# Overview and recap of Dustin's talk on quantization

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As throughout the last semester, let us begin by fixing a smooth projective curve $X$ of genus $g > 1$ over a field $k$, and let $G$ be a reductive group. Our discussion started from the **classical Hitchin map**:

$$T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X) = \mathrm{Sect}(X, C \times_{\mathbb{G}_m} \omega_X).$$

The actors here are $\mathrm{Bun}_G$ (the moduli stack of principal $G-$bundles over $X$), $C = \mathfrak{g}^*//G$ (the affine quotient of $\mathfrak{g}^*$ with respect to the adjoint action of $G$) and $\omega_X$ (the sheaf of regular differentials on $X$). Passing to the level of rings of functions, we get a map:

$$\mathfrak{z}^{cl}(X) := \mathcal{O}(\mathrm{Hitch}(X)) \xrightarrow{h^{cl}} \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}). \tag{1}$$

The connected components of $\mathrm{Bun}_G$ are $\mathrm{Bun}_G^\gamma$, indexed by elements $\gamma \in \pi_1(G)$. In Andrei's Oct 22 lecture, we proved the following:

**Proposition 1** *The map $h^{cl}$ becomes an isomorphism when we restrict it to any connected component $Bun_G^\gamma \subset Bun_G$.*

**Proposition 2** *The algebra $\Gamma(T^*Bun_G, \mathcal{O})$ has trivial Poisson bracket.*

Our main focus last semester was to quantize the map $h^{cl}$, i.e. to prove the following theorem:

**Theorem 1** *There exists a filtered commutative algebra $\mathfrak{z}(X)$ such that $\mathrm{gr}\ \mathfrak{z}(X) \cong \mathfrak{z}^{cl}(X)$, and a map:*

$$\mathfrak{z}(X) \xrightarrow{h} \Gamma(Bun_G, D'),$$

*such that the vertical maps in the following diagram are isomorphisms, and the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{gr}\ \mathfrak{z}(X) & \xrightarrow{gr\ h} & \mathrm{gr}\ \Gamma(Bun_G, D') \\ \cong\downarrow & & \downarrow \\ \mathfrak{z}^{cl}(X) & \xrightarrow{h^{cl}} & \Gamma(T^* Bun_G, \mathcal{O}) \end{array} \tag{2}$$

*In the above, $D'$ denotes the sheaf appropriately twisted differential operators on the stack $Bun_G$.*

Of course, one can restrict the above to any connexted component $\mathrm{Bun}_G^\gamma$:

$$h_\gamma : \mathfrak{z}(X) \xrightarrow{h} \Gamma(\mathrm{Bun}_G, D'_{\mathrm{Bun}_G}) \xrightarrow{\mathrm{rest}} \Gamma(\mathrm{Bun}_G^\gamma, D'),$$

and:

$$\begin{array}{ccccc} \mathrm{gr}\ h_\gamma : \mathrm{gr}\ \mathfrak{z}(X) & \xrightarrow{\mathrm{gr}\ h} & \mathrm{gr}\ \Gamma(\mathrm{Bun}_G, D') & \xrightarrow{\mathrm{rest}} & \mathrm{gr}\ \Gamma(\mathrm{Bun}_G^\gamma, D') \\ \cong\downarrow & & \downarrow & & \downarrow \\ \mathrm{gr}\ h_\gamma^{cl} : \mathfrak{z}^{cl}(X) & \xrightarrow{h^{cl}} & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}) & \xrightarrow{\mathrm{rest}} & \Gamma(T^*\mathrm{Bun}_G^\gamma, \mathcal{O}) \end{array}$$

Then we have the following corollaries:

**Corollary 1** *The morphism gr $h_\gamma$ is an isomorphism.*

**Corollary 2** *The morphism $h_\gamma$ is a filtered isomorphism (the quantization of Proposition 1).*

**Corollary 3** *The algebra $\Gamma(Bun_G, D')$ is commutative (the quantization of Proposition 2).*

**Corollary 4** *The vertical morphism on the right in* (2)*, while a priori just injective, is actually an isomorphism.*

The Theorem was ultimately proved in Dustin's Dec 3 talk, and today we will review both the construction of $\mathfrak{z}(X)$ and the proof of the theorem. First, we will recall how we proved the "classical" Propositions 1 and 2, via the local-to-global principle.

Take any closed point $x \in X$, and consider the ind-scheme $\mathrm{Bun}_G^{\infty,x}$ of principal $G-$bundles on $X$ with level structure at $x$ (i.e. with a fixed trivialization on the formal neighborhood Spec $\mathcal{O}_x$). The group ind-scheme $G(\mathcal{K}_x)$ acts on $\mathrm{Bun}_G^{\infty,x}$ by changing the trivialization.

Whenever we have an action of a group scheme $H$ on a stack $\mathcal{Y}$, this induces an "infinitesimal action" $\mathfrak{h} = \mathrm{Lie}\ H \longrightarrow \mathrm{Vect}(\mathcal{Y})$. Taking the dual of this, we get a "moment map" $T^*\mathcal{Y} \longrightarrow \mathfrak{h}^*$. In our case, this construction provides a map:

$$T^*\mathrm{Bun}_G^{\infty,x} \longrightarrow (\mathfrak{g} \otimes \mathcal{K}_x)^* \cong \mathfrak{g}^* \otimes \omega_{\mathcal{K}_x}. \tag{3}$$

On the rings of functions, this corresponds to a map:

$$\overline{\mathrm{Sym}(\mathfrak{g} \otimes \mathcal{K}_x)} \xrightarrow{\widetilde{\widetilde{h_x^{cl}}}} \Gamma(T^*\mathrm{Bun}_G^{\infty,x}, \mathcal{O}). \tag{4}$$

Modding out by the $G(\mathcal{O}_x)$ action means forgetting the trivialization, and therefore $\mathrm{Bun}_G^{\infty,x}/G(\mathcal{O}_x) = \mathrm{Bun}_G$. This implies that the subscheme:

$$T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{\infty,x} \hookrightarrow T^*\mathrm{Bun}_G^{\infty,x}$$

consists of cotangent vectors that are killed by the $G(\mathcal{O}_x)-$action. Therefore, the restriction of (3) gives:

$$T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{\infty,x} \longrightarrow (\mathfrak{g} \otimes \mathcal{O}_x)^{\perp} \cong (\mathfrak{g} \otimes \mathcal{K}_x/\mathcal{O}_x)^* \cong \mathfrak{g}^* \otimes \omega_{\mathcal{O}_x}. \tag{5}$$

Passing to rings of functions, we get:

$$\mathrm{Sym}(\mathfrak{g} \otimes \mathcal{K}_x/\mathcal{O}_x) \xrightarrow{\widetilde{h_x^{cl}}} \Gamma(T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{\infty,x}, \mathcal{O}). \tag{6}$$

Now we take $G(\mathcal{O}_x)-$invariants in the above, which corresponds to the following map on spaces:

$$T^*\mathrm{Bun}_G \longrightarrow (\mathfrak{g} \otimes \mathcal{K}_x/\mathcal{O}_x)^*//G(\mathcal{O}_x) \longrightarrow \mathrm{Sect}(\mathrm{Spec}\ \mathcal{O}_x, C \times_{\mathbb{G}_m} \omega_{\mathcal{O}_x}) =: \mathrm{Hitch}_x. \tag{7}$$

The second map was proved to be an isomorphism in the lectures. Then, the above gives rise to the following morphism on rings:

$$\mathfrak{z}_x^{cl} := \mathrm{Sym}(\mathfrak{g} \otimes \mathcal{K}_x/\mathcal{O}_x)^{G(\mathcal{O}_x)} \xrightarrow{h_x^{cl}} \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}). \tag{8}$$

The map (7) is called the **local Hitchin map**. The natural inclusion $\mathrm{Hitch}(X) \hookrightarrow \mathrm{Hitch}_x$ has the property that the following composition is precisely the local Hitchin map:

$$T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X) \hookrightarrow \mathrm{Hitch}_x.$$

At the level of functions, we just reverse all arrows:

$$h_x^{cl} : \mathfrak{z}_x^{cl} \twoheadrightarrow \mathfrak{z}^{cl}(X) \xrightarrow{h^{cl}} \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}). \tag{9}$$

As $x$ varies, the local Hitchin maps can be "glued" together, by means of the $D_X$−scheme:

$$\begin{array}{ccc} \mathrm{Hitch} & = & \mathrm{Jets}(C \times_{\mathbb{G}_m} \omega_X) \\ \downarrow & & \\ X & & \end{array}$$

The fiber of Hitch over $x$ is just the local $\mathrm{Hitch}_x$, while the scheme of all horizontal sections $\mathrm{HorSect}(X, \mathrm{Hitch})$ coincides with the global $\mathrm{Hitch}(X)$. We will write $\mathfrak{z}^{cl} = \mathcal{O}(\mathrm{Hitch})$, and then the compositions (9) patch up over all $x$ to give a global morphism:

$$h_{gl}^{cl} : \mathfrak{z}^{cl} \twoheadrightarrow \mathfrak{z}^{cl}(X) \otimes \mathcal{O}_X \xrightarrow{h^{cl}} \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}) \otimes \mathcal{O}_X. \tag{10}$$

The above composition merely reflects the properties of conformal blocks: recall that for a $D_X$−algebra $\mathcal{B}$, there exists an algebra $H_\nabla(X, \mathcal{B})$ of **conformal blocks** and a horizontal morphism:

$$\phi_{\mathcal{B}} : \mathcal{B} \twoheadrightarrow H_\nabla(X, \mathcal{B}) \otimes \mathcal{O}_X,$$

which is universal in the following sense: any horizontal surjection $\mathcal{B} \twoheadrightarrow B \otimes \mathcal{O}_X$ factors through $\phi_{\mathcal{B}}$. In other words, the functor $H_\nabla(X, \cdot)$ is left adjoint to the functor $\cdot \otimes \mathcal{O}_X$. Last semester, we proved the following:

**Lemma 1** *The map* $\mathfrak{z}^{cl} \twoheadrightarrow \mathfrak{z}^{cl}(X) \otimes \mathcal{O}_X$ *of* (10) *is horizontal, and*

$$H_\nabla(X, \mathfrak{z}^{cl}) = \mathfrak{z}^{cl}(X).$$

*Therefore,* (10) *merely reflects the left-adjointness of the functor* $H_\nabla$*.*

Let us present the general strategy for quantizing the above discussion (as in Sam's third lecture), emphasizing the places where we run into trouble. Back up to the group $G(\mathcal{K}_x)$ acting on $\mathrm{Bun}_G^{\infty,x}$. From this, we get an infinitesimal action:

$$\mathfrak{g} \otimes \mathcal{K}_x \longrightarrow \Gamma(\mathrm{Bun}_G^{\infty,x}, \mathrm{Vect}) \hookrightarrow \Gamma(\mathrm{Bun}_G^{\infty,x}, D_{\mathrm{Bun}_G^{\infty,x}}),$$

where $D_{\mathrm{Bun}_G^{\infty,x}}$ denotes the sheaf of differential operators on $\mathrm{Bun}_G^{\infty,x}$. Since $D_{\mathrm{Bun}_G^{\infty,x}}$ is a sheaf of algebras, we get a map:

$$\overline{U(\mathfrak{g} \otimes \mathcal{K}_x)} \xrightarrow{\widetilde{\widetilde{h_x}}} \Gamma(\mathrm{Bun}_G^{\infty,x}, D_{\mathrm{Bun}_G^{\infty,x}}). \tag{11}$$

Modding out by the $G(\mathcal{O}_x)$ vector fields gives us a map:

$$\mathbb{V}_x := \overline{U(\mathfrak{g} \otimes \mathcal{K}_x)} \bigotimes_{\overline{U(\mathfrak{g} \otimes \mathcal{O}_x)}} \mathbb{C} \xrightarrow{\widetilde{h_x}} \Gamma(\mathrm{Bun}_G^{\infty,x}, \pi^* D_{\mathrm{Bun}_G}), \tag{12}$$

where $\pi : \mathrm{Bun}_G^{\infty,x} \longrightarrow \mathrm{Bun}_G$ is just the map that quotients out the $G(\mathcal{O}_x)$ action. Here, $\mathbb{V}_x$ denotes the **vacuum module**, defined by the property:

$$\mathrm{Hom}_{\mathfrak{g} \otimes \mathcal{K}_x}(\mathbb{V}_x, M) \cong M^{G(\mathcal{O}_x)}.$$

Therefore, take $G(\mathcal{O}_x)-$invariants in (12):

$$\mathbb{V}_x^{G(\mathcal{O}_x)} \xrightarrow{h_x} \Gamma(\mathrm{Bun}_G, D_{\mathrm{Bun}_G}). \tag{13}$$

One would like this map to be the quantization of (8), but alas! It turns out that both the left and the right hand side of (13) are trivial: they are equal to $\mathbb{C}$. To get some non-trivial objects, we must twist both $\mathbb{V}_x$ and $D_{\mathrm{Bun}_G}$, as in Dustin's first talk. Let's describe how this works.

Take the canonical line bundle $\mathcal{L}_{\det}$ of $\mathrm{Bun}_G$, whose fiber over a principal $G-$bundle $P_G$ is canonically:

$$\mathcal{L}_{\det}|_{P_G} \cong \det(R\Gamma(X, \mathfrak{g}_{P_G})).$$

On the representation-theoretic side, take the central extension:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{G(\mathcal{K}_x)} \longrightarrow G(\mathcal{K}_x) \longrightarrow 1.$$

The line bundle $\pi^*\mathcal{L}_{\det}$ on $\mathrm{Bun}_G^{\infty,x}$ is not $G(\mathcal{K}_x)-$ equivariant, but it is $\widehat{G(\mathcal{K}_x)}-$ equivariant, where the central $\mathbb{G}_m$ acts fiberwise by homotheties. Taking Lie algebras, we obtain a map:

$$\widehat{\mathfrak{g} \otimes \mathcal{K}_x} \longrightarrow \Gamma(\mathrm{Bun}_G^{\infty,x}, D(\pi^*\mathcal{L}_{\det}, \pi^*\mathcal{L}_{\det})), \tag{14}$$

But this is not exactly what we need. In Sam's talk, we showed how to define the sheaf $D(\mathcal{L}_{\det}^{\lambda}, \mathcal{L}_{\det}^{\lambda})$ for any complex number $\lambda$. It is called the algebra of **twisted differential operators**. We will use $\lambda = \frac{1}{2}$, so define:

$$D_{\mathrm{Bun}_G}^{crit} := D(\mathcal{L}_{\det}^{\frac{1}{2}}, \mathcal{L}_{\det}^{\frac{1}{2}})$$

Together with this, we also define the Kac-Moody extension $\widehat{\mathfrak{g}}^{crit}$ to be "half" of the extension $\widehat{\mathfrak{g} \otimes \mathcal{K}_x}$, i.e. constructed using $\frac{1}{2}$ times the Killing form. As in (14), we obtain a map:

$$\overline{U(\widehat{\mathfrak{g}}^{crit})} \xrightarrow{\widetilde{\widetilde{h_x}}} \Gamma(\mathrm{Bun}_G^{\infty,x}, D(\pi^*\mathcal{L}_{\det}^{\frac{1}{2}}, \pi^*\mathcal{L}_{\det}^{\frac{1}{2}})).$$

This is the correct twist of the map (11). Now it's time to go through the usual story: mod out by the $G(\mathcal{O}_x)$ directions:

$$\mathbb{V}_x^{crit} := \overline{U(\widehat{\mathfrak{g}}^{crit})} \bigotimes_{\overline{U(\mathfrak{g} \otimes \mathcal{O}_x \oplus \mathbb{C})}} \mathbb{C} \xrightarrow{\widetilde{h_x}} \Gamma(\mathrm{Bun}_G^{\infty,x}, \pi^* D_{\mathrm{Bun}_G}^{crit}).$$

The critical twisted vacuum $\widehat{\mathfrak{g}}^{crit}-$module $\mathbb{V}_x^{crit}$ is defined by the property:

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}^{crit}}(\mathbb{V}_x^{crit}, M) \cong M^{G(\mathcal{O}_x)}.$$

Therefore taking $G(\mathcal{O}_x)-$invariants, we obtain:

$$\mathfrak{z}_x := \mathrm{End}_{\widehat{\mathfrak{g}}^{crit}}(\mathbb{V}_x^{crit}) = (\mathbb{V}_x^{crit})^{G(\mathcal{O}_x)} \xrightarrow{h_x} \Gamma(\mathrm{Bun}_G, D_{\mathrm{Bun}_G}^{crit}). \tag{15}$$

This is the correct quantization of the map (8). As in the classical case, these maps can be glued as $x$ ranges over $X$. Namely, there exists a **commutative**

$D_X-$algebra $\mathfrak{z}$ whose fiber over $x \in X$ is just $\mathfrak{z}_x$ defined above. Moreover, the morphisms (15) glue and give rise to a morphism:

$$\mathfrak{z} \xrightarrow{h_{gl}} \Gamma(\mathrm{Bun}_G, D^{crit}_{\mathrm{Bun}_G}) \otimes \mathcal{O}_X. \qquad (16)$$

We claim (and will later argue) that this morphism is horizontal. Therefore, we are led to define:

$$\mathfrak{z}(X) = H_\nabla(X, \mathfrak{z}),$$

which is the correct quantization of the Poisson algebra $\mathfrak{z}^{cl}(X)$ of (1). From the left-adjointness of $H_\nabla$ and the horizontality of the map $h$, we deduce the existence of an algebra morphism:

$$\mathfrak{z}(X) \xrightarrow{h} \Gamma(\mathrm{Bun}_G, D^{crit}_{\mathrm{Bun}_G}), \qquad (17)$$

which is the correct quantization of the map $h^{cl}$ from (1), as stated in Theorem 1. Now let us try to justify the claim we just made: why is the morphism $h_{gl}$ from (16) horizontal? This can be sketched in several sentences:

1. The assignment $x \longrightarrow V \otimes \mathcal{K}_x$ defines a crystal of l.l.c.v.s over $X$, for any finite-dimensional vector space $V$.

2. The assignment $x \longrightarrow \mathfrak{g} \otimes \mathcal{K}_x$ defines a crystal of Lie algebras of l.l.c.v.s over $X$.

3. The assignment $x \longrightarrow \mathbb{V}^{crit}_x$ defines a crystal of $\mathfrak{g} \otimes \mathcal{K}_x$ modules over $X$.

4. The assignment $x \longrightarrow \mathrm{End}_{\widehat{\mathfrak{g}}^{crit}}(\mathbb{V}^{crit}_x) = \mathfrak{z}_x$ defines a crystal of associative algebras over $X$. In particular, Jacob's talk on crystals implies the existence of the $\mathcal{D}_X-$algebra $\mathfrak{z}$.

5. The assignment $x \longrightarrow \mathrm{Bun}^{\infty,x}_G$ defines a crystal of schemes over $X$.

6. The assignment $x \longrightarrow G(\mathcal{K}_x)$ defines a crystal of group ind-schemes over $X$, and its action on $\mathrm{Bun}^{\infty,x}_G$ is compatible with the crystal structure.

7. The assignment $x \longrightarrow \widehat{G}^{crit}_x$ defines a crystal of group ind-schemes over $X$, and its action on $\pi^*_x \mathcal{L}_{\det}$ is compatible with the crystal structure.

8. The maps $\widetilde{\widetilde{h_x}}, \widetilde{h_x}, h_x$ are compatible with the crystal structure. In other words, the morphism (16) is horizontal.

9. Finally, the filtration on the vacuum modules $\mathbb{V}_x^{crit}$ and the filtration on the algebras $\mathfrak{z}_x$ are compatible with the crystal structure. Therefore, we obtain a filtration on the $\mathcal{D}_X-$algebra $\mathfrak{z}$ and on its algebra of conformal blocks $\mathfrak{z}(X)$.

The canonical injections $\mathrm{gr}\ \mathfrak{z}_x \hookrightarrow \mathfrak{z}_x^{cl}$ are also compatible with the crystal structure, so they induce an injection $\mathrm{gr}\ \mathfrak{z} \hookrightarrow \mathfrak{z}^{cl}$. It was proved by Feigin and Frenkel that this injection is actually an isomorphism:

$$\mathrm{gr}\ \mathfrak{z} \cong \mathfrak{z}^{cl} \Rightarrow H_\nabla(X, \mathrm{gr}\ \mathfrak{z}) \cong H_\nabla(X, \mathfrak{z}^{cl}).$$

Moreover, the canonical morphism $\mathfrak{z} \twoheadrightarrow H_\nabla(X, \mathfrak{z}) \otimes \mathcal{O}_X$ induces a surjection:

$$\mathrm{gr}\ \mathfrak{z} \twoheadrightarrow \mathrm{gr}\ H_\nabla(X, \mathfrak{z}) \otimes \mathcal{O}_X = \mathrm{gr}\ \mathfrak{z}(X) \otimes \mathcal{O}_X.$$

By the left-adjointness of conformal blocks, this yields a surjection:

$$H_\nabla(X, \mathrm{gr}\ \mathfrak{z}) \twoheadrightarrow \mathrm{gr}\ \mathfrak{z}(X).$$

So let's see where we stand: the map (17) induces the commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}\ \mathfrak{z}(X) & \xrightarrow{\mathrm{gr}\ h} & \mathrm{gr}\ \Gamma(\mathrm{Bun}_G, D^{crit}_{\mathrm{Bun}_G}) \\ {\scriptstyle a}\uparrow & & \downarrow{\scriptstyle b} \\ H_\nabla(X, \mathrm{gr}\ \mathfrak{z}) & & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}) \\ {\scriptstyle \cong}\downarrow & & \uparrow{\scriptstyle h^{cl}(X)} \\ H_\nabla(X, \mathfrak{z}^{cl}) & = & \mathfrak{z}^{cl}(X) \end{array}$$

As we previosuly said, $a$ is surjective and $b$ is injective, while the map $h^{cl}(X)$ is an injection (it becomes an isomorphism only when we restrict to a connected component). Therefore, we deduce that $a$ must be injective, and thus an isomorphism. This proves Theorem 1.