# FACTORIZABLE GERBES: CORRECTIONS OF THE TALK

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## 1. Gerbes with a $\mathbb{Z}_2$-equivariance

**1.1.** Let $X$ be a smooth curve, $A$ an abelian group, and consider the following two 2-categories:

$\mathrm{G}(X^2/\mathbb{Z}_2)$: $\mathbb{Z}_2$-equivariant $A$-gerbes on $X^2$.

$\mathrm{G}(X^{(2)})$: $A$-gerbes on $X^{(2)}$.

There is a natural functor $\mathrm{G}(X^{(2)}) \to \mathrm{G}(X^2/\mathbb{Z}_2)$ given by pullback along

$$\pi : X^2 \to X^{(2)}.$$

but it is *not* an equivalence. In fact, it is not even fully faithful.

**1.2.** First, let us recall that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two 2-categories, a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is called fully faithful if for every $c_1', c_1'' \in \mathcal{C}_1$, the induced functor of 1-categories

$$\mathrm{Hom}_{\mathcal{C}_1}(c_1', c_1'') \to \mathrm{Hom}_{\mathcal{C}_2}(F(c_1'), F(c_1'')) \tag{1.1}$$

is an equivalence.

We shall say that $F$ is 1-fully faithful if (1.1) is fully faithful. In particular "fully faithful" (="0-fully faithful") implies "1-fully faithful".

**1.3.** We claim that the above functor $\mathrm{G}(X^{(2)}) \to \mathrm{G}(X^2/\mathbb{Z}_2)$ is 1-fully faithful, but not fully faithful. Indeed, both $\mathrm{G}(X^2/\mathbb{Z}_2)$ and $\mathrm{G}(X^{(2)})$ are Picard 2-categories (there is a monoidal structure, in which every object is invertible). So, it is enough to compare the categories of endomorphisms of the unit objects in both cases.

For $\mathrm{G}(X^2/\mathbb{Z}_2)$ we are dealing with the category of $\mathbb{Z}_2$-equivariant $A$-local systems on $X^2$. In case $\mathrm{G}(X^{(2)})$ we are dealing with the category of $A$-local systems on $X^{(2)}$. Pullback by $\pi$ defines a fully faithful functor from the latter to the former. Its essential image is the full subcategory described as follows:

For a $\mathbb{Z}_2$-equivariant $A$-local system $\mathcal{L}$ on $X^2$, consider its restriction $\mathcal{L}|_X$, where $X \overset{\Delta}{\hookrightarrow} X^2$. The datum of $\mathbb{Z}_2$-equivariance defines a homomorphism $\mathbb{Z}_2 \to A$ (for each connected component of $X$).

The sought-for essential image consists of those $\mathcal{L}$, for which the above homomorphism is trivial.

**1.4.** Here is now a variant of $\mathrm{G}(X^2/\mathbb{Z}_2)$, denoted $\mathrm{G}(X^2/\mathbb{Z}_2)'$, which is equivalent to $\mathrm{G}(X^{(2)})$:

We consider the category of triples $(\mathcal{G}_{X^2}, \mathcal{G}_X, \alpha)$, where $\mathcal{G}_{X^2}$ is a $\mathbb{Z}_2$-equivariant $A$-gerbe on $X^2$, $\mathcal{G}_X$ is an $A$-gerbe on $X$, and $\alpha$ is a $\mathbb{Z}_2$-equivariant isomorphism of $A$-gerbes

$$\mathcal{G}_{X^2}|_X \simeq \mathcal{G}_X.$$

*Date*: February 16, 2012.

**1.5.** Let us now consider the following variants of $\mathrm{G}(X^2/\mathbb{Z}_2)$, $\mathrm{G}(X^2/\mathbb{Z}_2)'$ and $\mathrm{G}(X^{(2)})$, denoted $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)$, $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)'$ and $\mathrm{G}(X^{(2)}; X^{(2)}-X)$, respectively: in each case we add a datum of a $\mathbb{Z}_2$-equivariant trivialization of the restriction of our gerbe to $X^2-\Delta(X)$.

Each of the above 2-categories $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)$, $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)'$ and $\mathrm{G}(X^{(2)}; X^{(2)}-X)$ is discrete: the categories of endomorphisms of objects are trivial. I.e., $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)$, $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)'$ and $\mathrm{G}(X^{(2)}; X^{(2)}-X)$ are sets (and in fact, abelian groups, due to the Picard structure). By the above, $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)'$ is equivalent to $\mathrm{G}(X^{(2)}; X^{(2)}-X)$.

Since every gerbe on a smooth algebraic variety $Y$ trivialized off a smooth divisor $D$ is canonically of the form $\mathcal{O}(D)^{log(a)}$ for $a \in A$, we obtain that $\mathrm{G}(X^{(2)})$ identifies with the abelian group $A$.

Similarly, $\mathrm{G}(X^2/\mathbb{Z}_2)$ identifies with the category of triples $(a_1, \xi, \beta)$, where $a_1 \in A$, $\xi$ is a $\mathbb{Z}/2\mathbb{Z}$-equivariant structure on the gerbe $\mathcal{O}(D)^{log(a_1)}$ and $\beta$ is the identification of this $\mathbb{Z}/2\mathbb{Z}$-equivariant structure with the tautological one over $X^2-\Delta(X)$. However, it is easy to see that, given $a_1$, the data of $(\xi, \beta)$ is uniquely recovered. So, we obtain that $\mathrm{G}(X^2/\mathbb{Z}_2)$ also identifies with the abelian group $A$.

Now, the map $\mathrm{G}(X^{(2)}; X^{(2)}-X) \to \mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2)$ is $a \mapsto a_1 = 2a$.

**1.6.** If we take $X$ to be the affine line, then the forgetful functors

$$\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2) \to \mathrm{G}(X^2/\mathbb{Z}_2) \text{ and } \mathrm{G}(X^{(2)}; X^{(2)}-X) \to \mathrm{G}(X^{(2)})$$

are essentially surjective at the level of objects.

For any $X$, pullback by $\pi$ maps the fibers of the functor $\mathrm{G}(X^{(2)}; X^{(2)}-X) \to \mathrm{G}(X^{(2)})$ isomorphically to the fibers of $\mathrm{G}(X^2/\mathbb{Z}_2; X^2-X/\mathbb{Z}_2) \to \mathrm{G}(X^2/\mathbb{Z}_2)$. These fibers are 1-categories, each being a principal homogenous space for the Picard category of $A$-local system on $X^{(2)}-\Delta(X)$.

## 2. Factorizable gerbes

**2.1.** We fix a lattice $\Lambda$. A factorizable $A$-gerbe on $(X, \Lambda)$ is by definition an assignment for every finite set $I$ equipped with a map $\lambda^I : I \to \Lambda$ of an $A$-gerbe $\mathcal{G}_{\lambda^I}$ on $X^I$, equipped with the following identifications.

**2.2. Compatibility on the diagonals.** For every surjection (including the case of isomorphisms) of finite sets $\phi : I \twoheadrightarrow J$, and $\lambda^I : I \to \Lambda$, let $\lambda^J : J \to \Lambda$ be the induced map:

$$\lambda^J(j) = \sum_{\phi(i)=j} \lambda^I(i).$$

The first piece of data on the collection

$$(I, \lambda^I) \mapsto \mathcal{G}_{\lambda^I}$$

is that of an equivalence of gerbes on $X^J$:

$$\alpha(\lambda^I, \phi) : \mathcal{G}_{\lambda^I}|_{X^J} \simeq \mathcal{G}_{\lambda^J},$$

where $X^J \to X^I$ is the closed embedding induced by $\phi$.

Furthermore, to a composition

$$I \overset{\phi}{\twoheadrightarrow} J \overset{\psi}{\twoheadrightarrow} K$$

we specify an isomorphism between the resulting two isomorphisms

$$\alpha(\lambda^I, \psi \circ \phi) \overset{\beta(\lambda^I, \phi, \psi)}{\Longrightarrow} \alpha(\lambda^J, \psi) \circ \alpha(\lambda^I, \phi)|_{X^K}$$

between the gerbes

$$\mathcal{G}_{\lambda^I}|_{X^K} \text{ and } \mathcal{G}_{\lambda^K}.$$

We require that the data of $\beta(\lambda^I, \phi, \psi)$ be compatible with three-fold compositions

$$I \twoheadrightarrow J \twoheadrightarrow K \twoheadrightarrow L$$

in a natural sense.

**2.3. The symmetrized version.** We claim that the data of

$$\{\mathcal{G}_{\lambda^I}, \alpha(\lambda^I, \phi), \beta(\lambda^I, \phi, \psi)\}$$

automatically define the symmetrized version of the gerbes mentioned in the talk.

For example, for an element $\lambda \in \Lambda$, consider $I = \{1, 2\}$ and $\lambda^I : I \to \Lambda$ be such that it sends both elements 1 and 2 to $\lambda$. Consider the corresponding gerbe $\mathcal{G}_{\lambda^I}$ on $X^2$. We claim that it canonically descends to a gerbe on $X^{(2)}$.

Indeed, the $\mathbb{Z}_2$-equivariant structure on $\mathcal{G}_{\lambda^I}$ is provided by the data of $\alpha(\lambda^I, \phi)$ and $\beta(\lambda^I, \phi, \psi)$ for $\phi, \psi \in \mathrm{Aut}(I)$.

Now, by Sect. 1.4, in order to descend $\mathcal{G}_{\lambda^I}$ to $X^{(2)}$ we need to specify a $\mathbb{Z}_2$-equivariant equivalence between $\mathcal{G}_{\lambda^I}|_X$ and a gerbe on $X$.

The required equivalence is provided by $\alpha(\lambda^I, \phi)$ for $\phi : \{1, 2\} \to \{1\}$. The data of equivariance on this equivalence is provided by the data of $\beta(\lambda^I, \phi, \psi)$.

The upshot is, that contrary to what was said at the talk, if the data of isomorphisms $\alpha(\lambda^I, \phi)$ is specified correctly, we automatically, obtain gerbes also in the symmetrized setting.

In particular, if we have a string of elements of $\Lambda$

$$\underbrace{\lambda_1, ..., \lambda_1}_{m_1}, \underbrace{\lambda_2, ..., \lambda_2}_{m_2}, ...., \underbrace{\lambda_n, ..., \lambda_n}_{m_n},$$

we obtain a gerbe on the corresponding partically symmetrized power of the curve

$$X^{(m_1)} \times X^{(m_2)} \times ... \times X^{(m_n)}.$$

**2.4. Factorization.** Let $\phi$ be again a surjective map $I \twoheadrightarrow J$ of finite sets, and let

$$X^I_{\phi,disj} \subset X^I$$

be the open subset consisting of points $x^I : I \to X$ for which

$$x^I(i_1) \neq x^I(i_2) \text{ if } \phi(i_1) \neq \phi(i_2).$$

Note that we have a tautological isomorphism of varieties

$$X^I \simeq \prod_{j \in J} X^{I_j},$$

where $I_j := \phi^{-1}(j)$. For a map $\lambda^I : I \to \Lambda$, and $j \in J$, let

$$\lambda^{I_j} : I_j \to \Lambda$$

be a map obtained by restriction.

The data of factorization is that of an isomorphism

$$\gamma(\lambda^I, \phi) : \mathcal{G}_{\lambda^I}|_{X^I_{\phi,disj}} \simeq \left( \underset{j \in J}{\boxtimes} \mathcal{G}_{\lambda^{I_j}} \right) |_{X^I_{\phi,disj}}.$$

Furthermore, to a composition

$$I \overset{\phi}{\twoheadrightarrow} J \overset{\psi}{\twoheadrightarrow} K$$

we need to specify an isomorphism $\delta(\lambda^I, \phi, \psi)$ between two resulting equivalences of the gerbes

$$\mathcal{G}_{\lambda^I}|_{X^I_{\psi\circ\phi, disj}} \text{ and } \left(\underset{k\in J}{\boxtimes} \mathcal{G}_{\lambda^{I_k}}\right)|_{X^I_{\psi\circ\phi, disj}}.$$

We require that the data of $\delta(\lambda^I, \phi, \psi)$ be compatible with three-fold compositions

$$I \twoheadrightarrow J \twoheadrightarrow K \twoheadrightarrow L$$

in a natural sense.

Now, we have to specify isomorphisms of equivalences that express the compatibility condition between the data of

$$\alpha(\lambda^I, \phi) \text{ and } \gamma(\lambda^{I'}, \phi'), \tag{2.1}$$

and these isomorphisms must satisfy natural conditions, separately, for compositions $\psi \circ \phi$ and $\psi' \circ \phi'$ with respect to the data of

$$\beta(\lambda^I, \phi, \psi) \text{ and } \delta(\lambda^{I'}, \phi', \psi'),$$

respectively. We leave it to the reader to write down whar the above equivalences are.

**2.5. Unit.** Suppose that $I = J \sqcup K$, and $\lambda^I : I \to \Lambda$ equal to $\lambda_0^J \sqcup \lambda^K$, where

$$\lambda_0^J : J \to \{0\} \subset \Lambda.$$

The last piece of data consists of an equivalence $\epsilon(\lambda^K, J)$ between $\mathcal{G}_{\lambda^I}$ and the pullback of the gerbe $\mathcal{G}_{\lambda^K}$ along $X^I \to X^K$.

Furthermore, for $I = J \sqcup K \sqcup L$, with $\lambda^I = \lambda_0^{J\sqcup K} \sqcup \lambda^L$, we need to specify an isomorphism $\zeta(\lambda^L, K, J)$ between the resulting two equivalences of the gerbe $\mathcal{G}_{\lambda^I}$ and the pullback of $\mathcal{G}_{\lambda^L}$ along $X^I \to X^L$. These isomorphisms must satisfy a natural condition with respect to unions

$$I = J \sqcup K \sqcup L \sqcup M.$$

**2.6. Further compatibilities.** We need to specify isomorphisms of equivalences that express the compatibility conditions between the data of

$$\alpha(\lambda^I, \phi) \text{ and } \epsilon(\lambda^{K''}, J''), \tag{2.2}$$

(and these equivalences must satisfy natural conditions, separately for compositions $\psi \circ \phi$ and unions $I'' = J'' \sqcup K'' \sqcup L''$).

We need to specify isomorphisms of equivalences that express the compatibility conditions between the data of

$$\gamma(\lambda^{I'}, \phi') \text{ and } \epsilon(\lambda^{K''}, J''), \tag{2.3}$$

(and these equivalences must also satisfy natural conditions, separately for compositions $\psi' \circ \phi'$ and unions $I = J \sqcup K \sqcup L$).

Finally, there is a last condition that involves the compatibility of the isomorphisms of the pairwise equivalences (2.1), (2.2) and (2.3).

## 3. The quadratic form

**3.1.** As was mentioned in the talk (with the correction from Sect. 1.5), to every datum of a factorizable gerbe there corresponds a quadratic form

$$Q : \Lambda \to A.$$

Let us describe the fibers of this map. However, since factorizable gerbes form a Picard 2-category, it suffices to describe the fiber over the 0 quadratic form, i.e., the 2-category of those factorizable gerbes, for which the associated quadratic form is 0.

**Lemma 3.2.** *The 2-category of factorizable gerbes whose associated quadratic form is 0 is canonically equivalent to the 2-category of gerbes on $X$ with respect to the abelian group $A \underset{\mathbb{Z}}{\otimes} \Lambda$.*

In a subsequent talk we will describe how to explicitly construct *a* gerbe on $X$ corresponding to a given quadratic form. The construction will not be completely canonical, but will depend on a choice that has to do with 2-torsion in $A$. The nature of this additional choice will be explained in the next subsection.

**3.3.** Consider the following complex of abelian groups

$$T^0 \to T^1 \to T^2, \tag{3.1}$$

where $T^2$ is the group of bilinear maps $\Lambda \otimes \Lambda \to A$; $T^1$ is the group of 2-cocyles on $\Lambda$ with coefficients in $A$; $T^0$ is the group of 1-cochains on $\Lambda$ with coefficients in $A$.

The map $T^0 \to T^1$ is the differential in the chain complex computing $H^\bullet(\Lambda, A)$. The map $T^1 \to T^2$ sends a cocycle $\psi$ to the bilinear form $B$ defined by

$$B(\lambda, \mu) = \psi(\lambda, \mu) - \psi(\mu, \lambda).$$

It is easy to see that the above complex is acyclic in degree 1; its 0-th cohomology identifies with

$$H^1(\Lambda, A) \simeq \mathrm{Hom}(\Lambda, A).$$

Its 2-nd cohomology identifies with the set of quadratic forms

$$\{Q : \Lambda \to A, \}$$

where a bilinear form $B$ gets sent to $Q$ with $Q(\lambda) = B(\lambda, \lambda)$.

In fact, we can replace the complex (3.1) by a quasi-isomorphic one, where we replace $T^1$ by the set $'T^1$ of bilinear forms $\Lambda \otimes \Lambda \to A$ (any such is a cocyle), and $T^0$ by the preimage of $'T^1$ in $T^0$.

**3.4.** As to any complex of amplitude $[0, 2]$, to the complex (3.1) there corresponds a canonically defined 2-category.

In a subsequent talk, we will prove:

**Theorem 3.5.** *Suppose $X = \mathbb{A}^1$. Then the 2-category corresponding to* (3.1) *is canonically equivalent to that of factorizable gerbes.*

Thus, according to the above theorem, in order to construct a gerbe on $X = \mathbb{A}^1$, corresponding to a given quadratic form $Q$ we need to do the following:

(1) For any choice of bilinear form $B$ with $Q(\lambda) = B(\lambda, \lambda)$ we obtain a factorizable gerbe $\mathcal{G}(B)$.

(2) For any two $B', B''$ with the same associated quadratic form, there exists $\psi \in T^1$, such that $B' - B'' = d(\psi)$, and for every choice of $\psi$ we obtain an equivalence
$$e(\psi) : \mathcal{G}(B') \to \mathcal{G}(B'').$$
(3) For any two $\psi', \psi'' \in T^1$ for which $\psi' - \psi'' = d(\phi)$ for $\phi \in T^0$, there exists an isomorphism between equivalences $e(\psi') \to e(\psi'')$.
(4) However, the choice of $\phi$ as above is not unique; the ambiguity is given by $\mathrm{Hom}(\Lambda, A)$. So, we obtain that in order to obtain a factorizable gerbe associated to $Q$ we need to trivialize a certain canonical $\mathrm{Hom}(\Lambda, A)$-gerbe over the point.

## 4. Braided monoidal categories attached to quadratic forms

In this section we take our group of coefficients $A$ to be the group of invertible elements $R^\times$ in a commutative ring $R$.

**4.1. Associated monoidal category.** As was mentioned in the talk, given a factorizable gerbe $\mathcal{G}$ on $X$, for each $(I, \lambda^I)$ we can consider the category
$$D(X^I)_{\mathcal{G}_{\lambda^I}}$$
of sheaves of $R$-modules twisted by the gerbe $\mathcal{G}_{\lambda^I}$.

As will be explained in the subsequent talks by Jacob, the data of
$$(I, \lambda^I) \mapsto D(X^I)_{\mathcal{G}_{\lambda^I}}$$
forms a factorizable category on $X$.

Moreover, as will be also explained by Jacob, when $X = \mathbb{A}^1$, the data of a factorizable category is equivalent to that of a braided monoidal category. We shall describe this braided monoidal category explicitly, following a letter by Deligne to Lusztig.

**4.2.** Thus, according to Theorem 3.5 there should be a map from the 2-category corresponding to the complex (3.1) to that of braided monoidal categories. We shall now describe this map explicitly.

Let $R\text{-mod}^\Lambda$ be the category of $\Lambda$-graded $R$-modules, with its natural symmetric monoidal structure.

Consider the complex
$$T^0 \to T^1.$$

By construction, it corresponds to a the Picard category of central extensions
$$1 \to R^\times \to \widetilde{\Lambda} \to \Lambda \to 1,$$
which can be also identified with the Picard category $\mathrm{Aut}(R\text{-mod}^\Lambda, \otimes)$ of $R$-linear automorphisms of $R\text{-mod}^\Lambda$ as a *monoidal* category, which induce the identity map at the level of isomorphism classes of objects.

Now, it is easy to see that the group of bilinear maps
$$\{B : \Lambda \otimes \Lambda \to R^\times\}$$
identifies with the (discrete) 2-category of braided monoidal structures extending the monoidal structure on $R\text{-mod}^\Lambda$.

Hence, the complex (3.1), which represents the 2-category
$$\mathrm{Hom}(\Lambda \otimes \Lambda \to R^\times)/(T^0 \to T^1),$$

maps to the 2-category of braided monoidal categories, by forgetting the data of monoidal equivalence with $R$-mod$^{\Lambda}$.