_K, and obtain a reduction. This will be done using a geometric device known as Higgs bundles.

2. Higgs bundles

2.1. Let Y be any scheme. We introduce a new functor $\mathbf{Sch}^{op}_{/k} \to \mathbf{Groupoids}$, denoted $\mathrm{Higgs}(Y)$ that assigns to a scheme S the category of pairs (P_G, f), where P_G is a G-bundle on $S \times Y$, and f is a section of the associated bundle $\mathfrak{g}_{P_G}$ of Lie algebras.

We call points of $\mathrm{Higgs}(Y)$ "Higgs bundles on Y", or "G-bundles on Y with a Higgs field", the latter being the data of f.

Exercise 2.1.1. *Show that* $\mathrm{Higgs}(Y)$ *is nothing but* $\mathrm{Maps}(Y, \mathfrak{g}/G)$.[1]

Notation $\mathrm{Maps}(Y, -)$ in the above exercise is as in [Sept17].

Let $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$ be the open subscheme of regular elements (recall that an element of $\mathfrak{g}$ is called regular, if the dimension of its centralizer equals the rank of $\mathfrak{g}$).

We let $\mathrm{Higgs}^{\mathrm{reg}}(Y)$ be the corresponding subfunctor of $\mathrm{Higgs}(Y)$. Let $\overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}^{\mathrm{reg}}$ be the subset of regular semi-simple elements. We let $\overset{\circ}{\mathrm{Higgs}}(Y)$ be the corresponding subfunctor of $\mathrm{Higgs}^{\mathrm{reg}}(Y)$. We'll call points of $\mathrm{Higgs}^{\mathrm{reg}}(Y)$ (resp., $\overset{\circ}{\mathrm{Higgs}}(Y)$) regular (resp., regular semi-simple) Higgs bundles.

Exercise 2.1.2. *Show that if Y is a proper scheme, then the maps*

$$\overset{\circ}{\mathrm{Higgs}}(Y) \hookrightarrow \mathrm{Higgs}^{\mathrm{reg}}(Y) \hookrightarrow \mathrm{Higgs}(Y)$$

are open embeddings of functors.

2.2. Let $\mathfrak{c}$ denote the Chevalley space, i.e., $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^*)^G)$, the GIT quotient of $\mathfrak{g}$ by G, i.e.,

$$\mathfrak{c} = \mathfrak{g}//G \simeq \mathfrak{t}//W,$$

where $\mathfrak{t}$ is the Cartan subalgebra and W is the Weyl group. (As was mentioned several times at the seminar, $\mathfrak{c}$ is actually isomorphic to the affine space $\mathbb{A}^r$, where r is the rank of $\mathfrak{g}$.)

Let ϖ denote the Chevalley map $\mathfrak{g} \to \mathfrak{c}$. Recall (Kostant's theorem) that ϖ is flat, and its restriction to $\mathfrak{g}^{\mathrm{reg}}$ is smooth.

Let $\overset{\circ}{\mathfrak{c}} \subset \mathfrak{c}$ be the open subscheme equal to the image of $\overset{\circ}{\mathfrak{g}}$ under $\mathfrak{c}$. We call the closed subset $\mathfrak{c} - \overset{\circ}{\mathfrak{c}}$ the discriminant locus.

Exercise 2.2.1. *Take $G = GL_n$. Identify $\mathfrak{c}$ with the variety of monic polynomials of degree n, and explain the terminology "discriminant locus".*

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[1] Here and elsewhere, the notation Z/H means "the stack-theoretic quotient". This is to distinguish it from the GIT quotient $Z//H$, which for Z affine means $\mathrm{Spec}(\Gamma(Z, \mathcal{O}_Z)^H)$.

2.2.2. For Y as above we set $\mathrm{Hitch}(Y)$ be the functor $\mathbf{Sch}^{op}_{/k} \to \mathbf{Sets}$ that we earlier denoted $\mathrm{Maps}(Y, \mathfrak{c})$. I.e., $\mathrm{Hom}(S, \mathrm{Hitch}(Y)) = \mathrm{Hom}(S \times Y, \mathfrak{c})$.

Let $\overset{\circ}{\mathrm{Hitch}}(Y)$ be the subfunctor corresponding to maps to $\overset{\circ}{\mathfrak{c}}$.

If Y is a scheme, we'll consider another subfunctor, denoted $\mathrm{Hitch}^\sharp(Y) \subset \mathrm{Hitch}(Y)$ that corresponds to those maps $S \times Y \to \mathfrak{c}$, such that for any point $s \in S$, the corresponding map $Y_s \to \mathfrak{c}$ generically maps to $\overset{\circ}{\mathfrak{c}}$ (i.e., the preimage of $\overset{\circ}{\mathfrak{c}}$ is a dense subset). We won't use $\mathrm{Hitch}^\sharp(Y)$ in this talk, but it will be important for the next one describing the work of Ngo.

2.2.3. The map $\varpi : \mathfrak{g} \to \mathfrak{c}$ factors through a map $\varpi/G : \mathfrak{g}/G \to \mathfrak{c}$, and hence gives rise to a map $h : \mathrm{Higgs}(Y) \to \mathrm{Hitch}(Y)$, which we'll refer to as the Hitchin map.

For a fixed k-point $\sigma \in \mathrm{Hitch}(Y)$, we let $\mathrm{Higgs}(Y)_\sigma$ be its preimage in $\mathrm{Higgs}(Y)$, i.e.,

$$\mathrm{pt} \underset{\mathrm{Hitch}(Y)}{\times} \mathrm{Higgs}(Y).$$

2.2.4. Since

$$\begin{array}{ccc} \overset{\circ}{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ \overset{\circ}{\mathfrak{c}} & \longrightarrow & \mathfrak{c} \end{array}$$

is Cartesian, so is

$$\begin{array}{ccc} \overset{\circ}{\mathrm{Higgs}}(Y) & \longrightarrow & \mathrm{Higgs}(Y) \\ \downarrow & & \downarrow \\ \overset{\circ}{\mathrm{Hitch}}(Y) & \longrightarrow & \mathrm{Hitch}(Y). \end{array}$$

In addition, we'll denote by $\mathrm{Higgs}^\sharp(Y)$ the pull-back of $\mathrm{Hitch}^\sharp(Y)$ under h.

2.2.5. *Twisting.* We'll now discuss variants of the above constructions in the presence of a line bundle. So, let $\mathfrak{L}$ be a line bundle on Y. Again, we won't need this for the purposes of proving Theorem 1.1.1, but we'll need it for the next talk.

Note that both $\mathfrak{g}$ and $\mathfrak{c}$ acted on by $\mathbb{G}_m$ with the map ϖ being equivariant. Therefore, we can make sense of $\mathrm{Higgs}_{\mathfrak{L}}(Y)$ so that

$$\mathrm{Hom}(S, \mathrm{Higgs}_{\mathfrak{L}}(Y)) = (P_G, f \in \Gamma(S \times Y, \mathfrak{g}_{P_G} \otimes \mathfrak{L})).$$

We shall denote by

$$\overset{\circ}{\mathrm{Higgs}}_{\mathfrak{L}}(Y) \subset \mathrm{Higgs}^{\mathrm{reg}}_{\mathfrak{L}}(Y) \subset \mathrm{Higgs}_{\mathfrak{L}}(Y)$$

the corresponding subfunctors.

We define $\mathrm{Hitch}_{\mathfrak{L}}(Y)$ by

$$\mathrm{Hom}(S, \mathrm{Hitch}_{\mathfrak{L}}(Y)) = \mathrm{Hom}_Y(S \times Y, \mathfrak{c}_{\mathfrak{L}}),$$

where $\mathfrak{c}_{\mathfrak{L}}$ is the twist of $\mathfrak{c}$ by $\mathfrak{L}$, i.e., $\mathfrak{c} \overset{\mathbb{G}_m}{\times} (\mathfrak{L} - 0)$, where $\mathfrak{L} - 0$ is the $\mathbb{G}_m$-torsor over Y corresponding to $\mathfrak{L}$.

Note that when Y is a complete smooth curve X, and $\mathfrak{L} := \Omega_X$, we have an isomorphism

$$\mathrm{Higgs}_\Omega(X) \simeq T^* \mathrm{Bun}_G,$$

once we choose a $G \times \mathbb{G}_m$-invariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$.

As above, we have a map $h_{\mathcal{L}} : \mathrm{Higgs}_{\mathcal{L}}(Y) \to \mathrm{Hitch}_{\mathcal{L}}(Y)$. For Y being a complete smooth curve X and $\mathcal{L} := \Omega_X$, the map $h_{\mathcal{L}}$ is the Hitchin map discussed in the previous talks.

2.3. Let us explain what is the relevance of Higgs bundles to the proof of Theorem 1.1.1. The idea is that if for a given G-bundle P_G on Y we supplement it with a structure of regular Higgs bundle, i.e., Higgs field $f \in \Gamma(Y, \mathfrak{g}_{P_G} \otimes \mathcal{L})$, so that $(P_G, f) \in \mathrm{Higgs}^{\mathrm{reg}}(Y)$, this would allow to reduce P_G to a commutative group sub-scheme J_Y of $G_Y := G \times Y$. If moreover, (P_G, f) is regular semi-simple, then the group-scheme J_Y is etale-locally isomorphic to a torus. This will allow to carry out the main step in the proof of Theorem 1.1.1, see Section 1.3.

In order to see how a structure of Higgs bundle on a given P_G allows to obtain such a reduction, we shall first consider the case of $G = GL_n$. However, of course, in this case Theorem 1.1.1 is evident, as it is just Hilbert's 90 (a.k.a. faithfully flat descent for vector bundles).

3. The case of $G = GL_n$

3.1. For GL_n, we have $\mathfrak{c} = (\mathbb{A}^1)^{(n)}$ the symmetric power of $\mathbb{A}^1$. Let $\mathfrak{c}' \to \mathfrak{c}$ be the canonical n-sheeted cover [2] of $\mathfrak{c}$. I.e., if $\mathfrak{c} \simeq \mathrm{Spec}\left((k[a_1, ..., a_n])^{S_n}\right)$, then

$$\mathfrak{c}' := \mathrm{Spec}\left(k[a_1, ..., a_n, b])^{S_n} / \prod_{i=1,\dots,n} (b - a_i)\right).$$

3.1.1. Let us fix a k-point $\sigma \in \mathrm{Hitch}(Y)$, i.e., a map $Y \to \mathfrak{c}$. Set

$$Y' := Y \underset{\mathfrak{c}}{\times} \mathfrak{c}'.$$

We call Y' "the spectral cover" of Y corresponding to σ. Let p denote the map $Y' \to Y$.

By the definition of $\mathfrak{c}'$, we have:

Proposition 3.1.2. *For a scheme S the groupoid $\mathrm{Hom}(S, \mathrm{Higgs}^{\mathrm{reg}}(Y)_\sigma)$ is equivalent to that of line bundles on $S \times Y'$.*

Exercise 3.1.3. *Deduce this proposition from the Cayley-Hamilton theorem.*

3.1.4. To explain the terminology "spectral cover" assume that $\sigma \in \overset{\circ}{\mathrm{Hitch}}(Y)$; in this case Y'_σ is etale over Y.

Let $(M, f : M \to M)$ be a point in $\mathrm{Higgs}(Y)_\sigma$. Let y be a k-point of Y.

Exercise 3.1.5. *Deduce from Prop 3.1.2 that the set $p^{-1}(y)$ identifies with the set of eigenvalues of $f_y : M_y \to M_y$.*

3.2. Note that we can rephrase the above exercise as follows: if for a fixed M we have chosen an $f : M \to M$ corresponding to a $\sigma \in \overset{\circ}{\mathrm{Hitch}}(Y)$, then for each $y \in Y$, we can canonically decompose the fiber M_y into a direct sum of 1-dimensional subspaces (the eigenspaces of f_y). However, this decomposition is *unordered*. In particular, we cannot do it globally over M: we don't know which line bundle is the first and which is the second, and so on.

For a vector bundle M, to decompose it as a direct sum of line bundles $M \simeq M_1 \oplus ... \oplus M_n$ means to reduce its structure group from GL_n to its Cartan subgroup $\mathbb{G}_m \times ... \times \mathbb{G}_m$.

However, what does our ability to decompose every fiber of M into an unordered sum of lines mean?

[2] By a cover here we mean a finite flat (but necessarily etale) map

3.2.1. We claim that the data of an etale spectral cover $Y' \to Y$ defines a group subscheme $J_Y \subset (GL_n)_Y = GL_n \times Y$, such that etale-locally J_Y is isomorphic to the Cartan group T_Y. And we claim that $M \in \mathrm{Higgs}(Y)_\sigma$ admits a canonical reduction to this group subscheme.

3.2.2. Namely, by Prop 3.1.2, the structure sheaf $\mathcal{O}_{Y'}$ gives rise to a k-point of $\mathrm{Higgs}(Y)_\sigma$, which we shall denote by (M^0_σ, f^0_σ). Note also that by construction M^0_σ is the trivial vector bundle M^0.

We define J_Y as follows: for a scheme Y_1 over Y,

$$\mathrm{Hom}_Y(Y_1, J_Y) := \mathrm{Aut}((M^0, f^0_\sigma)_{Y_1}) \subset \mathrm{Aut}(M^0)_{Y_1}.$$

Now, for $(M, f) \in \mathrm{Hom}(S, \mathrm{Higgs}(Y)_\sigma)$, the desired reduction to J_Y is given by:

$$\mathrm{Isom}_{S\times Y}\left((M^0, f^0_\sigma), (M, f)\right) \subset \mathrm{Isom}_{S\times Y}\left(M^0, M\right).$$

3.2.3. Let us now describe the sheaf of groups J_Y more explicitly.

Note that by construction, for $Y_1 \to Y$, the group $\mathrm{Hom}(Y_1, J_Y)$ identifies with the group of invertible elements in the ring $\Gamma(Y_1 \underset{Y}{\times} Y', \mathcal{O})$. Thus, we can write

$$J_Y \simeq \mathrm{Res}^{Y'}_Y(\mathbb{G}_m),$$

where $\mathrm{Res}^{Y'}_Y$ is Weil's restriction of scalars functor (by the definition of the latter).

However, this description of J_Y is specific for GL_n. For a general G it will have a different flavor, which for GL_n plays out as follows:

3.2.4. Let $\tilde{\mathfrak{c}} := (\mathbb{A}^1)^n = \mathbb{A}^n$, which is an $n!$-sheeted ramified cover of $\mathfrak{c} \simeq (\mathbb{A}^1)^{(n)}$. In fact, we have that $\mathfrak{c}'$ is the GIT quotient $\tilde{\mathfrak{c}}//S_{n-1}$, where $S_{n-1} \subset S_n$.

In the above situation, set

$$\tilde{Y} := Y \underset{\mathfrak{c}}{\times} \tilde{\mathfrak{c}}.$$

We have a natural S_n-action on $\tilde{Y}$, which make it an S_n-etale cover of Y if $\sigma \in \overset{\circ}{\mathrm{Hitch}}(Y)$.

Exercise 3.2.5. *Show that $\tilde{Y} \underset{Y}{\times} J_Y$ identifies as a group-scheme with $(\mathbb{G}_m^{\times n})_{\tilde{Y}}$.*

3.3. Now the question is, how should we generalize the above discussion so that it makes sense for any group G?

3.3.1. Our basic ingredients would be as follows:

For any Y and any $\sigma : Y \to \mathfrak{c}$ we'll want to find a "model" element $(P^0_G, f^0_\sigma) \in \mathrm{Higgs}^{\mathrm{reg}}(Y)_\sigma$, with P^0_G being as usual the trivial G-bundle, i.e., $f^0_\sigma \in \mathrm{Hom}(Y, \mathfrak{g}^{\mathrm{reg}})$.

If we require that the assignment $\sigma \mapsto f^0_\sigma$ behave functorially in Y, the above amounts to a map between schemes

$$\upsilon : \mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}},$$

which is a section of $\varpi : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c}$.

3.3.2. Now, given $\sigma : Y \to \mathfrak{c}$, what would the group subscheme $J_Y \subset G_Y$ be? This is just as in the GL_n-case: for $Y_1 \to Y$

$$\mathrm{Hom}_Y(Y_1, J_Y) := \mathrm{Aut}((P^0_G, f^0_\sigma)_{Y_1}) \subset \mathrm{Aut}((P^0_G)_{Y_1}) = \mathrm{Hom}_Y(Y_1, G_Y).$$

We'll see that when σ maps to $\overset{\circ}{\mathfrak{c}}$, such J_Y is indeed a non-split torus, i.e., it will become isomorphic to the constant group scheme corresponding to the Cartan T after an etale base change.

Finally, for any Higgs bundle $(P_G, f) \in \mathrm{Hom}(S, \mathrm{Higgs}^{\mathrm{reg}}(Y)_\sigma)$, we consider

$$\mathrm{Isom}_{S\times Y}\left((P^0_G, f^0_\sigma), (P_G, f)\right) \subset \mathrm{Isom}_{S\times Y}\left(P^0_G, P_G\right),$$

which is a torsor with respect to J_Y, by construction. This provides the required reduction of P_G as a G-bundle to J_Y.

3.3.3. This is the untwisted story (i.e., when there is no line bundle $\mathcal{L}$ present), and it will be sufficient for the purposes of this talk (i.e., proving Theorem 1.1.1). However, for Ngo's work [Ngo], we'll need to discuss also the twisted version.

Harking back at the definitions, this would mean that we'll need to choose υ so that it is $\mathbb{G}_m$-equivariant. However, it is easy to see that this is impossible.

However, we can ask for less: we can ask for an existence of a section

$$s_{\mathcal{L}} : \mathrm{Hitch}_{\mathcal{L}}(Y) \to \mathrm{Higgs}^{\mathrm{reg}}_{\mathcal{L}}(Y),$$

of the map $h_{\mathcal{L}}$, and which behaves functorially in $(Y, \mathcal{L})$. In other words, for $\sigma \in \mathrm{Hitch}_{\mathcal{L}}(Y)$ we'll want to construct a distinguished pair $(P_{G,\sigma}, f^0_\sigma)$, but without insisting that $P_{G,\sigma}$ be the trivial G-bundle.

Such a datum would be equivalent to constructing a map of stacks

$$\mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}}/G,$$

which is an inverse to $\varpi/G : \mathfrak{g}^{\mathrm{reg}}/G \to \mathfrak{c}$, and which is equipped with an equivariant structure with respect to $\mathbb{G}_m$, acting on both sides. However, even this is not always possible for all groups G (although it is possible for, say, GL_n). Instead, we'll have a map

$$(\upsilon/\mathbb{G}_m)' : (\mathfrak{c}/\mathbb{G}_m) \underset{\mathrm{pt}/\mathbb{G}_m}{\times} \mathrm{pt}/\mathbb{G}_m \to \mathfrak{g}^{\mathrm{reg}}/(G \times \mathbb{G}_m),$$

where $\mathrm{pt}/\mathbb{G}_m \to \mathrm{pt}/\mathbb{G}_m$ is the map corresponding to $\mathbb{G}_m \overset{x\mapsto x^2}{\longrightarrow} \mathbb{G}_m$. In other words, we'll be able to construct a section $s_{\mathcal{L}}$ as long as we choose a square root of $\mathcal{L}$ on Y, i.e., a line bundle $\mathcal{L}'$, such that $\mathcal{L}'^{\otimes 2} \simeq \mathcal{L}$.

4. The Kostant section

4.1. In this section we'll construct the sought-for section $\upsilon : \mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}}$.

4.1.1. Let $\phi : SL_2 \to G$ be a map, such that $\phi(e)$ is a regular nilpotent element (and, hence, $\phi(f)$ is also regular nilpotent). Here (e, h, f) is the standard basis for the Lie algebra $\mathfrak{sl}_2$. Let $\mathfrak{b}^+$ (resp., $\mathfrak{b}^-$) be the Borel Lie subalgebra corresponding to positive (resp., negative) eigenvalues of h.

Consider the affine subspace $(\phi(f) + \mathfrak{b}^+) \subset \mathfrak{g}$. It is clearly preserved by the adjoint action of N^+ on $\mathfrak{g}$.

Lemma 4.1.2. (Kostant)
(1) *The subvariety* $\phi(f) + \mathfrak{b}^+$ *is contained in* $\mathfrak{g}^{\text{reg}}$.
(2) *The restriction of the map* ϖ *to* $\phi(f) + \mathfrak{b}^+$ *makes*

$$(\phi(f) + \mathfrak{b}^+) \to \mathfrak{c}$$

into an N^+*-torsor. (I.e., the map* $(\phi(f) + \mathfrak{b}^+)/N^+ \to \mathfrak{c}$ *is an isomorphism of stacks, in particular, the LHS is a scheme.)*

Proof. Exercise. □

4.1.3. Let now $\mathfrak{a}$ be the affine subspace of $\phi(f) + \mathfrak{b}^+$ equal to $\phi(f) + \ker(\mathrm{ad}_{\phi(e)})$.

Proposition 4.1.4. (Kostant) *The restriction of* ϖ *to* $\mathfrak{a}$ *is an isomorphism.*

Proof. Exercise. □

Thus,

$$\mathfrak{c} \to \mathfrak{a} \hookrightarrow (\phi(f) + \mathfrak{b}^+) \hookrightarrow \mathfrak{g}^{\text{reg}}$$

provides a section of the torsor $(\phi(f) + \mathfrak{b}^+) \to \mathfrak{c}$, and also the desired map υ. It's called the Kostant section.

4.1.5. Here is a surprise, however:

Exercise 4.1.6. *Identify explicitly the Kostant section* $\mathfrak{c} \to \mathfrak{g}^{\text{reg}}$ *in the case* $G = GL_n$*, and convince yourself that it's actually different from the one used in Section 3.2.1.*

So, for an arbitrary G, our "canonical" representative of a Higgs bundle for a given point of $\mathrm{Hitch}(Y)$ is such that when specialized to $G = GL_n$, it's different from the one we used before. But this is OK: for what we are about to do, it doesn't matter what choice of a map υ we use.

Remark. It is shown in [Ngo] that any two choices of υ are conjugate by means of a map $\mathfrak{c} \to G$, which implies that whatever constructions we perform, all choices of υ are equivalent.

4.2. For the purposes of the next talk, let us discuss the equivariant properties of the map υ with respect to $\mathbb{G}_m$, and, in particular, comment on the construction of the map $(\upsilon/\mathbb{G}_m)'$, see Section 3.3.3.

4.2.1. Consider the standard torus $\mathbb{G}_m \subset SL_2$. The following results from the definitions:

Lemma 4.2.2.
(1) The action of $\mathbb{G}_m$ *on* $\mathfrak{g}$ *given by* $(\lambda, x) \mapsto \lambda^2 \cdot \mathrm{Ad}_{\phi(\lambda)}(x)$ *preserves the subscheme* $\mathfrak{a}$.
(2) The map $\varpi|_{\mathfrak{a}}$ *is equivariant with respect to the action of* $\mathbb{G}_m$ *on* $\mathfrak{a}$ *given by point (1) above, and the square of the natural action of* $\mathbb{G}_m$ *on* $\mathfrak{c}$.

Exercise 4.2.3.
(1) *Show that the above lemma implies that the composition*

$$\mathfrak{c} \xrightarrow{\upsilon} \mathfrak{g}^{\text{reg}} \to \mathfrak{g}^{\text{reg}}/G$$

is naturally equivariant with respect to the action of $\mathbb{G}_m$ *on both sides, equal to the square of the natural action.*
(2) *Deduce from point (1) the existence of a map*

$$(\upsilon/\mathbb{G}_m)' : (\mathfrak{c}/\mathbb{G}_m) \underset{\mathrm{pt}/\mathbb{G}_m}{\times} \mathrm{pt}/\mathbb{G}_m \to \mathfrak{g}^{\text{reg}}/(G \times \mathbb{G}_m),$$

where $\mathrm{pt}/\mathbb{G}_m \to \mathrm{pt}/\mathbb{G}_m$ *is the squaring map.*

5. Regular centralizers and reduction

5.1. Thus, to complete our program, it remains to show that for Y and a map $\sigma : Y \to \overset{\circ}{\mathfrak{c}}$, the composed map

$$f^0_\sigma : Y \overset{\sigma}{\to} \overset{\circ}{\mathfrak{c}} \overset{\upsilon}{\to} \overset{\circ}{\mathfrak{g}}$$

is such that the functor on the category of schemes over Y

$$(Y_1 \to Y) \mapsto \{g \in \mathrm{Hom}_Y(Y_1, G) \,|\, \mathrm{Ad}_g(f^0_\sigma|_{Y_1}) = f^0_\sigma|_{Y_1}\}$$

is representable by a group-sceme J_Y, which is etale-locally isomorphic to the Cartan group T.

We shall accomplish this in a slightly greater generality, as it will be necessary also for the next talk.

5.2. Let $Z_\mathfrak{g}$ be the group sub-scheme of $G_\mathfrak{g} = G \times \mathfrak{g}$ of centarlizers. I.e.,

$$\mathrm{Hom}(S, Z_\mathfrak{g}) = \{x : S \to \mathfrak{g}, g : S \to G \,|\, \mathrm{Ad}_g(x) = x\}.$$

Let $Z_{\mathfrak{g}^{\mathrm{reg}}}$ (resp., $Z_{\overset{\circ}{\mathfrak{g}}}$) be the restriction of $Z_\mathfrak{g}$ to the open subset $\mathfrak{g}^{\mathrm{reg}}$ (resp., $\overset{\circ}{\mathfrak{g}}$).

Lemma 5.2.1. *The group-scheme $Z_{\mathfrak{g}^{\mathrm{reg}}}$ is commutative, and smooth over $\mathfrak{g}^{\mathrm{reg}}$.*

Proof. Exercise. □

Proposition 5.2.2. (B.C. Ngo) *There exists a smooth group-scheme $J_\mathfrak{c}$ over $\mathfrak{c}$ endowed with an isomorphism $J_\mathfrak{c} \underset{\mathfrak{c}}{\times} \mathfrak{g}^{\mathrm{reg}} \simeq Z_{\mathfrak{g}^{\mathrm{reg}}}$, as group-schemes over $\mathfrak{g}^{\mathrm{reg}}$, which is equivariant with respect to G acting on $\mathfrak{g}^{\mathrm{reg}}$ by conjugation.*

Proof. Exercise. □

5.2.3. Thus, for any Y and $\sigma : Y \to \mathfrak{c}$ and f^0_σ defined as above, we obtain that the functor

$$(Y_1 \to Y) \mapsto \{g \in \mathrm{Hom}(Y_1, G) \,|\, \mathrm{Ad}_g((f^0_\sigma)|_{Y_1}) = f^0_\sigma|_{Y_1}\}$$

is representable by $J_Y := J_\mathfrak{c} \underset{\mathfrak{c}}{\times} Y$.

Thus, in order to show that whenever σ lands in $\overset{\circ}{\mathfrak{c}}$, the group-scheme J_Y is etale locally isomorphic to T, it is enough to show the corresponding fact for $J_\mathfrak{c}$.

5.3. Identification of $J_\mathfrak{c}$–the regular semi-simple case. Let $\mathfrak{t}$ be the Lie algebra of the Cartan subgroup T, and recall that we have a canonical map

$$\mathfrak{t} \to \mathfrak{c},$$

which is an etale cover $\overset{\circ}{\mathfrak{c}}$ with structure group W–the Weyl group of G. Let $\overset{\circ}{\mathfrak{t}}$ denote the preimage of $\overset{\circ}{\mathfrak{c}}$, i.e., the complement to the root hyperplanes in $\mathfrak{t}$.

Lemma 5.3.1.

(1) $J_{\overset{\circ}{\mathfrak{c}}} \underset{\overset{\circ}{\mathfrak{c}}}{\times} \overset{\circ}{\mathfrak{t}} \simeq T_{\overset{\circ}{\mathfrak{t}}}$.

(2) *The W-equivariant structure on $J_{\overset{\circ}{\mathfrak{c}}} \underset{\overset{\circ}{\mathfrak{c}}}{\times} \overset{\circ}{\mathfrak{t}}$ corresponds to the canonical W-action on T.*

Proof. This follows from the definition of $J_\mathfrak{c}$: the left-hand side is $Z_\mathfrak{g}|_{\overset{\circ}{\mathfrak{t}}}$, and we know that the centralizer of a regular element of a given Cartan subalgebra $\mathfrak{t}$ is the corresponding Cartan subgroup T.

□

5.3.2. For a map $\sigma : Y \to \overset{\circ}{\mathfrak{c}}$ let $\tilde{Y}$ denote

$$\tilde{Y} := \overset{\circ}{\mathfrak{t}} \underset{\overset{\circ}{\mathfrak{c}}}{\times} Y.$$

This is an etale W-cover of Y.

By Lemma 5.3.1 above, we obtain that the pull-back of J_Y to $\tilde{Y}$ indeed identifies with $T_{\tilde{Y}}$, with the W-equivariant structure given by the the canonical W-action on T.

This recovers the picture for J_Y that we had for GL_n in Section 3.2.4.

5.4. Identification of $J_{\mathfrak{c}}$–the general case. For the next talk, let us say a few words how the group-scheme $J_{\mathfrak{c}}$ looks like over the entire $\mathfrak{c}$, i.e., outside the open subset $\overset{\circ}{\mathfrak{c}}$.

5.4.1. Consider the following group-scheme over $\mathfrak{c}$:

$$J'_{\mathfrak{c}} := \left(\mathrm{Res}^{\mathfrak{t}}_{\mathfrak{c}}(T_{\mathfrak{t}})\right)^W,$$

i.e., for $S \to \mathfrak{c}$,

$$\mathrm{Hom}_{\mathfrak{c}}(S, J'_{\mathfrak{c}}) = \{\phi \in \mathrm{Hom}_{\mathfrak{c}}(S \underset{\mathfrak{c}}{\times} \mathfrak{t}, T) \,|\, \phi \text{ is } W-\text{equivariant}\}.$$

Note that by Lemma 5.3.1, we have an isomorphism:

$$J_{\overset{\circ}{\mathfrak{c}}} \simeq J'_{\overset{\circ}{\mathfrak{c}}} := J'_{\mathfrak{c}}|_{\overset{\circ}{\mathfrak{c}}}.$$

Proposition 5.4.2. ([DonGa], Theorem 11.6) *The above isomorphism over $\overset{\circ}{\mathfrak{c}}$ extends to a homomorphism of group-schemes over $\mathfrak{c}$:*

$$J_{\mathfrak{c}} \to J'_{\mathfrak{c}}.$$

Moreover, the latter map is an open embedding.

This proposition implies that the difference between $J'_{\mathfrak{c}}$ and $J_{\mathfrak{c}}$ is given by a finite sheaf of groups in the etale topology, which vanishes over $\overset{\circ}{\mathfrak{c}}$.

Remark. In fact, using Sect. 6 of [DonGa] a complete description of $J_{\mathfrak{c}}$ can be given as a subfunctor of $J'_{\mathfrak{c}}$

5.4.3. For an arbitrary Y and a map $\sigma : Y \to \mathfrak{c}$ we let

$$\tilde{Y} := \mathfrak{t} \underset{\mathfrak{c}}{\times} Y.$$

We call $\tilde{Y}$ "the cameral cover" corresponding to σ. We let J'_Y be the pull-back of $J'_{\mathfrak{c}}$ by means of σ.

Note that for a scheme S, a J'_Y-torsor on $S \times Y$ is the same as a W-equivariant T-torsor on $S \times \tilde{Y}$. Using [DonGa] one can give a complete description of J_Y-torsors on $S \times Y$ in terms of W-equivariant T-torsors on $S \times \tilde{Y}$.

6. Summary and proof of Theorem 1.1.1

6.1. Let Y be a scheme and $\sigma : Y \to \mathfrak{c}$ be a map. Let J_Y be the corresponding group-scheme over Y.

Let J_Y-Tors denote the functor $\mathbf{Sch}^{op}_{/k} \to \mathbf{Groupoids}$ that assigns to a scheme S the (Picard) groupoid of J_Y-torsors on $S \times Y$.

Exercise 6.1.1. *Deduce from the work we have done that the exists an isomorphism of functors*

$$\mathrm{Hitch}(Y)^{\mathrm{reg}}_{\sigma} \simeq J_Y\text{-Tors},$$

6.2. Assume now that $Y = \mathrm{Spec}(K)$, where K is a field containing k. We claim that any G-bundle P_G on $\mathrm{Spec}(K)$, i.e., a k-point of $\mathrm{Bun}_G(Y)$, can be lifted to a k-point of $\overset{\circ}{\mathrm{Higgs}}(Y)$.

Indeed, we consider $\Gamma(\mathrm{Spec}(K), \mathfrak{g}_{P_G})$ as a K-vector space. This is the set of all liftings of P_G to a k-point of $\mathrm{Higgs}(Y)$. Now, $\overset{\circ}{\mathrm{Higgs}}(Y) \subset \mathrm{Higgs}(Y)$ corresponds to a Zariski open subvariety of $\Gamma(\mathrm{Spec}(K), \mathfrak{g}_{P_G})$, considered as an affine space over K, and, since K is infinite, it is non-empty.

Thus, by the above, any G-bundle on $\mathrm{Spec}(K)$ admits a reduction to a group subscheme $J_K \subset G_K$, such that J_K becomes isomorphic to $(\mathbb{G}_m)^{\times r}$ after an etale base change $K \mapsto \tilde{K}$.

6.3. Finally, we claim:

Lemma 6.3.1. *Let K be a field such that $H^2(\mathrm{Gal}(K), F)$ vanishes for any (continuous, discrete) $\mathrm{Gal}(K)$-module F. Then for any J_K as above, any J_K-torsor over $\mathrm{Spec}(K)$ is trivial.*

The lemma will imply Theorem 1.1.1. Indeed, we have Tsen's theorem that says that for K being the field of rational functions on an algebraic curve over an algebraically closed ground field, $\mathrm{Gal}(K)$ has cohomological dimension 1, i.e., that it satisfies the condition of Lemma 6.3.1 above.

Proof. (of Lemma 6.3.1)
Note that if J_K was a split torus, i.e., a product of copies of $\mathbb{G}_m$, the group $H^1(\mathrm{Spec}(K), J_K)$ would vanish with no other assumptions on K, by Hilbert's 90.

Exercise 6.3.2. *Show that for an separable field extension K_1/K and J_K a group scheme over K, we have a surjection of etale sheaves:*

$$\mathrm{Res}_K^{K_1}(J_K|_{K_1}) \twoheadrightarrow J_K.$$

Exercise 6.3.3. *Finish the proof of the lemma, and therefore, theorem.*

□

References

[DonGa] R. Donagi, D. Gaitsgory, *The Gerbe of Higgs Bundles*, arXiv:math/0005132.
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