# PARAMETERS AND DUALITY FOR THE METAPLECTIC GEOMETRIC LANGLANDS THEORY

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Abstract. This is a corrected version of the paper, and it differs substantially from the original one.

We introduce the space of parameters for the metaplectic Langlands theory as *factorization gerbes* on the affine Grassmannian, and develop metaplectic Langlands duality in the incarnation of the metaplectic geometric Satake functor.

We formulate a conjecture in the context of the global metaplectic Langlands theory, which is a metaplectic version of the "vanishing theorem" of [Ga5, Theorem 4.5.2].

## Introduction

0.1. **What is this paper about?** The goal of this paper is to provide a summary of the metaplectic Langlands theory. Our main objectives are:

–Description of the set (rather, space) of parameters for the metaplectic Langlands theory;

–Construction of the *metaplectic Langlands dual* (see Sect. 0.1.6 for what we mean by this);

–The statement of the *metaplectic geometric Satake*.

0.1.1. *The metaplectic setting.* Let $\mathbf{F}$ be a local field and $G$ an algebraic group over $\mathbf{F}$. The classical representation theory of locally compact groups studies (smooth) representations of the group $G(\mathbf{F})$ on vector spaces over another field $E$. Suppose now that we are given a central extension

$$1 \to E^\times \to \widetilde{G(\mathbf{F})} \to G(\mathbf{F}) \to 1. \tag{0.1}$$

We can then study representations of $\widetilde{G(\mathbf{F})}$ on which the central $E^\times$ acts by the tautological character. We will refer to (0.1) as a *local metaplectic extension* of $G(\mathbf{F})$, and to the above category of representations as *metaplectic representations* of $G(\mathbf{F})$ corresponding to the extension (0.1).

Let now $\mathbf{F}$ be a global field, and let $\mathbb{A}_\mathbf{F}$ be the corresponding ring of adèles. Let us be given a central extension

$$1 \to E^\times \to \widetilde{G(\mathbb{A}_\mathbf{F})} \to G(\mathbb{A}_\mathbf{F}) \to 1, \tag{0.2}$$

equipped with a splitting over $G(\mathbf{F}) \hookrightarrow G(\mathbb{A}_\mathbf{F})$.

We can then study the space of $E$-valued functions on the quotient $\widetilde{G(\mathbb{A}_\mathbf{F})}/G(\mathbf{F})$, on which the central $E^\times$ acts by the tautological character. We will refer to (0.2) as a *global metaplectic extension* of $G(\mathbf{F})$, and to the above space of functions as *metaplectic automorphic functions* on $G(\mathbf{F})$ corresponding to the extension (0.2).

There has been a renewed interest in the study of metaplectic representations and metaplectic automorphic functions, e.g., by B.Brubaker–D.Bump–S.Friedberg, P.McNamara, W.T.Gan–F.Gao.

M. Weissman has initiated a program of constructing the L-groups corresponding to metaplectic extensions, to be used in the formulation of the Langlands program in the metaplectic setting, see [We].

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0.1.2. *Parameters for metaplectic extensions.* In order to construct metaplectic extensions, in both the local and global settings, one starts with a datum of algebro-geometric nature. Namely, one usually takes as an input what we call a *Brylinski-Deligne datum*, by which we mean a central extension

$$1 \to (K_2)_{\mathrm{Zar}} \to \widetilde{G} \to G \to 1, \tag{0.3}$$

of sheaves of groups on the big Zariski site of $\mathbf{F}$, where $(K_2)_{\mathrm{Zar}}$ is the sheafification of the sheaf ofabelian groups that assigns to an affine scheme $S = \mathrm{Spec}(A)$ the group $K_2(A)$.

For a local field $\mathbf{F}$, let $\mathbf{f}$ denote its residue field and let us choose a homomorphism

$$\mathbf{f}^\times \to E^\times. \tag{0.4}$$

Then taking the group of $\mathbf{F}$-points of $\widetilde{G}$ and pushing out with respect to

$$K_2(\mathbf{F}) \overset{\text{symbol}}{\longrightarrow} \mathbf{f}^\times \to E^\times,$$

we obtain a central extension (0.1). A similar procedure applies also in the global setting.

0.1.3. *The geometric theory.* Let $k$ be a ground field and let $G$ be a reductive group over $k$.

In the local geometric Langlands theory one considers the loop group $G(\!(t)\!)$ along with its action on various spaces, such as the affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. Specfically one studies the behavior of categories of sheaves[1] on such spaces with respect to this action.

In the global geometric Langlands theory one considers a smooth proper curve $X$, and one studies the stack $\mathrm{Bun}_G$ that classifies principal $G$-bundles on $X$. The main object of investigation is the category of sheaves on $\mathrm{Bun}_G$.

There are multiple ways in which the local and global theories interact. For example, given a ($k$-rational) point $x \in X$, and identifying the local ring $\mathcal{O}_x$ of $X$ at $x$ with $k[\![t]\!]$, we have the map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G, \tag{0.5}$$

where we interpret $\mathrm{Gr}_G$ as the moduli space of principal $G$-bundles on $X$, trivialized over $X - x$.

0.1.4. *The setting of metaplectic geometric Langlands theory.* Let $E$ denote the field of coefficients of the sheaf theory that we consider. Recall (see Sect. 1.7.4) that if $\mathcal{Y}$ is a space[2] and $\mathcal{G}$ is a $E^\times$-gerbe on $\mathcal{Y}$, we can twist the category of sheaves on $\mathcal{Y}$, and obtain a new category, denoted

$$\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}).$$

In the local metaplectic Langlands theory, the input datum (which is an analog of a central extension (0.1)) is an $E^\times$-gerbe over the loop group $G(\!(t)\!)$ that behaves *multiplicatively*, i.e., one that is compatible with the group-law on $G(\!(t)\!)$.

Similarly, whenever we consider an action of $G(\!(t)\!)$ on $\mathcal{Y}$, we equip $\mathcal{Y}$ with $E^\times$-gerbe that is compatible with the given multiplicative gerbe on $G(\!(t)\!)$. In this case we say that the category $\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})$ carries a *twisted* action of $G(\!(t)\!)$, where the parameter of the twist is our gerbe on $G(\!(t)\!)$.

In the global setting we consider a gerbe $\mathcal{G}$ over $\mathrm{Bun}_G$, and the corresponding category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$ of twisted sheaves.

Now, if we want to consider the local vs. global interaction, we need a compatibility structure on our gerbes. For example, we need that for every point $x \in X$, the pullback along (0.5) of the given gerbe on $\mathrm{Bun}_G$ be a gerbe compatible with some given multiplicative gerbe on $G(\!(t)\!)$.

So, it is natural to seek an algebro-geometric datum, akin to (0.3), that would provide such a compatible family of gerbes.

[1] See Sect. 1.5 for what we mean by the category of sheaves.

[2] By a "space" we mean a scheme, stack, ind-scheme, or more generally a *prestack*, see Sect. 1.2 for what the latter word means.

0.1.5. *Geometric metaplectic datum.* It turns out that such a datum (let us call it "the geometric metaplectic datum") is not difficult to describe, see Sect. 2.4.1 below. It amounts to the datum of a *factorization gerbe* with respect to $E^\times$ on the *affine Grassmannian*[3] $\mathrm{Gr}_G$ of the group $G$.

In a way, this answer is more elementary than (0.3) in that we are dealing with étale cohomology rather than $K$-theory.

Moreover, in the original metaplectic setting, if the global field $\mathbf{F}$ is the function field corresponding to the curve $X$ over a finite ground field $k$, a geometric metaplectic datum gives rise directly to an extension (0.2).

Finally, a Brylinski-Deligne datum (i.e., an extension (0.3)) and a choice of a character $k^\times \to E^\times$ gives rise to a geometric metaplectic datum, see Sect. 3.4.

Thus, we could venture into saying that a geometric metaplectic datum is a more economical way, sufficient for most purposes, to encode also the datum needed to set up the classical metaplectic representation/automorphic theory.

0.1.6. *The metaplectic Langlands dual.* Given a geometric metaplectic datum, i.e., a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_G$, we attach to it a certain reductive group $H$, a gerbe $\mathcal{G}_{Z_H}$ on $X$ with respect to the center $Z_H$ of $H$, and a character $\epsilon : \pm 1 \to Z_H$. We refer to the triple

$$(H, \mathcal{G}_{Z_H}, \epsilon)$$

as the *metaplectic Langlands dual* datum corresponding to $\mathcal{G}$.

The datum of $\mathcal{G}_{Z_H}$ determines the notion of twisted $H$-local system of $X$. Such twisted local systems are supposed to play a role vis-à-vis metaplectic representations/automorphic functions of $G$ parallel to that of usual $\check{G}$-local systems vis-à-vis usual representations/automorphic functions of $G$.

For example, in the context of the global geometric theory (in the setting of D-modules), we will propose a conjecture (namely, Conjecture 9.6.2) that says that the monoidal category $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$ of quasi-coherent sheaves on the stack $\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}$ classifying such twisted local systems, *acts* on the category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$.

The geometric input for such an action is provided by the metaplectic geometric Satake functor, see Sect. 9.

Presumably, in the arithmetic context, the above notion of twisted $H$-local system coincides with that of homomorphism of the (arithmetic) fundamental group of $X$ to Weissman's L-group.

## 0.2. "Metaplectic" vs "Quantum".

In the paper [Ga4], a program was proposed towards the *quantum Langlands theory.* Let us comment on the terminological difference between "metaplectic" and "quantum", and how the two theories are supposed to be related.

0.2.1. If $\mathcal{Y}$ is a scheme (resp., or more generally, a prestack) we can talk about $E^\times$-gerbes on it. As was mentioned above, such gerbes on various spaces associated with the group $G$ and the geometry of the curve $X$ are parameters for the metaplectic Langlands theory.

Let us now assume that $k$ has characteristic 0, and let us work in the context of D-modules. Then, in addition to the notion of $E^\times$-gerbe on $\mathcal{Y}$, there is another one: that of *twisting* (see [GR1, Sect. 6]).

There is a forgetful map from twistings to gerbes. Roughly speaking, a gerbe $\mathcal{G}$ on $\mathcal{Y}$ defines the corresponding twisted category of sheaves (=D-modules) $\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}) = \mathrm{D\text{-}mod}_{\mathcal{G}}(\mathcal{Y})$, while if we lift our gerbe to a twsiting, we also have a forgetful functor

$$\mathrm{D\text{-}mod}_{\mathcal{G}}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

[3]Here the affine Grassmannian appears in its factorization (a.k.a, Beilinson-Drinfeld) incarnation. I.e., it is a prestack mapping to the Ran space of $X$, rather than $G((t))/G[\![t]\!]$, which corresponds to a particular point of $X$.

0.2.2. For the *quantum* Langlands theory, our parameter will be a factorizable *twisting* on the affine Grassmannian, which one can also interpret as a *Kac-Moody level*; we will denote it by $\kappa$.

Thus, for example, in the global quantum geometric Langlands theory, we consider the category

$$\text{D-mod}_\kappa(\text{Bun}_G),$$

which is the same as $\text{Shv}_{\mathcal{G}}(\text{Bun}_G)$, where $\mathcal{G}$ is the gerbe corresponding to $\kappa$.

As was mentioned above, the additional piece of datum that the twisting "buys" us is the forgetful functor

$$\text{D-mod}_\kappa(\text{Bun}_G) \to \text{QCoh}(\text{Bun}_G).$$

In the TQFT interpretation of geometric Langlands, this forgetful functor is called "the big brane". It allows us to relate the category $\text{D-mod}_\kappa(\text{Bun}_G)$ to representations of the Kac-Moody algebra attached to $G$ and the level $\kappa$.

0.2.3. Consider the usual Langlands dual group $\check{G}$ of $G$, and if $\kappa$ is non-degenerate, it gives rise to a twisting, denoted $-\kappa^{-1}$, on the affine Grassmannian $\text{Gr}_{\check{G}}$ of $\check{G}$.

In the global quantum geometric theory one expects to have an equivalence of categories

$$\text{D-mod}_\kappa(\text{Bun}_G) \simeq \text{D-mod}_{-\kappa^{-1}}(\text{Bun}_{\check{G}}). \tag{0.6}$$

We refer to (0.6) as the *global quantum Langlands equivalence*.

0.2.4. *How are the two theories related?* The relationship between the equivalence (0.6) and the metaplectic Langlands dual is the following:

Let $\mathcal{G}$ (resp., $\check{\mathcal{G}}$) be the gerbe on $\text{Gr}_G$ (resp., $\text{Gr}_{\check{G}}$) corresponding to $\kappa$ (resp., $-\kappa^{-1}$). We conjecture that the metaplectic Langlands dual data $(H, \mathcal{G}_{Z_H}, \epsilon)$ corresponding to $\mathcal{G}$ and $\check{\mathcal{G}}$ *are isomorphic*.

Furthermore, we conjecture that the resulting actions of

$$\text{QCoh}\left(\text{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$$

on $\text{D-mod}_\kappa(\text{Bun}_G)$ and $\text{D-mod}_{-\kappa^{-1}}(\text{Bun}_{\check{G}})$, respectively (see Sect. 0.1.6 above) are intertwined by the equivalence (0.6).

0.3. **What is actually done in this paper?** Technically, our focus is on the geometric metaplectic theory, with the goal of constructing the *metaplectic geometric Satake* functor.

0.3.1. The mathematical content of this paper is the following:

–We define a geometric metaplectic datum to be a factorization gerbe on the (factorization version) of affine Grassmannian $\text{Gr}_G$. This is done in Sect. 2.

–We formulate the classification result that describes factorization gerbes on $\text{Gr}_G$ in terms of étale cohomology on the classifying stack $BG$ of $G$. This is done in Sect. 3.

This classification result is inspired by an analogous one in the topological setting, explained to us by J. Lurie.

–We make an explicit analysis of the space of factorization gerbes in the case when $G = T$ is a torus. This is done in Sect. 4.

–We study the relationship between factorization gerbes on $\text{Gr}_G$ and those on $\text{Gr}_M$, where $M$ is the Levi quotient of a parabolic $P \subset G$. This is done in Sect. 5.

The main point is that the naive map from factorization gerbes on $\text{Gr}_G$ to those on $\text{Gr}_M$ needs to be corrected by a gerbe that has to do with signs. It is this correction that is responsible for the fact that the usual geometric Satake does not quite produce the category $\text{Rep}(\check{G})$, but rather its modification where we alter the commutativity constraint by the element $2\rho(-1) \in Z(\check{G})$.

–We define the notion of *metaplectic Langlands dual* datum, denoted $(H, \mathcal{G}_{Z_H}, \epsilon)$, attached to a given geometric metaplectic datum $\mathcal{G}$. We introduce the notion of $\mathcal{G}_{Z_H}$-twisted $H$-local system on $X$; when we work with D-modules, these local systems are $k$-points of a (derived) algebraic stack, denoted $\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}$. This is done in Sect. 6.

–We show that a factorization gerbe on $\mathrm{Gr}_G$ gives rise to a *multiplicative* gerbe over the loop group $G((t))$ for every point $x \in X$. Moreover, these multiplicative gerbes also admit a natural factorization structure when instead of a single point $x$ we consider the entire Ran space. This is done in Sect. 7.

–We introduce the various twisted versions of the category of representations of a reductive group, and the associated notion of twisted local system. This is done in Sect. 8.

–We define metaplectic geometric Satake as a functor between *factorization categories* over the Ran space. This is done in Sect. 9.

–We formulate a conjecture about the action of the monoidal category $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_{Z_H}}\right)$ on $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Bun}_G)$. This is also done in Sect. 9.

0.3.2. *A disclaimer.* Although most of the items listed in Sect. 0.3.1 have not appeared in the previously existing literature, this is mainly due to the fact that these earlier sources, specifically the paper [FL] of M. Finkelberg and the second-named author and the paper [Re] of R. Reich, did not use the language of $\infty$-categories, while containing most of the relevant mathematics.

So, one can regard the present paper as a summary of results that are "almost known", but formulated in the language that is better adapted to the modern take on the geometric Langlands theory[4].

We felt that there was a need for such a summary in order to facilitate further research in this area.

Correspondingly, our focus is on statements, rather than proofs. Most of the omitted proofs can be found in either [FL] or [Re], or can be obtained from other sources cited in the paper.

Below we give some details on the relation of contents of this paper and some of previously existing literature.

0.3.3. *Relation to other work: geometric theory.* As was just mentioned, a significant part of this paper is devoted to reformulating the results of [FL] and [Re] in a way tailored for the needs of the geometric metaplectic theory.

The paper [Re] develops the theory of factorization gerbes on $\mathrm{Gr}_G$ (in *loc. cit.* they are called "symmetric factorizable gerbes"). One caveat is that in the setting of [Re] one works with schemes over $\mathbb{C}$ and sheaves in the analytic topology, while in the present paper we work over a general ground field and étale sheaves.

The main points of the theory developed in [Re] are the description of the *homotopy groups* of the space of factorization gerbes (but not of the space itself; the latter is done in Sect. 3 of the present paper), and the fact that a factorization gerbe on $\mathrm{Gr}_G$ gives rise to a multiplicative gerbe on (the factorization version of) the loop group (we summarize this construction in Sect. 7 of the present paper).

The proofs of the corresponding results in [Re] are obtained by reducing assertions for a reductive group $G$ to that for its Cartan subgroup, and an explicit analysis for tori. We do not reproduce these proofs in the present paper.

In both [FL] and [Re], metaplectic geometric Satake is stated as an equivalence of certain abelian categories. In [FL], this is an equivalence of symmetric monoidal categories (corresponding to a chosen point $x \in X$), for a particular class of gerbes (namely, ones obtained from the determinant line bundle).

In [Re] more general gerbes are considered and the factorization structure on both sides of the equivalence is taken into account. Our version of metaplectic geometric Satake is a statement at the level of DG categories; it is no longer an equivalence, but rather a functor in one direction, between *monoidal factorization categories*. In this form, our formulation is a simple consequence of that of [Re].

[4]This excludes, however, the material in Sect. 9.5 and the statement of Conjecture 9.6.2 (the latter is new, to the best of our knowledge)

0.3.4. *Relation to other work: arithmetic theory.* As was already mentioned above, our notion of the metaplectic Langlands dual datum is probably equivalent to the datum constructed by M. Weissman in [We] for his definition of the L-group.

## 0.4. **Conventions.**

0.4.1. *Algebraic geometry.* In the main body of the paper we will be working over a fixed ground field $k$, assumed algebraically closed.

For arithmetic applications one would also be interested in the case of $k$ being a finite field $\mathbb{F}_q$. However, since all the constructions in this paper are canonical, the results over $\mathbb{F}_q$ can be deduced from those over $\overline{\mathbb{F}}_q$ by Galois descent.

We will denote by $X$ a smooth connected algebraic curve over $k$ (we *do not* need $X$ to be complete).

For the purposes of this paper, we *do not need* derived algebraic geometry, with the exception of Sects. 8.4 and 9.6 (where we discuss the stack of local systems, which is a derived object).

In the main body of the paper we will make an extensive use of algebro-geometric objects more general than schemes, namely, prestacks. We recall the definition of prestacks in Sect. 1.2, and refer the reader to [GR2, Vol. 1, Chapter 2] for a more detailed discussion.

0.4.2. *Coefficients.* In the main body of the paper, we will work with the *sheaf theory* of D-modules. Yet, we would like to separate notationally the round field, denoted $k$, and the field of coefficients, denoted $E$ (assumed algebraically closed and of characteristic 0).

0.4.3. *Groups.* We will work with a fixed connected algebraic group $G$ over $k$; our main interest is the case when $G$ is reductive.

We will denote by $\Lambda$ the coweight lattice of $G$ and by $\check{\Lambda}$ its dual, i.e., the weight lattice.

We will denote by $\alpha_i \in \Lambda$ (resp., $\check{\alpha}_i \in \check{\Lambda}$) the simple coroots (resp., roots), where $i$ runs over the set of vertices of the Dynkin diagram of $G$.

If $G$ is reductive, we denote by $\check{G}$ its Langlands dual, viewed as a reductive group over $E$.

0.4.4. *The usage of higher category theory.* Although, as we have said above, we do not need derived algebraic geometry, we do need higher category theory. However, we only really need $\infty$-categories for one type of manipulation: in order to define the notion of the *category of sheaves* on a given prestack (and a related notion of a *sheaf of categories* over a prestack); we will recall the corresponding definitions in Sects. 1.2 and 1.6), respectively. These definitions involve the procedure of taking the limit, and the language of higher categories is the adequate framework for doing so.

In their turn, sheaves of categories on prestacks appear for us as follows: the metaplectic spherical Hecke category, which is the recipient of the metaplectic geometric Satake functor (and hence is of primary interest for us), is a sheaf of categories over the Ran space.

Thus, the reader who is only interested in the notion of geometric metaplectic datum (and does not wish to proceed to metaplectic geometric Satake) *does not* higher category theory either.

0.4.5. *Glossary of $\infty$-categories.* We will now recall several most common pieces of notation, pertaining to $\infty$-categories, used in this paper. We refer the reader to [Lu1, Lu2] for the foundations of the theory, or [GR2, Vol. 1, Chapter 1] for a concise summary.

We denote by Spc the $\infty$-category of spaces. We denote by $*$ the point-space. For a space $\mathcal{S}$, we denote by $\pi_0(\mathcal{S})$ its *set of connected components*. If $\mathcal{S}$ is a space we can view it as an $\infty$-category; its objects are also called the *points* of $\mathcal{S}$.

For an $\infty$-category $\mathbf{C}$ and two objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we let $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Spc}$ denote the mapping space between them.

For an object $\mathbf{c} \in \mathbf{C}$ we let $\mathbf{C}_{\mathbf{c}/}$ (resp., $\mathbf{C}_{/\mathbf{c}}$) denote the corresponding under-category (resp., over-category).

In several places in the paper we will need the notion of left (resp., right) Kan extension. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, and let $\mathbf{E}$ is an $\infty$-category with colimits. Then the functor

$$\text{Funct}(\mathbf{D}, \mathbf{E}) \xrightarrow{\circ F} \text{Funct}(\mathbf{C}, \mathbf{E}) \tag{0.7}$$

admits a left adjoint, called the functor of *left Kan extension* along $F$.

For $\Phi \in \text{Funct}(\mathbf{C}, \mathbf{E})$, the value of its left Kan extension on $\mathbf{d} \in \mathbf{D}$ is calculated by the formula

$$\underset{(\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}) \in \mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{D}_{/\mathbf{d}}}{\text{colim}} \Phi(\mathbf{c}).$$

The notion of *right Kan extension* is obtained similarly: it is the right adjoint of (0.7); the formula for it is given by

$$\underset{(\mathbf{c}, \mathbf{d} \to F(\mathbf{c})) \in \mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{D}_{\mathbf{d}/}}{\lim} \Phi(\mathbf{c}).$$

0.4.6. *DG categories.* We let DGCat denote the $\infty$-category of DG categories over $E$, see [GR2, Vol. 1, Chapter 1, Sect. 10.3.3] (in *loc.cit.* it is denoted $\text{DGCat}_{\text{cont}}$). I.e., we will assume all our DG categories to be *cocomplete* and we allow only colimit-preserving functors as 1-morphisms.

For example, let $R$ be a DG associative algebra over $k$. Then we let $R$-mod denote the corresponding DG category of $R$-modules (i.e., its homotopy category is the usual derived category of the abelian category of $R$-modules, without any boundedness conditions).

For an algebraic group $H$ over $E$, we let $\text{Rep}(H)$ denote the DG category of representations of $H$, see, e.g., [DrGa, Sects. 6.4.3-6.4.4].

The piece of structure on DGCat that we will exploit extensively is the operation of tensor product, which makes DGCat into a symmetric monoidal category.

For a pair of DG associative algebras $R_1$ and $R_2$, we have:

$$(R_1\text{-mod}) \otimes (R_2\text{-mod}) \simeq (R_1 \otimes R_2)\text{-mod}.$$

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## 1. Preliminaries

This section is included for the reader's convenience: we review some constructions in algebraic geometry that involve higher category theory. The reader having a basic familiarity with this material should feel free to skip it.

1.1. **Some higher algebra.** To facilitate the reader's task, in this subsection we will review some notions from higher algebra that will be used in this paper. The main reference for this material is [Lu2].

We should emphasize that for the purposes of studying geometric metaplectic data, we only need higher algebra in $\infty$-categories that are $(n, 1)$-categories for small values of $n$. The corresponding objects can be studied in a hands-on way (i.e., we do not need the full extent of higher category theory).

The only place where we really need higher categories is for working with categories of sheaves on prestacks.

1.1.1. *Monoids and groups.* In any $\infty$-category $\mathbf{C}$ that contains finite products (including the empty finite product, i.e., a final object), it makes sense to consider the category $\mathrm{Monoid}(\mathbf{C})$ of monoid-objects in $\mathbf{C}$. This is a full subcategory in the category of *simplicial objects* of $\mathbf{C}$ (i.e., $\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C})$) that consists of objects, satisfying the Segal condition.

One defines the category commutative monoids $\mathrm{ComMonoid}(\mathbf{C})$ in $\mathbf{C}$ similarly, but using the category $\mathrm{Fin}_s$ of pointed finite sets instead of $\mathbf{\Delta}^{\mathrm{op}}$.

For example, take $\mathbf{C} = \infty\text{-Cat}$. In this way we obtain the notion of monoidal (resp., symmetric monoidal) category.

1.1.2. The $\infty$-category $\mathrm{Monoid}(\mathbf{C})$ (resp., $\mathrm{ComMonoid}(\mathbf{C})$) contains the full subcategory of group-like objects, denoted $\mathrm{Grp}(\mathbf{C})$ (resp., $\mathrm{ComGrp}(\mathbf{C})$).

Let $\mathrm{Ptd}(\mathbf{C})$ be the category of pointed objects in $\mathbf{C}$, i.e., $\mathbf{C}_{*/}$, where $*$ denotes the final object in $\mathbf{C}$. We have the loop functor

$$\Omega : \mathrm{Ptd}(\mathbf{C}) \to \mathrm{Grp}(\mathbf{C}), \quad (* \to \mathbf{c}) \mapsto * \underset{\mathbf{c}}{\times} *.$$

The left adjoint of this functor (if it exists) is called the functor of the *classifying space* and is denoted

$$H \mapsto B(H).$$

1.1.3. For $\mathbf{C} = \mathrm{Spc}$ (or $\mathbf{C} = \mathrm{Funct}(\mathbf{D}, \mathrm{Spc})$ for some other category $\mathbf{D}$), the functor $B$ does exist and is fully faithful. The essential image of $B : \mathrm{Grp}(\mathrm{Spc}) \to \mathrm{Ptd}(\mathrm{Spc})$ consists of *connected* spaces.

For an object $\mathcal{S} \in \mathrm{Ptd}(\mathrm{Spc})$, its $i$-th homotopy group $\pi_i(\mathcal{S})$ is defined to be

$$\pi_0(\Omega^i(\mathcal{S})),$$

where $\Omega^i(\mathcal{S})$ is viewed as a plain object of Spc.

1.1.4. For $k \geq 0$, we introduce the category $\mathbb{E}_k(\mathbf{C})$ of $\mathbb{E}_k$-objects in $\mathbf{C}$ inductively, by setting

$$\mathbb{E}_0(\mathbf{C}) = \mathrm{Ptd}(\mathbf{C})$$

and

$$\mathbb{E}_k(\mathbf{C}) = \mathrm{Monoid}(\mathbb{E}_{k-1}(\mathbf{C})).$$

Let $\mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \subset \mathbb{E}_k(\mathbf{C})$ the full subcategory of group-like objects, defined to be the preimage of

$$\mathrm{Grp}(\mathbf{C}) \subset \mathrm{Monoid}(\mathbf{C}) = \mathbb{E}_1(\mathbf{C})$$

under any of the $k$ possible forgetful functors $\mathbb{E}_k(\mathbf{C}) \to \mathbb{E}_1(\mathbf{C})$.

The functor $B : \mathrm{Grp}(\mathrm{Spc}) \to \mathrm{Ptd}(\mathrm{Spc})$ (if it exists) induces a functor

$$B : \mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \rightleftarrows \mathbb{E}_{k-1}^{\text{grp-like}}(\mathbf{C}) : \Omega$$

for $k \geq 2$, which is the left adjoint of

$$\Omega : \mathbb{E}_{k-1}^{\text{grp-like}}(\mathbf{C}) \to \mathbb{E}_k^{\text{grp-like}}(\mathbf{C}).$$

For $i \leq k$ we let $B^i$ denote the resulting functor

$$\mathbb{E}_k^{\text{grp-like}}(\mathbf{C}) \to \mathbb{E}_{k-i}^{\text{grp-like}}(\mathbf{C}).$$

1.1.5. One shows that the forgetful functor

$$\mathrm{Monoid}(\mathrm{ComMonoid}(\mathbf{C})) \to \mathrm{ComMonoid}(\mathbf{C})$$

is an equivalence.

This implies that for every $k$ we have a canonically defined functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbb{E}_k(\mathbf{C}),$$

and these functors are compatible with the forgetful functors $\mathbb{E}_k(\mathbf{C}) \to \mathbb{E}_{k-1}(\mathbf{C})$. Thus, we obtain a canonically defined functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbb{E}_\infty(\mathbf{C}) := \underset{\longleftarrow}{\lim}\, \mathbb{E}_k(\mathbf{C}). \tag{1.1}$$

It is known (see [Lu2, Remark 5.2.6.26]) that the functor (1.1) is an equivalence.

1.1.6. The category

$$\mathrm{ComGrp}(\mathrm{Spc}) \simeq \mathbb{E}_\infty^{\text{grp-like}}(\mathrm{Spc})$$

identifies with that of connective spectra.

For any $i \geq 0$, we have the mutually adjoint endo-functors

$$B^i : \mathrm{ComGrp}(\mathrm{Spc}) \rightleftarrows \mathrm{ComGrp}(\mathrm{Spc}) : \Omega^i$$

with $B^i$ being fully faithful.

1.1.7. Let $A$ be an object of $\mathbb{E}_2^{\text{grp-like}}(\mathrm{Spc})$, so that $B(A)$ is an object of $\mathrm{Grp}(\mathrm{Spc})$.

By an action of $A$ on an $\infty$-category $\mathbf{C}$ we shall mean an action of $B(A)$ on $\mathbf{C}$ as an object of $\infty$-Cat.

For example, taking $A = E^\times \in \mathrm{ComGrp}(\mathrm{Spc})$, we obtain an action of $E^\times$ on any DG category. Explicitly, we identify $B(E^\times)$ with the space of $E^\times$-torsors, i.e., lines, and the action in question sends a line $\ell$ to the endofunctor

$$\mathbf{c} \mapsto \ell \otimes \mathbf{c}.$$

1.2. **Prestacks.**

1.2.1. Let $\mathrm{Sch}^{\mathrm{aff}}$ be the category of *classical* affine schemes over $k$.

We let PreStk denote the category of all (accessible) functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

We shall say that an object of PreStk is $n$-truncated if it takes values in the full subcategory of Spc that consists of $n$-truncated spaces[5].

The $\infty$-category of $n$-truncated prestacks is in fact an $(n+1, 1)$-category. For small values of $n$, one can work with it avoiding the full machinery of higher category theory.

*Remark* 1.2.2. There will be two types of prestacks in this paper: the "source" type and the "target" type. The source type will be various geometric objects associated to the group $G$ and the curve $X$, such as the Ran space, affine Grassmannian $\mathrm{Gr}_G$, the loop group $\mathfrak{L}(G)$, etc. These prestacks re 0-truncated, i.e., they take values in the full subcategory

$$\mathrm{Sets} \subset \mathrm{Spc}\,.$$

There will be a few other source prestacks (such as $\mathrm{Bun}_G$ or quotients of $\mathrm{Gr}_G$ by groups acting on it) and they will be 1-truncated (i.e., they take values in the full subcategory of Spc spanned by ordinary groupoids).

When we talk about the category of sheaves on a prestack, the prestack in question will be typically of the source type.

[5]An object of Spc is said to be truncated, if for any choice of a base point, its homotopy groups $\pi_{n'}$ vanish for $n' > n$.

The target prestacks will be of the form $B^n(\mathcal{A})$ (see Sect. 1.1.5), where $\mathcal{A}$ is a prestack that takes a constant value $A$, where $A$ is a *discrete* abelian group (or its sheafification in, say, the étale topology, denoted $B^n_{\mathrm{et}}(\mathcal{A})$, see below). Such a prestack is $n$-truncated. When $n$ is small, they can be described in a hands-on way by specifying objects, 1-morphisms, 2-morphisms, etc; in this paper $n$ will be $\leq 4$, and in most cases $\leq 2$.

For example, we will often use the notion of a *multiplicative* $A$-gerbe on a group-prestack $\mathcal{H}$. Such an object is the same as a map of group-prestacks

$$\mathcal{H} \to B^2_{\mathrm{et}}(A).$$

1.2.3. Let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset \mathrm{Sch}^{\mathrm{aff}}$ denote the full subcategory of affine schemes of finite. Functorially, thus subcategory can be characterized as consisting of *co-compact* objects, i.e., $S \in \mathrm{Sch}^{\mathrm{aff}}$ if and only if the functor

$$S' \mapsto \mathrm{Hom}(S', S)$$

commutes with filtered limits.

Moreover, every object of $\mathrm{Sch}^{\mathrm{aff}}$ can be written as a filtered limit of objects of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

The two facts mentioned above combine to the statement that we can identify $\mathrm{Sch}^{\mathrm{aff}}$ with the pro-completion of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

1.2.4. We let

$$\mathrm{PreStk}_{\mathrm{lft}} \subset \mathrm{PreStk}$$

denote the full subcategory consisting of functors that preserve filtered colimits. I.e., $\mathcal{Y} \in \mathrm{PreStk}$ is locally of finite type if for

$$S = \lim_{\alpha} S_\alpha,$$

the map

$$\mathrm{Maps}(S, \mathcal{Y}) \to \underset{\alpha}{\mathrm{colim}}\, \mathrm{Maps}(S_\alpha, \mathcal{Y})$$

is an isomorphism in Spc.

The functors of restriction and left Kan extension along

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \tag{1.2}$$

define an equivalence between $\mathrm{PreStk}_{\mathrm{lft}}$ and the category of all functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

If $\mathcal{F} \in \mathrm{PreStk}_{\mathrm{lft}}$ is such that its restriction to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ takes values in $n$-truncated spaces, then $\mathcal{Y}$ itself is $n$-truncated.

1.2.5. In this paper we will work with the étale topology on $\mathrm{Sch}^{\mathrm{aff}}$. Let

$$\mathrm{Stk} \subset \mathrm{PreStk}$$

be the full subcategory consisting of objects that satisfy descent for Čech nerves of étale morphisms, see [GR2, Vol. 1, Chapter 2, Sect. 2.3.1].

The inclusion $\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$ admits a left adjoint, called the functor of étale sheafification, denoted $L_{\mathrm{et}}$.

This functor sends $n$-truncated objects to $n$-truncated objects.

1.2.6. Denote

$$\mathrm{Stk}_{\mathrm{lft}} := \mathrm{Stk} \cap \mathrm{PreStk}_{\mathrm{lft}} \subset \mathrm{PreStk}\,.$$

However, we can consider a different subcategory of $\mathrm{PreStk}_{\mathrm{lft}}$, denoted $\mathrm{NearStk}_{\mathrm{lft}}$. Namely, identifying $\mathrm{PreStk}_{\mathrm{lft}}$ with $\mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc})$, we can consider the full subcategory consisting of functors that satisfy descent for Čech covers of étale morphisms (within $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$).

Restriction along (1.2) sends $\mathrm{Stk}_{\mathrm{lft}}$ to $\mathrm{NearStk}_{\mathrm{lft}}$. However, it is not true that the functor of left Kan extension along (1.2) sends $\mathrm{NearStk}_{\mathrm{lft}}$ to $\mathrm{Stk}_{\mathrm{lft}}$. However, the following weaker statement holds (see [GR2, Vol. 1, Chapter 2, Proposition 2.7.7]):

**Lemma 1.2.7.** *Assume that $\mathcal{Y} \in \mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc})$ is $n$-truncated for some $n$. Then the left Kan extension of $\mathcal{Y}$ along* (1.2) *belongs to* $\mathrm{Stk}_{\mathrm{lft}}$.

This formally implies:

**Corollary 1.2.8.** *If $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ is $n$-truncated for some $n$, then $L_{\mathrm{et}}(\mathcal{Y})$ belongs to* $\mathrm{Stk}_{\mathrm{lft}}$.

1.3. **Gerbes.**

1.3.1. Let $\mathcal{Y}$ be a prestack, and let $\mathcal{A}$ be a group-like $\mathbb{E}_n$-object in the category $\mathrm{PreStk}_{/\mathcal{Y}}$, for $n \geq 1$. In other words, for a given $(S \overset{y}{\to} \mathcal{Y}) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$, the space

$$\mathrm{Maps}(S, \mathcal{A}) \underset{\mathrm{Maps}(S,\mathcal{Y})}{\times} \{y\} \tag{1.3}$$

is a group-like $\mathbb{E}_n$-object of Spc, in a way functorial in $(S, y)$.

We include the case of $n = \infty$, when we stipulate that $\mathcal{A}$ is a commutative group-object of $\mathrm{PreStk}_{/\mathcal{Y}}$. I.e., (1.3) should be a commutative group-object of Spc, i.e., a connective spectrum.

For any $0 \leq i \leq n$, we let $B^i(\mathcal{A})$ denote the $i$-fold classifying space of $\mathcal{A}$. This is a group-like $\mathbb{E}_{n-i}$-object in $\mathrm{PreStk}_{/\mathcal{Y}}$. For $i = 1$ we simply write $B(\mathcal{A})$ instead of $B^1(\mathcal{A})$.

1.3.2. We let $B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})$ (resp., $B^i_{\mathrm{Zar},/\mathcal{Y}}(\mathcal{A})$) denote the étale (resp., Zariski) sheafification of $B^i(\mathcal{A})$ in the category $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ (see [GR2, Vol. 1, Chapter 2, Sect. 2.3]). We will be interested in spaces of the form

$$\mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})), \tag{1.4}$$

where $\mathrm{Maps}_{/\mathcal{Y}}(-,-)$ is short-hand for $\mathrm{Maps}_{\mathrm{PreStk}_{/\mathcal{Y}}}(-,-)$.

Note that (1.4) is naturally a group-like $\mathbb{E}_{n-i}$-space (resp., a commutative group object in Spc if $n = \infty$).

1.3.3. In most examples, we will take $\mathcal{A}$ to be of the form $A \times \mathcal{Y}$, where $A$ is a torsion abelian group, considered as a constant prestack. In this case

$$\mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^i_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})) \simeq \mathrm{Maps}(\mathcal{Y}, B^i_{\mathrm{et}}(A)).$$

Note that

$$\pi_j\left(\mathrm{Maps}(\mathcal{Y}, B^i_{\mathrm{et}}(A))\right) = \begin{cases} H^{i-j}_{\mathrm{et}}(\mathcal{Y}, A), & j \leq i; \\ 0, & j > i. \end{cases}$$

Here $H^\bullet_{\mathrm{et}}(\mathcal{Y}, A)$ refers to the étale cohomology of $\mathcal{Y}$ with coefficients in $A$. In other words, it is the cohomology of the object

$$\mathrm{C}^\bullet_{\mathrm{et}}(\mathcal{Y}, A) := \underset{(S,y)\in \mathrm{Sch}^{\mathrm{aff}}/\mathcal{Y}}{\underset{\longleftarrow}{\lim}} \mathrm{C}^\bullet_{\mathrm{et}}(S, A),$$

see [GL2, Construction 3.2.1.1].

1.3.4. Note also that in this case the functor

$$S \mapsto \mathrm{Maps}(S, B^i_{\mathrm{et}}(A)), \quad (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$

identifies with the *left Kan extension* of its restriction to $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}$. I.e., if an affine scheme $S$ is written as a filtered limit

$$S = \underset{\underset{\alpha}{\longleftarrow}}{\lim} S_\alpha, \quad S_\alpha \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},$$

then the map

$$\underset{\underset{\alpha}{\longrightarrow}}{\mathrm{colim}}\, \mathrm{Maps}(S_\alpha, B^i_{\mathrm{et}}(A)) \to \mathrm{Maps}(S, B^i_{\mathrm{et}}(A))$$

is an isomorphism (this latter assertion means that $B^i_{\mathrm{et}}(A)$ is locally of finite type as a prestack), see Corollary 1.2.8.

1.3.5. For $k = 1$, the points of the space

$$\mathrm{Tors}_{\mathcal{A}}(\mathcal{Y}) := \mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})) \tag{1.5}$$

are by definition $\mathcal{A}$-torsors on $\mathcal{Y}$.

1.3.6. Our primary interest is the cases of $k = 2$. We will call objects of the space

$$\mathrm{Ge}_{\mathcal{A}}(\mathcal{Y}) := \mathrm{Maps}_{/\mathcal{Y}}(\mathcal{Y}, B^2_{\mathrm{et},/\mathcal{Y}}(\mathcal{A})). \tag{1.6}$$

$\mathcal{A}$-gerbes on $\mathcal{Y}$.

When $\mathcal{A}$ is of the form $A \times \mathcal{Y}$ (see Sect. 1.3.3 above), we will simply write $\mathrm{Ge}_A(\mathcal{Y})$.

1.4. **Gerbes coming from line bundles.** In this subsection we will be studying gerbes for a constant commutative group-prestack, corresponding to a torsion abelian group $A$. In what follows, we will be assuming that the orders of elements of $A$ are co-prime to $\mathrm{char}(k)$.

1.4.1. Let $A(-1)$ denote the group

$$\underset{n\in\mathbb{N}}{\mathrm{colim}}\, \mathrm{Hom}(\mu_n, A) \text{ for any } n \gg 1.$$

In the above formula we regard $\mathbb{N}$ as a poset via

$$n' \geq n \Leftrightarrow n \mid n',$$

and in forming the above colimit the transition maps are given by

$$\mu_{n'} \overset{x \mapsto x^{\frac{n'}{n}}}{\twoheadrightarrow} \mu_n, \quad \text{for } n \mid n'. \tag{1.7}$$

For future reference, denote also

$$A(1) = \underset{n\in\mathbb{N}}{\mathrm{colim}} \left( \mu_{n'} \underset{\mathbb{Z}/n'\mathbb{Z}}{\otimes} A_{n\text{-tors}} \right),$$

where $A_{n\text{-tors}} \subset A$ is the subgroup of $n$-torsion elements, and in the above formula $n'$ is any integer divisible by $n$.

1.4.2. We claim that to any line bundle $\mathcal{L}$ on a prestack $\mathcal{Y}$ and an element $a \in A(-1)$ one can canonically associate an $A$-gerbe, denoted $\mathcal{L}^a$, over $\mathcal{Y}$.

It suffices to perform this construction for $A = \mu_n$ and $a$ coming from the identity map $\mu_n \to \mu_n$. In this case, the corresponding $\mu_n$-gerbe will be denoted $\mathcal{L}^{\frac{1}{n}}$.

By definition, for an affine test scheme $S$ over $\mathcal{Y}$, the value of $\mathcal{L}^{\frac{1}{n}}$ on $S$ is the groupoid of pairs

$$(\mathcal{L}', (\mathcal{L}')^{\otimes n} \simeq \mathcal{L}|_S),$$

where $\mathcal{L}'$ is a line bundle on $S$.

Note that if $\mathcal{L}$ admits an $n$-th root $\mathcal{L}'$, then this $\mathcal{L}'$ determines a trivialization of $\mathcal{L}^{\frac{1}{n}}$.

*Remark* 1.4.3. We emphasize the notational difference between the $\mu_n$-gerbe $\mathcal{L}^{\frac{1}{n}}$, and the line bundle $\mathcal{L}^{\otimes \frac{1}{n}}$, when the latter happens to exist. Namely, a choice of $\mathcal{L}^{\otimes \frac{1}{n}}$ defines a trivialization of the gerbe $\mathcal{L}^{\frac{1}{n}}$.

1.4.4. Let $Y$ be a smooth scheme, and let $Z \subset Y$ be a subvariety of codimension one. Let $Z_i, i \in I$ denote the irreducible components of $Z$. For every $i$, let $\mathcal{O}(Z_i)$ denote the corresponding line bundle on $Y$, trivialized away from $Z$.

We obtain a homomorphism

$$\text{Maps}(I, A(-1)) \to \text{Ge}_A(Y) \underset{\text{Ge}_A(Y-Z)}{\times} *, \quad (I \mapsto a_i) \rightsquigarrow \bigotimes_i \mathcal{O}(Z_i)^{a_i} \tag{1.8}$$

**Lemma 1.4.5.** *Assume that the orders of elements in $A$ are* prime *to* $\text{char}(k)$*, i.e., that $A$ has no $p$-torsion, where $p = \text{char}(k)$. Then the map* (1.8) *is an isomorphism in* Spc.

*Proof.* The assertion follows from the fact that the étale cohomology group $H^i_{\text{et},Z}(Y, A)$ identifies with $\text{Maps}(I, A(-1))$ for $i = 2$ and vanishes for $i = 1, 0$. □

1.5. **The sheaf-theoretic context.** Most of this paper is devoted to the discussion of gerbes. However, in the last two sections, we will apply this discussion in order to formulate metaplectic geometric Satake. The latter involves *sheaves* and more generally *sheaves of categories* on various geometric objects.

1.5.1. There are several possible sheaf-theoretic contexts (for schemes of finite type):

(a) For any ground field $k$ one can consider the derived category of $\ell$-adic sheaves with constructible cohomology.

(b) When the ground field is $\mathbb{C}$, then for an arbitrary algebraically closed field $E$ of characteristic 0, we can consider sheaves of complexes of $E$-vector spaces with constructible cohomology.

(c) When the ground field $k$ has characteristic 0, we can consider the derived category of D-modules.

Since our view is to towards quantum geometric Langlands, we will limit the discussion to the third case. That said, we will keep the notational distinction between the ground field, denoted $k$, and the field of coefficients for our sheaves, denoted $E$, although in the D-module case they are the same.

1.5.2. When discussing sheaves (and sheaves of categories) we will only need to consider algebro-geometric objects that are *locally of finite type*, i.e., prestacks that belong to $\text{PreStk}_{\text{lft}}$, see Sect. 1.2.4.

In what follows, in order to simplify the notation, we will omit the subscripts ft and lft.

1.5.3. We will denote by

$$\text{Shv} : (\text{Sch}^{\text{aff}}_{\text{ft}})^{\text{op}} \to \text{DGCat} \tag{1.9}$$

the functor constructed in [GR2, Vol. 1, Chapter 5, Sect. 3.1] that associates to an affine scheme $S$ of finite type the DG category

$$\text{Shv}(S) := \text{D-mod}(S)$$

(whose homotopy category is the unbounded derived category of quasi-coherent D-modules on $S$), and to a morphism $f : S_1 \to S_2$ the functor

$$f^! : \text{Shv}(S_2) \to \text{Shv}(S_1).$$

A basic feature of this functor (and which distinguishes the D-module context from the constructible ones) is that the functor (1.9) carries a natural *symmetric monoidal structure*. In particular, for $S_1, S_2 \in \text{Sch}^{\text{aff}}_{\text{ft}}$ we have a canonical equivalence

$$\text{Shv}(S_1) \otimes \text{Shv}(S_2) \to \text{Shv}(S_1 \times S_2).$$

1.5.4. Yoneda embedding is a fully faithful functor

$$\text{Sch}^{\text{aff}}_{\text{ft}} \hookrightarrow \text{PreStk}_{\text{lft}}.$$

The right Kan extension of Shv along the (opposite of the) Yoneda embedding $(\text{Sch}^{\text{aff}}_{\text{ft}})^{\text{op}} \to (\text{PreStk}_{\text{lft}})^{\text{op}}$ defines a functor

$$\text{Shv} : (\text{PreStk}_{\text{lft}})^{\text{op}} \to \text{DGCat}.$$

Thus, if $\mathcal{Y} \in \text{PreStk}_{\text{lft}}$ is written as

$$\mathcal{Y} = \underset{i}{\underset{\longrightarrow}{\text{colim}}}\, S_i, \quad S_i \in \text{Sch}^{\text{aff}}_{\text{ft}},$$

we have by definition

$$\text{Shv}(\mathcal{Y}) = \underset{i}{\underset{\longleftarrow}{\lim}}\, \text{Shv}(S_i).$$

1.6. **Sheaves of categories.** Sheaves of categories appear in this paper as a language in which we formulate the metaplectic geometric Satake functor. The reader can skip this subsection on the first pass, and return to it when necessary.

The discussion in this section is essentially borrowed from [Ga1].

1.6.1. Note that the diagonal morphism for affine schemes defines on every object of $(\text{Sch}^{\text{aff}}_{\text{ft}})^{\text{op}}$ a canonical structure of co-commutative co-algebra.

Hence, the symmetric monoidal structure on Shv (see [GR2, Vol. 2, Chapter 3, Corollary 6.1.2]) naturally gives rise to a functor

$$(\text{Sch}^{\text{aff}}_{\text{ft}})^{\text{op}} \to \text{ComAlg}(\text{DGCat}) =: \text{DGCat}^{\text{SymMon}}.$$

In particular, for every $S \in \text{Sch}^{\text{aff}}_{\text{ft}}$, the category $\text{Shv}(S)$ has a natural symmetric monoidal structure, and for every $f : S_1 \to S_2$, the functor $f^! : \text{Shv}(S_2) \to \text{Shv}(S_1)$ is symmetric monoidal.

1.6.2. By a sheaf of DG categories $\mathcal{C}$ over $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ we will mean a functorial assignment

$$(S \overset{y}{\to} \mathcal{Y}) \in ((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}})^{\mathrm{op}} \rightsquigarrow \mathcal{C}(S,y) \in \mathrm{Shv}(S)\text{-}\mathbf{mod}, \tag{1.10}$$

where $\mathrm{Shv}(S)$-**mod** denotes the category of modules in the (symmetric) monoidal category DGCat for the (commutative) algebra object $\mathrm{Shv}(S)$. We impose the following quasi-coherence condition:

For a morphism of affine schemes $f : S_1 \to S_2$, $y_2 : S_2 \to \mathcal{Y}$ and $y_1 = y_2 \circ f$, consider the corresponding functor

$$\mathcal{C}(S_2, y_2) \to \mathcal{C}(S_1, y_1). \tag{1.11}$$

Part of the data of (1.10) is that the functor (1.11) should be $\mathrm{Shv}(S_2)$-linear. Hence, it gives rise to a functor of $\mathrm{Shv}(S_1)$-module categories

$$\mathrm{Shv}(S_1) \underset{\mathrm{Shv}(S_2)}{\otimes} \mathcal{C}(S_2, y_2) \to \mathcal{C}(S_1, y_1), \tag{1.12}$$

where $\otimes$ is the operation of tensor product of DG categories (see, e.g., [GR2, Vol. 1, Chapter 1, Sect. 10.4]).

We require that (1.12) should be an isomorphism.

*Remark* 1.6.3. What we defined as a sheaf of categories over $\mathcal{Y}$ would in the language of [Ga1] be rather called a *crystals of categories*. More precisely, [Ga1, Theorem 2.6.3] guarantees that our notion of a sheaf of categories over $\mathcal{Y}$ coincides with the notion of a sheaf of categories over $\mathcal{Y}_{\mathrm{dR}}$ in the terminology of [Ga1].

1.6.4. A basic example of a sheaf of categories is denoted $\mathrm{Shv}_{/\mathcal{Y}}$; it is defined by setting

$$\mathrm{Shv}_{/\mathcal{Y}}(S,y) := \mathrm{Shv}(S).$$

Let $Z$ be a prestack locally of finite type over $\mathcal{Y}$. We define a sheaf of categories $\mathrm{Shv}(Z)_{/\mathcal{Y}}$ over $\mathcal{Y}$ by setting for $S \overset{y}{\to} \mathcal{Y}$,

$$\mathrm{Shv}(Z)_{/\mathcal{Y}}(S,y) = \mathrm{Shv}(S \underset{\mathcal{Y}}{\times} Z).$$

The fact that for $f : S_1 \to S_2$, the functor

$$\mathrm{Shv}(S_1) \underset{\mathrm{Shv}(S_2)}{\otimes} \mathrm{Shv}(S_2 \underset{\mathcal{Y}}{\times} Z) \to \mathrm{Shv}(S_1 \underset{\mathcal{Y}}{\times} Z)$$

is an equivalence follows from [Ga1, Theorem 2.6.3].

1.6.5. *Descent.* Forgetting the module structure, a sheaf of DG categories $\mathcal{C}$ over $\mathcal{Y}$ defines a functor

$$((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}\,. \tag{1.13}$$

It follows from [Ga1, Theorem 2.6.3.] that the assignment (1.13) satisfies étale descent (in fact, it satisfies h-descent).

1.6.6. Applying to the functor (1.13) the procedure of right Kan extension along

$$((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}})^{\mathrm{op}} \to ((\mathrm{PreStk}_{\mathrm{lft}})_{/\mathcal{Y}})^{\mathrm{op}},$$

we obtain that for every prestack $Z$ over $\mathcal{Y}$ there is a well-defined DG category $\mathcal{C}(Z)$.

Namely, if

$$Z \simeq \underset{i}{\underset{\longrightarrow}{\mathrm{colim}}}\, S_i, \quad (S_i, y_i) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}},$$

then

$$\mathcal{C}(Z) = \underset{i}{\underset{\longleftarrow}{\lim}}\, \mathcal{C}(S_i, y_i).$$

We will refer to $\mathcal{C}(Z)$ as the "category of sections of $\mathcal{C}$ over $Z$". By construction the DG category $\mathcal{C}(Z)$ is naturally an object of $\mathrm{Shv}(Z)$-**mod**.

When $Z$ is $\mathcal{Y}$ itself, we will refer to $\mathcal{C}(\mathcal{Y})$ as the "category of global sections of $\mathcal{C}$".

1.6.7. *Example.* For $\mathcal{C} = \mathrm{Shv}(Z)_{/Y}$ as in Sect. 1.6.4, we have

$$\mathcal{C}(\mathcal{Y}) \simeq \mathrm{Shv}(Z).$$

1.6.8. The construction in Sect. 1.6.6 defines a functor

$$\{\text{Sheaves of categories over } \mathcal{Y}\} \to \mathrm{Shv}(\mathcal{Y})\text{-}\mathbf{mod}. \tag{1.14}$$

The functor (1.14) admits a left adjoint given by sending

$$\mathcal{C} \rightsquigarrow \left((S \to \mathcal{Y}) \mapsto \mathrm{Shv}(S) \underset{\mathrm{Shv}(\mathcal{Y})}{\otimes} \mathcal{C}\right). \tag{1.15}$$

We have the following assertion from [Ga1, Theorem 1.5,2] states:

**Theorem 1.6.9.** *For $\mathcal{Y}$ that is an ind-scheme of ind-finite type, the mutually adjoint functors* (1.14) *and* (1.15) *are equivalences.*

*Remark* 1.6.10. For the purposes of the present paper one can make do avoiding (the somewhat non-trivial) Theorem 1.6.9. However, allowing ourselves to use it simplifies a lot of discussions related to sheaves of categories.

1.7. **Some twisting constructions.** The material in this subsection may not have proper references in the literature, so we provide some details. The reader is advised to skip it and return to it when necessary.

1.7.1. *Twisting by a torsor.* Let $\mathcal{Y}$ be a prestack, and let $\mathcal{H}$ (resp., $\mathcal{F}$) a group-like object in $\mathrm{PreStk}_{/\mathcal{Y}}$ (resp., an object in $\mathrm{PreStk}_{/\mathcal{Y}}$, equipped with an action of $\mathcal{H}$). In other words, these are functorial assignments

$$(S,y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \rightsquigarrow \mathcal{H}(S,y) \in \mathrm{Grp}(\mathrm{Spc}), \quad (S,y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \rightsquigarrow \mathcal{F}(S,y) \in \mathrm{Spc},$$

and an action of $\mathcal{H}(S,y)$ on $\mathcal{F}(S,y)$.

Let $\mathcal{T}$ be an $\mathcal{H}$-torsor on $\mathcal{Y}$. In this case, we can form a $\mathcal{T}$-twist of $\mathcal{F}$, denoted $\mathcal{F}_{\mathcal{T}}$, and which is an étale *sheaf*. Here is the construction[6]:

Consider the subcategory $\mathrm{Split}(\mathcal{T}) \subset (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ formed by $(S,y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ for which the torsor $\mathcal{T}|_S$ *admits a splitting*. This subcategory forms a basis of the étale topology, so it is sufficient to specify the restriction of $\mathcal{F}_{\mathcal{T}}$ to $\mathrm{Split}(\mathcal{T})$.

The sought-for functor $\mathcal{F}_{\mathcal{T}}|_{\mathrm{Split}(\mathcal{T})}$ is given by sending $(S,y)$ to

$$\left(* \underset{\mathrm{Maps}_{/\mathcal{Y}}(S, B_{\mathrm{et}}(\mathcal{H}))}{\times} *\right) \underset{\mathrm{Maps}_{/\mathcal{Y}}(S,\mathcal{H})}{\otimes} \mathcal{F},$$

where the two maps

$$* \to \mathrm{Maps}_{/\mathcal{Y}}(S, B_{\mathrm{et}}(\mathcal{H})) \leftarrow *$$

are the trivial map, and the one given by the composition

$$S \to \mathcal{Y} \overset{\mathcal{T}}{\to} B_{\mathrm{et}}(\mathcal{H}),$$

and we note that

$$* \underset{\mathrm{Maps}_{/\mathcal{Y}}(S, B_{\mathrm{et}}(\mathcal{H}))}{\times} *$$

is a groupoid equipped with a simply-transitive action of the group $\mathrm{Maps}_{/\mathcal{Y}}(S,\mathcal{H})$.

---

[6]Note that when $\mathcal{T}$ is the trivial torsor, the output of this construction is the étale sheafification of $\mathcal{F}$.

1.7.2. *A twist of a sheaf of categories by a gerbe.* Let now $\mathcal{C}$ be a sheaf of DG categories over $\mathcal{Y}$, and let $\mathcal{A}$ be a group-like $\mathbb{E}_2$-object in $(\text{PreStk}_{\text{lft}})_{/\mathcal{Y}}$.

Let us be given an action of $\mathcal{A}$ on $\mathcal{C}$. In other words, we are given a functorial assignment for every $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ of an action of $\mathcal{A}(S,y)$ on $\mathcal{C}(S,y)$, see Sect. 1.1.7.

Let $\mathcal{G}$ be an étale $\mathcal{A}$-gerbe on $\mathcal{Y}$. Repeating the construction of Sect. 1.7.1, we obtain that we can form the twist $\mathcal{C}_{\mathcal{G}}$ of $\mathcal{C}$ by $\mathcal{G}$, which is a new sheaf of DG categories over $\mathcal{Y}$.

In more detail, for $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ such that $\mathcal{G}|_S$ *admits a splitting*, we define the value of $\mathcal{C}_{\mathcal{G}}$ on $(S,y)$ to be

$$\left(* \underset{\text{Maps}_{/\mathcal{Y}}(S, B^2_{\text{et}}(\mathcal{A}))}{\times} *\right) \underset{\text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{A}))}{\otimes} \mathcal{C}(S,y),$$

where the two maps

$$* \to \text{Maps}_{/\mathcal{Y}}(S, B^2_{\text{et}}(\mathcal{A})) \leftarrow *$$

are the trivial map, and the one given by the composition

$$S \to \mathcal{Y} \overset{\mathcal{G}}{\to} B^2_{\text{et}}(\mathcal{A}),$$

and we note that

$$* \underset{\text{Maps}_{/\mathcal{Y}}(S, B^2_{\text{et}}(\mathcal{A}))}{\times} *$$

is a groupoid equipped with a simply-transitive action of the group $\text{Maps}_{/\mathcal{Y}}(S, B_{\text{et}}(\mathcal{A}))$.

Concretely, for every $(S \overset{y}{\to} \mathcal{Y}) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ and a trivialization of $\mathcal{G}|_S$ we have an identification

$$\mathcal{C}_{\mathcal{G}}(S,y) \simeq \mathcal{C}(S,y).$$

The effect of change of trivialization by a point $a \in B_{\text{et}}(\mathcal{A})(S,y)$ has the effect of action of

$$a \in \text{Funct}(\mathcal{C}(S,y), \mathcal{C}(S,y)).$$

1.7.3. Let $A$ be a torsion subgroup of $E^{\times}$.

Let us take $\mathcal{A}$ to be the constant group-prestack $\mathcal{Y} \times A$. In this case, the embedding $A \to E^{\times}$ gives rise to an action of $\mathcal{A}$ on *any* sheaf of DG categories.

Thus, for every $\mathcal{G} \in \text{Ge}_A(\mathcal{Y})$ and any sheaf of categories $\mathcal{C}$ over $\mathcal{Y}$, we can form its twisted version $\mathcal{C}_{\mathcal{G}}$.

1.7.4. *The category of sheaves twisted by a gerbe.* Let $A$ and $\mathcal{G}$ be as in Sect. 1.7.3.

We apply the above construction to $\mathcal{C} := \text{Shv}_{/\mathcal{Y}}$. Thus, for any $(S,y) \in (\text{Sch}^{\text{aff}}_{\text{ft}})_{/\mathcal{Y}}$ we have the twisted version of the category $\text{Shv}(S)$, denoted $\text{Shv}_{\mathcal{G}}(S)$.

As in Sect. 1.6.6, the procedure of Kan extension defines the category

$$\text{Shv}_{\mathcal{G}}(Z)$$

for any $Z \in \text{PreStk}_{/\mathcal{Y}}$.

## 2. Factorization gerbes on the affine Grassmannian

In this section we introduce our main object of study: factorization gerbes on the affine Grassmannian, which we stipulate to be the parameters for the metaplectic Langlands theory.

2.1. **The Ran space.** The Ran space of a curve $X$ is an algebro-geometric device (first suggested in [BD1]) that allows us to talk about *factorization structures* relative to our curve.

2.1.1. Let $X$ be a fixed smooth algebraic curve. We let $\mathrm{Ran} \in \mathrm{PreStk}$ be the Ran space of $X$. By definition, for an affine test scheme $S$, the space $\mathrm{Maps}(S, \mathrm{Ran})$ is discrete (i.e., is a set), and equals the set of finite non-empty subsets of the (set) $\mathrm{Maps}(S, X)$.

For a finite set $J$ we have a map

$$\mathrm{Ran}^J \to \mathrm{Ran} \tag{2.1}$$

given by the union of the corresponding finite subsets.

This operation makes Ran into a (non-unital) semi-group object in $\mathrm{PreStk}_{\mathrm{lft}}$ (see [Lu2, Definition 5.4.1.1] for what this means).

2.1.2. The Ran space admits the following explicit description as a colimit (as an object of PreStk):

$$\mathrm{Ran} = \underset{\overrightarrow{I}}{\mathrm{colim}}\, X^I,$$

where $I$ runs through the category opposite to that of non-empty finite sets and surjective maps[7]. For a surjection $\phi : I_1 \to I_2$, the corresponding map $X^{I_2} \to X^{I_1}$ is the corresponding diagonal morphism, denoted $\Delta_\phi$.

This presentation makes it manifest that $\mathrm{Ran} \in \mathrm{PreStk}_{\mathrm{lft}}$.

2.1.3. We denote by

$$(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}} \subset \mathrm{Ran} \times \mathrm{Ran}$$

the open substack corresponding to the following condition:

For an affine test scheme $S$, and two points

$$I_1, I_2 \in \mathrm{Maps}(S, \mathrm{Ran}),$$

the point $I_1 \times I_2 \in \mathrm{Maps}(S, \mathrm{Ran} \times \mathrm{Ran})$ belongs to $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$ if the corresponding subsets

$$I_1, I_2 \subset \mathrm{Maps}(S, X)$$

satisfy the following condition: for every $i_1 \in I_1, i_2 \in I_2$, the corresponding two maps $S \rightrightarrows X$ have non-intersecting images.

2.1.4. We give a similar definition for any power: for a finite set $J$ we let

$$\mathrm{Ran}^J_{\mathrm{disj}} \subset \mathrm{Ran}^J$$

be the open substack corresponding to the following condition:

An $S$-point of $\mathrm{Ran}^J$, given by

$$I_j \subset \mathrm{Maps}(S, X), \quad j \in J$$

belongs to $\mathrm{Ran}^J_{\mathrm{disj}}$ if for every $j_1 \neq j_2$ and $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$, the corresponding two maps $S \rightrightarrows X$ have non-intersecting images.

2.2. **Factorization patterns over the Ran space.** Let $Z$ be a prestack over Ran. At the level of $k$-points, a factorization structure on $Z$ is the following system of isomorphisms:

For a $k$-point $\underline{x}$ of Ran corresponding a finite set $x_1, ..., x_n$ of $k$-points of $X$, the fiber $Z_{\underline{x}}$ of $Z$ over the above point is supposed to be identified with

$$\prod_i Z_{\{x_i\}},$$

where $\{x_i\}$ are the corresponding singleton points of Ran.

We will now spell this idea, and some related notions, more precisely.

[7] We note that this category is *not filtered*, and hence Ran is *not* an ind-scheme.

2.2.1. By a factorization structure on $Z$ we shall mean an assignment for any finite set $J$ of an isomorphism

$$Z^J \underset{\mathrm{Ran}^J}{\times} \mathrm{Ran}^J_{\mathrm{disj}} \overset{\gamma_J}{\simeq} Z \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^J_{\mathrm{disj}}, \tag{2.2}$$

where the morphism $\mathrm{Ran}^J \to \mathrm{Ran}$ is given by (2.1).

We require the isomorphisms (2.2) to be compatible with surjections of finite sets in the sense that for $I \overset{\phi}{\twoheadrightarrow} J$ the diagram

$$\begin{array}{ccc}
Z^I \underset{\mathrm{Ran}^I}{\times} \mathrm{Ran}^I_{\mathrm{disj}} & \overset{\gamma_I}{\longrightarrow} & Z \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} \\
\sim\downarrow & & \uparrow\sim \\
\left(\prod\limits_{j\in J} Z^{I_j} \underset{\mathrm{Ran}^{I_j}}{\times} \mathrm{Ran}^{I_j}_{\mathrm{disj}}\right) \underset{\prod\limits_{j\in J} \mathrm{Ran}^{I_j}_{\mathrm{disj}}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} & & (Z \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^J_{\mathrm{disj}}) \underset{\mathrm{Ran}^J_{\mathrm{disj}}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} \\
\prod\limits_{j\in J}\gamma_{I_j}\downarrow & & \uparrow\gamma_J \\
\left(\prod\limits_{j\in J} Z \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{I_j}_{\mathrm{disj}}\right) \underset{\prod\limits_{j\in J} \mathrm{Ran}^{I_j}_{\mathrm{disj}}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} & & (Z^J \underset{\mathrm{Ran}^J}{\times} \mathrm{Ran}^J_{\mathrm{disj}}) \underset{\mathrm{Ran}^J_{\mathrm{disj}}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} \\
\sim\downarrow & & \uparrow\sim \\
\left(Z^J \underset{\mathrm{Ran}^J}{\times} \prod\limits_{j\in J} \mathrm{Ran}^{I_j}_{\mathrm{disj}}\right) \underset{\prod\limits_{j\in J} \mathrm{Ran}^{I_j}_{\mathrm{disj}}}{\times} \mathrm{Ran}^I_{\mathrm{disj}} & \overset{\sim}{\longrightarrow} & Z^J \underset{\mathrm{Ran}^J}{\times} \mathrm{Ran}^I_{\mathrm{disj}},
\end{array} \tag{2.3}$$

where $I_j := \phi^{-1}(j)$, is required to commute. Furthermore, if $Z$ takes values in $\infty$-groupoids (rather than sets), we require a homotopy-coherent system of compatibilities for higher order compositions, see [Ras1, Sect. 6].

2.2.2. Let $\mathcal{C}$ be a sheaf of DG categories over Ran (recall that this means that we are working over a ground field of characteristic 0 and in the context of D-modules).

By a *factorization structure* on $\mathcal{C}$ we shall mean a functorial assignment for any finite set $J$ and an $S$-point of $\mathrm{Ran}^J_{\mathrm{disj}}$, given by

$$I_j \subset \mathrm{Maps}(S, X), \quad j \in J$$

of an identification

$$\underset{j,\mathrm{Shv}(S)}{\bigotimes} \mathcal{C}(S, I_j) \to \mathcal{C}(S, I), \tag{2.4}$$

where $I = \underset{j\in J}{\sqcup} I_j$.

We require the functors (2.4) to be compatible with surjections $J_1 \twoheadrightarrow J_2$ via the commutative diagrams analogous to (2.3). A precise formulation of these compatibilities is given in [Ras1, Sect. 6].

2.2.3. Let $Z$ be a factorization prestack over Ran. Assume that for every finite set $I$, the category $\mathrm{Shv}(X^I \underset{\mathrm{Ran}}{\times} Z)$ is dualizable. We claim that in this case the sheaf of categories $\mathrm{Shv}(Z)_{/\mathrm{Ran}}$, i.e.,

$$(S, I \subset \mathrm{Maps}(S, X)) \rightsquigarrow \mathrm{Shv}(S \underset{\mathrm{Ran}}{\times} Z),$$

has a natural factorization structure.

Indeed, for any $Z$ we have a canonically defined system of functors

$$\underset{j,\mathrm{Shv}(S)}{\bigotimes} \mathrm{Shv}(S \underset{I_j,\mathrm{Ran}}{\times} Z) \to \mathrm{Shv}\left(\prod_{j,S} (S \underset{I_j,\mathrm{Ran}}{\times} Z)\right) = \mathrm{Shv}(S \underset{\mathrm{Ran}^J}{\times} Z^J) \overset{(2.2)}{\simeq} \mathrm{Shv}(S \underset{I,\mathrm{Ran}}{\times} Z)$$

for a map $S \to \operatorname{Ran}^J_{\mathrm{disj}}$. We claim that the first arrow is an equivalence if each $\operatorname{Shv}(X^I \underset{\operatorname{Ran}}{\times} Z)$ is dualizable.

To prove this, it suffices to can consider the universal case when $S = X^I_{\mathrm{disj}}$ for a finite set $I$ and a surjection $I \twoheadrightarrow J$. We have

$$\bigotimes_{j, \operatorname{Shv}(S)} \operatorname{Shv}(S \underset{I_j, \operatorname{Ran}}{\times} Z) \simeq \left( \bigotimes_{j \in J} \operatorname{Shv}(X^{I_j} \underset{\operatorname{Ran}}{\times} Z) \right) \underset{\operatorname{Shv}(X^I)}{\otimes} \operatorname{Shv}(X^I_{\mathrm{disj}}) \to$$
$$\to \operatorname{Shv}\left( \prod_{j \in J} (X^{I_j} \underset{\operatorname{Ran}}{\times} Z) \right) \underset{\operatorname{Shv}(X^I)}{\otimes} \operatorname{Shv}(X^I_{\mathrm{disj}}) \simeq \operatorname{Shv}\left( \left( \prod_{j \in J} (X^{I_j} \underset{\operatorname{Ran}}{\times} Z) \right) \underset{X^I}{\times} X^I_{\mathrm{disj}} \right) = \operatorname{Shv}(S \underset{\operatorname{Ran}^J}{\times} Z^J),$$

where the second are is an isomorphism due to the assumption that the categories $\operatorname{Shv}(X^{I_j} \underset{\operatorname{Ran}}{\times} Z)$ are dualizable.

2.2.4. Let $Z$ be a factorization prestack over Ran, and let $A$ be a torsion abelian group. Let $\mathcal{G}$ be an $A$-gerbe on $Z$. By a factorization structure on $\mathcal{G}$ we shall mean a system of identifications

$$\mathcal{G}^{\boxtimes J}|_{Z^J \underset{\operatorname{Ran}^J}{\times} \operatorname{Ran}^J_{\mathrm{disj}}} \simeq \mathcal{G}|_{Z \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^J_{\mathrm{disj}}}, \tag{2.5}$$

where the underlying spaces are identified via (2.2).

The identifications (2.5) are required to be compatible with surjections $J_1 \twoheadrightarrow J_2$ via the commutative diagrams (2.3). Note that since gerbes form a 2-groupoid, we only need to specify the datum of (2.5) up to $|J| = 3$, and check the relations up to $|J| = 4$.

Factorization gerbes over $Z$ naturally form a space (in fact, a 2-groupoid), equipped with a structure of commutative group in Spc (i.e., connective spectrum), to be denoted $\operatorname{FactGe}_A(Z)$.

*Remark* 2.2.5. Note that the diagrams (2.3) include those corresponding to automorphisms of finite sets. I.e., the datum of factorization gerbe includes equivariance with respect to the action of the symmetric group. For this reason what we call "factorization gerbe" in [Re] was called "symmetric factorizable gerbe".

2.2.6. *Variant.* Let $Z$ be a factorization prestack over Ran, and let $\mathcal{G}$ be a factorization $A$-gerbe over it for $A \subset E^\times$. Assume that for every finite set $I$, the category $\operatorname{Shv}_{\mathcal{G}}(X^I \underset{\operatorname{Ran}}{\times} Z)$ is dualizable. Then the sheaf of categories $\operatorname{Shv}_{\mathcal{G}}(Z)_{/\operatorname{Ran}}$ defined by

$$(S, I \subset \operatorname{Maps}(S, X)) \rightsquigarrow \operatorname{Shv}_{\mathcal{G}}(S \underset{\operatorname{Ran}}{\times} Z)$$

has a natural factorization structure.

2.2.7. By a similar token, we can consider factorization line bundles over factorization prestacks, and also $\mathbb{Z}$- or $\mathbb{Z}/2\mathbb{Z}$-graded line bundles[8].

If $\mathcal{L}$ is a (usual, i.e., not graded) factorization line bundle and $a \in A(-1)$, we obtain a factorization gerbe $\mathcal{L}^a$.

2.3. **The Ran version of the affine Grassmannian.** In this subsection we introduce the Ran version of the affine Grassmannian, which plays a crucial role in the geometric Langlands theory.

[8]Note that in the latter case, the compatibility involved in the factorization structure (arising from the diagrams (2.3) for automorphisms of finite sets $J$) involves *sign rules*. I.e., a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle *does not* give rise to a factorization line bundle by forgetting the grading.

2.3.1. For an algebraic group $G$, we define the Ran version of the affine Grassmannian of $G$, denoted $\mathrm{Gr}_G$, to be the following prestack.

For an affine test scheme $S$, the groupoid (in fact, set) $\mathrm{Maps}(S, \mathrm{Gr}_G)$ consists of triples

$$(I, \mathcal{P}_G, \alpha),$$

where $I$ is an $S$-point of Ran, $\mathcal{P}_G$ is a $G$-bundle on $S \times X$, and $\alpha$ is a trivialization of $\mathcal{P}_G$ over the open subset $U_I \subset S \times X$ equal to the complement of the union of the graphs of the maps $S \to X$ corresponding to the elements of $I \subset \mathrm{Maps}(S, X)$.

2.3.2. It is known that for every finite set $I$, the prestack $X^I \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G$ is an ind-scheme of ind-finite type. This implies, in partircular, that the dualizability assumptions in Sects. 2.2.3 and 2.2.6 are satisfied.

2.3.3. The basic feature of the prestack $\mathrm{Gr}_G$ is that it admits a natural factorization structure over Ran, obtained by gluing bundles.

Hence, for a torsion abelian group $A$, it makes sense to talk about factorization $A$-gerbes over $\mathrm{Gr}_G$. We denote the the resulting space (i.e., in fact, a connective 2-truncated spectrum) by

$$\mathrm{FactGe}_A(\mathrm{Gr}_G).$$

2.3.4. *An example.* Let $\mathcal{L}$ be a factorization line bundle on $\mathrm{Gr}_G$, and let $a$ be an element of $A(-1)$. Then the $A$-gerbe

$$\mathcal{L}^a$$

of Sect. 1.4.1 is naturally a factorization gerbe on $\mathrm{Gr}_G$.

This example is important because there is a canonical factorization line bundle on $\mathrm{Gr}_G$, denoted $\det_{\mathfrak{g}}$; we will encounter it in Sect. 5.2.1.

2.3.5. Assume for a moment that $X$ is proper.

Let $\mathrm{Bun}_G$ denote the moduli stack of $G$-bundles on $X$. Note that we have a tautological projection

$$\mathrm{Gr}_G \to \mathrm{Bun}_G\,. \tag{2.6}$$

Recall now that [GL2, Theorem 3.2.13] says[9] that the map (2.6) is a *universal homological equivalence*. This implies that any gerbe on $\mathrm{Gr}_G$ uniquely descends to a gerbe on $\mathrm{Bun}_G$.

In particular, this is the case for factorization gerbes.

## 2.4. **The space of geometric metaplectic data.**

2.4.1. Let $E^{\times,\mathrm{tors}}$ denote the group of roots of unity in $E$ of orders co-prime with $\mathrm{char}(k)$.

We stipulate that the space

$$\mathrm{FactGe}_{E^{\times,\mathrm{tors}}}(\mathrm{Gr}_G)$$

is the space of parameters for the metaplectic Langlands theory. We also refer to it as the space *geometric metaplectic data.*

This includes both the global case (when $X$ is complete), and the local case when we take $X$ to be a Zariski neighborhood of some point $x$.

2.4.2. Given an $E^{\times,\mathrm{tors}}$-factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_G$, we can thus talk about the factorization sheaf of categories, denoted

$$\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}},$$

whose value on $S, I \subset \mathrm{Maps}(S, X)$ is

$$\mathrm{Shv}_{\mathcal{G}}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G).$$

[9]This assertion was proved in *loc.cit.* under the additional assumption that $G$ be semi-simple and simply connected. However, in the case of constant groups-schemes, the statement is known to hold in general: see [Ga3, Theorem 4.1.6].

## 3. Parameterization of factorization gerbes

From now on we let $A$ be a torsion abelian group whose elements have orders prime to $\mathrm{char}(k)$. The main example is $A = E^{\times,\mathrm{tors}}$.

The goal of this section is to describe the set of isomorphism classes (and, more ambitiously, the *space*) of $A$-factorization gerbes on $\mathrm{Gr}_G$ in terms of more concise algebro-geometric objects.

3.1. **Parameterization via étale cohomology.** In this subsection we will create a space, provided by the theory of étale cohomology, that maps to the space $\mathrm{FactGe}_A(\mathrm{Gr}_G)$, thereby giving a parameterization of geometric metaplectic data.

3.1.1. Let $B_{\mathrm{et}}(G) := \mathrm{pt}/G$ be the stack of $G$-torsors. I.e., this is the sheafification in the étale topology of the prestack $B(G)$ that attaches to an affine test scheme $S$ the groupoid

$$*/\,\mathrm{Maps}(S, G).$$

3.1.2. Consider the space of maps

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1)) \times X),$$

which is the same as $\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(G) \times X, B^4_{\mathrm{et}}(A(1)))$.

I.e., this is the space of maps

$$B_{\mathrm{et}}(G) \times X \to B^4_{\mathrm{et}}(A(1)), \tag{3.1}$$

equipped with an identification of the composite map

$$X = \mathrm{pt} \times X \to B_{\mathrm{et}}(G) \times X \to B^4_{\mathrm{et}}(A(1)) \tag{3.2}$$

with

$$X \to \mathrm{pt} \to B^4_{\mathrm{et}}(A(1)).$$

We claim that there is a naturally defined map

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B_{\mathrm{et}}(G) \times X, B^4_{\mathrm{et}}(A(1)) \times X) \to \mathrm{FactGe}_A(\mathrm{Gr}_G), \tag{3.3}$$

3.1.3. The construction of the map (3.3) proceeds as follows. Let us be given a map (3.1) equipped with a trivialization of the composition (3.2).

For an affine test scheme $S$ and an $S$-point $(I, \mathcal{P}_G, \alpha)$ of $\mathrm{Gr}_G$, we need to construct a $A$-gerbe $\mathcal{G}_I$ on $S$.

Moreover, for $\phi : I \twoheadrightarrow J$, such that the point

$$\{\phi^{-1}(j) \subset \mathrm{Maps}(S, \mathrm{Ran}^J), \quad j \in J\}$$

hits $\mathrm{Ran}^J_{\mathrm{disj}}$, we need to be given an identification

$$\mathcal{G}_I \simeq \bigotimes_{j \in J} \mathcal{G}_{I_j}. \tag{3.4}$$

3.1.4. Let us interpret the datum of $\mathcal{P}_G$ as a map

$$S \times X \to B_{\mathrm{et}}(G) \times X.$$

Composing with (3.1), we obtain a map

$$S \times X \to B^4_{\mathrm{et}}(A(1)), \tag{3.5}$$

and a trivialization of the resulting map

$$U_I \to B^4_{\mathrm{et}}(A(1)), \tag{3.6}$$

where $U_I$ is as in Sect. 2.3.1.

We claim that such a datum indeed gives rise to a $A$-gerbe $\mathcal{G}_I$ on $S$, equipped with identifications (3.4).

3.1.5. First off, since

$$H^i_{\mathrm{et}}(S \times X, A(1)) \text{ and } H^{i-1}_{\mathrm{et}}(U_I, A(1))$$

for $i = 3$ and $i = 4$ vanish étale-locally on $S$, we obtain that the prestack that sends $S$ to the space of maps (3.5), equipped with a trivialization of (3.6), identifies with $B^2_{\mathrm{et}}$ of the prestack that sends $S$ to the space of maps

$$S \times X \to B^2_{\mathrm{et}}(A(1)), \tag{3.7}$$

equipped with a trivialization of

$$U_I \to B^2_{\mathrm{et}}(A(1)). \tag{3.8}$$

3.1.6. Thus, given a map (3.7), equipped with a trivialization of (3.8), we need to construct a locally constant map

$$S \to A$$

whose dependence on (3.7) and the trivialization of (3.8) respects the structure of commutative group on $A(1)$. Let $\Gamma_I$ denote the complement of $U_I$ (the scheme structure on $\Gamma_I$ is irrelevant). Thus, we need to construct the trace map

$$H^2_{\mathrm{et},\Gamma_I}(S \times X, A(1)) \to H^0_{\mathrm{et}}(S, A). \tag{3.9}$$

3.1.7. Consider the maps

$$\begin{array}{ccc} \Gamma_I & \xrightarrow{\iota} & S \times X \\ {\scriptstyle\pi}\big\downarrow & & \\ S. & & \end{array}$$

Let $p_X$ denote the projection $X \to \mathrm{pt}$. We have a canonical identification

$$(p_X)!(A) \simeq A_X(1)[2],$$

where for a scheme $Y$ we denote by $A_Y$ the constant étale sheaf with value $A$, and hence

$$(\mathrm{id}_S \times p_X)^!(A_S) \simeq A_{S\times X}(1)[2].$$

From here we obtain an isomorphism

$$\iota^!(A_{S\times X})(1)[2] \simeq \pi^!(A_S),$$

and by the $(\pi_*, \pi^!)$-adjunction, a morphism

$$\pi_* \circ \iota^!(A_{S\times X})(1)[2] \to A_S, \tag{3.10}$$

The sought-for morphism (3.9) is obtained from (3.10) by applying $H^0_{\mathrm{et}}(S, -)$.

3.1.8. We have the following assertion that results from [Re, Theorem II.7.3][10] and the computation of the homotopy groups of the left-hand side of (3.3) (the latter is given in Sect. 3.3 below):

**Proposition 3.1.9.** *The map* (3.3) *is an isomorphism.*

*Remark* 3.1.10. As was explained to us by J. Lurie, the assertion of Proposition 3.1.9 is nearly tautological if one works over the field of complex numbers and in the context of sheaves in the analytic topology.

3.1.11. From Proposition 3.1.9 we will obtain that

$$\pi_i(\mathrm{FactGe}_A(\mathrm{Gr}_G)) = H^{4-i}_{\mathrm{et}}(B(G) \times X; \mathrm{pt} \times X, A(1)).$$

Below we will analyze what these cohomology groups look like.

## 3.2. Digression: étale cohomology of B(G).

[10]The statement of *loc.cit.* needs to be corrected by replacing the group denoted there by $Q(\Lambda_T, A)^W_{\mathbb{Z}}$ by $\mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$, introduced below. This is what [Re, Theorem II.7.3] actually proves.

3.2.1. Let $\pi_{1,\mathrm{alg}}(G)$ denote the algebraic fundamental group of $G$. Explicitly, $\pi_{1,\mathrm{alg}}(G)$ can be described as follows:

Choose a short exact sequence

$$1 \to T_2 \to \widetilde{G}_1 \to G \to 1,$$

where $T_2$ is a torus and $[\widetilde{G}_1, \widetilde{G}_1]$ is simply connected. Set $T_1 = \widetilde{G}_1/[\widetilde{G}_1, \widetilde{G}_1]$. Let $\Lambda_1$ and $\Lambda_2$ be the coweight lattices of $T_1$ and $T_2$, respectively. Then $\pi_{1,\mathrm{alg}}(G) \simeq \Lambda_1/\Lambda_2$.

Equivalently, $\pi_{1,\mathrm{alg}}(G)$ is the quotient of $\Lambda$ by the root lattice.

3.2.2. For an abelian group $A$, let $\mathrm{Quad}(\Lambda, A)^W$ denote the set of $W$-invariant quadratic forms on $\Lambda$ with values in $A$. For any such form, denoted $q$, let $b$ denote the associated symmetric bilinear form:

$$b(\lambda_1, \lambda_2) = q(\lambda_1 + \lambda_2) - q(\lambda_1) - q(\lambda_2).$$

Let $\mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, A)^W$ be the subset consisting of forms $q$ that satisfy the following additional condition: for every coroot $\alpha \in \Lambda$ and any $\lambda \in \Lambda$

$$b(\alpha, \lambda) = \langle \check{\alpha}, \lambda\rangle \cdot q(\alpha), \tag{3.11}$$

where $\check{\alpha}$ is the root corresponding to $\alpha$.

*Remark* 3.2.3. Note that the identity

$$2b(\alpha, \lambda) = 2\langle \check{\alpha}, \lambda\rangle \cdot q(\alpha)$$

holds automatically.

Moreover, (3.11) itself holds automatically if $\frac{\alpha}{2} \in \Lambda$.

3.2.4. Note that we have an injective map

$$\mathrm{Quad}(\Lambda, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A \to \mathrm{Quad}(\Lambda, A)^W,$$

whose image belongs to $\mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$.

Assume for a moment that $A$ is divisible. Then for any element $q \in \mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$ there exists an element $q_{\mathbb{Z}} \in \mathrm{Quad}(\Lambda, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A$ such that $q - q_{\mathbb{Z}}$ comes from a quadratic form on $\pi_{1,\mathrm{alg}}(G)$ under the projection $\Lambda \to \pi_{1,\mathrm{alg}}(G)$.

In particular, the inclusion

$$\mathrm{Quad}(\Lambda, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A \hookrightarrow \mathrm{Quad}(\Lambda, A)^W_{\mathrm{restr}}$$

is an equality when the derived group of $G$ is simply connected.

3.2.5. We claim:

**Theorem 3.2.6.** *Let $A$ be a torsion abelian group $A$ whose elements have orders co-prime with* $\mathrm{char}(p)$. *Assume also that $A$ is divisible. Then:*

$$H^i_{\mathrm{et}}(B(G), A(1)) = 0 \textit{ for } i = 1, 3;$$
$$H^2_{\mathrm{et}}(B(G), A(1)) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A);$$
$$H^4_{\mathrm{et}}(B(G), A(1)) \simeq \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}}.$$

*Remark* 3.2.7. When $A$ is not divisible, the only difference will be that $H^3_{\mathrm{et}}(B(G), A(1)) \simeq \mathrm{Ext}^1(\pi_{1,\mathrm{alg}}(G), A)$; in particular it will vanish if the derived group of $G$ is simply-connected.

As we could not find a reference for this statement in the literature, we will supply the proof in Sect. A.

*Remark* 3.2.8. In fact, this is the same computation as in the context of algebraic topology, where we calculate singular cohomology with coefficients in $\mathbb{Q}/\mathbb{Z}$ of the classifying space of a compact connected Lie group, for which we could not find a reference either (the cohomology with $\mathbb{Z}$ coefficients is well-known of course).

3.3. **Analysis of homotopy groups of the space of factorization gerbes.** In this subsection we will assume that $A$ is divisible (this assumption is only necessary when the derived group of $G$ is simply-connected, see Remark 3.2.7 above).

3.3.1. Let $p_{B(G)}$ denote the projection $B(G) \to \text{pt}$. Consider the object

$$K := \tau^{\geq 1, \leq 4}(R(p_{B(G)})_*(A(1))) \in \text{Shv}(\text{pt}).$$

By the Leray spectral sequence and smooth base change

$$H^{4-i}_{\text{et}}(B(G) \times X; \text{pt} \times X, A(1)) \simeq H^{4-i}_{\text{et}}(X, p_X^*(K)).$$

3.3.2. By Theorem 3.2.6, we have a distinguished triangle

$$\text{Hom}(\pi_{1,\text{alg}}(G), A)[-2] \to K \to \text{Quad}(\Lambda, A(-1))^W_{\text{restr}}[-4].$$

From here we obtain that

$$\pi_0(\text{FactGe}_A(\text{Gr}_G)) \simeq H^4_{\text{et}}(B(G) \times X; \text{pt} \times X, A(1))$$

identifies with

$$\text{Quad}(\Lambda, A(-1))^W_{\text{restr}} \times H^2_{\text{et}}(X, \text{Hom}(\pi_{1,\text{alg}}(G), A)),$$

while

$$\pi_1(\text{FactGe}_A(\text{Gr}_G)) \simeq H^3_{\text{et}}(B(G) \times X; \text{pt} \times X, A(1))$$

identifies with

$$H^1_{\text{et}}(X, \text{Hom}(\pi_{1,\text{alg}}(G), A))$$

and

$$\pi_2(\text{FactGe}_A(\text{Gr}_G)) \simeq H^0_{\text{et}}(B(G) \times X; \text{pt} \times X, A(1))$$

identifies with

$$H^0_{\text{et}}(X, \text{Hom}(\pi_{1,\text{alg}}(G), A)).$$

3.3.3. In particular, we obtain a map (of spectra)

$$\text{FactGe}_A(\text{Gr}_G) \to \text{Quad}(\Lambda, A(-1))^W_{\text{restr}}.$$

Let $\text{FactGe}^0_A(\text{Gr}_G)$ denote its fiber.

3.3.4. By Proposition 3.1.9, $\text{FactGe}^0_A(\text{Gr}_G)$ receives an isomorphism from the space classifying objects in $\text{Maps}_{\text{Ptd}(\text{PreStk}_{/X})}(B_{\text{et}}(G) \times X, B^4_{\text{et}}(A(1)) \times X)$ are trivial étale-locally on $X$.

In other words, this is the space classifying maps from $X$ to

$$B_{\text{et}}\left(\text{Maps}_{\text{Ptd}(\text{PreStk})}(B(G), B^3_{\text{et}}(A(1)))\right).$$

However, since $H^3_{\text{et}}(B(G), A(1)) = 0$, we obtain that the map

$$B_{\text{et}}\left(\text{Maps}_{\text{Ptd}(\text{PreStk})}(B(G), B^2_{\text{et}}(A(1)))\right) \to \text{Maps}_{\text{Ptd}(\text{PreStk})}(B(G), B^3_{\text{et}}(A(1)),$$

is an isomorphism, where we note that

$$\text{Maps}_{\text{Ptd}(\text{PreStk})}(B(G), B^2_{\text{et}}(A(1)) \simeq \text{Hom}(\pi_{1,\text{alg}}(G), A).$$

Thus, we obtain:

**Corollary 3.3.5.** *The map* (3.3) *induces an isomorphism*

$$\text{Maps}(X, B^2_{\text{et}}(\text{Hom}(\pi_{1,\text{alg}}(G), A))) \simeq \text{FactGe}^0_A(\text{Gr}_G). \tag{3.12}$$

3.4. **Parametrization of factorization line bundles.** This subsection is included for the sake of completeness, in order to make contact with the theory of metaplectic extensions developed in [We].

Recall from Sect. 2.3.4 that given a factorization line bundle $\mathcal{L}$ on $\text{Gr}_G$ and an element $a \in A(-1)$ we can produce a factorization gerbe $\mathcal{L}^a$. In this subsection we will describe a geometric data that gives rise to factorization line bundles[11] on $\text{Gr}_G$.

[11] We emphasize that this construction produces just factorization line bundles, and *not* $\mathbb{Z}/2\mathbb{Z}$-graded ones.

3.4.1. Let $K_2$ denote the prestack over $X$ that associates to an affine scheme $S = \operatorname{Spec}(A)$ mapping to $X$ the abelian group $K_2(A)$. Let $(K_2)_{\text{Zar}}$ be the sheafification of $K_2$ in the Zariski topology.

On the one hand, we consider the space $\operatorname{CExt}(G, (K_2)_{\text{Zar}})$ (in fact, an ordinary groupoid) of *Brylinski-Deligne data*, which are by definition *central* extensions

$$1 \to (K_2)_{\text{Zar}} \to \widetilde{G} \to G \times X \to 1$$

of the constant group-scheme $G \times X$ by $(K_2)_{\text{Zar}}$.

The operation of Baer sum makes $\operatorname{CExt}(G, (K_2)_{\text{Zar}})$ into a commutative group in spaces, i.e., into a Picard category.

On the other hand, consider the Picard category

$$\operatorname{FactPic}(\operatorname{Gr}_G)$$

of factorizable line bundles on $\operatorname{Gr}_G$.

In the paper [Ga6] a map of Picard groupoids is constructed:

$$\operatorname{CExt}(G, (K_2)_{\text{Zar}}) \to \operatorname{FactPic}(\operatorname{Gr}_G), \tag{3.13}$$

and the following conjecture is stated (this is Conjecture 6.1.2 in *loc.cit.*:

**Conjecture 3.4.2.** *The map* (3.13) *is an isomorphism.*

*Remark* 3.4.3. One can show that it follows from [BrDe, Theorem 3.16] combined with Sect. 4.1.5 that Conjecture 3.4.2 holds when $G = T$ is a torus.

3.4.4. Let us fix an integer $\ell$ of order prime to $\operatorname{char}(p)$. In [Ga6, Sect. 6.3.6] the following map was constructed

$$\operatorname{CExt}(G, (K_2)_{\text{Zar}}) \to \operatorname{Maps}_{\operatorname{Ptd}(\operatorname{PreStk}_{/X})}\left(B_{\text{et}}(G) \times X, B^4_{\text{et}}(\mu_\ell^{\otimes 2} \times X)\right). \tag{3.14}$$

Let us take $A = \mu_\ell$, and note that $A(1) \simeq \mu_\ell^{\otimes 2}$. Note that the construction in Sect. 1.4.2 gives rise to a canonical map

$$\operatorname{FactPic}(\operatorname{Gr}_G) \to \operatorname{FactGe}_{\mu_\ell}(\operatorname{Gr}_G). \tag{3.15}$$

The following is equivalent to Conjecture 6.3.8 of *loc.cit.*:

**Conjecture 3.4.5.** *The following diagram commutes:*

$$\begin{array}{ccc}
\operatorname{CExt}(G, (K_2)_{\text{Zar}}) & \xrightarrow{(3.14)} & \operatorname{Maps}_{\operatorname{Ptd}(\operatorname{PreStk}_{/X})}\left(B_{\text{et}}(G) \times X, B^4_{\text{et}}(\mu_\ell^{\otimes 2} \times X)\right) \\
{\scriptstyle (3.13)}\downarrow & & \downarrow{\scriptstyle (3.3)} \\
\operatorname{FactPic}(\operatorname{Gr}_G) & \xrightarrow{(3.15)} & \operatorname{FactGe}_{\mu_\ell}(\operatorname{Gr}_G).
\end{array}$$

## 4. The case of tori

In this section we let $G = T$ be a torus. We will perform an explicit analysis of factorization gerbes on the affine Grassmannian $\operatorname{Gr}_T$, and introduce related objects (multiplicative factorization gerbes) that will play an important role in the sequel.

4.1. **Factorization Grassmannian for a torus.** In this section we will show that the affine Grassmannian of a torus can be approximated by a prestack assembled from (=written as a colomit of) powers of $X$.

4.1.1. Recall that $\Lambda$ denotes the coweight lattice of $G = T$. Consider the index category whose objects are pairs $(I, \lambda^I)$, where $I$ is a finite non-empty set and $\lambda^I$ is a map $I \to \Lambda$; in what follows we will denote by $\lambda_i \in \Lambda$ is the value of $\lambda^I$ on $i \in I$.

A morphism $(J, \lambda^J) \to (I, \lambda^I)$ is a surjection $\phi : I \twoheadrightarrow J$ such that

$$\lambda_j = \underset{i \in \phi^{-1}(j)}{\Sigma} \lambda_i. \tag{4.1}$$

Consider the prestack

$$\mathrm{Gr}_{T,\mathrm{comb}} := \underset{(I,\lambda^I)}{\mathrm{colim}}\, X^I.$$

The prestack $\mathrm{Gr}_{T,\mathrm{comb}}$ endowed with its natural forgetful map to Ran, also has a natural factorization structure.

There is a canonical map

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T, \tag{4.2}$$

compatible with the factorization structures.

Namely, for each $(I, \lambda^I)$ the corresponding $T$-bundle on $X^I \times X$ is

$$\bigotimes_{i \in I} \lambda_i \cdot \mathcal{O}(\Delta_i),$$

where $\Delta_i$ is the divisor on $X^I \times X$ corresponding to the $i$-th coordinate being equal to the last one.

4.1.2. As in [Ga2, Sect. 8.1] one shows that the map (4.2) induces an isomorphism of the sheafifications in the topology generated by finite surjective maps. In particular, for any $S \to \mathrm{Ran}$, the map

$$\mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T) \to \mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T,\mathrm{comb}})$$

is an isomorphism, and hence, so is the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_{T,\mathrm{comb}}).$$

Furthermore, for a given $\mathcal{G} \in \mathrm{FactGe}_{E^\times,\mathrm{tors}}(\mathrm{Gr}_T)$, the corresponding map of sheaves of categories

$$\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_{T,\mathrm{comb}})_{/\,\mathrm{Ran}}$$

is also an isomorphism.

4.1.3. The datum of a factorization gerbe on $\mathrm{Gr}_{T,\mathrm{comb}}$ can be explicitly described as follows:

For a finite set $I$ and a map

$$\lambda^I : I \to \Lambda$$

we specify a gerbe $\mathcal{G}_{\lambda^I}$ on $X^I$.

For a surjection of finite sets $I \overset{\phi}{\twoheadrightarrow} J$ such that (4.1) holds, we specify an identification

$$(\Delta_\phi)^*(\mathcal{G}_{\lambda^I}) \simeq \mathcal{G}_{\lambda^J}. \tag{4.3}$$

The identifications (4.3) must be compatible with compositions of maps of finite sets in the natural sense.

Let now $I \overset{\phi}{\twoheadrightarrow} J$ be a surjection of finite sets, and let

$$X^I_{\phi,\mathrm{disj}} \subset X^I, \quad x_{i_1} \neq x_{i_2} \text{ whenever } \phi(i_1) \neq \phi(i_2)$$

be the corresponding open subset. For $j \in J$, let $\lambda^{I_j}$ be the restriction of $\lambda^I$ to $I_j$.

We impose the structure of factorization that consists of isomorphisms

$$(\mathcal{G}_{\lambda^I})|_{X^I_{\phi,\mathrm{disj}}} \simeq \left(\bigotimes_{j \in J} \mathcal{G}_{\lambda^{I_j}}\right)|_{X^I_{\phi,\mathrm{disj}}}. \tag{4.4}$$

The isomorphisms (4.4) must be compatible with compositions of maps of finite sets in the natural sense.

In addition, the isomorphisms (4.4) and (4.3) must be compatible in the natural sense.

4.1.4. For a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_{T,\mathrm{comb}}$, the value of the category $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}_{T,\mathrm{comb}})_{/\,\mathrm{Ran}}$ on $X^I \to \mathrm{Ran}$ can be explicitly described as follows:

It is the limit over the index category

$$(J, \lambda^J, I \twoheadrightarrow J)$$

of the categories $\mathrm{Shv}_{\mathcal{G}_{\lambda^J}}(X^J)$.

4.1.5. *The case of factorization line bundles.* The datum of a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on $\mathrm{Gr}_{T,\mathrm{comb}}$ can be described in a way similar to that of factorization gerbes. This description recovers the notion of what in [BD1, Sect. 3.10.3] is called a $\theta$-datum.

We note that a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle is evenly (i.e., trivially) graded if and only if the corresponding $\theta$-datum is even, i.e., if the corresponding symmetric bilinear $\mathbb{Z}$-valued form on $\Lambda$ comes from a $\mathbb{Z}$-valued quadratic form.

We also note that [BD1, Proposition 3.10.7] says that restriction along

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_T$$

defines an equivalence between the Picard categories of factorization ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundles.

4.2. **Making the parameterization explicit for tori.** In this subsection we will show explicitly how a factorization $A$-gerbe on $\mathrm{Gr}_T$ gives rise to an $A$-valued quadratic form

$$q : \Lambda \to A(-1)$$

and also give a more hands-on proof of Proposition 3.1.9 in this case.

4.2.1. We first describe the bilinear form

$$b : \Lambda \times \Lambda \to A(-1).$$

Given two elements $\lambda_1, \lambda_2 \in \Lambda$, consider $I = \{1, 2\}$ and the map

$$\lambda^I : I \to \Lambda; \quad 1 \mapsto \lambda_1, 2 \mapsto \lambda_2.$$

Consider the corresponding gerbe

$$\mathcal{G}_{\lambda_1,\lambda_2} := \mathcal{G}_{\lambda^I}$$

over $X^2$. By (4.4) it is identified with $\mathcal{G}_{\lambda_1} \boxtimes \mathcal{G}_{\lambda_2}$ over $X^2 - \Delta(X)$. By Lemma 1.4.5, there exists a well-defined element $a \in A(-1)$ such that

$$\mathcal{G}_{\lambda_1,\lambda_2} \simeq (\mathcal{G}_{\lambda_1} \boxtimes \mathcal{G}_{\lambda_2}) \otimes \mathcal{O}(\Delta(X))^a.$$

We let

$$a =: b(\lambda_1, \lambda_2).$$

4.2.2. The fact that $b(-,-)$ is bilinear can be seen as follows. For a triple of elements $\lambda_1, \lambda_2, \lambda_3$ consider the corresponding gerbes

$$\mathcal{G}_{\lambda_1,\lambda_2,\lambda_3} \text{ and } (\mathcal{G}_{\lambda_1,\lambda_2} \boxtimes \mathcal{G}_{\lambda_3}) \otimes \mathcal{O}(\Delta_{1,3})^{\otimes b(\lambda_1,\lambda_3)} \otimes \mathcal{O}(\Delta_{2,3})^{\otimes b(\lambda_2,\lambda_3)}$$

over $X^3$.

They are identified away from the main diagonal $\Delta_{1,2,3}$, and hence this identification extends to all of $X^3$, since $\Delta_{1,2,3}$ has codimension 2. Restricting to $\Delta_{1,2}$, we obtain an identification

$$\mathcal{G}_{\lambda_1+\lambda_2,\lambda_3} \simeq (\mathcal{G}_{\lambda_1+\lambda_2} \boxtimes \mathcal{G}_{\lambda_3}) \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_1,\lambda_3)} \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_2,\lambda_3)}$$

as gerbes over $X^2$. Comparing with the identification

$$\mathcal{G}_{\lambda_1+\lambda_2,\lambda_3} \simeq (\mathcal{G}_{\lambda_1+\lambda_2} \boxtimes \mathcal{G}_{\lambda_3}) \otimes \mathcal{O}(\Delta)^{\otimes b(\lambda_1+\lambda_2,\lambda_3)},$$

we obtain the desired

$$b(\lambda_1, \lambda_3) + b(\lambda_2, \lambda_3) = b(\lambda_1 + \lambda_2, \lambda_3).$$

4.2.3. It is easy to see that the resulting map

$$b : \Lambda \times \Lambda \to A(-1)$$

is symmetric. In fact, we have a canonical datum of commutativity for the diagram

$$\begin{array}{ccc} \sigma^*(\mathcal{G}_{\lambda_1,\lambda_2}) & \longrightarrow & \sigma^*((\mathcal{G}_{\lambda_1} \boxtimes \mathcal{G}_{\lambda_2}) \otimes \mathcal{O}(\Delta(X))^{b(\lambda_1,\lambda_2)}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}_{\lambda_2,\lambda_1} & \longrightarrow & (\mathcal{G}_{\lambda_2} \boxtimes \mathcal{G}_{\lambda_1}) \otimes \mathcal{O}(\Delta(X))^{b(\lambda_2,\lambda_1)} \end{array} \tag{4.5}$$

that extends the given one over $X \times X - \Delta(X)$ (in the above formula, $\sigma$ denotes the transposition acting on $X \times X$):

Indeed, the measure of *non-commutativity* of the above diagonal is an étale $A$-torsor over $X \times X$, which is trivialized over $X \times X - \Delta(X)$, and hence this trivialization uniquely extends to all of $X \times X$.

For the sequel we will need to understand in more detail the behavior of the restriction of the above diagram to the diagonal.

4.2.4. We start with the following observation. We claim that to an element $a \in A(-1)$ one can canonically attach an $A$-torsor $(-1)^a$.

The Kummer cover

$$\mathbb{G}_m \overset{x \mapsto x^n}{\longrightarrow} \mathbb{G}_m$$

defines a group homomorphism

$$\mathbb{G}_m \to B_{\text{et}}(\mu_n). \tag{4.6}$$

From here we obtain a homomorphism

$$A(-1) \times \mathbb{G}_m \to B_{\text{et}}(A), \tag{4.7}$$

i.e., an element $a \in A(-1)$ defines an étale $A$-torsor $\chi_a$ over $\mathbb{G}_m$, which behaves multiplicatively. We let $(-1)^a$ denote the fiber of $\chi_a$ at $-1 \in \mathbb{G}_m$.

The multiplicativity of (4.7) along $\mathbb{G}_m$ implies that we have a canonical trivialization

$$((-1)^a)^{\otimes 2} \simeq \text{triv}\,. \tag{4.8}$$

The multiplicativity of (4.7) along $A(-1)$ implies that a choice of $a' \in A(-1)$ such that $2a' = a$ defines a trivialization of $(-1)^a$. Moreover, this trivialization is compatible with (4.8). This construction is a morphism (and hence an *isomorphism*) of $A_{2\text{-tors}}$-torsors:

$$\{a' \in A\,,\, 2a' = a\} \to \{\text{trivializations of } (-1)^a \text{ compatible with (4.8)}\}.$$

(By enlarging $A$ if needed, one shows that the LHS is empty if and only if the RHS is.)

4.2.5. Consider now the $A$-gerbe $\mathcal{O}(\Delta)^a$ on $X \times X$, equipped with the natural identification

$$\sigma^*(\mathcal{O}(\Delta)^a) \simeq \mathcal{O}(\Delta)^a \tag{4.9}$$

that uniquely extends the tautological one over $X \times X - \Delta(X)$.

Restricting (4.9) to the diagonal, and using the fact that $\sigma|_{\Delta(X)}$ is trivial, we obtain an identification of $A$-gerbes

$$\phi_a : \mathcal{O}(\Delta)^a|_{\Delta(X)} \simeq \mathcal{O}(\Delta)^a|_{\Delta(X)},$$

whose square is the identity map.

Hence, $\phi_a$ is given by tensoring by an $A$-torsor that squares to the trivial one. It is easy to see that this torsor is constant along $X$ and identifies canonically with $(-1)^a$ in a way compatible with (4.8).

4.2.6. We now return to the diagram (4.5). Restricting to the diagonal, we obtain that we have a canonical datum of commutativity for the diagram

$$
\begin{array}{ccc}
\mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & (\mathcal{G}^{\lambda_1}\otimes\mathcal{G}^{\lambda_2})\otimes\mathcal{O}(\Delta)^{b(\lambda_1,\lambda_2)}|_{\Delta(X)} \\
\downarrow & & \downarrow{\scriptstyle \text{tautological}\otimes\phi_{b(\lambda_1,\lambda_2)}} \\
\mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & (\mathcal{G}^{\lambda_2}\otimes\mathcal{G}^{\lambda_1})\otimes\mathcal{O}(\Delta)^{b(\lambda_2,\lambda_2)}|_{\Delta(X)}
\end{array}
\tag{4.10}
$$

that squares to the tautological one.

4.2.7. We are finally ready to recover the quadratic form

$$q:\Lambda\to A(-1).$$

Namely, in (4.10), let us set $\lambda_1=\lambda=\lambda_2$. We obtain a datum of commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{G}^{2\lambda} & \longrightarrow & (\mathcal{G}^{\lambda}\otimes\mathcal{G}^{\lambda})\otimes\mathcal{O}(\Delta)^{b(\lambda,\lambda)}|_{\Delta(X)} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi_{b(\lambda,\lambda)}} \\
\mathcal{G}^{2\lambda} & \longrightarrow & (\mathcal{G}^{\lambda}\otimes\mathcal{G}^{\lambda})\otimes\mathcal{O}(\Delta)^{b(\lambda,\lambda)}|_{\Delta(X)},
\end{array}
$$

where the upper and lower horizontal arrows are canonically identified, and which squares to the tautological one.

By Sects. 4.2.4 and 4.2.5, such a datum is equivalent to that of an element $q(\lambda)\in A(-1)$ such that $2q(\lambda)=b(\lambda,\lambda)$. This is the value of our quadratic form on $\lambda$.

4.2.8. The relation

$$q(\lambda_1+\lambda_2)=q(\lambda_1)+q(\lambda_2)+b(\lambda_1,\lambda_2)$$

is verified in a way similar to Sect. 4.2.2.

4.2.9. We will now give an alternative proof of Proposition 3.1.9 in the special case of tori.

First, we claim that the diagram

$$
\begin{array}{ccc}
\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(T)\times X, B^4_{\mathrm{et}}(A(1))\times X) & \xrightarrow{(3.3)} & \mathrm{FactGe}_A(\mathrm{Gr}_T) \\
\downarrow & & \downarrow \\
\mathrm{Quad}(\Lambda,A(-1)) & \xrightarrow{\mathrm{Id}} & \mathrm{Quad}(\Lambda,A(-1))
\end{array}
$$

commutes, where the left vertical arrow corresponds to the projection

$$H^4_{\mathrm{et}}(B(T)\times X;\mathrm{pt}\times X, A(1))\to H^4_{\mathrm{et}}(B(T),A(1))\simeq \mathrm{Quad}(\Lambda,A(-1)),$$

and the right vertical arrow is one constructed above.

As in Sect. 3.3.4, the fiber of the left vertical arrow in the above diagram identifies canonically with $\mathrm{Maps}\left(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda,A))\right)$. Hence, it remains to show that the induced map

$$\mathrm{Maps}\left(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda,A))\right)\to \mathrm{FactGe}^0_A(\mathrm{Gr}_T) \tag{4.11}$$

is an isomorphism.

4.2.10. We claim, however, that the latter follows from the description of factorizable gerbes in Sect. 4.1.3.

Namely, the groupoid $\mathrm{FactGe}_A^0(\mathrm{Gr}_T)$ is isomorphic to that of assignments

$$\lambda \mapsto \mathcal{G}^\lambda \in \mathrm{Ge}_A(X),$$

equipped with the following pieces of data:

- One is *multiplicativity*, i.e., we must be given isomorphisms of gerbes
$$\mathcal{G}^{\lambda_1+\lambda_2} \simeq \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}$$
that are associative in the natural sense.
- The other one is that of *commutativity*, i.e., we must be given the data of commutativity for the squares

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & \mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1} \end{array} \tag{4.12}$$

that satisfy the hexagon axiom.

In addition, the following conditions must be satisfied:

(1) The datum of commutativity for the outer square in

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2+\lambda_1} & \longrightarrow & \mathcal{G}^{\lambda_2} \otimes \mathcal{G}^{\lambda_1} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_1+\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} \end{array} \tag{4.13}$$

is the identity one.

(2) The datum of commutativity in (4.12) for $\lambda_1 = \lambda = \lambda_2$

$$\begin{array}{ccc} \mathcal{G}^{2\lambda} & \longrightarrow & \mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{G}^{2\lambda} & \longrightarrow & \mathcal{G}^{\lambda} \otimes \mathcal{G}^{\lambda} \end{array} \tag{4.14}$$

is the identity one.

In particular, for $T \simeq T_1 \times T_2$, the natural map

$$\mathrm{FactGe}_A^0(\mathrm{Gr}_{T_1}) \times \mathrm{FactGe}_A^0(\mathrm{Gr}_{T_2}) \to \mathrm{FactGe}_A^0(\mathrm{Gr}_T)$$

is an isomorphism, and for $T = \mathbb{G}_m$, we have

$$\mathrm{FactGe}_A^0(\mathrm{Gr}_T) \simeq \mathrm{Ge}_A(X).$$

This makes the isomorphism (4.11) manifest.

4.3. **The notion of *multiplicative* factorization gerbe.** In order to be able to state the metaplectic version of geometric Satake, we will need to discuss the notion of *multiplicative* factorization gerbe, first on $\mathrm{Gr}_T$, and then when the lattice $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ is replaced by a general finitely generated abelian group.

4.3.1. Note that since $T$ is commutative, $\mathrm{Gr}_T$ is naturally a factorization *group*-prestack over Ran. Hence, along with $\mathrm{FactGe}_A(\mathrm{Gr}_T)$, we can consider the corresponding space (in fact, commutative group in spaces)

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) \tag{4.15}$$

that corresponds to gerbes that respect the group structure on $\mathrm{Gr}_T$ over Ran.

We have the evident forgetful map

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_T). \tag{4.16}$$

Explicitlly, a multiplicative structure on a gerbe $\mathcal{G}$ is an identification

$$\mathrm{mult}(\mathcal{G}) \simeq \mathcal{G} \boxtimes \mathcal{G}$$

as *factorization gerbes* on $\mathrm{Gr}_T \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T$ (in the above formula mult denotes the multiplication map $\mathrm{Gr}_T \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T \to \mathrm{Gr}_T$), equipped with the a compatibility datum over triple product $\mathrm{Gr}_T \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_T$, and an identity satisfied over the quadruple product.

We will prove:

**Proposition 4.3.2.** *The forgetful map*

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A(\mathrm{Gr}_T)$$

*is fully faithful. Its essential image is the preimage under*

$$\mathrm{FactGe}_A(\mathrm{Gr}_T) \to \mathrm{Quad}(\Lambda, A(-1))$$

*of the subset consisting of those quadratic forms, whose associated bilinear form is zero.*

*Proof.* We will use the description of factorization on gerbes on $\mathrm{Gr}_T$ given in Sect. 4.1.3. In these terms, the multiplicative structure on $\mathcal{G}$ amounts to specifying isomorphisms

$$\mathcal{G}^{\lambda_1,\lambda_2} \simeq \mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2} \tag{4.17}$$

equipped with an associativity constraint, and equipped with the datum of the identification of (4.17) with the factorization isomorphism over $X \times X - \Delta(X)$.

In other words, we need that the factorization isomorphisms

$$\mathcal{G}^{\lambda_1,\lambda_2}|_{X\times X-\Delta(X)} \simeq \mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2}|_{X\times X-\Delta(X)}$$

extend to all of $X \times X$. If they extend, they do so uniquely, and the extended isomorphisms are automatically equipped with an associativity constraint.

Thus, by Sect. 4.2.1, we obtain that the category $\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T)$ identifies with the full subcategory of $\mathrm{FactGe}_A(\mathrm{Gr}_T)$, consisting of objects for which the bilinear form $b(-,-)$ vanishes.

□

*Remark* 4.3.3. Note that the set of quadratic forms $q : \Lambda \to A$ whose associated bilinear form vanishes, is in bijection with the set of *linear* maps $\Lambda \to A_{2\text{-tors}}$.

Note also that we have a tautological identification $A(-1)_{2\text{-tors}} \simeq A_{2\text{-tors}}$, since $\mu_2 \simeq \pm 1 \simeq \mathbb{Z}/2\mathbb{Z}$ canonically.

4.3.4. Note that $\mathrm{Gr}_T$ is not just a group-prestack over Ran, but a *commutative* group-prestack. Hence, along with

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) =: \mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T),$$

we can consider the spaces $\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ for any $k \geq 1$ and also

$$\mathrm{FactGe}_A^{\mathrm{com}}(\mathrm{Gr}_T) \simeq \mathrm{FactGe}_A^{\mathbb{E}_\infty}(\mathrm{Gr}_T) := \lim_k \mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T).$$

We claim, however, that the forgetful maps

$$\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T)$$

are all equivalences.

First off, the maps $\mathrm{FactGe}_A^{\mathbb{E}_{k+1}}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ are automatically equivalences for $k \geq 3$ because $A$-gerbes are 1-categorical objects. Similarly, the forgetful map $\mathrm{FactGe}_A^{\mathbb{E}_3}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T)$ is automatically fully faithful.

An $\mathbb{E}_2$-structure on a multiplicative gerbe $\mathcal{G}$ translates as a datum of commutativity for the squares

$$\begin{array}{ccc} \sigma^*(\mathcal{G}^{\lambda_1,\lambda_2}) & \xrightarrow{\sigma^*(4.17)} & \sigma^*(\mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2}) \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_2,\lambda_1} & \xrightarrow{(4.17)} & \mathcal{G}^{\lambda_2} \boxtimes \mathcal{G}^{\lambda_1} \end{array} \tag{4.18}$$

that coincides with the one coming from factorization over $X \times X - \Delta(X)$.

Thus, we are already given the datum of commutation of (4.18) over $X \times X - \Delta(X)$. Therefore, this datum automatically uniquely extends to all of $X \times X$. This implies that

$$\mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T) \to \mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_T)$$

is an equivalence.

An object in $\mathrm{FactGe}_A^{\mathbb{E}_2}(\mathrm{Gr}_T)$ comes from $\mathrm{FactGe}_A^{\mathbb{E}_3}(\mathrm{Gr}_T)$ if and only if the diagrams (4.18) square to the identity, in the sense that the datum of commutativity for the outer square in

$$\begin{array}{ccc} \mathcal{G}^{\lambda_1,\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2} \\ \downarrow & & \downarrow \\ \sigma^* \circ \sigma^*(\mathcal{G}^{\lambda_1,\lambda_2}) & \longrightarrow & \sigma^* \circ \sigma^*(\mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2}) \\ \downarrow & & \downarrow \\ \sigma^*(\mathcal{G}^{\lambda_2,\lambda_1}) & \longrightarrow & \sigma^*(\mathcal{G}^{\lambda_2} \boxtimes \mathcal{G}^{\lambda_1}) \\ \downarrow & & \downarrow \\ \mathcal{G}^{\lambda_1,\lambda_2} & \longrightarrow & \mathcal{G}^{\lambda_1} \boxtimes \mathcal{G}^{\lambda_2} \end{array}$$

is the tautological one. But this is automatic because this condition holds over $X \times X - \Delta(X)$.

*Remark* 4.3.5. Note that from Proposition 3.1.9 we obtain the following a priori description of the groupoid $\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_T)$ as

$$\mathrm{Maps}_{\mathbb{E}_k(\mathrm{PreStk}_{/X})}(B(T) \times X, B^4_{\mathrm{et}}(A(1)) \times X) \simeq \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B^{1+k}(T) \times X, B^{4+k}_{\mathrm{et}}(A(1)) \times X).$$

From Sect. 4.3.4 we obtain that the looping map

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B^{1+k}(T), B^{4+k}_{\mathrm{et}}(A(1))) \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk})}(B(T), B^4_{\mathrm{et}}(A(1)))$$

has the following properties:

- It induces an isomorphism
$$H^{2+k}_{\mathrm{et}}(B^{1+k}(T), A(1)) \to H^2_{\mathrm{et}}(B(T), A(1))$$
for any $k \geq 1$ (note that the RHS identifies with $\mathrm{Hom}(\Lambda, A)$).
- It induces an isomorphism
$$H^{3+k}_{\mathrm{et}}(B^{1+k}(T), A(1)) \to H^3_{\mathrm{et}}(B(T), A(1))$$
for any $k \geq 1$ (note that the RHS is zero).
- For any $k \geq 1$, the induced map
$$H^{4+k}_{\mathrm{et}}(B^{1+k}(T), A(1)) \to H^4_{\mathrm{et}}(B(T), A(1))$$
is injective with the image being the subset of $\mathrm{Quad}(\Lambda, A(-1))$, consisting of those quadratic forms, whose associated bilinear form is zero.

*Remark* 4.3.6. For the sake of completeness, let us reprove the isomorphisms of Remark 4.3.5 directly in the context of *algebraic topology*, where we take $T$ to be the corresponding Lie group. In this case we will think of $B(T)$ as $B^2(\Lambda)$.

We start with the groupoid
$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B(\Lambda), B^3(A)) \simeq \mathrm{Maps}_{\mathrm{Grp(Spc)}}(\Lambda, B^2(A)).$$

We can think of its objects as monoidal categories $\mathcal{C}$ that are groupoids such that $\pi_0(\mathcal{C}) = \Lambda$ (as monoids) and $\pi_1(\mathbf{1}_{\mathcal{C}}) = A$ (as groups).

A datum of lifting of such a point to a point of
$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^2(\Lambda), B^4(A)) \simeq \mathrm{Maps}_{\mathbb{E}_1(\mathrm{Spc})}(B(\Lambda), B^3(A))$$
amounts to endowing the monoidal category $\mathcal{C}$ with a braiding. A further lift to an object of
$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A)) \simeq \mathrm{Maps}_{\mathbb{E}_{k+1}(\mathrm{Spc})}(B(\Lambda), B^3(A))$$
for $k \geq 1$ amounts to the *condition* that the resulting braided monoidal category be *symmetric*. This already implies that the forgetful map
$$\mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k+1}(\Lambda), B^{4+k+1}(A)) \to \mathrm{Maps}_{\mathrm{Ptd(Spc)}}(B^{2+k}(\Lambda), B^{4+k}(A))$$
is an isomorphism for $k \geq 1$ and is fully faithful for $k = 0$.

Moreover, $\pi_2$ of these spaces identifies with
$$\mathrm{Maps}_{\mathrm{Grp(Spc)}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{Ab}}(\Lambda, A),$$
and $\pi_1$ identifies with
$$\mathrm{Maps}_{\mathbb{E}_2(\mathrm{Spc})}(\Lambda, B(A)) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B(A)) = \mathrm{Ext}^1_{\mathrm{Ab}}(\Lambda, A) = 0.$$

Finally, the set of isomorphism classes of braided monoidal categories as above is in bijection with $\mathrm{Quad}(\Lambda, A)$. Indeed, for a given $\mathcal{C}$, the corresponding bilinear form $b(\lambda_1, \lambda_2)$ is recovered as the square of the braiding
$$c^{\lambda_1} \otimes c^{\lambda_2} \to c^{\lambda_2} \otimes c^{\lambda_1} \to c^{\lambda_1} \otimes c^{\lambda_2},$$
and the quadratic form $q(\lambda)$ is recovered as the value of the braiding
$$c^{\lambda} \otimes c^{\lambda} \to c^{\lambda} \otimes c^{\lambda}.$$

In particular, this braided monoidal category is symmetric if and only if $b(-,-) = 0$.

4.3.7. The description of $\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T)$ given in Sect. 4.3.4 implies that we can describe this category as

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)).$$

Since $\pi_2(\mathrm{Ge}_A(X)) \simeq A$, as in Remark 4.3.6, we have a fiber sequence

$$\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \to \mathrm{Maps}_{\mathbb{E}_\infty}(\Lambda, \mathrm{Ge}_A(X)) \to \mathrm{Hom}(\Lambda, A_{2\text{-tors}}).$$

This fiber sequence identifies with the fiber sequence which is the top line in the diagram

$$\begin{array}{ccccc} \mathrm{FactGe}_A^0(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) & \longrightarrow & \mathrm{Hom}(\Lambda, A_{2\text{-tors}}) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \mathrm{FactGe}_A^0(\mathrm{Gr}_T) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_T) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1)). \end{array}$$

Let us note that we can also identify the groupoid $\mathrm{Maps}_{\mathbb{E}_\infty}(\Lambda, \mathrm{Ge}_A(X))$ with

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \overset{B^2(\mathrm{Hom}(\Lambda,A))}{\times} \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))),$$

where $\overset{?}{\times}$ means "divided by the diagonal action of ?".

4.3.8. In Sect. 4.5 we will see that if $A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$, there a *canonical* identification

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, B^2(A)) \simeq B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A)) \times \mathrm{Hom}(\Lambda, \mathbb{Z}/2\mathbb{Z}).$$

This implies that for $A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$, we have a canonical identification

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_T) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Lambda, \mathrm{Ge}_A(X)) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Lambda, A))) \times \mathrm{Hom}(\Lambda, \mathbb{Z}/2\mathbb{Z}).$$

4.4. **More general abelian groups.** In this section we generalize the discussion of Sect. 4.3 to the case when instead of a lattice $\Lambda$ (thought of as a lattice of cocharacters of a torus) we take a general finitely generated abelian group. We need this in order to state the metaplectic version of geometric Satake.

4.4.1. Let $\Gamma$ be a finitely generated abelian group. We define the commutative group-prestack over Ran

$$\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$$

as follows. Write $\Gamma$ as $\Lambda_1/\Lambda_2$, where $\Lambda_1 \supset \Lambda_2$ are lattices. Let $T_1$ and $T_2$ be the corresponding tori. We define $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ as a quotient of $\mathrm{Gr}_{T_1}$ by $\mathrm{Gr}_{T_2}$, viewed as commutative group-prestacks over Ran, sheafified in the topology of finite surjective maps.

It is easy to see that this definition (as well as other constructions we are going to perform) is canonically independent of the presentation of $\Gamma$ as a quotient.

The group-prestack $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ has a natural factorization structure over Ran.

4.4.2. Since $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ is a (commutative) group-prestack over Ran, along with $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$, we can consider the space (in fact, commutative group in spaces)

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}), \tag{4.19}$$

that correspond to gerbes that respect that group structure on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$ over Ran.

*Remark* 4.4.3. Along with $\mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$, one can also consider its variants

$$\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}),\ \mathrm{FactGe}_A^{\mathbb{E}_\infty}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{FactGe}_A^{\mathrm{com}}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}).$$

However, as in Sect. 4.3.4, one shows that the forgetful maps

$$\mathrm{FactGe}_A^{\mathbb{E}_k}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A^{\mathbb{E}_1}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) = \mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

are equivalences for all $k \geq 1$.

4.4.4. The following results from Proposition 4.3.2:

**Corollary 4.4.5.** *Let $\Gamma$ be written as a quotient of two lattices as in Sect. 4.4.1. Let $\mathcal{G}_1$ be a factorization $A$-gerbe on $\mathrm{Gr}_{T_1}$, and let $b_1$ and $q_1$ be the associated bilinear and quadratic forms on $\Lambda_1$, respectively.*

(a) *The gerbe $\mathcal{G}_1$ can be descended to a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$* only if $b_1(\Lambda_2,-)=0$.

(a') *In the situation of* (a), *a descent exists étale-locally on $X$ if and only if the restriction of $q_1$ to $\Lambda_2$ is trivial.*

(a") *In the situation of* (a'), *a descent datum is equivalent to the trivialization of $\mathcal{G}_2 := \mathcal{G}_1|_{\mathrm{Gr}_{T_2}}$ as a factorization gerbe on $\mathrm{Gr}_{T_2}$.*

(b) *In the situation of* (a"), *the descended gerbe $\mathcal{G}$ admits a multiplicative structure if and only if $b_1$ is trivial. In the latter case, the multiplicative structure is unique up to a unique isomorphism.*

From here, we obtain:

**Corollary 4.4.6.**

(a) *The forgetful map*

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$$

*is fully faithful.*

(b) *There are canonically defined maps*

$$\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Quad}(\Gamma, A(-1)) \text{ and } \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Hom}(\Gamma, A)_{2\text{-tors}}$$

*that fit into the pullback square*

$$\begin{array}{ccc} \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Gamma, A)_{2\text{-tors}} & \longrightarrow & \mathrm{Quad}(\Gamma, A(-1)). \end{array}$$

(c) *There is a canonical equivalences*

$$\mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, \mathrm{Ge}_A(X))$$

(d) *The fiber of the map* $\mathrm{FactGe}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Quad}(\Gamma, A(-1))$, *denoted* $\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$, *consists of objects that are trivial étale-locally on $X$.*

*Assume now that $A$ is divisible (unless $\Gamma$ is torison-free). Then we furthermore have:*

(e) *We have a canonical isomorphism*

$$\mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma, A))) \simeq \mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}).$$

(f) *There is a canonical equivalence*

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, \mathrm{Ge}_A(X)) \simeq \simeq \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \underset{}{\overset{B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma,A))}{\times}} \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma, A))).$$

4.4.7. Let now $G$ be a connective reductive group. Let $\Gamma = \pi_{1,\mathrm{alg}}(G)$. From Sect. 3.2.1 we obtain that there is a canonically defined map

$$\mathrm{Gr}_G \to \mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}, \tag{4.20}$$

compatible with the factorization structure.

Consider the resulting map

$$\mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) \to \mathrm{FactGe}_A(\mathrm{Gr}_G). \tag{4.21}$$

It is easy to see that it makes the diagram

$$\begin{array}{ccc} \mathrm{FactGe}_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) & \longrightarrow & \mathrm{FactGe}_A(\mathrm{Gr}_G) \\ \downarrow & & \downarrow \\ \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A(-1)) & \longrightarrow & \mathrm{Quad}(\Lambda, A(-1))^W_{\mathrm{restr}} \end{array}$$

commute, thereby inducing a map

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m}) \to \mathrm{FactGe}^0_A(\mathrm{Gr}_G). \tag{4.22}$$

From Corollaries 4.4.6 and 3.3.5 we obtain:

**Corollary 4.4.8.** *The map* (4.22) *is an isomorphism.*

From here we obtain:

**Corollary 4.4.9.** *The map* (4.21) *is fully faithful.*

4.5. **Splitting multiplicative gerbes.** In this subsection we will assume that $A_{2\text{-tors}} \subset \mathbb{Z}/2\mathbb{Z}$. (Note that this happens, e.g., if $A \subset E^\times$.)

We will show that in this case the fiber sequence

$$\mathrm{FactGe}^0_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{FactGe}^{\mathrm{mult}}_A(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \to \mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$$

admits a canonical splitting, functorial in $\Gamma$.

4.5.1. According to Corollary 4.4.6, it suffices to construct a splitting of the fiber sequence

$$B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma, A)) \to \mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\Gamma, B^2(A)) \to \mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}).$$

By functoriality, it suffices to treat the universal case: i.e., when $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and we need to construct an object of

$$\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(\mathbb{Z}/2\mathbb{Z}))$$

that projects to the identity map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

4.5.2. We will construct the sought-for object in $\mathrm{Maps}_{\mathbb{E}_\infty(\mathrm{Spc})}(\mathbb{Z}/2\mathbb{Z}, B^2(\mathbb{Z}/2\mathbb{Z}))$ as a symmetric monoidal groupoid $\mathcal{C}$ with $\pi_0(\mathbb{Z}/2\mathbb{Z}) \simeq \pi_1(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

As a monoidal groupoid, we set $\mathcal{C}$ be the product $\mathbb{Z}/2\mathbb{Z} \times B(\mathbb{Z}/2\mathbb{Z})$. A braided monoidal structure on such a $\mathcal{C}$ is equivalent to a choice of a bilinear form $b'$ on $\mathbb{Z}/2\mathbb{Z}$ with values in $\mathbb{Z}/2\mathbb{Z}$. We set it to be

$$b'(\overline{1}, \overline{1}) = \overline{1}.$$

The resulting braided monoidal structure is automatically symmetric, and the associated quadratic form $q : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the identity map, as required.

4.5.3. In what follows, for a given element $\epsilon \in \mathrm{Hom}(\Gamma, A)_{2\text{-tors}}$, we will denote by $\mathcal{G}^\epsilon$ the resulting multiplicative factorization gerbe on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$.

For $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and the identity map, we will denote this gerbe by $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$. We will refer to it as the *sign gerbe*.

*Remark* 4.5.4. Note that $\mathcal{G}^\epsilon$, viewed as a gerbe on $\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}$, equipped with the multiplicative structure, admits a canonical trivialization. However, this trivialization is *not* compatible with the factorization structure.

Similarly, this trivialization is *not* compatible with the *commutative structure* on $\mathcal{G}^\epsilon$.

4.5.5. For a given object $\mathcal{G} \in \mathrm{FactGe}_A^{\mathrm{mult}}(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$ let us denote by $\epsilon$ the map

$$\Gamma \to A_{2\text{-tors}} \simeq \mathbb{Z}/2\mathbb{Z}$$

that measures the obstruction of $\mathcal{G}$ to belong to $\mathrm{FactGe}_A^0(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m})$.

We obtain that, canonically attached to $\mathcal{G}$, there exists an object

$$\mathcal{G}^0 \in \mathrm{FactGe}_A^0(\mathrm{Gr}_{\Gamma\otimes\mathbb{G}_m}) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\Gamma, A))),$$

such that

$$\mathcal{G} \simeq \mathcal{G}^0 \otimes \mathcal{G}^\epsilon,$$

where $\mathcal{G}^\epsilon$ is as in Sect. 4.5.3.

4.6. **More on the sign gerbe.** In this subsection we will make a digression, needed for the sequel, in which we will discuss several manipulations with the gerbe $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$ introduced in Sect. 4.5.

4.6.1. Let $Z$ be a prestack over Ran, equipped with a factorization structure, and equipped with a map to $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$, compatible with the factorization structures.

Let $\mathcal{L}$ be a line bundle on $Z$. We equip it with a $\mathbb{Z}/2\mathbb{Z}$-graded structure as follows: for an affine test scheme $S$ and a map $S \to Z$ such that the composite $S \to Z \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ maps to the even/odd connected component, the grading on $\mathcal{L}|_S$ is even/odd.

Let us be given a factorization structure on $\mathcal{L}$, viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle. Note that $\mathcal{L}^{\otimes 2}$ is then a plain factorization line bundle. Assume that $\mathrm{char}(k) \neq 2$, and consider the $\mathbb{Z}/2\mathbb{Z}$-gerbe on $Z$ given by $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$, equipped with its natural factorization structure.

It is easy to see, however, that $(\mathcal{L}^{\otimes 2})^{\frac{1}{2}}$ identifies canonically with $\mathcal{G}^{\epsilon_{\mathrm{taut}}}|_Z$. Indeed, both gerbes are canonically trivialized as plain gerbes, and the factorization structures on both are given by the sign rules.

4.6.2. *An example.* Let us take $Z = \mathrm{Gr}_{\mathbb{G}_m}$. We can take as $\mathcal{L}$ the *determinant line bundle* on $\mathrm{Gr}_{\mathbb{G}_m}$, denoted $\det_{\mathbb{G}_m,\mathrm{St}}$, corresponding to the standard one-dimensional representation of $\mathbb{G}_m$.

4.6.3. Let $\mathcal{C}$ be a sheaf of categories over $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$, and let $\mathcal{C}$ be equipped with a factorization structure, compatible with the factorization structure on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$.

Viewing $\mathbb{Z}/2\mathbb{Z}$ as 2-torsion in $E^\times$, and using the twisting construction of Sect. 1.7.2, we can twist $\mathcal{C}$ by $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$ and obtain a new factorization sheaf of categories, denoted $\mathcal{C}^{\epsilon_{\mathrm{taut}}}$.

Suppose that in the above situation $\mathcal{C}$ is endowed with a monoidal (symmetric monoidal) monoidal structure, compatible with the group structure on $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$. Then $\mathcal{C}^{\epsilon_{\mathrm{taut}}}$ also acquires a monoidal (resp., symmetric monoidal) structure.

*Remark* 4.6.4. Note that for any $S \to \mathrm{Ran}$, the corresponding categories $\mathcal{C}(S)$ and $\mathcal{C}^{\epsilon_{\mathrm{taut}}}(S)$ are canonically identified (since $\mathcal{G}^{\epsilon_{\mathrm{taut}}}$, viewed as a plain gerbe, is trivial). However, the factorization structures on $\mathcal{C}(S)$ and $\mathcal{C}^{\epsilon_{\mathrm{taut}}}(S)$ that are different. The same applies to the monoidal situation, but *not* to the symmetric monoidal one.

4.6.5. Assume for a moment that the structure of sheaf over $\mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m}$ on $\mathcal{C}$ has been refined to that of sheaf over $\mathrm{Gr}_{\mathbb{G}_m}$, also compatible with the factorization structures.

Note that $\mathrm{Gr}_{\mathbb{G}_m}$ carries a locally constant function, denoted $d$, given by the degree. Hence, we have a well-defined endo-functor on $\mathcal{C}$,

$$c \mapsto c[d]. \tag{4.23}$$

Note that this functor is *not* compatible with the factorization structure, due to sign rules. However, when we view (4.23) as a functor

$$\mathcal{C} \to \mathcal{C}^{\epsilon_{\mathrm{taut}}},$$

it is an equivalence of factorization categories.

## 5. Jacquet functors for factorization gerbes

In this section we take $G$ to be reductive. We will study the interaction between factorization gerbes on $\mathrm{Gr}_G$ and those on $\mathrm{Gr}_M$, where $M$ is the Levi quotient of a parabolic of $G$.

5.1. **The naive Jacquet functor.** Let $P$ be a parabolic subgroup of $G$, and we let $P \twoheadrightarrow M$ be its Levi quotient. Let $N_P$ denote the unipotent radical of $P$.

5.1.1. Consider the diagram of the Grassmannians

$$\mathrm{Gr}_G \overset{\mathsf{p}}{\longleftarrow} \mathrm{Gr}_P \overset{\mathsf{q}}{\longrightarrow} \mathrm{Gr}_M.$$

We claim that pullback along $\mathsf{q}$ defines an equivalence,

$$\mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_M) \to \mathrm{Ge}_A(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P) \tag{5.1}$$

for any $S \to \mathrm{Ran}$, in particular, inducing an equivalence

$$\mathrm{FactGe}_A(\mathrm{Gr}_M) \to \mathrm{FactGe}_A(\mathrm{Gr}_P).$$

5.1.2. To show that (5.1) is an equivalence, let us choose a splitting $M \hookrightarrow P$ of the projection $P \twoheadrightarrow M$. In particular, we obtain an adjoint action of $M$ on $N_P$. Hence, we obtain an action of the group-prestack $\mathfrak{L}^+(M)$ (see Sect. 7.1.3 for the definition of this group-prestack) over Ran on $\mathrm{Gr}_{N_P}$.

We can view $\mathrm{Gr}_M$ as a quotient $\mathfrak{L}(M)/\mathfrak{L}^+(M)$ (see Sect. 7.2.2), and hence we can view the map

$$\mathfrak{L}(M) \to \mathrm{Gr}_M$$

as a $\mathfrak{L}^+(M)$-torsor. Then $\mathrm{Gr}_P$, when viewed as a prestack over $\mathrm{Gr}_M$ is obtained by twisting $\mathrm{Gr}_{N_P}$ by the above $\mathfrak{L}^+(M)$-torsor.

Now, the equivalence in (5.1) follows from the fact that for any $S \to \mathrm{Ran}$, pullback defines an isomorphism

$$H^i_{\mathrm{et}}(S, A) \to H^i_{\mathrm{et}}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{N_P}, A)$$

for all $i$.

5.1.3. In terms of the parameterization given by Proposition 3.1.9, the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A(\mathrm{Gr}_M)$$

can be interpreted as follows:

It corresponds to the map

$$\begin{aligned}&\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(G) \times X, B^4_{\mathrm{et}}(A(1))) \to \\ &\quad \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(P) \times X, B^4_{\mathrm{et}}(A(1))) \overset{\sim}{\longleftarrow} \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{PreStk}_{/X})}(B(M) \times X, B^4_{\mathrm{et}}(A(1))),\end{aligned}$$

where the second arrow is an isomorphism since the map $B(P) \to B(M)$ induces an isomorphism an étale cohomology with constant coefficients.

Thus, if $\mathcal{G}^G$ is a factorization $A$-gerbe on $\mathrm{Gr}_G$, and $\mathcal{G}^M$ is the corresponding the factorization $A$-gerbe on $\mathrm{Gr}_M$, the corresponding quadratic forms

$$q : \Lambda \to A(-1)$$

coincide.

5.1.4. We now take $A := E^{\times,\mathrm{tors}}$. Given a factorization $E^{\times,\mathrm{tors}}$-gerbe $\mathcal{G}^G$ over $\mathrm{Gr}_G$, consider its pullback to $\mathrm{Gr}_P$, denoted $\mathcal{G}^P$. We let $\mathcal{G}^M$ denote the canonically defined factorization gerbe on $\mathrm{Gr}_M$, whose pullback to $\mathrm{Gr}_P$ gives $\mathcal{G}^P$.

By construction, for any $S \to \mathrm{Ran}$, we have a well-defined pullback functor

$$\mathsf{p}^! : \mathrm{Shv}_{\mathcal{G}^P}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G) \to \mathrm{Shv}_{\mathcal{G}^P}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P).$$

Furthermore, since the morphism $\mathsf{q}$ is ind-schematic, we have a well-defined push-forward functor

$$\mathsf{q}_* : \mathrm{Shv}_{\mathcal{G}^P}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_P) \to \mathrm{Shv}_{\mathcal{G}^M}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_M).$$

Thus, the composite $\mathsf{q}_* \circ \mathsf{p}^!$ defines a map between factorization sheaves of categories

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^M}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}. \tag{5.2}$$

We will refer to (5.2) as the *naive Jacquet functor*.

5.2. **The critical twist.** The functor (5.2) is not quite what we need for the purposes of geometric Satake. Namely, we will need to correct this functor by a cohomological shift that depends on the connected component of $\mathrm{Gr}_M$ (this is needed in order to arrange that the corresponding functor on the spherical categories maps perverse sheaves to perverse sheaves). However, this cohomological shift will destroy the compatibility of the Jacquet functor with factorization, due to sign rules. In order to compensate for this, we will apply an additional twist of our categories by the square root of the determinant line bundle.

The nature of this additional twist will be explained in the present subsection.

For the rest of this subsection we will assume that $\mathrm{char}(k) \neq 2$.

5.2.1. Let $\det_{\mathfrak{g}}$ denote the determinant line bundle on $\mathrm{Gr}_G$, corresponding to the adjoint representation. It is constructed as follows. For an affine test scheme $S$ and an $S$-point $I \subset \mathrm{Maps}(S, X)$ of Ran, consider the corresponding $G$-bundle $\mathcal{P}_G$ on $S \times X$, equipped with an isomorphism

$$\alpha : \mathcal{P}_G \simeq \mathcal{P}^0_G$$

over $U_I \subset S \times X$. Consider the corresponding vector bundles associated with the adjoint representation

$$\mathfrak{g}_{\mathcal{P}_G}|_{U_I} \simeq \mathfrak{g}_{\mathcal{P}^0_G}|_{U_I}.$$

Then

$$\det.\mathrm{rel.}(\mathfrak{g}_{\mathcal{P}_G}, \mathfrak{g}_{\mathcal{P}^0_G}) \tag{5.3}$$

is a well-defined line bundle[12] on $S$.

This construction is compatible with pullbacks under $S' \to S$, thereby giving rise to the sought-for line bundle $\det_{\mathfrak{g}}$ on $\mathrm{Gr}_G$.

It is easy to see that $\det_{\mathfrak{g}}$ is equipped with a factorization structure over Ran.

[12] Note that the line bundle (5.3) is a priori $\mathbb{Z}$-graded, but since $G$ is reductive, and in particular, unimodular, this grading is actually trivial (i.e., concentrated in degree 0).

5.2.2. Consider the factorization $\mathbb{Z}/2\mathbb{Z}$-gerbe $\det_{\mathfrak{g}}^{\frac{1}{2}}$ over $\mathrm{Gr}_G$.

From now on we will choose a square root, denoted $\omega_X^{\otimes\frac{1}{2}}$ of the canonical line bundle $\omega_X$ on $X$ (see again Remark 1.4.3 for our notational conventions).

Let $P$ be again a parabolic of $G$. Consider the factorization gerbes $\det_{\mathfrak{g}}^{\frac{1}{2}}|_{\mathrm{Gr}_P}$ and $\det_{\mathfrak{m}}^{\frac{1}{2}}|_{\mathrm{Gr}_P}$ over $\mathrm{Gr}_P$. We claim that the choice of $\omega_X^{\otimes\frac{1}{2}}$ gives rise to an identification of the gerbes

$$\det_{\mathfrak{g}}^{\frac{1}{2}}|_{\mathrm{Gr}_P} \simeq \det_{\mathfrak{m}}^{\frac{1}{2}}|_{\mathrm{Gr}_P} \otimes \mathcal{G}^{\epsilon_P}|_{\mathrm{Gr}_P}, \tag{5.4}$$

where $\mathcal{G}^{\epsilon_P}$ is the $\mathbb{Z}/2\mathbb{Z}$-gerbe on $\mathrm{Gr}_{M/[M,M]}$ corresponding to the map $\epsilon_P$

$$\Lambda_{M/[M,M]} \overset{2\check{\rho}_{G,M}}{\longrightarrow} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z},$$

where $2\check{\rho}_{G,M} : M/[M,M] \to \mathbb{G}_m$ is the determinant of the action of $M$ on $\mathfrak{n}(P)$.

In fact, we claim that the ratio of the line bundles $\det_{\mathfrak{g}}|_{\mathrm{Gr}_P}$ and $\det_{\mathfrak{m}}|_{\mathrm{Gr}_P}$, i.e.,

$$\det_{\mathfrak{g}}|_{\mathrm{Gr}_P} \otimes (\det_{\mathfrak{m}}|_{\mathrm{Gr}_P})^{\otimes -1},$$

admits a square root, to be denoted $\det_{\mathfrak{n}(P)}$, which is a $\mathbb{Z}$-graded (and, in particular, $\mathbb{Z}/2\mathbb{Z}$-graded) factorization line bundle on $\mathrm{Gr}_P$, with the grading given by the map

$$\mathrm{Gr}_P \to \mathrm{Gr}_M \to \mathrm{Gr}_{M/[M,M]} \overset{2\check{\rho}_{G,M}}{\longrightarrow} \mathrm{Gr}_{\mathbb{G}_m}, \tag{5.5}$$

see Sects. 4.6.5 and 4.6.1.

*Remark* 5.2.3. In fact, more is true: the construction of [BD2, Sect. 4] defines a square root of $\det_{\mathfrak{g}}$ itself, again viewed as a graded factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle, where the grading is given by the map

$$\mathrm{Gr}_G \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G)\otimes\mathbb{G}_m} \to \mathrm{Gr}_{\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{G}_m},$$

where $\pi_{1,\mathrm{alg}}(G) \to \mathbb{Z}/2\mathbb{Z}$ is is the canonical map that fits into the diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \pi_{1,\mathrm{alg}}(G) \\ {\scriptstyle 2\check{\rho}}\big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z}, \end{array}$$

where $2\check{\rho}$ is the sum of positive roots.

5.2.4. The graded line bundle $\det_{\mathfrak{n}(P)}$ is constructed as follows. For an $S$-point $(I, \mathcal{P}_P, \mathcal{P}_G|_{U_I} \simeq \mathcal{P}^0_G|_{U_I})$ of $\mathrm{Gr}_P$ we set the value of $\det_{\mathfrak{n}(P)}$ on $S$ to be

$$\mathrm{rel.\,det.}(\mathfrak{n}(P)_{\mathcal{P}_P}, \mathfrak{n}(P)_{\mathcal{P}^0_P}).$$

Let us construct the isomorphism

$$(\det_{\mathfrak{n}(P)})^{\otimes 2} \otimes \det_{\mathfrak{m}}|_{\mathrm{Gr}_P} \simeq \det_{\mathfrak{g}}|_{\mathrm{Gr}_P}.$$

Let us identify the vector space $\mathfrak{g}/\mathfrak{p}$ with the dual of $\mathfrak{n}(P)$ (say, using the Killing form). For an $S$-point $(I, \mathcal{P}_P, \mathcal{P}_G|_{U_I} \simeq \mathcal{P}^0_G|_{U_I})$ of $\mathrm{Gr}_P$, denote

$$\mathcal{E} := \mathfrak{n}(P)_{\mathcal{P}_P} \text{ and } \mathcal{E}_0 := \mathfrak{n}(P)_{\mathcal{P}^0_P}.$$

Then the ratio of $\det_{\mathfrak{g}}|_S$ and $\det_{\mathfrak{m}}|_S$ identifies with the line bundle

$$\mathrm{rel.\,det.}(\mathcal{E}, \mathcal{E}_0) \otimes \mathrm{rel.\,det.}(\mathcal{E}^\vee, \mathcal{E}_0^\vee).$$

Note, however, that for any line bundle $\mathcal{L}$ on $S \times X$, we have

$$\mathrm{rel.\,det.}(\mathcal{E}, \mathcal{E}_0) \otimes \mathrm{rel.\,det.}(\mathcal{E}^\vee, \mathcal{E}_0^\vee) \simeq \mathrm{rel.\,det.}(\mathcal{E}\otimes\mathcal{L}, \mathcal{E}_0\otimes\mathcal{L}) \otimes \mathrm{rel.\,det.}(\mathcal{E}^\vee\otimes\mathcal{L}, \mathcal{E}_0^\vee\otimes\mathcal{L}).$$

Letting $\mathcal{L}$ be the pullback of $\omega_X^{\otimes\frac{1}{2}}$, we thus need to construct an isomorphism

$$\mathrm{rel.\,det.}(\mathcal{E}\otimes\omega_X^{\otimes\frac{1}{2}}, \mathcal{E}_0\otimes\omega_X^{\otimes\frac{1}{2}}) \simeq \mathrm{rel.\,det.}(\mathcal{E}^\vee\otimes\omega_X^{\otimes\frac{1}{2}}, \mathcal{E}_0^\vee\otimes\omega_X^{\otimes\frac{1}{2}}).$$

However, this follows from the (relative to $S$) local Serre duality on $S\times X$:

$$\mathbb{D}^{\mathrm{Serre}}_{/S}(\mathcal{E}\otimes\omega_X^{\otimes\frac{1}{2}})\simeq\mathcal{E}^\vee\otimes\omega_X^{\otimes\frac{1}{2}}[1] \text{ and } \mathbb{D}^{\mathrm{Serre}}_{/S}(\mathcal{E}_0\otimes\omega_X^{\otimes\frac{1}{2}})\simeq\mathcal{E}_0^\vee\otimes\omega_X^{\otimes\frac{1}{2}}[1].$$

5.3. **The corrected Jacquet functor.** We will now use the square root gerbe $\det_P^{\frac{1}{2}}$ from the previous subsection in order to introduce a correction to the naive Jacquet functor from Sect. 5.1.4.

5.3.1. Let $d_{G,M}:\mathrm{Gr}_P\to\mathbb{Z}$ be locally constant function on $\mathrm{Gr}_P$ corresponding to the map (5.5), see Sect. 4.6.5.

Given a factorization $E^{\times,\mathrm{tors}}$-gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_G$ and the corresponding factorization gerbe $\mathcal{G}^M$ on $\mathrm{Gr}_M$ (see Sect. 5.1.4), we will now define the corrected Jacquet functor as a map between factorization sheaves of categories:

$$J^G_M:\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}}\to\mathrm{Shv}_{\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}}}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}. \tag{5.6}$$

5.3.2. Namely, $J^G_M$ is the composition of the following four factorizable operations:

(i) The pullback functor

$$\mathsf{p}^!:\mathrm{Shv}_{\mathcal{P}_G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)_{/\,\mathrm{Ran}}\to\mathrm{Shv}_{(\mathcal{P}_G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}};$$

(ii) The identification

$$\mathrm{Shv}_{(\mathcal{P}_G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}}\simeq\mathrm{Shv}_{(\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}}\otimes\mathcal{G}^{\epsilon_P})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}},$$

given by the isomorphism of gerbes (5.4);

(iii) The cohomological shift functor $\mathcal{F}\mapsto\mathcal{F}[-d_{G,M}]$

$$\mathrm{Shv}_{(\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}}\otimes\mathcal{G}^{\epsilon_P})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}}\to\mathrm{Shv}_{(\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}},$$

see Sect. 4.6.5.

(iv) The pushforward functor

$$\mathsf{q}_*:\mathrm{Shv}_{(\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}})|_{\mathrm{Gr}_P}}(\mathrm{Gr}_P)_{/\,\mathrm{Ran}}\to\mathrm{Shv}_{\mathcal{G}^M\otimes\det_{\mathfrak{m}}^{\frac{1}{2}}}(\mathrm{Gr}_M)_{/\,\mathrm{Ran}}.$$

## 6. The metaplectic Langlands dual datum

In section we take $G$ to be reductive. Given a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_G$, we will define the *metaplectic Langlands dual datum* attached to $\mathcal{G}$, and the corresponding notion of twisted local system on $X$.

6.1. **The metaplectic Langlands dual *root datum*.** The first component of the metaplectic Langlands dual datum is purely combinatorial and consists of a certain root datum that only depends on the root datum of $G$ and $q$. This is essentially the same as the root datum defined by G. Lusztig as a recipient of the quantum Frobenius.

6.1.1. Given a factorization $A$-gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_G$, let

$$q:\Lambda\to A(-1)$$
$$b:\Lambda\times\Lambda\to A(-1)$$

be the associated quadratic and bilinear forms, respectively. Let $\Lambda^\sharp\subset\Lambda$ be the kernel of $b$. Let $\check{\Lambda}^\sharp$ be the dual of $\Lambda^\sharp$. Note that the inclusions

$$\Lambda^\sharp\subset\Lambda \text{ and } \check{\Lambda}\subset\check{\Lambda}^\sharp$$

induce isomorphisms after tensoring with $\mathbb{Q}$.

Following [Lus], we will now define a new root datum

$$(\Delta^\sharp\subset\Lambda^\sharp,\check{\Delta}^\sharp\subset\check{\Lambda}^\sharp). \tag{6.1}$$

6.1.2. We let $\Delta^\sharp$ be equal to $\Delta$ as an *abstract set*. For each element $\alpha \in \Delta$, we let the corresponding element $\alpha^\sharp \in \Delta^\sharp$ be equal to

$$\mathrm{ord}(q(\alpha)) \cdot \alpha \in \Lambda,$$

and the corresponding element $\check{\alpha}^\sharp \in \check{\Delta}^\sharp$ be

$$\frac{1}{\mathrm{ord}(q(\alpha))} \cdot \check{\alpha} \in \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

The fact that $q$ lies in $\mathrm{Quad}(\Lambda, A(-1))_{\mathrm{restr}}$ implies that $\alpha^\sharp$ and $\check{\alpha}^\sharp$ defined in this way indeed belong to $\Lambda^\sharp \subset \Lambda$ and $\check{\Lambda}^\sharp \subset \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, respectively.

6.1.3. Since $q$ was $W$-invariant, the action of $W$ on $\Lambda$ preserves $\Lambda^\sharp$. Moreover, for each $\alpha \in \Delta$, the action of the corresponding reflection $s_\alpha \in W$ on $\Lambda^\sharp$ equals that of $s_{\alpha^\sharp}$.

This implies that restriction defines an isomorphism from $W$ to the group $W^\sharp$ of automorphisms of $\Lambda^\sharp$ generated by the elements $s_{\alpha^\sharp}$.

Hence, (6.1) is a finite root system with Weyl group $W^\sharp$, isomorphic to the original Weyl group $W$.

It follows from the constriction that if $\alpha_i$ are the simple coroots of $\Delta$, then the corresponding elements $\alpha_i^\sharp \in \Lambda^\sharp$ form a set of simple roots of $\Delta^\sharp$.

6.1.4. We let $G^\sharp$ denote the reductive group (over $k$) corresponding to (6.1).

6.2. **The "$\pi_1$-gerbe".** Let $\mathcal{G}^G$ be as above. In this subsection we will show that in addition to the reductive group $G^\sharp$, the datum of $\mathcal{G}^G$ defines a certain *multiplicative* factorization gerbe on the affine Grassmannian attached to the abelian group $\pi_{1,\mathrm{alg}}(G^\sharp)$.

6.2.1. Let $\mathcal{G}^T$ be the factorization gerbe on $\mathrm{Gr}_T$, corresponding to $\mathcal{G}^G$ via Sect. 5.1.4. Consider the corresponding torus $T^\sharp$.

Let $\mathcal{G}^{T^\sharp}$ be the factorization gerbe on $\mathrm{Gr}_{T^\sharp,\mathrm{Ran}}$ equal to the pullback of $\mathcal{G}^T$ under $T^\sharp \to T$. By Proposition 4.3.2(b), the gerbe $\mathcal{G}^{T^\sharp}$ carries a canonical multiplicative structure.

Consider the algebraic fundamental group $\pi_{1,\mathrm{alg}}(G^\sharp)$ of $G^\sharp$, and the projection $\Lambda^\sharp \to \pi_{1,\mathrm{alg}}(G^\sharp)$. Consider the corresponding map

$$\mathrm{Gr}_{T^\sharp} \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}. \tag{6.2}$$

We claim that there exists a canonically defined multiplicative factorization $A$-gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$, whose pullback under (6.2) identifies with $\mathcal{G}^{T^\sharp}$.

6.2.2. By Corollary 4.4.5, we need to show that for every simple coroot $\alpha_i$, the pullback of $\mathcal{G}^T$ to $\mathrm{Gr}_{\mathbb{G}_m}$ under

$$\mathbb{G}_m \xrightarrow{\alpha_i^\sharp} T$$

is trivialized.

By the transitivity of the construction in Sect. 5.1.4, we can replace $G$ by its Levi subgroup $M_i$ of semi-simple rank 1, corresponding to $\alpha_i$. Furthermore, using the map $SL_2 \to M_i$, we can assume that $G = SL_2$.

6.2.3. Note that by Sect. 3.3.2, any factorizable $A$-gerbe on $\mathrm{Gr}_{SL_2}$ is *canonically* of the form $(\det_{SL_2,\mathrm{St}})^a$ for some element $a \in A(-1)$, where $\det_{SL_2,\mathrm{St}}$ is the determinant line bundle on $\mathrm{Gr}_{SL_2}$ corresponding to the action on the *standard* representation.

Let us first calculate the resulting $A$-gerbe on $\mathrm{Gr}_{\mathbb{G}_m}$, where we think of $\mathbb{G}_m$ as the Cartan subgroup of $SL_2$.

For an integer $k$ let $\det_{\mathbb{G}_m,k}$ denote the determinant line bundle on $\mathrm{Gr}_{\mathbb{G}_m}$ associated with the action of $\mathbb{G}_m$ on the one-dimensional vector space given by the $k$-th power of the tautological character. This is a $\mathbb{Z}$-graded factorization line bundle, and we note that the grading is even if $k$ is even.

The restriction of $\det_{SL_2,\mathrm{St}}$ to $\mathrm{Gr}_{\mathbb{G}_m}$ identifies with $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$, and hence the restriction of $(\det_{SL_2,\mathrm{St}})^a$ to $\mathrm{Gr}_{\mathbb{G}_m}$ identifies with $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$. The associated quadratic form

$$q : \mathbb{Z} \to A$$

takes value $a$ on the generator $1 \in \mathbb{Z}$. Let $n := \mathrm{ord}(a)$.

We need to show that the pullback of $(\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1})^a$ under the isogeny

$$\mathbb{G}_m \overset{x \mapsto x^n}{\longrightarrow} \mathbb{G}_m \tag{6.3}$$

is canonically trivial as a factorization gerbe on $\mathrm{Gr}_{\mathbb{G}_m}$. For this, it suffices to show that the pullback of the factorization line bundle $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ under the above isogeny admits a canonical $n$-th root.

6.2.4. Note that a line bundle on $X$ gives rise to a *multiplicative* factorization line bundle on $\mathrm{Gr}_{\mathbb{G}_m}$ (see [BD1, Lemma 3.10.3]) Denote this construction by

$$\mathcal{L} \mapsto \mathrm{Fact}(\mathcal{L}).$$

Explicitly, when we think of factorization line bundles on $\mathrm{Gr}_{\mathbb{G}_m}$ in terms of $\mathrm{Gr}_{\mathbb{G}_m,\mathrm{comb}}$, the value of $\mathrm{Fact}(\mathcal{L})$ on $X^I$ corresponding to a given map $\lambda^I : I \to \Lambda \simeq \mathbb{Z}$ is given by

$$\underset{i \in I}{\boxtimes} \mathcal{L}^{\otimes \lambda_i}.$$

6.2.5. By local Serre duality, the factorization line bundle $\det_{\mathbb{G}_m,1} \otimes \det_{\mathbb{G}_m,-1}$ identifies canonically with $\mathrm{Fact}(\omega_X^{-1})$.

Now, the multiplicative structure on $\mathrm{Fact}(\omega_X^{-1})$ implies that its pullback under the isogeny (6.3) admits a canonical $n$-th root, given by $\mathrm{Fact}(\omega_X^{-1})$ itself.

6.2.6. *Example.* Suppose that $\mathcal{G}^G$ is trivial, in which case $T^\sharp = T$ and $G^\sharp = G$. In this case $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp) \otimes \mathbb{G}_m}$ is also trivial.

6.3. **The metaplectic Langlands dual datum as a triple.** In this subsection we take $A := E^{\times,\mathrm{tors}}$.

6.3.1. By Sect. 4.5.5, to $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp) \otimes \mathbb{G}_m}$ we can canonically attach an object

$$(\mathcal{G}^{\pi_{1,\mathrm{alg}} \otimes \mathbb{G}_m})^0 \in \mathrm{FactGe}^0_A(\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp) \otimes \mathbb{G}_m}) \simeq \mathrm{Maps}(X, B^2_{\mathrm{et}}(\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G^\sharp), E^{\times,\mathrm{tors}})))$$

and a map

$$\epsilon : \pi_{1,\mathrm{alg}}(G^\sharp) \to \mathbb{Z}/2\mathbb{Z}.$$

6.3.2. Let $H$ denote the Langlands dual of $G^\sharp$, viewed as an algebraic group over $E$. Note that

$$\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G^\sharp), E^\times)$$

identifies with $Z_H(E)$, where $Z_H$ denotes the center of $H$.

Hence, we can think of $(\mathcal{G}^{\pi_{1,\mathrm{alg}} \otimes \mathbb{G}_m})^0$ as a $Z_H$-gerbe on $X$, to be denoted $\mathcal{G}_Z$. Furthermore, we interpret the above map $\epsilon$ as a homomorphism

$$\epsilon : \mathbb{Z}/2\mathbb{Z} \to Z_H(E). \tag{6.4}$$

6.3.3. We will refer to the triple

$$(H, \mathcal{G}_Z, \epsilon) \tag{6.5}$$

as the *metaplectic Langlands dual datum* corresponding to $\mathcal{G}^G$.

## 7. Factorization gerbes on loop groups

In this section we will perform a crucial geometric construction that will explain why our definition of geometric metaplectic datum was "the right thing to do":

We will show that a factorization gerbe on $\mathrm{Gr}_G$ give rise to a (factorization) gerbe on (the factorization version of) the loop group of $G$.

7.1. **Digression: factorization loop and arc spaces.** Up until this point, the geometric objects that have appeared in this paper were all locally of finite type, considered as prestacks. However, the objects that we will introduce below *do not* have this property.

7.1.1. For an affine test scheme $S$ and an $S$-point of Ran, given by a finite set $I \subset \mathrm{Maps}(S, X)$, let $\hat{\mathcal{D}}_I$ be the corresponding relative formal disc:

By definition, $\hat{\mathcal{D}}_I$ is the formal scheme equal to the completion of $S \times X$ along the union of the graphs of the maps $S \to X$ corresponding to the elements of $I$.

Note that for a finite set $J$ and a point

$$\{I_j, j \in J\} \in \mathrm{Ran}^J_{\mathrm{disj}},$$

we have

$$\hat{\mathcal{D}}_I \simeq \underset{j}{\sqcup}\, \hat{\mathcal{D}}_{I_j}, \tag{7.1}$$

where $I = \underset{j}{\sqcup}\, I_j$.

7.1.2. Since $S$ was assumed affine, $\hat{\mathcal{D}}_I$ is an ind-object in the category $\mathrm{Sch}^{\mathrm{aff}}$. Let $\mathcal{D}_I$ be the affine scheme corresponding to the formal scheme $\hat{\mathcal{D}}_I$, i.e., the image of $\hat{\mathcal{D}}_I$ under the functor

$$\mathrm{colim} : \mathrm{Ind}(\mathrm{Sch}^{\mathrm{aff}}) \to \mathrm{Sch}^{\mathrm{aff}}.$$

In other words, if

$$\hat{\mathcal{D}}_I \simeq \underset{\alpha}{\mathrm{colim}}\, Z_\alpha,$$

where $Z_\alpha = \mathrm{Spec}(A_\alpha)$ and the colimit is taken in PreStk, then $\mathcal{D}_I = \mathrm{Spec}(A)$, where

$$A = \underset{\alpha}{\lim}\, A_\alpha.$$

Let $\overset{\circ}{\mathcal{D}}_I$ be the open subscheme of $\mathcal{D}_I$, obtained by removing the closed subscheme $\Gamma_I$ equal to the union of the graphs of the maps $S \to X$ corresponding to the elements of $I$.

7.1.3. Let $Z$ be a prestack. We define the prestacks $\mathfrak{L}^+(Z)$ (resp., $\mathfrak{L}(Z)$) over Ran as follows.

For an affine test scheme $S$ and an $S$-point of Ran, given by a finite set $I \subset \mathrm{Maps}(S, X)$, its lift to an $S$-point of $\mathfrak{L}^+(Z)$ (resp., $\mathfrak{L}(Z)$) is the datum of a map $\mathcal{D}_I \to Z$ (resp., $\overset{\circ}{\mathcal{D}}_I \to Z$).

The isomorphisms (7.1) imply that $\mathfrak{L}^+(Z)$ and $\mathfrak{L}(Z)$ are naturally factorization prestacks over Ran.

7.1.4. Assume for a moment that $Z$ is an affine scheme. Note that in this case the definition of $\mathfrak{L}^+(Z)$, the datum of a map $\mathcal{D}_I \to Z$ is equivalent to that of a map of prestacks $\hat{\mathcal{D}}_I \to Z$.

Assume now that $Z$ is a smooth scheme of finite type (but not necessarily affine). Then one shows that for every $S \to \mathrm{Ran}$, the fiber product

$$S \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(Z)$$

is a projective limit (under smooth maps) of smooth affine schemes over $S$.

7.2. **Factorization loop and arc groups.**

7.2.1. Let us recall that the Beauville-Laszlo Theorem says that the definition of $\mathrm{Gr}_G$ can be rewritten in terms of the pair $\overset{\circ}{\mathcal{D}}_I \subset \mathcal{D}_I$.

Namely, given $I$ as above, the datum of its lift to a point of $\mathrm{Gr}_G$ is a pair $(\mathcal{P}_G, \alpha)$, where $\mathcal{P}_G$ is a $G$-bundle on $\mathcal{D}_I$, and $\alpha$ is the trivialization of $\mathcal{P}_G|_{\overset{\circ}{\mathcal{D}}_I}$. (Note that restriction along $\hat{\mathcal{D}}_I \to \mathcal{D}_I$ induces an equivalence between the category of $G$-bundles on $\mathcal{D}_I$ and that on $\hat{\mathcal{D}}_I$.)

In other words, the Beauville-Laszlo says that restriction along

$$(\overset{\circ}{\mathcal{D}}_I \subset \mathcal{D}_I) \to (U_I \subset S \times X)$$

induces a bijection on the corresponding pairs $(\mathcal{P}_G, \alpha)$. In the above formula, the notation $U_I$ is as in Sect. 2.3.1.

7.2.2. This interpretation of $\mathrm{Gr}_G$ shows that the group-prestack $\mathfrak{L}(G)$ acts naturally on $\mathrm{Gr}_G$, with the stabilizer of the unit section being $\mathfrak{L}^+(G)$. Furthermore, the natural map

$$\mathfrak{L}(G)/\mathfrak{L}^+(G) \to \mathrm{Gr}_G, \tag{7.2}$$

is an isomorphism, where the quotient is understood in the sense of stacks in the étale topology.

The isomorphism (7.2) implies that for every $S \to \mathrm{Ran}$, the fiber product

$$S \underset{\mathrm{Ran}}{\times} \mathfrak{L}(G),$$

is an ind-scheme over $S$.

7.2.3. Recall that given a group-prestack $\mathcal{H}$ over a base $Z$, we can talk about a gerbe over $\mathcal{H}$ being *multiplicative*, i.e., compatible with the group-structure.

In particular, we can consider the spaces

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \text{ and } \mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))$$

of multiplicative factorization gerbes on $\mathfrak{L}(G)$ and $\mathfrak{L}^+(G)$, respectively.

7.2.4. The isomorphism (7.2) defines a map

$$\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \underset{\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))}{\times} * \to \mathrm{FactGe}_A(\mathrm{Gr}_G). \tag{7.3}$$

The following result is established in [Re, Theorem III.2.10]:

**Proposition 7.2.5.** *The map* (7.3) *is an isomorphism.*

We will sketch the proof of this proposition in Sect. 7.5. It consists of explicitly constructing the inverse map.

7.2.6. Let us restate Proposition 7.2.5 in words. It says that, given a factorization gerbe on $\mathrm{Gr}_G$, its pullback under the projection

$$\mathfrak{L}(G) \to \mathrm{Gr}_G,$$

carries a uniquely defined multiplicative structure that is compatible with that of factorization and the trivialization of the further restriction of our gerbe to $\mathfrak{L}^+(G)$.

7.3. **The $\mathfrak{L}^+(G)$-equivariant structure.** The main step in constructing the map

$$\mathrm{FactGe}_A(\mathrm{Gr}_G) \to \mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}(G)) \underset{\mathrm{FactGe}_A^{\mathrm{mult}}(\mathfrak{L}^+(G))}{\times} *, \tag{7.4}$$

inverse to (7.3), consists of constructing a (canonical) structure of equivariance with respect to $\mathfrak{L}^+(G)$ on a given factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_G$. We will explain this construction in the present subsection.

7.3.1. For a non-negative integer $n$, let

$$\widetilde{\mathrm{Gr}}^n_G \to \mathrm{Ran}^n$$

be the $n$-fold convolution diagram. I.e., for an $S$-point of $\mathrm{Ran}^n$

$$\{I_j, 1 \leq j \leq n\} \in \mathrm{Ran}^n, \quad I_j \subset \mathrm{Hom}(S, X),$$

its lift to an $S$-point of $\widetilde{\mathrm{Gr}}^n_G$ consists of a string of $G$-bundles

$$\mathcal{P}^1_G, \mathcal{P}^2_G, ..., \mathcal{P}^n_G \tag{7.5}$$

on $S \times X$, together with identifications

$$\mathcal{P}^0_G|_{U_{I_1}} \overset{\alpha_1}{\simeq} \mathcal{P}^1_G|_{U_{I_1}}, \ \mathcal{P}^1_G|_{U_{I_2}} \overset{\alpha_2}{\simeq} \mathcal{P}^2_G|_{U_{I_2}}, ..., \mathcal{P}^{n-1}_G|_{U_{I_n}} \overset{\alpha_n}{\simeq} \mathcal{P}^n_G|_{U_{I_n}},$$

where $\mathcal{P}^0_G$ denotes the trivial $G$-bundle.

We have a naturally defined map

$$\widetilde{\mathrm{Gr}}^n_G \to \mathrm{Gr}_G \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^n \tag{7.6}$$

that sends the above data to

$$(I := \underset{j}{\sqcup} I_j, \ \mathcal{P}_G = \mathcal{P}^n_G, \alpha = \alpha_n \circ .... \circ \alpha_1).$$

This map is an isomorphism over $\mathrm{Ran}^n_{\mathrm{disj}}$.

7.3.2. In Sect. 7.3.3 below we will explain that for a decomposition $n = n_1 + n_2$, we can view $\widetilde{\mathrm{Gr}}^n_G$ as a *twisted product*

$$\widetilde{\mathrm{Gr}}^{n_1+n_2}_G \simeq \widetilde{\mathrm{Gr}}^{n_1}_G \widetilde{\times} \widetilde{\mathrm{Gr}}^{n_2}_G,$$

which identifies with the usual product $\widetilde{\mathrm{Gr}}^{n_1}_G \times \widetilde{\mathrm{Gr}}^{n_2}_G$ when restricted to $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$, where

$$\mathrm{Ran}^n \simeq \mathrm{Ran}^{n_1} \times \mathrm{Ran}^{n_2} \to \mathrm{Ran} \times \mathrm{Ran}$$

corresponds to the projection $\mathrm{Ran}^{n_1} \to \mathrm{Ran}$ on the last coordinate and the projection $\mathrm{Ran}^{n_2} \to \mathrm{Ran}$ on the first component.

This is a well-known construction and the reader familiar with it can safely skip it. For simplicity, we will take $n_1 = n_2 = 1$.

7.3.3. For a pair of $S$-points of Ran, given by $I_1, I_2 \subset \mathrm{Hom}(S, X)$, respectively, denote $I := I_1 \cup I_2$.

Using the Beauville-Laszlo theorem, we can think of a point of $\mathrm{Gr}_G \times \mathrm{Ran}$ to consist of a $G$-bundle $\mathcal{P}^1_G$ on $\mathcal{D}_{I_1}$ and its trivialization $\alpha_1$ on $\mathcal{D}_{I_1} - \Gamma_{I_1}$. Equivalently, we can take $\mathcal{P}^1_G$ to be defined over $\mathcal{D}_I$ and $\alpha_1$ to be defined on $\mathcal{D}_I - \Gamma_{I_1}$. The latter presentation implies that there is a canonical $\mathrm{Ran} \times \mathfrak{L}^+(G)$-torsor over $\mathrm{Gr}_G \times \mathrm{Ran}$, denoted $\mathcal{R}$, that classifies trivializations of $\mathcal{P}^1_G|_{\mathcal{D}_{I_2}}$. Note that $\mathcal{R}$ is canonically trivialized over $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$.

We claim that $\widetilde{\mathrm{Gr}}^2_G$ identifies with the twist of $\mathrm{Ran} \times \mathrm{Gr}_G$, viewed as a prestack over $\mathrm{Ran} \times \mathrm{Ran}$ by means of $\mathcal{R}$:

$$\left( (\mathrm{Ran} \times \mathrm{Gr}_G) \underset{\mathrm{Ran} \times \mathrm{Ran}}{\times} \mathcal{R} \right) / (\mathrm{Ran} \times \mathfrak{L}^+(G)). \tag{7.7}$$

Indeed, we can think of a point of (7.7) as a datum of $(\mathcal{P}^1_G, \alpha_1, \mathcal{P}^2_G, \alpha_2)$, where:

–$\mathcal{P}^1_G$ is a $G$-bundle on $\mathcal{D}_I$;

–$\alpha_1$ is a trivialization of $\mathcal{P}^1_G$ on $\mathcal{D}_{I_1} - \Gamma_{I_1}$;

–$\mathcal{P}^2_G$ is defined on $\mathcal{D}_{I_2}$;

– $\alpha_2$ is an identification $\mathcal{P}^1_G|_{\mathcal{D}_{I_2} - \Gamma_{I_2}} \simeq \mathcal{P}^2_G|_{\mathcal{D}_{I_2} - \Gamma_{I_2}}$.

However, again by the Beauville-Laszlo theorem, we can equivalently think of the pair $(\mathcal{P}^2_G, \alpha_2)$ as follows: we can take $\mathcal{P}^2_G$ to be defined over $\mathcal{D}_I$ and $\alpha_2$ to be defined over $\mathcal{D}_I - \Gamma_2$.

Finally, the Beauville-Laszlo theorem again implies that the latter reproduces $\widetilde{\mathrm{Gr}}^2_G$.

Note that we have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{\mathrm{Gr}}^2_G \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}} & \xrightarrow[\sim]{(7.7)} & (\mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}} \\
\Big\downarrow{\scriptstyle (7.6)}\ \sim & & \Big\downarrow{\scriptstyle \text{trivialization of } \mathcal{R} \text{ on } (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}}\ \sim \\
\mathrm{Gr}_G \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}} & & (\mathrm{Gr}_G \times \mathrm{Gr}_G) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}} \\
\Big\downarrow{\scriptstyle \text{factorization}} & & \Big\downarrow{\scriptstyle (7.6)}\ \sim \\
(\mathrm{Gr}_G \times \mathrm{Gr}_G) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}} & \xrightarrow{=} & (\mathrm{Gr}_G \times \mathrm{Gr}_G) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}.
\end{array}
\tag{7.8}
$$

7.3.4. The key observation (proved by reduction to the Cartan subgroup) is that a factorization gerbe $\mathcal{G}$ on $\mathrm{Gr}_G$ admits a *unique* structure of equivariance with respect to $\mathfrak{L}^+(G)$ that has the following property:

In the setting of Sect. 7.3.3 consider the twisted product $\mathcal{G}\widetilde{\boxtimes}\mathcal{G}$, which is a well-defined gerbe on $\widetilde{\mathrm{Gr}}^2_G$ due to the identification

$$\left((\mathrm{Ran}\times\mathrm{Gr}_G) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} \mathcal{R}\right)/(\mathrm{Ran}\times\mathfrak{L}^+(G)) \simeq \widetilde{\mathrm{Gr}}^2_G$$

and the chosen structure of equivariance with respect to $\mathfrak{L}^+(G)$ on $\mathcal{G}$.

We require that $\mathcal{G}\widetilde{\boxtimes}\mathcal{G}$ should admit an identification with the pullback of $\mathcal{G}$ under the map

$$\widetilde{\mathrm{Gr}}^2_G \overset{(7.6)}{\longrightarrow} \mathrm{Gr}_G \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran}) \to \mathrm{Gr}_G,$$

which extends the *already existing* identification over

$$\widetilde{\mathrm{Gr}}^2_G \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}},$$

given by the factorization structure on $\mathcal{G}$ via the diagram (7.8).

7.4. **Another view on the bilinear form.** The $\mathfrak{L}^+(G)$-equivariant structure on $\mathcal{G}$ gives rise to the following interpretation of the bilinear form attached to $\mathcal{G}$, when $G$ is a torus $T$.

7.4.1. Namely, choose an arbitrary point $x \in X$, and consider the restrictions

$$\mathrm{Gr}_{G,x} := \{x\} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G \text{ and } \mathfrak{L}^+(G)_x := \{x\} \underset{\mathrm{Ran}}{\times} \mathfrak{L}^+(G) \simeq G(\hat{\mathcal{O}}_x).$$

We obtain that the $A$-gerbe restriction $\mathcal{G}|_{\mathrm{Gr}_{G,x}}$ is equivariant with respect to $G(\hat{\mathcal{O}}_x)$.

7.4.2. For $G = T$, since $T$ is commutative, the action of $T(\hat{\mathcal{O}}_x)$ on $\mathrm{Gr}_{T,x}$ is trivial. Hence, for every $\lambda \in \Lambda$, the action of $T(\hat{\mathcal{O}}_x)$ on the corresponding point of $\mathrm{Gr}_{T,x}$ defines a multiplicative $A$-torsor on $T(\hat{\mathcal{O}}_x)$.

Since the elements of $A$ have orders prime to $\mathrm{char}(k)$, the above multiplicative $A$-torsor is pulled back from $T$, and by Kummer theory, the latter is given by a homomorphism

$$\Lambda \to A(-1).$$

Thus, we have constructed a map

$$\Lambda \to \mathrm{Hom}(\Lambda, A(-1)). \tag{7.9}$$

7.4.3. By unwinding the constructions, one shows that (7.9) equals one coming from the bilinear form attached to $\mathcal{G}$ and our chosen element $\lambda \in \Lambda$.

7.5. **Construction of the inverse map in Proposition 7.2.5.**

7.5.1. For a non-negative integer $n$, consider the prestack

$$Z^n := \mathfrak{L}^+(G)\backslash(\widetilde{\mathrm{Gr}}^n \underset{\mathrm{Ran}^n}{\times} \mathrm{Ran}), \tag{7.10}$$

where $\mathrm{Ran} \to \mathrm{Ran}^n$ is the diagonal map.

It is easy to see that as $n$ varies, the prestacks (7.10) form a simplicial object in $\mathrm{PreStk}_{/\,\mathrm{Ran}}$; denote it by $Z^\bullet$. Consider its geometric realization $|Z^\bullet|$, viewed as a prestack over Ran, equipped with a factorization structure.

By the construction in Sect. 7.3, a factorization $A$-gerbe on $\mathrm{Gr}_G$ gives rise to a *2-gerbe* on $|Z^\bullet|$ with respect to $A$, i.e., a map

$$|Z^\bullet| \to B^3_{\mathrm{et}}(A),$$

equipped with a trivialization of its restriction to

$$B_{\mathrm{et}}(\mathfrak{L}^+(G)) = Z^0 \to |Z^\bullet|.$$

Moreover, the above 2-gerbe is naturally equipped with the factorization structure.

7.5.2. Note now that we have the (simplicial) identification between (7.10) and the Čech nerve of the map

$$B_{\mathrm{et}}(\mathfrak{L}^+(G)) \to B_{\mathrm{et}}(\mathfrak{L}(G)).$$

Thus, we obtain a 2-gerbe on $B_{\mathrm{et}}(\mathfrak{L}(G))$, equipped with a trivialization of its restriction to $B_{\mathrm{et}}(\mathfrak{L}^+(G))$, and equipped with a factorization structure.

The latter datum is equivalent to that of a multiplicative gerbe on $\mathfrak{L}(G)$, equipped with a (multiplicative) trivialization of its restriction to $\mathfrak{L}^+(G)$.

## 8. Factorization category of representations

From now on, until the end of the paper we will assume that $k = E$ and we will work in the context of D-modules.

8.1. **Digression: factorization categories arising from symmetric monoidal categories.** In this subsection we will explain a procedure that produces a factorization sheaf of categories from a sheaf symmetric monoidal categories $X$. The source of the metaplectic geometric Satake functor will be a factorization sheaf of categories obtained in this way.

For a more detailed discussion see [Ras2, Sect. 6].

8.1.1. Let $\mathcal{C}$ be a sheaf of symmetric monoidal categories over $X$. To it we will associate a sheaf of symmetric monoidal categories over Ran, equipped with a factorization structure, denoted $\mathrm{Fact}(\mathcal{C})$.

We will construct $\mathrm{Fact}(\mathcal{C})$ as a family of sheaves of symmetric monoidal categories over $X^I$ for all finite non-empty sets $I$, compatible under surjections $I_1 \twoheadrightarrow I_2$. We will use Theorem 1.6.9 that says that the datum of sheaf of categories over $X^I$ is equivalent to that of a category acted on by $\mathrm{Shv}(X^I)$. So, we will produce system of symmetric monoidal categories $\mathrm{Fact}(\mathcal{C})(X^I)$, compatible under

$$\mathrm{Fact}(\mathcal{C})(X^{I_2}) \simeq \mathrm{Shv}(X^{I_2}) \underset{\mathrm{Shv}(X^{I_1})}{\otimes} \mathrm{Fact}(\mathcal{C})(X^{I_1}). \tag{8.1}$$

8.1.2. Let $\mathcal{C}_X$ denote the category of sections of $\mathcal{C}$ over $X$; this is a symmetric monoidal category over $\mathrm{Shv}(X)$. For a finite set $J$ we let $\mathcal{C}_X^{\otimes J}$ the $J$-fold tensor product of $\mathcal{C}$ *over* $\mathrm{Shv}(X)$.

Note that for a surjection of finite sets $I \twoheadrightarrow J$ we have a canonical isomorphism

$$\mathcal{C}_X^{\otimes J} \simeq \left(\bigotimes_{j\in J} \mathcal{C}_X^{I_j}\right) \underset{\mathrm{Shv}(X^J)}{\otimes} \mathrm{Shv}(X), \tag{8.2}$$

where $I_j$ denotes the preimage of $j \in J$ under $I \to J$.

In addition, for $I \twoheadrightarrow J$, the symmetric monoidal structure on $\mathcal{C}_X$ gives rise to the functors

$$\mathcal{C}_X^{\otimes I} \to \mathcal{C}_X^{\otimes J}. \tag{8.3}$$

*Remark* 8.1.3. We will be particularly interested in the case when $\mathcal{C}$ is constant, i.e., $\mathcal{C}_X \simeq \mathcal{C}_{\mathrm{pt}} \otimes \mathrm{Shv}(X)$ for a symmetric monoidal category $\mathcal{C}_{\mathrm{pt}}$. Note that in this case $\mathcal{C}_X^{\otimes J}$ is just $\mathcal{C}_{\mathrm{pt}}^{\otimes J} \otimes \mathrm{Shv}(X)$.

8.1.4. For a given $I$, let $\mathrm{Tw}(I)$ be the category whose objects are pairs

$$I \twoheadrightarrow J \twoheadrightarrow K \tag{8.4}$$

(here $J$ and $K$ are sets (automatically, finite and non-empty)), and where morphisms from $(J, K)$ to $(J', K')$ are commutative diagrams

$$\begin{array}{ccccc} I & \longrightarrow & J & \longrightarrow & K \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow & & \uparrow \\ I & \longrightarrow & J' & \longrightarrow & K'. \end{array} \tag{8.5}$$

(Note that the arrows between the $K$'s go in the opposite direction.)

8.1.5. Consider the functor

$$\mathrm{Tw}(I) \to \mathrm{DGCat} \tag{8.6}$$

that sends an object (8.4) to

$$\bigotimes_{k \in K} \mathcal{C}_X^{\otimes J_k},$$

where $J_k$ is the preimage under $J \to K$ of the element $k \in K$. The above tensor product is naturally a symmetric monoidal category over $\mathrm{Shv}(X^K)$.

For a morphism (8.5) in $\mathrm{Tw}(I)$, we let the corresponding functor

$$\bigotimes_{k \in K} \mathcal{C}_X^{\otimes J_k} \to \bigotimes_{k' \in K'} \mathcal{C}_X^{\otimes J'_{k'}}$$

be given by the composition

$$\bigotimes_{k \in K} \mathcal{C}_X^{\otimes J_k} \overset{(8.3)}{\longrightarrow} \bigotimes_{k \in K} \mathcal{C}_X^{\otimes J'_k} \overset{(8.2)}{\simeq} \bigotimes_{k \in K} \left( \left( \bigotimes_{k' \in K'_k} \mathcal{C}_X^{\otimes J'_{k'}} \right) \underset{\mathrm{Shv}(X^{K'_k})}{\otimes} \mathrm{Shv}(X) \right) =$$

$$= \left( \bigotimes_{k' \in K'} \mathcal{C}_X^{\otimes J'_{k'}} \right) \underset{\mathrm{Shv}(X^{K'})}{\otimes} \mathrm{Shv}(X^K) \to \bigotimes_{k' \in K'} \mathcal{C}_X^{\otimes J'_{k'}},$$

where the the last arrow is given by the direct image functor along $X^K \to X^{K'}$.

8.1.6. We let $\mathrm{Fact}(\mathcal{C})(X^I)$ on be the object of DGCat equal to the colimit of the functor (8.6) over $\mathrm{Tw}(I)$.

The compatibilities (8.1), as well as the factorization structure on $\mathrm{Fact}(\mathcal{C})$ follow from the construction.

8.1.7. Let $\mathrm{Fact}(\mathcal{C})(\mathrm{Ran})$ denote the category of global sections of $\mathrm{Fact}(\mathcal{C})$ over Ran.

As in [Ga5, Sect. 4.2], the (symmetric) monoidal structure on $\mathrm{Fact}(\mathcal{C})$ as a sheaf of categories over Ran and the operation of union of finite sets makes $\mathrm{Fact}(\mathcal{C})(\mathrm{Ran})$ into a *non-unital* (symmetric) monoidal category.

8.2. **Twisting procedures on the Ran space.** In this subsection we will start with a symmetric monoidal category $\mathcal{C}$ and some twisting data, and associate to it a sheaf of categories on the Ran space.

8.2.1. First, to $\mathcal{C}$ be associate the constant sheaf of symmetric monoidal categories over $X$, which, by a slight abuse of notation we denote by the same symbol $\mathcal{C}$; we have $\mathcal{C}_X = \mathcal{C} \otimes \mathrm{Shv}(X)$, see Remark 8.1.3.

Consider the corresponding factorization sheaf $\mathrm{Fact}(\mathcal{C})$ of symmetric monoidal categories over Ran.

8.2.2. Let now $A$ be a torsion abelian group that acts by automorphisms of the identity functor on $\mathcal{C}$ (viewed as a symmetric monoidal category), and let $\mathcal{G}_A$ be an $A$-gerbe on $X$.

Using Sect. 1.7.2, we can twist $\mathcal{C}$ by $\mathcal{G}_A$ and obtain a new sheaf of symmetric monoidal categories over $X$, denoted $\mathcal{C}_{\mathcal{G}_A}$.

In particular, we have the symmetric monoidal category $\mathcal{C}_{\mathcal{G}_A}(X)$ over $\operatorname{Shv}(X)$.

8.2.3. Applying to $\mathcal{C}_{\mathcal{G}_A}$ the construction from Sect. 8.1, we obtain a new sheaf of symmetric monoidal categories over Ran, denoted $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}$.

In particular, we obtain the symmetric monoidal category $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(\operatorname{Ran})$.

Note that the value of $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}$ on $X$ under the canonical map $X \to \operatorname{Ran}$ is the symmetric monoidal category $\mathcal{C}_{\mathcal{G}_A}(X)$

8.2.4. Let now $\epsilon$ be a 2-torsion element of $A$. Then we can further twist $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}$ to obtain a factorization sheaf of symmetric monoidal DG categories, denoted $\operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$.

Namely, the element $\epsilon$ can be used to modify the braiding on $\mathcal{C}$ and thereby obtain a *new* symmetric monoidal category, denoted $\mathcal{C}^{\epsilon}$. We wet

$$\operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A} := \operatorname{Fact}(\mathcal{C}^{\epsilon})_{\mathcal{G}_A}.$$

A key feature of the latter twist is that we have a canonical isomorphism

$$\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A} \simeq \operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}, \tag{8.7}$$

as sheaves of *monoidal* categories over Ran. But this identification is *not* compatible with either the symmetric monoidal nor factorization structure.

*Remark* 8.2.5. At the level of underlying triangulated categories, the modification

$$\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A} \rightsquigarrow \operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$$

can be described as follows[13]:

We let $\operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$ be the same as $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}$ as a plain sheaf of monoidal categories. We define the factorization structure on $\operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$ as follows:

The action of $\epsilon$ on $\mathcal{C}$ defines a direct sum decomposition

$$\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(S) \simeq \operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)^1 \oplus \operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)^{-1}$$

for any $S \to \operatorname{Ran}$.

Hence, for $S \to \operatorname{Ran}^J$ we have a direct sum decomposition

$$(\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A})^{\otimes J}(S) \simeq \underset{\gamma^J : J \to \pm 1}{\oplus} (\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A})^{\otimes J}(S)^{\gamma^J}. \tag{8.8}$$

For a given $\gamma^J$, let $J_{-1} \subset J$ be the preimage of the element $-1 \in \pm 1$.

We define the factorization functor for $\operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}(S)$ and $S \to \operatorname{Ran}^J_{\text{disj}}$ to be equal to the one for $\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(S)$ on each factor of (8.8), for every choice of an ordering on $J_{-1}$. A change of ordering will result in multiplication by the sign character of the group of permutations of $J_{-1}$.

*Remark* 8.2.6. A general framework that performs both twistings

$$\operatorname{Fact}(\mathcal{C}) \rightsquigarrow \operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}$$

in one shot is explained in Sect. B.

The construction in *loc.cit.* also makes the identification (8.7) as plain sheaves of monoidal categories over Ran, manifest. In particular, we have an identification

$$\operatorname{Fact}(\mathcal{C})_{\mathcal{G}_A}(\operatorname{Ran}) \simeq \operatorname{Fact}(\mathcal{C})^{\epsilon}_{\mathcal{G}_A}(\operatorname{Ran}), \tag{8.9}$$

[13] However, it may not be so straightforward to perform this construction at the level of $\infty$-categories as it involves "explicit formulas".

as monoidal (but *not* symmetric monoidal) categories.

8.3. **Twisting the category of representations.** In this subsection we will introduce a factorization sheaf of symmetric monoidal categories on the Ran space, which will appear as the source of the metaplectic geometric Satake functor.

8.3.1. Let $H$ be an algebraic group. We apply the discussion in Sect. 8.2 to the pair

$$\mathcal{C} = \mathrm{Rep}(H), \quad A = Z_H(E)^{\mathrm{tors}}.$$

Thus, let $\mathcal{G}_Z$ be a gerbe (of finite order) on $X$ with respect to $Z_H$, and let $\epsilon$ be an element of order 2 in $Z_H$.

8.3.2. Thus, we obtain the symmetric monoidal category $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X)$, and sheaves of symmetric monoidal categories over Ran:

$$\mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z} \text{ and } \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z},$$

and a *monoidal* equivalence

$$\mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \simeq \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}). \tag{8.10}$$

The case of interest for us is when the triple $(H, \mathcal{G}_Z, \epsilon)$ is the metaplectic datum attached to a geometric metaplectic datum of a reductive group $G$.

8.3.3. *Example of tori.* Consider the particular case when $G = T$ is a torus, and we start with a factorization gerbe $\mathcal{G}^T$ on $\mathrm{Gr}_T$ that is multiplicative[14]. In this case,

$$\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$$

is naturally a sheaf of symmetric monoidal DG categories on Ran, equipped with a factorization structure.

Note also that by Proposition 4.3.2(a), we have $T^\sharp = T$, and so $H \simeq \check{T}$. It is straightforward to show explicitly (see [Re, Proposition IV.5.2]) that we have a canonical isomorphism

$$\mathrm{Fact}(\mathrm{Rep}(\check{T}))^{\epsilon}_{\mathcal{G}_Z} \simeq \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}} \tag{8.11}$$

as sheaves of factorization monoidal categories.

8.4. **Twisted local systems.** Let $(H, \mathcal{G}_Z)$ be as in Sect. 8.3. In this subsection we will introduce the notion of *twisted local system* for $(H, \mathcal{G}_Z)$.

8.4.1. By definition, a $\mathcal{G}_Z$-twisted local system on $X$ with respect to $H$ is a t-exact symmetric monoidal functor

$$\mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to \mathrm{Shv}(X),$$

where the t-structure on $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X)$ is one for which the forgetful functor $\mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to \mathrm{Shv}(X)$ is t-exact for the perverse t-structure on $\mathrm{Shv}(X)$.

In Sect. 9.5 we will formulate a precise relationship between twisted local systems in the above sense and objects appearing in the global metaplectic geometric theory.

*Remark* 8.4.2. Presumably, twisted local systems as defined above are the same as Galois representations into the metaplectic L-group, as defined in [We].

8.4.3. Let $\sigma$ be a twisted local system on $X$ as defined as above. The functoriality of the construction in Sect. 8.1 defines a symmetric monoidal functor

$$\mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Shv}(\mathrm{Ran}).$$

In particular, we obtain a *monoidal* functor

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Shv}(\mathrm{Ran}).$$

[14]Recall that "multiplicative"="commutative", see Remark 4.4.3.

8.4.4. Assume now that $X$ is complete. Composing with the functor of direct image

$$\mathrm{Shv}(\mathrm{Ran}) \to \mathrm{Vect},$$

we thus obtain a functor

$$\mathrm{Ev}_\sigma : \mathrm{Fact}(\mathrm{Rep}(H))^\epsilon_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Vect}\,. \tag{8.12}$$

We will use the functor (8.12) for the definition of the notion of *twisted Hecke eigensheaf* with respect to $\sigma$.

8.4.5. Again that $X$ is complete. We will now construct the *derived* stack $\mathrm{LocSys}_H^{\mathcal{G}_Z}$ of $\mathcal{G}_Z$-twisted local systems on $X$. Its $k$-points will be the twisted local systems as defined in Sect. 8.4.1.

We follow the strategy of [AG, Sect. 10.2]. For a derived affine scheme $S$, we set

$$\mathrm{Maps}(S, \mathrm{LocSys}_H^{\mathcal{G}_Z})$$

to be the space of *right t-exact* symmetric monoidal functors

$$\mathrm{Rep}(H)_{\mathcal{G}_Z}(X) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}(X).$$

One shows that $\mathrm{LocSys}_H^{\mathcal{G}_Z}$ defined in this way is representable by a quasi-smooth derived algebraic stack (see [AG, Sect. 8.1] for what this means).

8.4.6. As in [Ga5, Sect. 4.3], we have a canonically defined (symmetric) monoidal functor

$$\mathrm{Loc} : \mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right). \tag{8.13}$$

The following is proved in the same way as [Ga5, Proposition 4.3.4][15]:

**Proposition 8.4.7.** *The functor* (8.13) *is a* localization*, i.e., it admits a fully faithful right adjoint.*

## 9. Metaplectic geometric Satake

We take $G$ to be a reductive group. We will define the metaplectic geometric Satake functor and formulate the "metaplectic vanishing conjecture" about the global Hecke action.

9.1. **The metaplectic spherical Hecke category.** In this subsection we introduce the metaplectic spherical Hecke category, which is the recipient of the metaplectic geometric Satake functor.

9.1.1. Let $\mathcal{G}^G$ be a factorization $E^{\times,\mathrm{tors}}$-gerbe on $\mathrm{Gr}_G$. We define the sheaf of categories $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ as follows. For an affine test scheme $S$ and an $S$-point of Ran, we define the corresponding category by

$$\mathrm{Sph}_{\mathcal{G}^G}(S) := \mathrm{Shv}_{\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}|_S}(S \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_G)^{\mathfrak{L}^+(G)|_S}. \tag{9.1}$$

In the above formula, $\mathfrak{L}^+(G)|_S$ denotes the value on $S$ of the factorization group-scheme $\mathfrak{L}^+(G)$. The superscript $\mathfrak{L}^+(G)|_S$ indicates the equivariant category with respect to that group-scheme. Note that the latter makes sense due to the structure of equivariance on the gerbe $\mathcal{G}^G \otimes \det_{\mathfrak{g}}^{\frac{1}{2}}|_S$ with respect to $\mathfrak{L}^+(G)|_S$, which was constructed in Sect. 7.3.

By Proposition 7.2.5, we obtain that the operation of convolution product defines on $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ a structure of sheaf of *monoidal* categories over Ran.

By construction, $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ carries a natural factorization structure, see Sect. 2.2.3.

[15] The proof is reproduced in [Ro, Sect. 1.3].

9.1.2. Let $P$ be a parabolic subgroup of $G$ with Levi quotient $M$. Let us denote by $\mathcal{G}^M$ the factorization gerbe on $\mathrm{Gr}_M$ corresponding to $\mathcal{G}^G$.

The functor (5.6) naturally upgrades to a functor between sheaves of categories

$$J_M^G : (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}} \to (\mathrm{Sph}_{\mathcal{G}^M})_{/\,\mathrm{Ran}}. \tag{9.2}$$

By construction, (9.2) respects the factorization structure, i.e., it is a functor between factorization sheaves of categories.

*Remark* 9.1.3. We note that the functor (9.2) is *not at all* compatible with the monoidal structures!

9.2. **The metaplectic geometric Satake functor.** Metaplectic geometric Satake is a canonically defined functor between factorization sheaves of monoidal DG categories

$$\mathrm{Sat} : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}. \tag{9.3}$$

We will now explain how to obtain this functor from [Re, Theorem IV.8.3][16].

9.2.1. By Sect. 2.1.2, the datum of a functor (9.3) amounts to a compatible collection of functors

$$\mathrm{Sat}(I) : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I) \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I), \tag{9.4}$$

where $I$ runs over the category of finite non-empty sets and surjective morphisms.

Both sides in (9.4) are equipped with t-structures; moreover one shows that $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)$ identifies with the *derived category* of its t-structure[17], i.e., the canonical map of [Lu2, Theorem 1.3.3.2]

$$D\left(\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit}\right) \to \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)$$

is an equivalence.

Now, [Re, Theorem IV.8.3] constructs an *equivalence* of abelian categories

$$\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit} \to \left((\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I)\right)^{\heartsuit}. \tag{9.5}$$

Applying [Lu2, Theorem 1.3.3.2] again, we obtain a canonically defined functor

$$D\left(\left(\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(X^I)\right)^{\heartsuit}\right) \to (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}(X^I),$$

thus giving rise to the desired functor (9.4).

The functoriality with respect to the finite sets $I$, as well as compatibility with factorization is built into the construction.

9.3. **Example: metaplectic geometric Satake for tori.** In this subsection we let $G = T$ be a torus.

9.3.1. Let $\Lambda^\sharp \subset \Lambda$ denote the kernel of $b$.

Direct image along the inclusion

$$\mathrm{Gr}_{T^\sharp} \to \mathrm{Gr}_T \tag{9.6}$$

is a fully faithful functor

$$\mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}, \tag{9.7}$$

where we denote by $\mathcal{G}^{T^\sharp}$ the restriction of $\mathcal{G}^T$ along (9.6).

In this case, it follows from Sect. 7.4 that the forgetful functor

$$(\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T)_{/\,\mathrm{Ran}}$$

[16] For a more detailed discussion on how to carry out this extension see [Ras2, Sect. 6], where the classical (i.e., non-metaplectic situation) is considered, but for this step, there is no difference between the two cases.

[17] Here, the derived category is understood as a DG category, see [Lu2, Sect. 1.3.2].

factors through the essential image of (9.7), thereby giving rise to a functor

$$(\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}} \to \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}}, \tag{9.8}$$

compatible with the factorization structures.

9.3.2. Furthermore, since the action of $\mathfrak{L}^+(T)$ on $\mathrm{Gr}_T$ is trivial, the functor (9.8) admits a canonically defined right inverse

$$\mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \to (\mathrm{Sph}_{\mathcal{G}^T})_{/\,\mathrm{Ran}}, \tag{9.9}$$

which is *monoidal* and compatible with the factorization structures.

9.3.3. By Proposition 4.3.2(b), the factorization gerbe $\mathcal{G}^{T^\sharp}$ carries a canonical multiplicative structure. Recall the equivalence

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} \simeq \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp})_{/\,\mathrm{Ran}} \tag{9.10}$$

of (8.11).

The geometric Satake functor for $T$ is the composite of (9.10) and (9.9).

## 9.4. **Compatibility with Jacquet functors.**

9.4.1. A key feature of the assignment

$$\mathcal{G}^G \rightsquigarrow \mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes \mathbb{G}_m}$$

of Sect. 6.2.1 is compatibility with parabolics in the following sense.

Note that for a parabolic $P$ of $G$ with Levi quotient $M$, the corresponding reductive group $M^\sharp$ identifies with the Levi subgroup of $G^\sharp$, attached to the same subset of the Dynkin diagram.

We have a canonical surjection

$$\pi_{1,\mathrm{alg}}(M^\sharp) \to \pi_{1,\mathrm{alg}}(G^\sharp), \tag{9.11}$$

and the corresponding map of factorization Grassmannians

$$\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m} \to \mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}. \tag{9.12}$$

Let $\mathcal{G}^M$ be the factorization gerbe on $\mathrm{Gr}_M$ that corresponds to $\mathcal{G}^G$ under the map of Sect. 5.1.4. Then the multiplicative gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(M^\sharp)\otimes\mathbb{G}_m}$ attached to $\mathcal{G}^M$ by Sect. 6.2.1 identifies with the pullback with respect to (9.12) of the multiplicative gerbe $\mathcal{G}^{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ on $\mathrm{Gr}_{\pi_{1,\mathrm{alg}}(G^\sharp)\otimes\mathbb{G}_m}$ attached to $\mathcal{G}^G$.

9.4.2. Let $M_H$ be the standard Levi quotient in $H$ corresponding to standard Levi $M^\sharp$ of $G^\sharp$. Corresponding to (9.11) we have the inclusion

$$Z_H \to Z_{M_H}.$$

By the above, this inclusion is compatible with the corresponding datum of

$$\epsilon : \pm 1 \to Z_H(E),\ \epsilon : \pm 1 \to Z_{M_H}(E)$$

and the corresponding $Z_H$- and $Z_{M_H}$-gerbes on $X$ (we denote both by $\mathcal{G}_Z$).

Therefore, restriction along $M_H \to H$ defines a monoidal functor

$$\mathrm{Res}^G_M : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} \to \mathrm{Fact}(\mathrm{Rep}(M_H))^{\epsilon}_{\mathcal{G}_Z},$$

compatible with the factorization structures.

9.4.3. The key feature of the monoidal functor (9.3) is that it makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z} & \xrightarrow{\mathrm{Sat}} & (\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}} \\ {\scriptstyle \mathrm{Res}^G_M}\downarrow & & \downarrow{\scriptstyle J^G_M} \\ \mathrm{Fact}(\mathrm{Rep}(M_H))^{\epsilon}_{\mathcal{G}_Z} & \xrightarrow{\mathrm{Sat}} & (\mathrm{Sph}_{\mathcal{G}^M})_{/\,\mathrm{Ran}}, \end{array}$$

where $J^G_M$ is the Jacquet functor of (9.2).

9.5. **Global Hecke action.** In this subsection we will assume that $X$ is complete. We will define the notion of Hecke eigensheaf on $\mathrm{Bun}_G$ with respect to a given twisted local system.

9.5.1. Consider category of *global sections* of $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$ over Ran (see Sect. 1.6.6), denote it by

$$\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran}),$$

and note that it identifies with

$$\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(G)}.$$

As in [Ga5, Sect. 4.4], the monoidal structure on $(\mathrm{Sph}_{\mathcal{G}^G})_{/\,\mathrm{Ran}}$, and the operation of union of finite sets, define a (non-unital) monoidal structure on $\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran})$.

Moreover, the Hecke action defines a monoidal action of $\mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran})$ on $\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$, where by a slight abuse of notation we denote by the same symbols $\mathcal{G}^G$ and $\det_{\mathfrak{g}}^{\frac{1}{2}}$ the corresponding $E^{\times,\mathrm{tors}}$-gerbes on $\mathrm{Bun}_G$, see Sect. 2.3.5.

9.5.2. Passing to global sections over Ran in (9.3), we obtain a monoidal functor

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Sph}_{\mathcal{G}^G}(\mathrm{Ran}),$$

where we remind that $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ denotes the monoidal category of global sections of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}$.

Thus, we obtain a monoidal action of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ on $\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$.

9.5.3. *Hecke eigensheaves.* Let $\sigma$ be a twisted local system on $X$, as defined in Sect. 8.4.1. Recall (see Sect. 8.4.3) that $\sigma$ gives rise to a (symmetric) monoidal functor

$$\mathrm{Ev}_\sigma : \mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Vect},$$

and hence, via the monoidal equivalence (8.10) to a monoidal functor

$$\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{Vect},$$

which we will denote by the same symbol $\mathrm{Ev}_\sigma$.

We define the category of twisted Hecke eigensheaves with respect to $\sigma$ to be the DG category of functors of $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$-module categories

$$\mathrm{Vect} \to \mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G),$$

where $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ acts on Vect via $\mathrm{Ev}_\sigma$ and on $\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ as in Sect. 9.5.2.

9.6. **The metaplectic vanishing conjecture.** We continue to assume that $X$ is complete. In this subsection we will assume that $k$ has characteristic 0, and that our sheaf theory is that of D-modules. Recall (see Sect. 8.4) that in this case we have the (derived) stack $\mathrm{LocSys}_H^{\mathcal{G}_Z}$.

We will state a conjecture to the effect that the (non-unital) monoidal category

$$\mathrm{QCoh}(\mathrm{LocSys}_H^{\mathcal{G}_Z})$$

acts on the category

$$\mathrm{Shv}_{\mathcal{G}^G\otimes\det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G).$$

9.6.1. Recall (see Proposition 8.4.7) that we have a (symmetric) monoidal functor

$$\mathrm{Loc} : \mathrm{Fact}(\mathrm{Rep}(H))_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$$

of (8.13) with a fully faithful right adjoint. Hence, by (8.10), we obtain a monoidal functor, denoted by the same symbol

$$\mathrm{Loc} : \mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran}) \to \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right),$$

also with a fully faithful right adjoint.

The following is an analog of [Ga5, Theorem 4.5.2] in the metaplectic case:

**Conjecture 9.6.2.** *If an object of* $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ *lies in the kernel of the functor* Loc*, then this object acts by zero on* $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$.

This conjecture can be restated as follows:

**Conjecture 9.6.3.** *The action of* $\mathrm{Fact}(\mathrm{Rep}(H))^{\epsilon}_{\mathcal{G}_Z}(\mathrm{Ran})$ *on* $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ *(uniquely) factors through an action of* $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

*Remark* 9.6.4. Using Fourier-Mukai transform, one can show that Conjecture 9.6.2 holds when $G = T$ is a torus, see [Lys].

9.6.5. Let us assume Conjecture 9.6.3, so that $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ becomes a module category over $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

As in the classical (i.e., non-metaplectic case), one expects that $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ is "almost" free of rank one, and the "almost" has to do with temperedness.

More precisely, one expects that the metaplectic geometric Satake functor (9.3) extends to a *derived metaplectic geometric Satake equivalence*, generalizing [Ga5, Sects. 4.6 and 4.7], which one can use in order to define the *tempered part* of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$, as in [AG, Sect. 12.8].

Now, one expects that the tempered subcategory of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ *is* free of rank one as a module over $\mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$.

However, it is not clear whether this module admits a distinguished generator.

9.6.6. Furthermore, one expects that the entire $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ is *non-canonically* equivalent to the category $\mathrm{IndCoh}_{\mathrm{nilp}}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right)$, where we refer the reader to [AG, Sect. 11.1] for the $\mathrm{IndCoh}_{\mathrm{nilp}}$ notation.

9.6.7. When $G = T$ is a torus, we have

$$\mathrm{IndCoh}_{\mathrm{nilp}}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right) = \mathrm{QCoh}\left(\mathrm{LocSys}_H^{\mathcal{G}_Z}\right).$$

In particular, the equivalence of Sect. 9.6.6 says that for each $\sigma \in \mathrm{LocSys}_H^{\mathcal{G}_Z}$, the corresponding category of Hecke eigensheaves is non-canonically equivalent to Vect. This equivalence can be made explicit as follows (see [Lys] for more details):

A point $\sigma \in \mathrm{LocSys}_H^{\mathcal{G}_Z}$ gives rise to a trivialization of the pullback of the gerbe $\mathcal{G}^T$ from $\mathrm{Bun}_T$ to $\mathrm{Bun}_{T^\sharp}$. Hence, it gives rise to a central extension

$$1 \to E^\times \to \mathrm{Heis}_\sigma \to \mathrm{Bun}_{\ker(T^\sharp \to T)} \to 1,$$

which is easily seen to be of Heinsenberg type, i.e., corresponding to a *non-degenerate* symplectic form on $\ker(T^\sharp \to T)$ with values in $E^\times$.

The category of Hecke eigensheaves with respect to $\sigma$ is *canonically* equivalent to

$$(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Bun}_T))^{\mathrm{Bun}_{T^\sharp}},$$

where the $\mathrm{Bun}_{T^\sharp}$-equivariance makes sense due to the above trivialization of $\mathcal{G}|_{\mathrm{Bun}_{T^\sharp}}$. This category is *canonically* equivalent to the category of representations of $\mathrm{Heis}_\sigma$, on which $E^\times$ acts by the standard character.

Since $\mathrm{Heis}_\sigma$ is of Heinsenberg type, the above category is *non-canonically* equivalent to Vect.

9.6.8. At the moment, we *do not* have a conjecture as to how to explicitly describe the category of Hecke eigensheaves in the tempered subcategory of $\mathrm{Shv}_{\mathcal{G}^G\otimes \det_{\mathfrak{g}}^{\frac{1}{2}}}(\mathrm{Bun}_G)$ with respect to a given $\sigma$ for a general reductive $G$.

## Appendix A. Calculation of the étale cohomology of $B(G)$

A.1. **The Leray spectral sequence.** The calculation is based on considering the Leray spectral sequence associated with the projection

$$\pi : B(B) \to B(G),$$

where $B \subset G$ is the Borel subgroup.

Namely, let $\underline{A}$ denote the constant étale sheaf on either $B(G)$ or $B(B)$ with coefficients in $A$, and let us consider the exact triangle

$$\underline{A} \to \pi_*(\underline{A}) \to \tau^{\geq 1}(R\pi_*(\underline{A})). \tag{A.1}$$

We note that each individual cohomology sheaf $R^i\pi_*(\underline{A})$ is constant with fiber $H^i_{\mathrm{et}}(G/B, A)$.

Note also that the projection $B(B) \to B(T)$ defines an isomorphism an étale cohomology, so we obtain:

$$H^i_{\mathrm{et}}(B(B), A) \simeq H^i_{\mathrm{et}}(B(T), A) \simeq \begin{cases} 0 \text{ for } i \text{ odd;} \\ \mathrm{Hom}(\Lambda, A(-1)) \text{ for } i = 2; \\ \mathrm{Quad}(\Lambda, A(-2)) \text{ for } i = 4. \end{cases} \tag{A.2}$$

A.2. **Cohomology in degrees $\leq 3$.** From the long exact cohomology sequence associated with (A.1) we immediately obtain that $H^1_{\mathrm{et}}(B(G), A) = 0$.

Next, the fact that $H^1_{\mathrm{et}}(G/B, A) = 0$ implies that the map

$$H^2_{\mathrm{et}}(B(G), A) \to H^2_{\mathrm{et}}(B(B), A)$$

is injective with image equal to the kernel of the map

$$H^2_{\mathrm{et}}(B(B), A) \to H^2_{\mathrm{et}}(G/B, A). \tag{A.3}$$

We identify $H^2_{\mathrm{et}}(G/B, A) = \mathrm{Hom}(\Lambda_{\mathrm{sc}}, A(-1))$, where $\Lambda_{\mathrm{sc}}$ is the coroot lattice in $\Lambda$, and the map (A.3) becomes the restriction map

$$\mathrm{Hom}(\Lambda, A(-1)) \to \mathrm{Hom}(\Lambda_{\mathrm{sc}}, A(-1)). \tag{A.4}$$

Since $\pi_{1,\mathrm{alg}}(G) = \Lambda/\Lambda_{\mathrm{sc}}$, we obtain the desired identification

$$H^2_{\mathrm{et}}(B(G), A) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), A).$$

Now, since $A$ was assumed divisible, the map (A.4) is surjective. Hence, the map

$$H^3_{\mathrm{et}}(B(G), A) \to H^3_{\mathrm{et}}(B(B), A)$$

is injective. Since $H^3_{\mathrm{et}}(B(B), A) = 0$, we obtain the desired $H^3_{\mathrm{et}}(B(G), A) = 0$.

A.3. **Cohomology in degree 4: injectivity.** We will now show that the map

$$H^4_{\text{et}}(B(G), A) \to H^4_{\text{et}}(B(B), A)$$

is injective.

For this, it suffices to show that

$$H^3_{\text{et}}(B(G), \tau^{\geq 1}(R\pi_*(\underline{A}))) = 0.$$

Since, $H^3_{\text{et}}(G/B, A) = 0$, we have

$$H^3_{\text{et}}(B(G), \tau^{\geq 1}(R\pi_*(\underline{A}))) = H^1_{\text{et}}(B(G), H^2_{\text{et}}(G/B, A)),$$

and the latter vanishes as $H^1_{\text{et}}(B(G), -) = 0$.

Thus, we obtain an injection

$$H^4_{\text{et}}(B(G), A) \hookrightarrow H^4_{\text{et}}(B(B), A) \simeq H^4_{\text{et}}(B(T), A) \simeq \text{Quad}(\Lambda, A(-2)),$$

and out task is to show that its image equals $\text{Quad}(\Lambda, A)^W_{\text{restr}}$.

A.4. **Containment in one direction.** We will first show that the image of $H^4_{\text{et}}(B(G), A)$ in $\text{Quad}(\Lambda, A(-2))$ is contained in $\text{Quad}(\Lambda, A(-2))^W_{\text{restr}}$.

For this, it suffices to show that for any $q \in \text{Quad}(\Lambda, A(-2))$ that lies in the image of the above map, and any *simple* coroot $\alpha_i$, we have

$$s_i(q) = q \text{ and } b(\alpha_i, \lambda) = \langle \check{\alpha}_i, \lambda\rangle \cdot q(\alpha_i) \text{ for any } \lambda \in \Lambda.$$

Let $P_i$ be the subminimal parabolic associated with $i$, and let $M_i$ be its Levi quotient. We have a commutative diagram

$$\begin{array}{ccc}
H^4_{\text{et}}(B(M_i), A) & \longrightarrow & H^4_{\text{et}}(B(B), A) \\
\sim\downarrow & & \downarrow= \\
H^4_{\text{et}}(B(P_i), A) & \longrightarrow & H^4_{\text{et}}(B(B), A) \\
\uparrow & & \uparrow= \\
H^4_{\text{et}}(B(G), A) & \longrightarrow & H^4_{\text{et}}(B(B), A),
\end{array}$$

which implies that it is sufficient to prove our claim for $G$ replaced by $M_i$, which is a reductive group of semi-simple rank 1.

A.5. **Calculation for groups of semi-simple rank** 1. Any group $G$ of semi-simple rank 1 is of the form

$$G' \times T',$$

where $G'$ is $SL_2$, $PGL_2$ or $GL_2$ and $T'$ is a torus.

If $G' = SL_2$, then

$$H^4_{\text{et}}(B(G), A) \simeq H^4_{\text{et}}(B(G'), A) \oplus H^4_{\text{et}}(B(T'), A) \simeq A(-2) \oplus \text{Quad}(\Lambda_{T'}, A(-2)).$$

Similarly, in this case, it is easy to see that in this case

$$\text{Quad}(\Lambda_G, A)^W_{\text{restr}} = \text{Quad}(\Lambda_{G'}, A)^W_{\text{restr}} \oplus \text{Quad}(\Lambda_{T'}, A) = A \oplus \text{Quad}(\Lambda_{T'}, A),$$

and the assertion follows.

In the two cases of $G' = PGL_2$ or $G' = GL_2$, it is easy to see that the inclusion

$$\text{Quad}(\Lambda_G, A)^W_{\text{restr}} \subset \text{Quad}(\Lambda_G, A)^W$$

is an equality, and there is nothing to prove.

A.6. **The opposite containment.** It remains to show that any element $q \in \mathrm{Quad}(\Lambda, A(-2))^W_{\mathrm{restr}}$ lies in the image of $H^4_{\mathrm{et}}(B(G), A)$ in $\mathrm{Quad}(\Lambda, A(-2))$.

According to Sect. 3.2.2:

(I) $q$ lies in the image of the map

$$\mathrm{Quad}(\Lambda, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-2) \to \mathrm{Quad}(\Lambda, A(-2))^W_{\mathrm{restr}}.$$

(II) $q$ comes from a quadratic form on $\pi_{1,\mathrm{alg}}(G)$.

We first deal with case II. Let $T_{\mathrm{sc}}$ be the Cartan of the simply connected cover of the derived group of $G$, so that $\Lambda_{\mathrm{sc}}$ is the coweight lattice of $T_{\mathrm{sc}}$. Consider the (2)-stack $B(T)/B(T_{\mathrm{sc}})$. We have a canonical isomorphism

$$H^4_{\mathrm{et}}(B(T)/B(T_{\mathrm{sc}}), A) \simeq \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A(-2))$$

which fits into the commutative diagram

$$\begin{array}{ccc} H^4_{\mathrm{et}}(B(T), A) & \xrightarrow{\sim} & \mathrm{Quad}(\Lambda, A(-2)) \\ \uparrow & & \uparrow \\ H^4_{\mathrm{et}}(B(T)/B(T_{\mathrm{sc}}), A) & \xrightarrow{\sim} & \mathrm{Quad}(\pi_{1,\mathrm{alg}}(G), A(-2)). \end{array}$$

Now the desired containment follows from the commutative diagram

$$\begin{array}{ccccc} H^4_{\mathrm{et}}(B(G), A) & \longrightarrow & H^4_{\mathrm{et}}(B(B), A) & \xleftarrow{\sim} & H^4_{\mathrm{et}}(B(T), A) \\ \uparrow & & & & \uparrow \mathrm{id} \\ H^4_{\mathrm{et}}(B(T)/B(T_{\mathrm{sc}}), A) & & \longrightarrow & & H^4_{\mathrm{et}}(B(T), A), \end{array}$$

where the left vertical arrow comes from the canonical projection $B(G) \to B(T)/B(T_{\mathrm{sc}})$.

In order to deal with case I, it suffices to show that for any $\ell$ comprime with $\mathrm{char}(k)$, the map

$$H^4_{\mathrm{et}}(B(G), \mathbb{Z}_\ell) \to \mathrm{Quad}(\Lambda, \mathbb{Z}_\ell(-2))^W$$

is an isomorphism.

A.7. **Computation of the integral cohomology.** From the long exact cohomology sequence associated with (A.1), we obtain that the image of

$$H^4_{\mathrm{et}}(B(G), \mathbb{Z}_\ell) \to H^4_{\mathrm{et}}(B(T), \mathbb{Z}_\ell) \tag{A.5}$$

equals

$$\ker\left(\ker(H^4_{\mathrm{et}}(B(T), \mathbb{Z}_\ell) \to H^4_{\mathrm{et}}(G/B, \mathbb{Z}_\ell)) \to H^2_{\mathrm{et}}(B(G), H^2_{\mathrm{et}}(G/B, \mathbb{Z}_\ell))\right).$$

Since both groups $H^4_{\mathrm{et}}(G/B, \mathbb{Z}_\ell)$ and

$$H^2_{\mathrm{et}}(B(G), H^2_{\mathrm{et}}(G/B, \mathbb{Z}_\ell)) \simeq \mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), H^2_{\mathrm{et}}(G/B, \mathbb{Z}_\ell))$$

are torsion-free, we obtain that the image of (A.5) equals

$$H^4_{\mathrm{et}}(B(T), \mathbb{Z}_\ell) \cap \mathrm{Im}(H^4_{\mathrm{et}}(B(G), \mathbb{Q}_\ell) \to H^4_{\mathrm{et}}(B(T), \mathbb{Q}_\ell).$$

However,

$$H^4_{\mathrm{et}}(B(T), \mathbb{Z}_\ell) \simeq \mathrm{Quad}(\Lambda, \mathbb{Z}_\ell(-2)),$$

and rationally, we know that

$$H^4_{\mathrm{et}}(B(G), \mathbb{Q}_\ell) \simeq \mathrm{Quad}(\Lambda, \mathbb{Q}_\ell(-2))^W.$$

Hence, the image of (A.5) equals

$$\mathrm{Quad}(\Lambda, \mathbb{Z}_\ell(-2)) \cap \mathrm{Quad}(\Lambda, \mathbb{Q}_\ell(-2))^W = \mathrm{Quad}(\Lambda, \mathbb{Z}_\ell(-2))^W,$$

as desired.

## Appendix B. Twisting of factorization categories by gerbes

B.1. **The context.** Let $\mathcal{C}$ be a symmetric monoidal category, and let $A$ be a torsion abelian group that acts by automorphisms of the identity functor on $\mathcal{C}$ (viewed as a symmetric monoidal functor). We will assume that the orders of elements in $A$ are co-prime with $\text{char}(k)$.

Up to passing to a colimit, we can write

$$A = \text{Hom}(\Gamma, E^{\times,\text{tors}}),$$

where $\Gamma$ is finitely generated group.

By Sect. 4.5.5, we can think of a pair $(\mathcal{G}_A \in \text{Ge}_A(X), \epsilon \in A_{2\text{-tors}})$ as a multiplicative factorization gerbe $\mathcal{G}$ on $\text{Gr}_{\Gamma\otimes\mathbb{G}_m}$ with respect to $E^{\times,\text{tors}}$. (Recall also that the multiplicative structure on $\mathcal{G}$ automatically lifts to a commutative one, see Remark 4.4.3.)

We will show how to perform a twist of $\text{Fact}(\mathcal{C})$ by means of $\mathcal{G}$ and obtain a new sheaf of symmetric monoidal categories over Ran, denoted $\text{Fact}(\mathcal{C})_{\mathcal{G}}$, equipped with a factorization structure.

This construction will contain both twisting constructions, mentioned in Sects. 8.2.3 and 8.2.4, respectively.

B.2. **Two symmetric monoidal structures on** $\text{Rep}(A)$**.** Consider the category $\text{Rep}(A)$ of representations of $A$ on $E$-vector spaces. Note that it is semi-simple: every representation $V$ canonically splits as

$$V \simeq \underset{\gamma\in\Gamma}{\oplus} V_\gamma \otimes E^\gamma,$$

where $V_\gamma$ are vector spaces and $E^\gamma$ is the 1-dimensional representation of $A$ corresponding to the character $A \overset{\gamma}{\to} E^{\times,\text{tors}}$.

The category $\text{Rep}(A)$ has two symmetric monoidal structures. One is given by the usual tensor product of $A$-representations (we denote it by $\otimes$). The other is given by induction along the diagonal map $A \times A \to A$ (we denote it by $*$). Explicitly,

$$k^{\gamma_1} * k^{\gamma_2} = \begin{cases} k^\gamma \text{ if } \gamma_1 = \gamma = \gamma_2, \\ 0 \text{ if } \gamma_1 \neq \gamma_2. \end{cases}$$

Note that these two symmetric monoidal structures are *lax-compatible* in the sense that there exists a natural transformation

$$(V_1 \star W_1) \otimes (V_2 \star W_2) \to (V_1 \otimes V_2) \star (W_1 \otimes W_2)$$

satisfying a homotopy-coherent system of compatibilities.

Note also that the action of $A$ on $\mathcal{C}$ gives rise to an action of $\text{Rep}(A)$, equipped with the $*$ monoidal structure, on $\mathcal{C}$. Explicitly, every object $c\,\mathcal{C}$ can be canonically written as

$$c \simeq \underset{\gamma\in\Gamma}{\oplus} c^\gamma,$$

where $A$ acts on $c^\gamma$ according to the character $\gamma$, and $k^\gamma$ acs on $c$ as a projector on $c^\gamma$.

This action is lax-compatible with the symmetric monoidal structure on $\mathcal{C}$ in the sense that we have a natural transformation

$$(V_1 \star c_1) \otimes (V_2 \star c_2) \to (V_1 \otimes V_2) \star (c_1 \otimes c_2),$$

satisfying a homotopy-coherent system of compatibilities.

B.3. **Creating factorization categories.** The assignment

$$\mathcal{C} \mapsto \text{Fact}(\mathcal{C}) \tag{B.1}$$

is functorial with respect to lax symmetric monoidal functors.

Hence, we obtain that $\text{Fact}(\text{Rep}(A)^{\otimes})$ (i.e., the sheaf of symmetric monoidal categories obtained from $\text{Rep}(A)$ in the $\otimes$ symmetric monoidal structure) acquires another symmetric monoidal structure given by $*$, which is lax-compatible with one given by $\otimes$.

Similarly, the action of $\text{Rep}(A)$ on $\mathcal{C}$ implies that $\text{Fact}(\text{Rep}(A)^{\otimes})$, viewed as a sheaf of monoidal categories over Ran with respect to $*$, acts on $\text{Fact}(\mathcal{C})$. This action is lax-compatible with $\otimes$-symmetric monoidal structure on $\text{Fact}(\text{Rep}(A)^{\otimes})$, and the given one on $\text{Fact}(\mathcal{C})$.

B.4. **Relation to the affine Grassmannian.** We note now that there exists a canonical equivalence of sheaves of categories over Ran

$$\text{Fact}(\text{Rep}(A)^{\otimes}) \simeq \text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}, \tag{B.2}$$

compatible with the factorization structures.

Under this equivalence, the $\otimes$-symmetric monoidal structure on $\text{Fact}(\text{Rep}(A)^{\otimes})$ corresponds to the symmetric monoidal structure on $\text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}$ given by convolution along the group structure on $\text{Gr}_{\Gamma\otimes\mathbb{G}_m}$. The $*$-symmetric monoidal structure on $\text{Fact}(\text{Rep}(A)^{\otimes})$ corresponds to the symmetric monoidal structure on $\text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}$ given by pointwise !-tensor product.

B.5. **The twisting construction.** We obtain that $\text{Fact}(\mathcal{C})$ acquires an action of $\text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}$, viewed as a symmetric monoidal category with respect to the pointwise !-tensor product.

Now, using Theorem 1.6.9, we obtain that we can upgrade $\text{Fact}(\mathcal{C})$ to a sheaf of categories over $\text{Gr}_{\Gamma\otimes\mathbb{G}_m}$, compatible with the factorization structure.

Hence, the construction of Sect. 1.7.2 allows to twist $\text{Fact}(\mathcal{C})$ by any factorization $E^{\times,\text{tors}}$-gerbe $\mathcal{G}$ on $\text{Gr}_{\Gamma\otimes\mathbb{G}_m}$, and obtain another factorization sheaf of categories, to be denoted $\text{Fact}(\mathcal{C})_{\mathcal{G}}$.

Since the action of $\text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}$ on $\text{Fact}(\mathcal{C})$ is lax-compatible with the symmetric monoidal structure on $\text{Shv}(\text{Gr}_{\Gamma\otimes\mathbb{G}_m})_{/\text{Ran}}$ given by convolution and the existing symmetric monoidal structure on $\text{Fact}(\mathcal{C})$, if $\mathcal{G}$ carries a *commutative* structure with respect to the group structure on $\text{Gr}_{\Gamma\otimes\mathbb{G}_m}$, the twisted sheaf of categories $\text{Fact}(\mathcal{C})_{\mathcal{G}}$ carries a symmetric monoidal structure.

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