$\mathcal{D}_X-$schemes, jets and comformal blocks (the commutative case)

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The purpose of this talk is to introduce $\mathcal{D}_X-$*schemes* (and the particular example of *jets*) and then link them to *comformal blocks*. In this talk, all algebras will be commutative and Sym will **always** denote $\text{Sym}_{\mathcal{O}_X}$. However, all Hom and $\otimes$ will be understood over the base field k.

1 $\mathcal{D}_X-$schemes

Fix a base field k and a smooth scheme X over k. A $\mathcal{D}_X-$scheme is a scheme equipped with a flat connection over X. For an affine scheme, this is equivalent to being the spectrum of a $\mathcal{D}_X-$algebra. For example, affine $\mathcal{D}_X-$schemes of finite type have the form:

$$\text{Spec}((\text{Sym } \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F})/\mathcal{I}), \tag{1}$$

for some coherent $\mathcal{O}_X-$sheaf $\mathcal{F}$ and some $\mathcal{D}_X-$ideal sheaf $\mathcal{I}$. Throughout this talk, we will often pass freely from $\mathcal{D}_X-$algebras to affine $\mathcal{D}_X-$schemes and vice-versa (the two categories are opposite in the usual sense).

A very important example of an affine $\mathcal{D}_X-$scheme is the following:

$$\text{Spec}(\text{Sym } \mathcal{M}),$$

for any $\mathcal{D}_X-$module $\mathcal{M}$. This suggests that $\mathcal{D}_X-$algebras are generalizations of $\mathcal{D}_X-$modules, which is supported by the following fact: $\mathcal{D}_X-$modules parametrize solutions of **linear** differential equations, while $\mathcal{D}_X-$algebras parametrize solutions of **nonlinear** differential equations.

More precisely, suppose we take the $\mathcal{D}_X-$scheme $(\mathrm{Sym}\ \mathcal{D}_X^n)/\mathcal{I}$, where the ideal $\mathcal{I}$ is generated (locally) by "polynomials" $P_1, ..., P_k \in \mathrm{Sym}\ \mathcal{D}_X^n$. Then giving a map of $\mathcal{D}_X-$modules:

$$(\mathrm{Sym}\ \mathcal{D}_X^n)/\mathcal{I} \longrightarrow \mathcal{O}_X$$

is the same as giving a collection of functions $f_1, ..., f_n$ which satisfy the system of nonlinear differential equations:

$$P_i(f_1, ..., f_n) = 0.$$

A map of $\mathcal{D}_X-$schemes is one which is a morphism of $\mathcal{D}_X-$algebras at the level of coordinate rings. A more involved notion is the following:

Definition 1 *Given a morphism of $\mathcal{D}_X-$schemes $\mathcal{Y} \to \mathcal{Z}$, the* functor of horizontal sections $\mathrm{HorSect}(\mathcal{Z}, \mathcal{Y})$ *is given by:*

$$S \in \mathrm{Sch} \longrightarrow \mathrm{HorHom}_{\mathcal{Z}}(\mathcal{Z} \times S, \mathcal{Y}).$$

HorHom *consists of* horizontal morphisms*, i.e. morphisms of $\mathcal{D}_X-$schemes.*

The above definition is completely analogous to that of the functor Sect defined by Dennis in his Sep 17 lecture, by replacing $\mathcal{O}_X$ with $\mathcal{D}_X$. Note that for a morphism of $\mathcal{O}_X-$algebras to be a morphism of $\mathcal{D}_X-$algebras is a closed condition. Since the functor of sections is representable (proved on Sep 17), it follows that the functor of horizontal sections is also representable. Moreover $\mathrm{HorSect}(\mathcal{Z}, \mathcal{Y}) \hookrightarrow \mathrm{Sect}(\mathcal{Z}, \mathcal{Y})$ is a closed embedding.

2 Jets

In this section, we will show that the forgetful functor $\mathcal{D}_X$-sch $\longrightarrow \mathcal{O}_X$-sch has a right adjoint, which is called the *Jet* functor:

$$\mathcal{J} : \mathcal{O}_X\text{-sch} \longrightarrow \mathcal{D}_X\text{-sch}, \qquad \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{Z}, \mathcal{J}\mathcal{Y}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{Z}, \mathcal{Y}), \qquad (2)$$

for any $\mathcal{O}_X-$scheme $\mathcal{Y}$ and any $\mathcal{D}_X-$scheme $\mathcal{Z}$. At the level of algebras, this functor will be a left adjoint to the forgetful functor:

$$\mathcal{J} : \mathcal{O}_X\text{-alg} \longrightarrow \mathcal{D}_X\text{-alg}, \qquad \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{J}\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}), \qquad (3)$$

for any $\mathcal{O}_X-$algebra $\mathcal{A}$ and any $\mathcal{D}_X-$algebra $\mathcal{B}$. Naturally, Spec $\mathcal{J}\mathcal{A} = \mathcal{J}(\text{Spec } \mathcal{A})$. Basically, there is only one natural construction which will make $\mathcal{J}$ into a left adjoint:

$$\mathcal{J}\mathcal{A} = \text{Sym}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{A})/\mathcal{I}, \tag{4}$$

where $\mathcal{I}$ is the $\mathcal{D}_X-$ideal generated by $\text{Ker}(\text{Sym } \mathcal{A} \to \mathcal{A})$. In other words, $\mathcal{J}\mathcal{A}$ is the $\mathcal{D}_X-$algebra generated by $\mathcal{A}$. Setting $\mathcal{Z} = X$ in (2) gives us the following:

Proposition 1 *For any $\mathcal{O}_X-$scheme $\mathcal{Y}$, we have:*

$$\text{HorSect}(X, \mathcal{J}\mathcal{Y}) = \text{Sect}(X, \mathcal{Y}).$$

Example 1 *For any $\mathcal{O}_X-$module $\mathcal{N}$, we have:*

$$\mathcal{J}(\text{Sym } \mathcal{N}) = \text{Sym}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{N}).$$

Example 2 *Let X be a smooth projective curve, $C = \text{Spec}((\text{Sym } \mathfrak{g})^G)$ as in our previous talks, and consider the fiber bundle $C_{\omega_X} = C \times_{k^*} \omega_X$ on X. Then we have:*

$$\text{HorSect}(X, \mathcal{J}C_{\omega_X}) = \text{Hitch}(X),$$

and

$$(\mathcal{J}C_{\omega_X})_x = \text{Hitch}_x(X),$$

for any closed point $x \in X$. The first equality follows from Proposition 1, while the second one follows from Proposition 2 below.

Let us now prove that the definition of jets in (4) is the correct one, i.e. that it satisfies property (3). For this, consider the following constructions:

$$(\phi : \mathcal{J}\mathcal{A} \to \mathcal{B}) \longrightarrow (\phi' : \mathcal{A} \to \mathcal{B}), \qquad \phi'(a) = \phi(1 \otimes a),$$

$$(\phi' : \mathcal{A} \to \mathcal{B}) \longleftarrow (\phi : \mathcal{J}\mathcal{A} \to \mathcal{B}), \qquad \phi(d \otimes a) = d \cdot \phi'(a),$$

where ϕ denotes any map of $\mathcal{D}_X-$algebras, while ϕ' denotes any map of $\mathcal{O}_X-$algebras. It's easy to check that the assignments $\phi \to \phi'$ and $\phi' \to \phi$ are well-defined, are inverses to each other and are natural in $\mathcal{A}$ and $\mathcal{B}$.

3 Why are they called Jets?

This section is not just motivated by ethymological questions, but will actually be very useful for us. Our purpose will be to prove the following result:

Proposition 2 *Pick a closed point $x \in X$, and let $\mathcal{Y}$ be any $\mathcal{O}_X-$scheme. Then the fiber of $\mathcal{J}\mathcal{Y}$ over x is given by:*

$$(\mathcal{J}\mathcal{Y})_x = \mathrm{Sect}(\mathrm{Spf}\ \widehat{\mathcal{O}}_x, \mathcal{Y}). \tag{5}$$

where $\widehat{\mathcal{O}}_x$ is the completed local ring of X at x.

Let us recall that for any $k-$scheme S, we define:

$$(\mathrm{Spf}\ \widehat{\mathcal{O}}_x) \times S = \varinjlim_n ((\mathrm{Spec}\ \mathcal{O}_x/\mathfrak{m}_x^n) \times S) \neq (\mathrm{Spec}\ \widehat{\mathcal{O}}_x) \times S.$$

Therefore, the structure ring of $\mathrm{Spf}\ \widehat{\mathcal{O}}_x \times \mathrm{Spec}\ C$ is

$$\widehat{\mathcal{O}}_x \widehat{\otimes} C := \varprojlim_n ((\mathcal{O}_x/\mathfrak{m}_x^n) \otimes C) \neq \widehat{\mathcal{O}}_x \otimes C.$$

The above proposition makes the terminology clear, since a section from the formal disk to $\mathcal{Y}$ is, by definition, an $\mathcal{Y}-$jet at x. By naturality, it will be enough to prove the proposition in the affine case $\mathcal{Y} = \mathrm{Spec}\ \mathcal{A}$. In the following, C will denote any algebra and $\mathcal{B}$ will denote any $\mathcal{D}_X-$ algebra. I claim the the following functorial bijections hold:

$$\mathrm{Hom}(\mathrm{Spec}\ C, \mathrm{Spec}\ \mathcal{B}_x) \cong \mathrm{Hom}(\mathcal{B}_x, C), \tag{6}$$

$$\mathrm{Hom}(\mathcal{B}_x, C) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{B}, \widehat{\mathcal{O}}_x \widehat{\otimes} C). \tag{7}$$

Specialize $\mathcal{B} = \mathcal{J}\mathcal{A}$, and we have:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{J}\mathcal{A}, \widehat{\mathcal{O}}_x \widehat{\otimes} C) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \widehat{\mathcal{O}}_x \widehat{\otimes} C), \tag{8}$$

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \widehat{\mathcal{O}}_x \widehat{\otimes} C) = \mathrm{Hom}(\mathrm{Spf}\ \widehat{\mathcal{O}}_x \times \mathrm{Spec}\ C, \mathrm{Spec}\ \mathcal{A}). \tag{9}$$

This sequence of identifications proves (5) on the level of $C-$points, and since they hold naturally in C, they are enough to establish Proposition 2.

- Relation (6) is just the bijection between morphisms of affine schemes and morphisms of algebras.
- To prove relation (7), it is enough to verify it in the bigger category of vector spaces and $\mathcal{D}_X-$modules. Then, we need to verify that for any $\mathcal{D}_X-$module $\mathcal{M}$ and any vector space V, we have

$$\mathrm{Hom}(\mathcal{M}_x, V) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_x \widehat{\otimes} V)$$

  The map going from right to left is just evaluation at x. Let's now define the map going from left to right: given any morphism of vector spaces $\phi : \mathcal{M}_x \to V$, what does it mean to assign to it a morphism $\mathcal{M} \to \widehat{\mathcal{O}}_x \widehat{\otimes} V$? It merely means to give morphisms $\phi_n : \mathcal{M} \to \mathcal{O}_x / \mathfrak{m}_x^n \otimes V$ for all n, which satisfy the inverse limit compatibilities. We start off with $\phi_1 = \phi$, and then there is a unique way to inductively define each ϕ_n such that the inverse limit is a morphism of $\mathcal{D}_X-$algebras.
- Relation (8) is just property (3).
- Relation (9) is just the bijection between morphisms of affine schemes and morphisms of algebras.

4 Conformal Blocks

The functor k-sch $\longrightarrow$ $\mathcal{D}_X$-sch sending a $k-$scheme S to the "constant" $\mathcal{D}_X-$scheme $X \times S$ (which has coordinate ring $\mathcal{O}_X \otimes_k \mathcal{O}_S$) has a right adjoint functor:

$$H_\nabla(X, \cdot) : \mathcal{D}_X-\mathrm{sch} \longrightarrow k-\mathrm{sch}, \qquad \mathrm{Hom}(S, H_\nabla(X, \mathcal{Z})) \cong \mathrm{Hom}_{\mathcal{D}_X}(X \times S, \mathcal{Z}),$$

for any $\mathcal{D}_X-$scheme $\mathcal{Z}$ and any $k-$scheme S. Alternatively, we can define this functor for algebras:

$$H_\nabla(X, \cdot) : \mathcal{D}_X-\mathrm{alg} \longrightarrow k-\mathrm{alg}, \qquad \mathrm{Hom}(H_\nabla(X, \mathcal{B}), C) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{B}, \mathcal{O}_X \otimes_k C),$$

for any $\mathcal{D}_X$-algebra $\mathcal{B}$ and any k-algebra C. Obviously, Spec $H_\nabla(X, \mathcal{B}) = H_\nabla(X, \text{Spec } \mathcal{B})$. The scheme $H_\nabla(X, \mathcal{Z})$ is called the *scheme of conformal blocks* of $\mathcal{Z}$, and it is tautologically the largest constant $\mathcal{D}_X$-subscheme of $\mathcal{Z}$.

Example 3 *For any $\mathcal{D}_X$-scheme $\mathcal{Z}$, we have:*

$$H_\nabla(X, \mathcal{Z}) \cong \text{HorSect}(X, \mathcal{Z}).$$

This follows easily by unraveling the definitions.

Example 4 *Setting $\mathcal{Z} = \mathcal{J}\mathcal{Y}$ in the above for some $\mathcal{O}_X$-scheme $\mathcal{Y}$, and combining with Proposition 1 gives us:*

$$H_\nabla(X, \mathcal{J}\mathcal{Y}) \cong \text{Sect}(X, \mathcal{Y}).$$

5 Why do we denote conformal blocks by H_∇?

In this section we restrict to X projective of dimension n, and to affine $\mathcal{D}_X$-algebras. The reason why we denote algebras of conformal blocks by $H_\nabla(X, \mathcal{B})$ is that they turn out to be some sort of "cohomology algebras" of the $\mathcal{D}_X$-algebra $\mathcal{B}$. In fact, Verdier duality implies the following natural bijection for $\mathcal{D}_X$-modules:

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes_k V) \cong \text{Hom}(H^n_{dR}(X, \mathcal{M}), V), \tag{10}$$

for any $\mathcal{D}_X$-module $\mathcal{M}$ and any vector space V. By definition, $H^\bullet_{dR}(X, \mathcal{M})$ are the cohomology groups of the complex of sheaves of k-vector spaces:

$$... \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^i T^* X \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^{i+1} T^* X \longrightarrow ...$$

These cohomology groups coincide with $R^\bullet \pi_*(\mathcal{M})$, where $\pi : X \to pt$ is the projection to a point. Note that (10) implies that

$$H_\nabla(X, \text{Sym } \mathcal{M}) = \text{Sym } H^n_{dR}(X, \mathcal{M}). \tag{11}$$

This can be further reinterpred as follows. Pick a closed point $x \in X$, let $i : x \hookrightarrow X$ be the closed embedding and $j : X - x \hookrightarrow X$ be the open embedding. Then for any $\mathcal{D}_X$-module $\mathcal{M}$ we have the exact triangle:

$$i_* \mathcal{M}_x[-n] \longrightarrow \mathcal{M} \to j_* j^* \mathcal{M}.$$

The shift by n happens when we pass from $\mathcal{D}_X$−modules to quasicoherent $\mathcal{O}_X$−modules, as we will be doing now. This induces a long exact sequence on cohomology:

$$... \longrightarrow H^{n-1}_{dR}(X-x, \mathcal{M}) \xrightarrow{\phi} \mathcal{M}_x \longrightarrow H^n_{dR}(X, \mathcal{M}) \longrightarrow H^n_{dR}(X-x, \mathcal{M}). \quad (12)$$

We claim that the last group is 0. To see this, recall that Lichtenbaum's theorem says that the Cech cohomological dimension of $X - x$ is at most $n - 1$, i.e. $H^n(X - x, \mathcal{F}) = 0$ for any quasicoherent $\mathcal{F}$. As the $\mathcal{D}_X$−module $\mathcal{M}$ is a quotient of the form:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \twoheadrightarrow \mathcal{M},$$

for some quasicoherent $\mathcal{F}$, and

$$H^n_{dR}(X - x, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) = H^n(X - x, \mathcal{F}) = 0,$$

it also follows that $H^n_{dR}(X - x, \mathcal{M}) = 0$. Therefore, (11) and (12) imply:

$$H_\nabla(X, \text{Sym } \mathcal{M}) = \text{Sym}(\mathcal{M}_x / \text{Im } \phi).$$

The above description applies equally well to $\mathcal{D}_X$−algebras, so we infer:

Corollary 1 *For any $\mathcal{D}_X$−algebra $\mathcal{B}$, we have:*

$$H_\nabla(X, \mathcal{B}) \cong \mathcal{B}_x / (\text{Im } \phi),$$

where $(\text{Im } \phi)$ *denotes the ideal generated by the image of the coboundary map* $\phi : H^{n-1}(X - x, \mathcal{B}) \longrightarrow \mathcal{B}_x$.

We can actually do all of this with any finite number of closed points $x_1, ..., x_k \in X$. The analogoue of the coboundary map is ϕ' given by:

$$... \longrightarrow H^{n-1}(X - \{x_1, ..., x_k\}, \mathcal{B}) \xrightarrow{\phi'} \mathcal{B}_{x_1} \oplus ... \oplus \mathcal{B}_{x_k} \longrightarrow H^n_{dR}(X, \mathcal{B}) \longrightarrow 0$$

We will need an algebra, not just a vector space, so define the map:

$$\tilde{\phi} : H^{n-1}(X - \{x_1, ..., x_k\}, \mathcal{B}) \longrightarrow \mathcal{B}_{x_1} \otimes ... \otimes \mathcal{B}_{x_k},$$
$$\tilde{\phi}(h) = \phi'_1(h) \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes \phi'_k(h).$$

In the above, ϕ'_i denotes the projection of the map ϕ' to the $i-$th factor.

Proposition 3 *We have the following natural isomorphism:*

$$\mathcal{B}_{x_1}/(\text{Im } \phi) \cong \mathcal{B}_{x_1} \otimes ... \otimes \mathcal{B}_{x_k}/(\text{Im } \tilde{\phi}), \tag{13}$$

where $(\text{Im } \tilde{\phi})$ *denotes the ideal generated by the image of the map* $\tilde{\phi}$*.*

To prove the proposition, take the natural morphism from left to right sending $b_1 \in \mathcal{B}_{x_1}$ to $b_1 \otimes 1 \otimes ... \otimes 1$. Its injectivity is immediate, and its surjectivity follows readily from the $k = 2$ case. Since it will also make the explanation clearer, let's just do $k = 2$. We have the following commutative diagram:

$$\begin{array}{ccccccc} H^{n-1}(X - x_1, \mathcal{B}) & \xrightarrow{\phi} & \mathcal{B}_{x_1} & \xrightarrow{\pi} & H^n(X, \mathcal{B}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow = & & \\ H^{n-1}(X - \{x_1, x_2\}, \mathcal{B}) & \xrightarrow{\phi'} & \mathcal{B}_{x_1} \oplus \mathcal{B}_{x_2} & \xrightarrow{\pi'} & H^n(X, \mathcal{B}) & \longrightarrow & 0 \end{array}$$

Take any $b_2 \in \mathcal{B}_{x_2}$, and look at $\pi'(b_2) \in H^n(X, \mathcal{B})$. By the above diagram, there exists $a \in \mathcal{B}_1$ such that $\pi(a) = \pi'(b_2)$. This means that $(-a, b_2) \in$ Ker $\pi' \Leftrightarrow (-a, b_2) = \phi'(h)$ for some h. Take any $b_1 \in \mathcal{B}_{x_1}$, and we have:

$$b_1 \otimes b_2 = b_1 \otimes \phi'_2(h) = (b_1 \otimes 1)(1 \otimes \phi'_2(h)) = (b_1 \otimes 1)(\phi'_1(h) \otimes 1 + 1 \otimes \phi'_2(h)) -$$
$$-(b_1 \cdot \phi'_1(h) \otimes 1) \in (\text{Im } \tilde{\phi}) + \mathcal{B}_1.$$

This implies that the map (13) is surjective, and concludes the proof of Proposition 3. Therefore, Corollary 1 implies the following:

Corollary 2 *For any* $\mathcal{D}_X-$*algebra* $\mathcal{B}$*, we have:*

$$H_\nabla(X, \mathcal{B}) \cong \mathcal{B}_{x_1} \otimes ... \otimes \mathcal{B}_{x_k}/(\text{Im } \tilde{\phi}).$$