THE GLOBAL NILPOTENT CONE

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The goal of this note is to reproduce Ginzburg's proof (cf. [G]) that the dimension of the global nilpotent cone is the same as the dimension of Bun_G. In what follows, everything is over a base field k, algebraically closed of characteristic zero[1]. We assume that X is a smooth projective curve, and G is a semisimple group. We will use ω to denote the cotangent sheaf of X, or its corresponding $\mathbb{G}_m$-torsor.

Recall that we have the Hitchin map

$$p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X), \tag{1}$$

where $\mathrm{Hitch}(X) = \Gamma(X, \omega \times^{\mathbb{G}_m} \mathfrak{t}/\!\!/W)$. There is a natural $\mathbb{G}_m$-action on $\mathrm{Hitch}(X)$, with a unique fixed point, denoted by 0.

Definition 0.1. The global nilpotent cone is

$$\mathcal{N}ilp = p^{-1}(0).$$

We will prove that

Theorem 0.1. *Notations are as above. Then*

$$\dim \mathcal{N}ilp = \dim \mathrm{Bun}_G.$$

We have the following corollaries, in which we assume that the genus of X is > 1.

Corollary 0.2. *The stack $T^*\mathrm{Bun}_G$ is good (in the sense of* [BD, §1.1.1]*) and therefore is a locally complete intersection.*

Remark 0.1. In [BD], it is proved that $T^*\mathrm{Bun}_G$ is indeed very good.

Proof. For any point $\eta \in \mathrm{Hitch}(X)$, the closure of the $\mathbb{G}_m$-orbit contains 0. Therefore, $\dim p^{-1}(\eta) \leq \dim p^{-1}(0) = \dim \mathrm{Bun}_G$. This implies that

$$\dim T^*\mathrm{Bun}_G \leq \dim \mathrm{Bun}_G + \dim \mathrm{Hitch}(X) = 2 \dim \mathrm{Bun}_G.$$

On the other hand, it is the general fact that $\dim T^*\mathrm{Bun}_G \geq 2 \dim \mathrm{Bun}_G$. This implies that $\dim T^*\mathrm{Bun}_G = 2 \dim \mathrm{Bun}_G$, and $T^*\mathrm{Bun}_G$ is good, which in term implies that $T^*\mathrm{Bun}_G$ is locally a complete intersection. □

Corollary 0.3. *The morphism p is flat.*

Remark 0.2. Recall that p is called flat if for any flat morphism $f : U \to T^*\mathrm{Bun}_G$, $p \circ f : U \to \mathrm{Hitch}(X)$ is flat.

Proof. Since $T^*\mathrm{Bun}_G$ is l.c.i., $\mathrm{Hitch}(X)$ is regular, and p has the relative dimension $\dim T^*\mathrm{Bun}_G - \dim \mathrm{Hitch}(X)$, the assertion follows from the local criterion of flatness. □

Theorem 0.1 is a consequence of the following theorem.

Theorem 0.4. *The stack $\mathcal{N}ilp$ is an isotropic substack of $T^*\mathrm{Bun}_G$.*

[1] Maybe one can only require the characteristic of k is good w.r.t. the group G.

We have to explain the meaning of the above sentence. First, let (M,ω) be a symplectic variety. A locally closed subscheme $N \subset M$ is called isotropic if every smooth subvariety $V \subset N$ is isotropic in M (i.e. $\omega|_V = 0$). Equivalently, this means $(N_{red})^{reg}$ is isotropic in M. In this case $2\dim N \leq \dim M$ (if $\dim M < \infty$).

If $\mathcal{Y}$ is a smooth (equidimensional) algebraic stack, then a locally closed substack $\mathcal{N} \subset T^*\mathcal{Y}$ is called isotropic if for some (and therefore any) smooth surjective map $S \to \mathcal{Y}$ (we always assume that S is locally of finite type), $S \times_{\mathcal{Y}} \mathcal{N} \subset S \times_{\mathcal{Y}} T^*\mathcal{Y} \subset T^*S$ is isotropic. In this case $\dim \mathcal{N} \leq \dim \mathcal{Y}$. (Proof: Assume that $S/\mathcal{Y}$ is of relative dimension d, then $\dim \mathcal{N} + d = \dim(S \times_{\mathcal{Y}} \mathcal{N}) \leq \dim S = \dim \mathcal{Y} + d$.)

Now, we show that Theorem 0.4 implies Theorem 0.1. Observe the natural morphism $\mathrm{Bun}_G \to T^*\mathrm{Bun}_G$ given by the zero section realizes Bun_G as a closed substack of $\mathcal{N}ilp$. Therefore, $\dim \mathcal{N}ilp \geq \dim \mathrm{Bun}_G$.

Now we prove Theorem 0.4, following the argument of Ginzburg (cf. [G]). We have the following obvious lemma.

Lemma 0.5. *Let $(M_1,\omega_1),(M_2,\omega_2)$ be symplectic varities, and Γ be a symplectic correspondence, (i.e. Γ is isotropic in $M_1 \times M_2$ with respect to the symplectic structure $-\mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$). Then for any $L \subset M_1$ isotropic, $\mathrm{pr}_2(\mathrm{pr}_1^{-1}L \cap \Gamma)$ is isotropic in M_2.*

Corollary 0.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of smooth algebraic stacks of finite type. Let $\mathcal{N} \subset T^*\mathcal{Y}$ be a closed substack. Let*

$$\mathcal{M} := \mathcal{X} \times_{T^*\mathcal{X}} (\mathcal{N} \times_{\mathcal{Y}} \mathcal{X}),$$

where the morphism $\mathcal{X} \to T^*\mathcal{X}$ is given by the zero section, and $\mathcal{N} \times_{\mathcal{Y}} \mathcal{X} \to T^*\mathcal{X}$ is the composition $\mathcal{N} \times_{\mathcal{Y}} \mathcal{X} \to T^*\mathcal{Y} \times_{\mathcal{Y}} \mathcal{X} \to T^*\mathcal{X}$. If the natural projection $\mathcal{M} \to \mathcal{N}$ is surjective, then $\mathcal{N}$ is isotropic in $T^*\mathcal{Y}$.

Proof. The assertion is true for $\mathcal{X},\mathcal{Y}$ being symplectic varieties by the above lemma. Now, let $V \to \mathcal{Y}$ be smooth surjective and $U = \mathcal{X} \times_{\mathcal{Y}} V$. We want to show that $\mathcal{N} \times_{\mathcal{Y}} V$ is isotropic in T^*V. But the surjectivity of $\mathcal{M} \to \mathcal{N}$ implies the surjectivity of $\mathcal{M} \times_{\mathcal{Y}} V \to \mathcal{N} \times_{\mathcal{Y}} V$. On the other hand,

$$\mathcal{M} \times_{\mathcal{Y}} V \cong U \times_{T^*\mathcal{X} \times_{\mathcal{X}} U} (U \times_V (\mathcal{N} \times_{\mathcal{Y}} V)) \cong U \times_{T^*U} (U \times_V (\mathcal{N} \times_{\mathcal{Y}} V)).$$

Therefore, the stack case follows from the scheme case. □

Remark 0.3. This corollary can be generalized a little bit provided k is uncountable. We can allow that X has countable many connected components, and f is of finite type when restricted to each connected component of X.

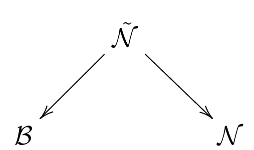
We want to plug in the above lemma with $\mathcal{X} = \mathrm{Bun}_B$, $\mathcal{Y} = \mathrm{Bun}_G$ and $\mathcal{N} = \mathcal{N}ilp$. The representability of the morphism $\mathrm{Bun}_B \to \mathrm{Bun}_G$ is shown in Dennis' early notes. Furthermore, it is locally of finite type. Therefore, it remains to show that

Proposition 0.7. *The natural morphism*

$$\mathrm{Bun}_B \times_{T^*\mathrm{Bun}_B} (\mathcal{N}ilp \times_{\mathrm{Bun}_G} \mathrm{Bun}_B) \to \mathcal{N}ilp$$

is surjective.

Proof. Let $\mathcal{F}$ be the universal G-bundle on $X \times \mathrm{Bun}_G$. Let



be the Springer correspondence between the flag variety $\mathcal{B}$ of $\mathfrak{g}$ and the nilpotent cone $\mathcal{N}$. The whole diagram is $G \times \mathbb{G}_m$-equivariant. Let $\widetilde{\mathcal{N}ilp} = \Gamma(X \times \mathrm{Bun}_G, \tilde{\mathcal{N}}_{\mathcal{F}\times\omega})$ be global Springer resolution, where $\mathcal{F}\times\omega$ denotes the $(G\times\mathbb{G}_m)$-torsor $\mathcal{F}\times(\omega\boxtimes\mathcal{O}_{\mathrm{Bun}_G})$. More precisely, $\widetilde{\mathcal{N}ilp}$ is the functor that associates every Bun_G-scheme S the set $\Gamma(X\times S,\tilde{\mathcal{N}}_{\mathcal{F}\times\omega}|_{X\times S})$. According to Dennis' note, $\widetilde{\mathcal{N}ilp}\to\mathrm{Bun}_G$ is representable. We thus have the following commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\mathcal{N}ilp} & & \\ & \swarrow & & \searrow & \\ \mathrm{Bun}_B\cong\Gamma(X\times\mathrm{Bun}_G,\mathcal{B}_{\mathcal{F}\times\omega}) & & & & \mathcal{N}ilp\cong\Gamma(X\times\mathrm{Bun}_G,\mathcal{N}_{\mathcal{F}\times\omega}) \\ & \searrow & & \swarrow & \\ & & \mathrm{Bun}_G & & \end{array} \qquad (2)$$

The proposition is the direct consequence of the following two lemmas.

Lemma 0.8. *The map $\widetilde{\mathcal{N}ilp}\to\mathcal{N}ilp$ is surjective.*

Lemma 0.9. *The map $\widetilde{\mathcal{N}ilp}\to\mathcal{N}ilp$ factors as*

$$\widetilde{\mathcal{N}ilp}\xrightarrow{\cong}\mathrm{Bun}_B\times_{T^*\mathrm{Bun}_B}(\mathcal{N}ilp\times_{\mathrm{Bun}_G}\mathrm{Bun}_B)\to\mathcal{N}ilp.$$

We begin with the proof of Lemma 0.8. It is enough to prove $\widetilde{\mathcal{N}ilp}(k)\to\mathcal{N}ilp(k)$ is surjective. Let $(\mathcal{E},\eta)\in\mathcal{N}ilp(k)$ be a k-point, where $\mathcal{E}$ is a G-bundle on X and $\eta\in\Gamma(X,\mathcal{N}_{\mathcal{E}\times\omega})$. The G-bundle $\mathcal{E}$ can be trivialized at the generic point ξ of X. We fix such a trivialization of $\mathcal{E}$, together with a trivialization of ω at the generic point. so that the restriction of η to the generic point gives rise to a point in $\mathcal{N}(K)$, where $K=k(X)$ is the function field of X. We claim that $\tilde{\mathcal{N}}(K)$ maps surjectively to $\mathcal{N}(K)$ so that η can be lifted to a section of $\tilde{\mathcal{N}}_{\mathcal{E}\times\omega}$ at the generic point of X. Then by the properness of the map $\tilde{\mathcal{N}}\to\mathcal{N}$, η can be lifted to a section of $\tilde{\mathcal{N}}_{\mathcal{E}\times\omega}$ over the whole X.

That $\tilde{\mathcal{N}}(K)\to\mathcal{N}(K)$ is surjective is equivalent to the fact that every nilpotent element $x\in\mathfrak{g}(K)$ is contained in a Borel subalgebra defined over K. One first observes that x is indeed contained in the nilpotent radical of a K-parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}_K$. This is because by the Jacobson-Morosov theorem, there is a $\mathfrak{sl}_2$-triple $(x,h,y)\subset\mathfrak{g}_K$ defined over K. Then h defines a grading on $\mathfrak{g}_K=\sum\mathfrak{g}_K^i$ such that $x\in\mathfrak{g}_K^2$, and $\mathfrak{p}=\sum_{i\geq 0}\mathfrak{g}_K^i$. So it remains to show that $\mathfrak{p}$ contains a Borel subalgebra defined over K. Let $\mathcal{P}$ be the variety of parabolic subalgebras of $\mathfrak{g}$ of type $\mathfrak{p}$. Then the lemma follows from the fact that $G(K)\to\mathcal{B}(K)\to\mathcal{P}(K)$ is surjective, which in turn follows from the fact that the fibration $G\to\mathcal{P}$ is Zariski locally trivial.

Finally, we prove Lemma 0.9. Recall that there is a short exact sequence of $G\times\mathbb{G}_m$-equivariant vector bundles

$$0\to\tilde{\mathcal{N}}=G\times^B\mathfrak{n}\to\mathcal{B}\times\mathfrak{g}=G\times^B\mathfrak{g}\to G\times^B(\mathfrak{g}/\mathfrak{n})\to 0.$$

Denote the last vector bundle by $\tilde{\mathfrak{g}}^{\perp}$. In other words, we have the following diagram

$$\begin{array}{ccccc} \tilde{\mathcal{N}} & \longrightarrow & \mathcal{B}\times\mathcal{N} & \longrightarrow & \mathcal{N}\ , \\ \downarrow & & \downarrow & & \\ \mathcal{B} & \xrightarrow{0} & \tilde{\mathfrak{g}}^{\perp} & & \end{array}$$

with the Cartesian square. By twisting the above diagram by $\mathcal{F}\times\omega$ and taking the global sections, we therefore obtain that

$$\begin{array}{ccccc} \widetilde{\mathcal{N}ilp} & \longrightarrow & \mathrm{Bun}_B \times_{\mathrm{Bun}_G} \mathcal{N}ilp & \longrightarrow & \mathcal{N}ilp\ , \\ \downarrow & & \downarrow & & \\ \mathrm{Bun}_B & \xrightarrow{0} & \Gamma(X\times \mathrm{Bun}_G, \tilde{\mathfrak{g}}^{\perp}_{\mathcal{F}\times\omega}) \cong T^*\mathrm{Bun}_B & & \end{array}$$

with the Cartesian square. The lemma follows. □

References

[BD] The book of Beilinson and Drinfeld.
[G] Ginzburg, Victor The global nilpotent variety is Lagrangian. Duke Math. J. 109 (2001), no. 3, 511–519.