Joint with Peter Scholze

Starting point: Naive question. Is there some "relative" variant of perverse sheaves?

Setup: Fix a prime $l$. Always work with $\mathbb{Z}/l\mathbb{Z}$-schemes. (All schemes g.c.q.s.)

Three scenarios we consider:
A) Let $\Lambda$ be an $\ell$-torsion ring, \( \mathbb{D}(\Lambda) = \text{left completion of } \mathbb{D}(\text{ét}, \Lambda) = \lim_{\leftarrow n} \mathbb{D}(\Lambda) \).

B) $\Lambda$ as in A, but look at $\mathbb{D}(\text{cons}(\Lambda))$.

C) Take $L/\mathbb{Q}_\ell$ an alg. extension, look at $\mathbb{D}(\text{cons}(\Lambda, L) \circ \Omega_L) \subset \mathbb{D}(\text{prét}(\Lambda))$.

Just write $\mathbb{D}(\Lambda)$ for the relevant category in each scenario.
Theorem (H. - Scholze). Let $X \to S$ be a finitely presented map. Let $D(X)$ be as in scenarios A) - C) (In scenario C, assume that all countr. subsets of $S$ have fin. many irreducible compnts.) Then there is a t-structure $(P/5D < 0(X), P/5D \geq 0(X))$ on $D(X)$ characterized by the condition that $A \in D(X)$ lies in $P/5D \leq 0$ resp. $P/5D \geq 0$ iff $V \to S$, with $\text{fib.} X_S \xrightarrow{h} X$

$h^* A$ lies in $P D < 0(X_S)$ resp. $P D \geq 0(X_S)$.

($h^*$ for both!!)
This $t$-structure interpolates between two extremes:

i) If $X \rightarrow S = X$ is the identity, just get the \underline{standard} $t$-structure on $D(X)$ or \underline{naive}

ii) If $S = Spec k$ is art, get the usual perverse $t$-structure on $D(X)$.

In general, this $t$-structure has no good "finite length" properties. However, it does have good properties along these lines after imposing another condition, namely the condition of being ULF.
Fix $f : X \to S$, $A \in \mathcal{D}(X)$.

Intuition: $A$ is universally locally acyclic (ULA) wrt $f$ if the cohom of $A$/slices of $A$ over small ball in $X$ is constant as the slices vary.

Definition: $A$ is ULA wrt $f$ if $\forall x \to x', \quad \exists \phi(x')$ specialization, the nat. map

$$\mathcal{R}(X_x, A) \to \mathcal{R}(X_{x'} S_f(x), A)$$

is an isom., and likewise after any base change on $S$.

Key things: 1) $X \to S$ smooth, $\Lambda$ is ULA.

2) In reasonable situations, "any" sheaf is ULA over a dense open in the target.
Thm. Fix $f: X \to S$ as before.

If $A \in \mathcal{D}(X)$ is $f$-ULA, then all relative perverse truncations of $A$ are $f$-ULA.

$\Rightarrow$ \text{Perv}(X/S) = \text{heart of rel. } \mathcal{t}\text{-structure on } \mathcal{D}(X)$

comes with full subcat. $\text{Perv}^{\text{ULA}}(X/S)$.

2) $\text{Perv}^{\text{ULA}}(X/S)$ is stable under relative Verdier duality (i.e. $R\mathcal{H}om(-, f^!\mathcal{A})$).

$\Rightarrow$ Eq. in case $C$ with $n=2$, $\text{Perv}^{\text{ULA}}(X/S)$ is noetherian and Artinian.

3) $\text{Perv}^{\text{ULA}}(X/S)$ stable under subquotients.
Key special case of main theorem:

$S = \text{Spec } V$, $V$ a rank one valuation ring with rank one algebraically closed fraction field ("IC valuation ring") absolutely integrally closed.

$|S| = \varnothing$, $X \to S$ as before, $j: X_\eta \to X$, $i: X_S \to X$ as usual.

By a result of Lurie, always can define a $t$-str.

whose connective part consists of sheaves which lie in $\mathcal{P}_{D^\infty}(\text{every fiber})$.

In the present case, get $\mathcal{P}_{D^{\geq 0}}(X) = A \in D(X)$ s.t. $j^*A \in \mathcal{P}_{D^\infty}(X_\eta)$ and $i^*A \in \mathcal{P}_{D^{\geq 0}}(X_S)$.

$\Rightarrow$ By general nonsense, $A \in D^\geq 0$ iff $j^*A \in \mathcal{P}_{D^{\geq 0}}(X_\eta)$ and $i^*A \in \mathcal{P}_{D^{\geq 0}}(X_S)$.

Claim: The latter pair of conditions is equivalent to: $j^*A$, $i^*A \in \mathcal{P}_{D^{\geq 0}}$. 
Key idea: Look at the exact triangle

\[ i_* A \to i_* A \to j_* j^* A \to \]

Assume \( j^* A \in D^+ \) \( \P_\leq 0 \). Then \( i^*_j \cdot D(X_0) \to D(X_3) \)

is the nearby cycles functor, which in particular

is perverse t-exact!

Tim of Gabber
(Illusie '94)

The idea in general case is to reduce to this (very!) special case by descent arguments.

For this, we need very fine topologies.
Recall: $V$-topology $\leq$ top. of universal submersions $\leq$ arc-topology

$\text{X} \rightarrow Y$ cover if $Y \text{ Spec } V \rightarrow Y$, $V$ val ring, can lift after replacing $V$ by some $V'/V$ faithfully flat

$Y \preceq \text{Spec } V' \\ \downarrow \\ \text{Spec } V$

$X \rightarrow Y$ univ. submersion if $|X| \rightarrow |Y|$ is a gr. map after any base change on $Y$.

Thm (Bhatt-Mathew, Gabber). In each of scenarios A)–c), $\text{Im } (\text{Bhatt-Mathew, Gabber})$. In each of scenarios A)–c), it is an arc-sheaf (Bhatt-Mathew)

In scenarios B)–c), it is an arc-sheaf (Bhatt-Mathew)

In scenario A), it is a universally submersive sheaf (Gabber).
Idea of Pf: 1) \( S = \text{Spec } V \) is aic valuation ring. OK.

2) \( S = \text{Spec } V \) aic val. ring. Reduce to the previous case by approx. \& descent.

3) \( S \) has each count'd comp. \( = \text{Spec } V, V \) aic val. ring.

Reduce to previous case by pure top. +

Lemma: "the perverse coh. amplitude is a constructible \( f_0 \) on the base."

4) General \( S \). Key pt: Can pick a \( U \)-hypercover \( S_0 \to S \) with all \( S_n \) as in 3). Then already have t-str. you want on all \( \mathcal{D}(X \times S_n) \). and the pullbacks \( \mathcal{D}(X \times S_n) \to \mathcal{D}(X \times S_m) \) are rel. perp. t-exact.

\( \implies \) Formal to get the desired t-str. on \( \mathcal{D}(X) = \lim \mathcal{D}(X \times S_n) \).
Second, if $S$ is a $\mathbb{G}_m$-scheme, or if dominates gen. pt.

$\text{Perv}^\text{ul}(X/S) \to \text{Perv}(X_s)$ fully faithful