Chiral algebra
Gauge theory tools for the Analytic Geometric Langlands program

Davide Gaiotto
Structure of the talk

• An intertwining kernel for Gaudin Hamiltonians
• The SL(2) addition kernel $K(a,b,c)$
• A Vertex Algebra interpretation of $K(a,b,c)$
• Hitchin Hamiltonians intertwining properties
• Hecke operators intertwining properties
• Arakawa’s Vertex Algebras and addition (multiplication) kernels
Appetizer: a Gaudin intertwiner

A peculiar formula

• Gaudin Hamiltonians for half-densities on \((\mathbb{C}P^1)^n\)

\[
H_i^{(a)} = \sum_{j \neq i} \frac{e_if_j + 2h_ih_j + f_ie_j}{z_i - z_j} \quad f_i = \partial_{a_i} \quad h_i = a_i\partial_{a_i} + \frac{1}{2} \quad e_i = -a_i^2\partial_{a_i} - a_i
\]

• Intertwining condition:

\[
H_i^{(a)} K(a, b, c) = H_i^{(b)} K(a, b, c) = H_i^{(c)} K(a, b, c)
\]

\[
\bar{H}_i^{(a)} K(a, b, c) = \bar{H}_i^{(b)} K(a, b, c) = \bar{H}_i^{(c)} K(a, b, c)
\]

• Algebraic solution:

\[
K(a, b, c) = \frac{1}{\det_{n \times n} A}
\]

\[
A^i_j = \frac{(a_i - a_j)(b_i - b_j)(c_i - c_j)}{z_i - z_j} \quad i \neq j
\]

\[
A^i_i = 0
\]
The SL(2) addition kernel
Gaudin $\rightarrow$ Hitchin

* Genus 0, parabolic points $z_i$, $\text{Bun}_0$

  * Lines $(1, a_i)$ at parabolic points $z_i$, modulo SL(2)

  * Gauge-fix three points: $a_1$, $a_2$, $a_3$, etc.

  \[
  \sqrt{d\mu_a} = |a_1 - a_2||a_1 - a_3||a_2 - a_3|d\mu_4 d\mu_5 \cdots d\mu_{n+3}|
  \]

  \[
  K(a, b, c) = \frac{1}{|\det A|} \sqrt{d\mu_a} \sqrt{d\mu_b} \sqrt{d\mu_c}
  \]

* $K(a,b,c)$ intertwines Hitchin’s Hamiltonians on $\text{Bun}_0 \times \text{Bun}_0 \times \text{Bun}_0$

* Claim: also intertwines aGL Hecke operators!
The Lame’ addition kernel
Genus 0, four points

- Fix \( z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z \)

- Gauge-fix \( a_1 = 0, a_2 = 1, a_3 = \infty, a_4 = a \) etcetera.

- Hitchin Hamiltonians -> Lame’ operator

\[
K(a, b, c) = \frac{1}{|det A|} |da||db||dc|
\]

\[
det A \sim 1 + \frac{a^2b^2c^2}{z^2} + \frac{(1-a)^2(1-b)^2(1-c)^2}{(1-z)^2} - 2\frac{abc}{z} - 2\frac{(1-a)(1-b)(1-c)}{1-z} - 2\frac{abc(1-a)(1-b)(1-c)}{1-z}
\]

- Why addition: \( det A=0 \) is support of addition along smooth fibers of Hitchin system, with zero at \( a=z \)

\[
x^2 = a(a-1)(a-z)u \quad \omega = \frac{dadu}{x} = \frac{dadx}{a(a-1)(a-z)}
\]
The SL(2) addition kernel
Genus 2 and higher, no parabolic points

\[ K(a, b, c) = \frac{1}{\left| \det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{1/2}} \right|} \]

- E: rank 2 associated bundle
- Singular at Theta divisor where \( E_a \otimes E_b \otimes E_c \otimes K^{1/2} \) has sections
- Parabolic points: build A from Green’s function

\[ K(a, b, c) = \frac{1}{\left| \det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{1/2}} \right|} \frac{1}{| \det A |} \]
Preliminary facts
The magic of $2 \times 2 \times 2 = 8$

- Hamiltonian action of $\text{SL}(2) \times \text{SL}(2) \times \text{SL}(2)$ on $\mathbb{C}^8$
- Quadratic moment maps $\mu_a, \mu_b, \mu_c$
- Unique invariant quartic polynomial: $\text{Tr} \, \mu_a^2 = \text{Tr} \, \mu_b^2 = \text{Tr} \, \mu_c^2$
- Notation: coordinates $s^{\pm \pm \pm}$
- Nice Lagrangian submanifold: $s^{\pm \pm \pm} = a^{\pm} b^{\pm} c^{\pm}$
- Vanishing moment maps
- Useful quantization

$$\delta^{(2)} \left( \frac{\det s^{\beta \gamma}}{\beta \gamma} \right) \simeq \delta^{(2)} \left( \frac{\det s^{\alpha + \gamma}}{\alpha \gamma} \right) \simeq \delta^{(2)} \left( \frac{\det s^{\alpha \beta +}}{\alpha \beta} \right)$$
Trifundamental Symplectic Bosons
Aka 4 beta-gamma systems

- Symplectic boson VOA attached to symplectic vector space $\mathbb{C}^8$
  \[ Z^{\alpha\beta\gamma}(z)Z^{\alpha'\beta'\gamma'}(w) \sim \frac{\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}}{z - w} \]
- $\hat{\mathfrak{sl}}(2)$ Kac-Moody subalgebras at critical level
  \[ J_a = \mu_a[Z], \quad J_b = \mu_b[Z], \quad J_c = \mu_c[Z] \]
- Centers of Kac-Moody subalgebras coincide!
  \[ \text{Tr} \, \mu_a^2[Z] = \text{Tr} \, \mu_b^2[Z] = \text{Tr} \, \mu_c^2[Z] \]
The non-chiral symplectic boson
A free theory

\[ Z(E_a, E_b, E_c) = \int DZ \, D\bar{Z} e^{\int_c (\bar{Z}, \partial Z) - (Z, \partial \bar{Z})} = \frac{1}{\left| \det \partial_{E_a \otimes E_b \otimes E_c \otimes K^{1/2}} \right|} \]

\[ \langle Z^{\alpha \beta \gamma}(z) Z^{\alpha' \beta' \gamma'}(w) \rangle_{E_a, E_b, E_c} = \int DZ \, D\bar{Z} e^{\int_c (\bar{Z}, \partial Z) - (Z, \partial \bar{Z})} Z^{\alpha \beta \gamma}(z) Z^{\alpha' \beta' \gamma'}(w) = \frac{G^{\alpha, \beta; \gamma; \alpha', \beta', \gamma'}(z, w)}{\left| \det \partial_{E_a \otimes E_b \otimes E_c \otimes K^{1/2}} \right|} \]

- \( Z \) is a section of \( E_a \otimes E_b \otimes E_c \otimes K^{1/2} \)
- \((\ , \ )\) is symplectic pairing
- Partition function is an half-density on Bun x Bun x Bun
Sugawara vectors and Hitchin Hamiltonians

A central point

\[ H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = \langle \text{Tr} \mu_a^2[Z] \rangle \]

\[ \begin{align*}
H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] &= H_b(\bar{z}) \cdot \mathcal{Z}[E_a, E_b, E_c] = H_c(z) \cdot \mathcal{Z}[E_a, E_b, E_c]
\end{align*} \]

• \( K(a,b,c) = Z(a,b,c) \) intertwines Hitchin’s Hamiltonians!

• Easy to generalize to real bundles: chiral algebra on surface with involution
Regular punctures
A nice singularity

\[ V(z, \bar{z}) = \int dv d\bar{v} e^{vZ^{+++}(z) - \bar{v}Z^{+++}(\bar{z})} \]

\[ H_a(z; w) \cdot \langle V(w, \bar{w}) \rangle_{E_a, E_b, E_c} = \langle \text{Tr} \mu_a^2[Z](z) V(w, \bar{w}) \rangle \]

- Insertion of \( V \) requires parabolic structure at \( z \)
- Insertion of \( V \) induces simple poles in \( \text{Tr} \mu_a^2[Z](\bar{z}) \)
- Addition kernel with unipotent regular singularities is a correlation function of \( V \)'s

\[ K(a, b, c) \equiv \langle V_{a_1, b_1, c_1}(z_1) \cdots V_{a_n, b_n, c_n}(z_n) \rangle = \int \prod_i |dv_i|^2 e^{v \cdot G(z_i, z_j) \cdot v - \text{c.c.}} \]
Regular punctures of general weight
A Mellin transform

- Deformed vertex operator

\[ V_{\lambda,\bar{\lambda}}(z, \bar{z}) = \int dv d\bar{v} \, v^\lambda \bar{v}^{-\bar{\lambda}} \, e^{vZ^{++} + \bar{v}\bar{Z}^{++}}(z) \]

- Correlation function is Mellin transform of Gaussian:

\[ K(a, b, c) = \int \prod_i |dv_i|^2 \, v_i^\lambda \bar{v}_i^{-\bar{\lambda}} \, e^{v_i \cdot G(z_i, z_j) \cdot v} - c.c. \]
Hecke operators and spectral flow modules
A role for Weyl modules

\[ H_a(z, \bar{z}) \cdot \mathcal{Z}[E_a, E_b, E_c] = \langle S_a(z, \bar{z}) \rangle_{E_a, E_b, E_c} \]

\[ S_a(z, \bar{z}) = S_b(z, \bar{z}) = S_c(z, \bar{z}) \]

\[ \partial_z^2 S_a(z, \bar{z}) + H_a(z) S_a(z, \bar{z}) = 0 \]

- Minimal Hecke modification makes \( \mathcal{Z}(z) \) anti-periodic.
- Hecke operator inserts a special Ramond module for symplectic bosons

\[ Z^{\alpha \beta \gamma}(z) S_a(0, 0)[v] \sim \frac{S_a(0, 0)[c^{\alpha \beta \gamma} v]}{z^{\frac{1}{2}}} + \cdots \]

\[ [c^{\alpha \beta \gamma}, c^{\alpha' \beta' \gamma'}] = \epsilon^{\alpha \alpha'} \epsilon_{\beta \beta'} \epsilon_{\gamma \gamma'} \]
A wonderful Weyl module
Quantizing the nice locus

• Non-chiral description:

$$S_a = S_a \left[ \int |dt|^2 e^t \det_{2 \times 2} u - c \cdot c \right]$$

$$c^{-} = u \cdot c^{+} = \partial_u$$

• Secretly symmetric

$$\mu_a[c] \circ \int |dt|^2 e^t \det_{2 \times 2} u - c \cdot c = 0$$

$$\mu_b[c] \circ \int |dt|^2 e^t \det_{2 \times 2} u - c \cdot c = 0$$

$$\mu_c[c] \circ \int |dt|^2 e^t \det_{2 \times 2} u - c \cdot c = 0$$
Arakawa’s VOA
Chiral algebras of class S, Sicilian boundary VOAs

• Chiral algebras $V_k[g]$ labelled by ADE Lie algebra and integer
  • $k$ copies of critical Kac-Moody subalgebras, glued along center
• Conformal blocks of $V_k[g]$ as twisted D-module on Bun $\times$ Bun $\ldots$ $\times$ Bun
  • GL dual to structure sheaf of diagonal
  • Obviously intertwines Hitchin Hamiltonians
• Non-chiral CFTs?
Operations on Arakawa’s VOA

Gluing or removing

- Whittaker reduction of one copy \( V_k[\mathfrak{g}] \to V_{k-1}[\mathfrak{g}] \)
- Diagonal quantum DS reduction \( V_k[\mathfrak{g}] \times V_{k'}[\mathfrak{g}] \to V_{k+k'-2}[\mathfrak{g}] \)
  - Non-chiral version of quantum DS reduction: 2d gauge theory
    \[
    Z_{k+k'-2,\mathfrak{g}}(a_1, \cdots, a_{k-1}, b_2, \cdots, b_{k'}) = \int_{\text{Bun}} d\mu_c Z_{k,\mathfrak{g}}(a_1, \cdots, a_{k-1}, c) Z_{k',\mathfrak{g}}(c, b_2, \cdots, b_{k'})
    \]
  - Still associative? Tested for \( V_4[5\mathfrak{l}_2] \) on 4pt sphere
- Wavefunction decomposition? \( Z_{k,\mathfrak{g}}(a_1, \cdots, a_k) = \sum_{\rho \sim \bar{\rho}} C_{\rho,\bar{\rho}}^{k-2} \prod_i \psi_{\rho,\bar{\rho}}(a_i) \)