GLC with restricted variation:

Conjecture: \( X = \text{curve}/k, \ G = \text{reductive}/E \)

\[
\text{Sh}_W(\text{Bun}_G) \simeq \text{IndCoh}_{\text{Nilp}}(\text{LocSys}^\text{vs}(G)).
\]

One of 3 contexts:

Spaces/\( k \), sheaves/\( E \):

1) \( l \)-adic: \( E = \overline{Q}_l, \ l \neq \text{char} \ k \)
2) de Rham: \( E = k, \ \text{char} \ k = 0 \)
3) Betti: \( k = \mathbb{C} \)
Automorphic Sheaves $(\text{Bun}_G)$:

Ind-completed derived category of sheaves on $\text{Bun}_G - / s.\text{supp} \subset \text{Nilp}$

Automatically holonomic

Galois

Ind$\text{Coh}_{\text{nilp}}(\text{LocSys}_G)$

$\text{Qcoh}$ renormalized.

Loc $\text{Sys}_G$ = moduli stack of local systems with restricted variation.

Kittens

Lagrangian $\text{Nilp}$

Stiiggs bundle?

$\mathcal{J} \times \text{Bun}_G$ $s$-equivalence classes $G$
Suppose $F \in \text{Shv}(\text{Bun}_G)$. For any $\nu \in \text{Rep}(G)$, we have $\mathcal{H}_\nu(F) \in \text{Shv}(\text{Bun}_G \times X)$.

Example: $F = \text{Hecke} \text{- eigensheaf}$ for eigenvalue $\varepsilon \in \text{LocSys}_G$.

$\mathcal{H}_\nu(F) = F \boxtimes V_\varepsilon$. 

By Satake, "universal" Hecke functor.
Theorem (Nadler-Yun):

If $F \in \text{Shv}(\text{Bun}_G)$, then

$$
\forall V : \text{Sing Supp}(\mathcal{H}_V(F)) \cap \text{Nil}_p \times \{0\} \subset T^* X.
$$

(i.e., $\mathcal{H}_V(F)$ is "lisse" along $X$)

Remark: It is in fact true that $\mathcal{H}_V(F) \in \text{Shv}_{\text{Nil}_p}(\text{Bun}_G) \otimes \text{Qlisse}(X)$

\[ F \quad \text{Bun}_G \]

\[ \mathcal{H}_V(F), \text{Bun}_G \times X \]

\[ \text{Nil}_p \times \{0\} \subset T^* X \]

"completed" l-systems on $X$. 
Theorem (converse claim)

Suppose \( F \in \text{Shv} (\text{Bun}_G) \) satisfies \( \forall V \in \text{Rep}(G) \)

\( \text{Sing Supp} \mathcal{H}_V (F) \subset T^* \text{Bun}_G \times \mathbb{P}^3 \) \quad \text{(Hecke-lisse)}

Then \( F \in \text{Shv}_{\text{Nilp}} (\text{Bun}_G) \).

Corollary All Hecke eigen-sheaves are in \( \text{Shv}_{\text{Nilp}} (\text{Bun}_G) \).
Proof of Thm: manipulations with $\text{Sing Supp}$ $\overline{S}$ $\overline{t}$$\overline{p}$$\overline{s}$ $\overline{u}$$\overline{n}$$\overline{g}$- $\overline{H}$$\overline{i}$$\overline{g}$$\overline{s}$ bundles.

Technical issue: need estimates on $\text{Sing Supp} \overline{H}^\cdot \overline{V}(F)$ (in terms of $\text{Sing Supp}(F)$) from below (while Nadler-Yun direction needs estimates from above).

Q.E.D.
Side remark

The "usual" Geometric Langlands conjecture (de Rham)

\[ \text{D-mod} (\mathcal{Bun}_G) \rightarrow \text{IndCoh}_\text{alg} (\text{locSys}_C) \]

must respect Hecke action (better: put together as action of \( \mathcal{O}_{\text{coh}}(\text{locSys}_C) \))

So (not surprisingly), "usual" conjecture \( \Rightarrow \) restricted conjecture

\( \text{de Rham} \) essentially equivalent.
2. **Spectral decomposition**: "Hecke-lisse things live over \( \text{LocSys} \)"

- **Reminder**: \( C = \text{category} \)
- **Bethi**: Action of \( \text{Rep}(G) \)
- **de Rham**: on \( C \) by Hecke functors
  
  (compatible with fusion)

- **Rem**: Pack this action as
  
  \( V_n \): act by Hecke functors at \( n \) points

  + compatibility with restriction to diagonals.
  
  \[ \mathbb{C}^n \to \mathbb{C}^n \]
Spectral decomposition. Theorem:

Let $C$ be a category together with a restricted action of $\text{Rep}(G)$.

1.Nice enough

2. $\forall I, J \in \mathcal{X}$, given $\text{Rep}(G)^{\text{op}} \otimes C \to C \otimes \text{Qlisse}(X)$ is compatible with $\otimes$.

3. Compatible with $I \otimes J$.

C-valued $\text{L}$-systems on $X$.

I.e., monoidal functor $\text{Rep}(G)^{\text{op}} \otimes \text{End}(C) \otimes \text{Qlisse}(X)$. 

(lisse action?)
Then $C$ acquires an $\mathcal{O}_{\text{loesys}}(\text{locSys}_G) 	o C$ (encodes all Hecke functors via tautological bundles: $\text{Rep}(G) \to \mathcal{O}_{\text{loesys}}(\text{locSys}_G(x)) \otimes \mathcal{Q}_{\text{lisser}}(x) \otimes$).

Proof Technical, but main idea is simple: (and familiar)

If $\text{locSys}_G$ were affine, we would need to construct $K[\text{locSys}_G]$ from input date (restricted action).
Technicalities:

\[ \operatorname{LocSys}_G \text{ is not affine.} \]

Instead, it is

\[ \bigcup \text{ formal affine schemes} / G \]
Why "Spectral decomposition"?

Re: \( \text{locSys}_G^{\text{res}} = \bigcup \text{locSys}_G^{\text{res}} \)_semisimple

Hence: Any \( C \) decomposes

\[ C = \bigoplus C^g \]

\( C^g \) lives over \( \text{locSys}_G^{\text{res}} \),

with restricted action of \( \text{Rep}(G) \).

E.g.: \( G \) is irreducible \( \rightarrow C^g \) is almost "eigenspace"
In particular, applying to

\[ C = \text{Shr}_{\text{Nilp}}(Bun_G) \]

 Automorphic.

So \( \text{Shr}_{\text{Nilp}}(Bun_G) \)

\( \subset \) \( (\text{Shr}_{\text{Nilp}}(Bun_G))^c \)

[Frankly: This is not immediate for technical reasons, but still true]
E.g., $\mathfrak{a}$ is irreducible:

**Galois side:**

\[ \text{LocSys}_{\mathfrak{a}, G} \] is formal:

\[ (\text{LocSys}_{\mathfrak{a}, G})^{\text{red}} = pt/\text{Aut}. \]

\[ \mathcal{Q} \text{Coh}(\text{LocSys}_{\mathfrak{a}, G}) \]

is generated by

\[ \mathcal{Q} \text{Coh}(\text{LocSys}_{\mathfrak{a}, G}^{\text{red}}) \]

**Automorphic side:**

\[ G \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_{G}) \]

generated by $G$-eigen objects.
Construction of spectral projector (Beilinson).

Fix $F \in \text{LocSys}_G$.

Define

$$R_F : \text{Shv}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)$$

s.t.

1) $R_F(-)$ is eigensheaf for $F$

2) $R_F$ is universal:

(left adjoint to

$$\begin{array}{ccc}
\text{Shv}(\text{Bun}_G) & \to & \text{Shv}(\text{Bun}_G) \\
F \text{-eigen} & \to & \text{("inclusion" of eigenspace)} \\
\end{array}$$

not really.}
$R_f$ defined explicitly via integrals of Hecke functors corresponding to regular rep. Since $R_f(-)$ has eigenproperty, $R_f(-) \in \text{Shv}_{\text{nilp}}(\text{Bun}_G)$.

Moreover: $R_f(-)$ for varying $F$ almost generate $\text{Shv}_{\text{nilp}}(\text{Bun}_G)$.
Imagine there's no $S$-equivalence and no reducible local systems, too.

**Galois side**

\[ \text{Loc Sys}_{\text{res}} = \coprod \text{Loc Sys}_{\text{res}}, \]

\[ \Omega \text{Loc} \left( \text{Loc Sys}_{\text{res}} \right) = \bigoplus \Omega \text{Loc} \left( \text{Loc Sys}_{\text{res}}^{\text{red}} \right) \]

\[ \bigoplus \Omega \text{Loc} \left( \text{Loc Sys}_{\text{res}}^{\text{red}} \right) \]

**Automorphic side**

\[ \bigoplus \text{Shr}_{\text{Nilp}} \left( \text{Bun}_G \right) \]

(Spectral decomposition)

\[ \bigoplus \text{Shr}_{\text{Nilp}} \left( \text{Bun}_G \right) \]

generated by Generalized eigenspaces

\[ \bigoplus \text{Shr}_{\text{Nilp}} \left( \text{Bun}_G \right)_{\text{6-eig}} \]

Eigenspaces.
Because they are left-adjoint.

In fact, this works even with \( S \)-equivalence:  
Correction: \((\text{LocSys}_{\text{res}}, \text{red}) \) is not discrete.

Replace \( F : \text{LocSys}_{\text{res}} \) with \( S \to \text{locSys}_{\text{res}} \) scheme.
Corollaries:

1. \( \text{Shv}_{\text{-nilp}}(\text{Bun}_G) \) is compactly generated by objects \( R(\delta_p), S - \text{locSys}_P, S \in \text{Bun}_G \)
In de Rham setting, $D_{\text{mod,}\text{nilp}}(\text{Bun}_G)$ has regular singularities (because generators do).

Remark: Restricted correspondence: $(\mathcal{G})$

\[ \text{Betti} \]
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) = \text{IndCoh}_{\text{Nilp}}(\text{locSys}^{\text{reg}}_{\text{Betti}}) \]

\[ \text{de Rham} \]
\[ D_{\text{mod,}\text{nilp}}(\text{Bun}_G) = \text{IndCoh}_{\text{nilp}}(\text{locSys}^{\text{reg}}_{\text{dR}}) \]
Rem (last time, Dennis):
In Betti contexts, it should be easier to show that $\text{Shr}(\text{Bun}_G)$ is a top. invariant of $X$.

Outline (conjectural):
Use $R_s(S)$ as generators in $\text{Shr}(\text{Bun}_G)$, $\text{Hom}$'s between them are locally constant.

($\text{then for co-dim constructible objects as in Ben-Zvi-Nadler}$)

$X$ varies

(Requires: using theory for variable $X$ (and f-dims!))