Multiplication kernels,
part IV
preview of talk by
Maxim Kontsevich
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let $C$ be a compact smooth curve over $C$
the compactness assumption is not really necessary at the beginning
let $S = T^*C$, symplectic variety.

We will define a certain partial compactification $T^*_C$ such that the complement in a disjoint $P \setminus T^*_C$
Union of affine lines, $\mathbb{A}^1$, and the symplectic form $\omega$ on $T^*C$ extends with first order poles on these $\mathbb{A}^1$ (so we have a Poisson structure on $\mathcal{P}$ with first order zeros on $\mathcal{O} - S$). These $\mathbb{A}^1$ components will correspond to pairs $(p \in \mathcal{C}, \text{"irregular term"})$

$$\exp\left(\sum_{i=1}^{s} c_i \frac{-X(i)}{r_i}\right)$$

where $c_i \in \mathbb{C}$ and $r_i > 0$ are rational numbers, $r_1 < r_2 < \ldots < r_s$. Here $X$ is a local coordinate on $C$ near $p$. 
Recall that such irregular terms classify connections on the punctured formal disk with possibly irregular singularities. (Hukuhara - Leveque - Turritin theorem). Namely, such a connection which is semisimple is generated as a D-module by

$$\exp\left(\sum_{i=1}^{n} c_i x^{z_i}\right) \times e^\lambda, \quad \lambda \in \mathbb{C}/\mathbb{Z}$$

where the rank of the connection is the common denominator of $z_i$.

**Example.** Consider $f = e^{-\frac{1}{2}z}$. 
Then \( \partial f = -\frac{1}{2} x^{-\frac{3}{2}} f. \) = \( g \)

Also \( \partial g = \frac{3}{4} x^{-\frac{5}{2}} f + \frac{1}{4} x^{-3} \),

\[ -\frac{3}{2} x^{-1} g + \frac{1}{4} x^{-3} f. \]

So we have

\[ \partial (f, g) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4 x^3} & -\frac{3}{2 x} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \]

Geometrically, this compactification is defined as follows.

Consider \( \overline{S} = \mathbb{P}(T^* C \oplus \mathcal{O}) \), the \( \mathbb{P}^1 \)-bundle on \( C \). We have the divisor at \( \infty \), \( \infty \in \overline{S} \), and the form \( \omega \) extends to \( \overline{S} \) with 2-nd order pole at \( \infty \). Now given
a point $p \in C$, let us blow up the corresponding point $p_0 \in C_{\infty}$. When we do, we will obtain an exceptional divisor on which the form will have a 1st order pole. We now keep blowing up.

In doing so, we should note that when we blow up the intersection of two curves where $C$ has poles of order $[m, n]$, then on the exceptional divisor we get it will have pole
of order $|m+n-1|$. So we have:

So after a few steps we can get a picture like this:

One can show that the "irregular terms" give rise to such a sequence of blow-ups. (Such term is attached to every component with 1)
(this will be explained below). Note that every component with first order pole is automatically isomorphic to \( A' \), it can intersect only with one component. This is easy to see by induction, since we will not blow up points of such components which are not intersections with another component; otherwise we’ll get a component with regular form \( w \), which we don’t want. 

**Definition** \( P = S \cup L_1 A^1 \) is the union of \( S \) with
these $A_1$'s (having 1st order pole of $w$). I.e., we throw away the components where $w$ has higher pole (including $C_0$, where it has a pole of order 2).

Let us now explain how these blow-ups are related to the functions

$$e^{\sum c_i x^{-r_i}} x,$$

mentioned above.

To this end, consider the symplectic form $\omega$ around such $A_1$, and let us note that this $A_1$ is given locally by the equation
Let $\lambda$ be the second coordinate such that the symplectic form locally has the form
\[ \omega = d\lambda \wedge \frac{dx}{x}. \]

However, the map $x : \mathbb{P} \to \mathbb{C}$ may be ramified around our index $[r]$. In this case, the coordinate $x$ should be replaced by $x' = u$. So we have
\[ \omega = d\lambda \wedge \frac{dx}{x} = \frac{1}{r} d\lambda \wedge \frac{du}{u}. \]

So the canonical 1-form
$\eta = pdx$ on $T^*C$ such that
$\eta = \omega$ extends as
$\eta = pdx = (\lambda + f(x^{1/\alpha})) \frac{dx}{x}$, where $f$ is a meromorphic function near 0. Note that we can replace $\lambda$ by $\lambda + g(u)$ for any holomorphic function of $u$, so we may uniquely specify $\lambda$ by the condition that $f(u) \in C[u^{-1}] - u^{-1}$ (only singular part). Thus we have $p = \lambda + f(x^{1/\alpha})$, non-neutal.
So the "area" function
\[ \exp \int_{x=x_0}^x p \, dx = \exp (\lambda \log x + \int f(x^{\frac{1}{\lambda}}) \, dx) \]

Exercise: Describe how to go back from \( f(u) \) to the blow-up algorithm.

So we see that
\[ \lambda \] is, in fact, a natural coordinate on our \( A' \)-component. This concludes the discussion of the local picture.

Now let us consider spectral curves. This is a global problem so it is now important that \( C \) is compact. For this purpose fix points \( z_1, \ldots, z_d \in O - S \).

Def. A spectral curve associated to this data is a compact curve.
\[ \Sigma \] which lies entirely in \( P \) and intersects the divisor \( P \sim S \) transversally at the points \( Z_i \) only. (if some \( Z_i \) coincide, the intersection index should equal the multiplicity.)

Let \( g \) be the genus of \( \Sigma \).

**Proposition.** Such curves \( \Sigma \) of genus \( g \) are parametrized by an affine \( g \)-dimensional space \( A^g \). (we'll consider smooth irreducible ones which will form a dense open set)

Note that \( \Sigma \) carries a \( 1 \)-form \( \varphi \) with singularities
at the points \( z_i \). Fix a line bundle \( L \) on \( \Sigma \). We have projection \( \pi: \Sigma \to C \) of degree \( r \). Consider the rank \( r \) vector bundle
\[
E = \mathcal{E}_{\Sigma, L} \text{ on } C, \quad E = \mathcal{E}|_{\Sigma, L}.
\]
This bundle carries a Higgs field \( \mathbf{\Sigma} \) obtained from \( \Sigma \), whose spectral curve is \( \Sigma \). The map \( (\Sigma, L) \to (E, \mathbf{\Sigma}) \) is generically a bijection. Thus we can think of pairs \((\Sigma, L)\) as points on an appropriate
Hitchin moduli space $\tilde{\mathcal{M}}_{\text{Higgs}} \cong T^* \text{Bun}_{\text{GL}_F}(C)$.

the moduli of bundles with appropriate level structure at a finite collection of points of $C$, defined by our irregular terms. Moreover, the map

$(\Sigma, L) \mapsto \Sigma$

is the Hitchin integrable system. Indeed, the set of possible $\Sigma$ is the Hitchin base (describing the spectrum at each point of $C$). The fiber
is the set of all bundles $L$ of degree $d$, which is the abelian variety $\text{Pic}_d(\Sigma)$. This is thus a complex integrable system (an irregular Hitchin system).

Remark. The simplest case is the regular case:

This is related to the Deligne–Simpson problem: describe $n$-tuples of $N \times N$ matrices $A_i$.
i=1, ... , n such that 
\[ A_1 + ... + A_n = 0 \]
and each \( A_i \) belongs to fixed conjugacy class 
\[ C_i \subset \mathfrak{o} \subset \mathfrak{o}^\mathfrak{g} \]
(adjoint orbit)

Let us now assume that one of the points \( Z_i \) is distinguished (call it \( Z_0 \)).

Then we can identify 
\[ \text{Pic}_d (\Sigma) \cong \text{Pic}_0 (\Sigma) \]
by tensoring with \( \mathcal{O}(Z_0) \).

Recall now that 
\[ \text{Pic}_g \Sigma \cong \text{Sym}^g \Sigma \]
by the Abel-Jacobi map 
\[ O (p_1) \otimes ... \otimes O (p_g) \mapsto (p_1, ... , p_g) \]
We claim that we have a birational symplectomorphism

\[ M^{\text{sing}}_{\text{Higgs}} = \frac{\Xi(\Sigma, L)^2}{\Xi} \sim \text{Sym}^g T^*_C \]

defined as follows. Given \((\Sigma, L)\), write \(L = \bigotimes_{i} O(p_i)^{\text{deg} = g}\) for \(p_i \in \Sigma\), then \(p_i \in T^*_C\) as \(\Sigma \subset T^*_C\), so \(\Xi \left( \Sigma, L \right) = (p_1, \ldots, p_g)\).

Now we can construct the quasiclassical 
"shift kernel" in
$\mathcal{Z}_{g+1}$, \( g \in \mathbb{N} \)

Namely, \( \mathcal{Z}_{g+1} \) is the set of \( g+1 \)-tuples

\[(p_0, p_1, \ldots, p_g, q_1, \ldots, q_g)\]

of points in \( T^* C \) such that they belong to the same (unique) spectral curve \( \Sigma \), and

\[0(p_0) \otimes \cdots \otimes 0(p_g) \cong 0(q_1) \otimes \cdots \otimes 0(q_g) \otimes 0(z_0)\]

We may also define the
quasiclassical
"multiplication kernel",
Lagrangian $Z_{g,g,g}$
by convolving $Z_{g,g,g}$ with itself $g$ times.

Remark. In fact, we never really used that $S$ was the cotangent bundle to $C$ in an essential way.
For example, one can take $S = \mathbb{C}^* \times \mathbb{C}^*$ and partially compactify with a disjoint union of copies of $\mathbb{C}^*$.
on which the symplectic form has first order poles. We can still define spectral curves \( \Sigma \subset \mathcal{P} \) (where \( \mathcal{P} \) is a compactification of \( \Sigma \)) and define shift and multiplication kernels is

\[
S^{g+1} \times S^g \to \mathcal{Z}_{g+1, g}
\]

\[
S^{2g} \times \overline{S}^g \to \mathcal{Z}_{2g, g}
\]

in the same way as before: e.g.,

\((p_0, p_1, \ldots, p_g, q_1, \ldots, q_g) \in Z_{g+1, g}\)
\[
\begin{align*}
(\Rightarrow) & \text{ they belong to the same spectral curve } \Sigma \\
\text{and } O(p_0) \otimes \cdots \otimes O(p_r) = \\
O(q_1) \otimes \cdots \otimes O(q_g) \otimes O(z_0).
\end{align*}
\]

One can also consider more general (rational) surfaces \( S \).

Finally, let us discuss the quantization of this picture. In fact, there is a canonical one.

To our configuration of irregular term corresponds an irregular oper: e.g. for \( r = 2 \).
\[(x-x_1) \cdots (x-x_m) \frac{d^2}{dx^2} + \cdots \] where the \( \lambda \)-coordinates of points \( z_i \) occur linearly as parameters.

One then needs to find at each \( A^1 \)-component the solution \( y_\lambda \) of this Opers which is our irregular term times an element \( y^0_\lambda \) of \( 1 + x \frac{\mathbb{C}[[x]]}{x^{1/2}} \)

**Example.**

\[- \frac{d^2}{dx^2} + x - \lambda \right) \psi_\lambda(x) = 0 \]

(\text{Airy equation})

\[ \text{Solution } \psi_\lambda(x) = \text{Ai} (x-\lambda) \]
\[ \text{as } x \to \infty \]

\[ \text{Ai} (x) = e^{-\frac{2}{3} x^{\frac{3}{2}}} \cdot x^{-\frac{1}{4}} \sum_{n=0}^{\infty} P_n(x) x^{-n} \]

where \( P_n(x) \) are polynomials of degree \( n \).

(There is a similar formula for a general differential equation).

i.e. \[ \psi_\lambda^0(x) = \sum_{n=0}^{\infty} P_n(x) x^{\frac{n}{\lambda}} \text{ near } x = 0 \]

where \( P_0(x) = 1 \), \( P_n(x) \) has degree \( n \).

\[ \psi_\lambda(x) = e^{\sum c_i x^{-\xi_i}} x^\lambda \psi_\lambda^0(x) \]

Now define structure constants of \( \mathfrak{c} \{ x \} \)
in the basis \( P_i(\lambda) \):

\[
P_i(\lambda) P_j(\lambda) = \sum_{k \leq i+j} C_{ij}^k P_k(\lambda).
\]

Now let \( K \) be the generating function of these structure constants:

\[
K = \sum c_{ij}^{\lambda} x^i y^j z^{-k} \frac{dz}{z}.
\]

This is an element of

\[
\mathcal{C}[x] \hat{\otimes} \mathcal{C}[y] \hat{\otimes} \mathcal{C}[z]^{\times}
\]

where \( \mathcal{C}[z]^{\times} = \mathcal{C}(z) \frac{dz}{z} \mathcal{C}[z]^{\times} \)

Claim. This Kernel satisfies a holonomic
differential equation, so generates a holonomic D-module. This should be the quantum addition kernel.

There is a similar formula for multidimensional case, where the space of operators has dimension $g$ and parameters $\lambda_i \rightarrow x_i$. In this case we get some bases of $\mathbb{C}[\lambda_1, \ldots, \lambda_g]$ out of asymptotic expansions.
as above, and define structure constants and their generating function, which should generate a holonomic $D$-module. Thus we have quantized our Lagrangians to a specific holonomic $D$-module, and moreover with a cyclic vector (defined up to scaling).

This story extends to more general rational symplectic surfaces $S$ (e.g. $\mathbb{C}^x \times \mathbb{C}^x$). In this
case instead of a holonomic D-module we get a holonomic module over a quantization of $S$. For example, for $C^\times \times C^\times$ we will get holonomic q-D-modules, i.e., modules over tensor powers of the algebra of q-difference operators,

$$A = \mathcal{C}\langle T, X \rangle / TX = qXT$$

(quantum forces). I.e. for shift kernel,
\((A^{\otimes g+1}, A^{\otimes g})\) - bimodule, holonomic and with a cyclic vector.