FUNCTORS GIVEN BY KERNELS, ADJUNCTIONS AND DUALITY

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Contents

Date: November 24, 2015.

Introduction

0.1. **The goals of this paper.** The goal of this paper is two-fold. One is to explain a certain phenomenon pertaining to adjoint functors between DG categories of D-modules on schemes of finite type. Two is to explain what this phenomenon generalizes to when instead of schemes we consider Artin stacks.

0.1.1. We begin by describing the situation with schemes.

We will be working over a ground field k of characteristic 0. By a scheme we shall mean a scheme of finite type over X.

For a scheme X we let D-mod(X) the DG category of D-modules on X; we refer the reader to [DrGa1, Sect. 5], where the basic properties of this category are discussed. In particular, the category D-mod(X) is compactly generated; the corresponding subcategory D-mod$(X)^c$ of compact objects identifies with D-mod$(X)_{\text{coh}}$ of cohomologically bounded objects with coherent cohomologies.

Let X_1 and X_2 be a pair of schemes, and let F be an (exact) functor

$$\text{D-mod}(X_1) \to \text{D-mod}(X_2).$$

Assume that F is *continuous*, i.e., commutes with colimits (which is equivalent to commuting with infinite direct sums). The (DG) category of such functors is equivalent to the category D-mod$(X_1 \times X_2)$. Namely, an object $\mathcal{Q} \in$ D-mod$(X_1 \times X_2)$ gives rise to the functor

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2), \quad \mathcal{M} \mapsto \text{pr}_{2\bullet}(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}).$$

Here for a morphism f, we denote by $f_\bullet$ the de Rham direct image functor between the corresponding DG categories of D-modules, and $\overset{!}{\otimes}$ is the usual tensor product functor on the DG category of D-modules on a scheme. We refer the reader to Sect. 1.2.2 for details.

In what follows we shall say that the functor $\mathsf{F} = \mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ is given by the kernel $\mathcal{Q}$.

0.1.2. It is a general theorem in the theory of DG categories that a functor F as above admits a right adjoint. However, this right adjoint need not be continuous. In fact, by Lemma 1.1.4, the right adjoint in question is continuous if and only if the functor F preserves compactness, i.e., maps $\text{D-mod}(X_1)^c$ to $\text{D-mod}(X_2)^c$.

Let us, however, assume that the right adjoint of F, denoted F^R, is continuous. Then, by the above, it is also given by a kernel

$$\mathcal{P} \simeq \text{D-mod}(X_2 \times X_1) = \text{D-mod}(X_1 \times X_2).$$

The question that we would like to address is the following: can we explicitly relate the kernels of F and F^R?

0.1.3. Before we give the answer in general, we consider the following well-known example (more details on this example are supplied in Sect. 1.8). For a k-vector space V considered as a scheme, take $X_1 = V$ and $X_2 = V^\vee$. We let F be the Fourier-Deligne transform functor $\text{D-mod}(V) \to \text{D-mod}(V^\vee)$. It is given by the kernel that we symbolically denote by

$$\exp \in \text{D-mod}(V \times V^\vee),$$

equal to the pullback of the exponential D-module on $\mathbb{A}^1$ under the evaluation map

$$V \times V^\vee \to \mathbb{A}^1.$$

We normalize exp so that it lives in cohomological degree $-\dim(V)$.

As is well-known, F is an equivalence of categories. Its inverse (and hence also the right adjoint) is the Fourier-Deligne transform $\text{D-mod}(V^\vee) \to \text{D-mod}(V)$, given by the kernel

$$-\exp := \mathbb{D}^{\text{Ve}}_{V \times V^\vee}(\exp)[2\dim(V)],$$

where $\mathbb{D}^{\text{Ve}}_X$ denotes the functor of Verdier duality on a scheme X.

0.1.4. The assertion of our main theorem in the case when X_1 is smooth and separated scheme is that the above phenomenon is not specific to the Fourier-Deligne transform, but holds for any functor F that preserves compactness. In fact, this generalization was one of the main initial motivations for this paper.

Namely, Theorem 1.3.4 says that the kernel $\mathcal{P}$ defining F^R is related to $\mathcal{Q}$ by the following formula:

$$\mathcal{P} = \mathbb{D}^{\text{Ve}}_{X_1 \times X_2}(\mathcal{Q})[2n_1], \tag{0.1}$$

where $n_1 = \dim(X_1)$, and where we remind that $\mathbb{D}^{\text{Ve}}_{X_1 \times X_2}$ is the Verdier duality functor on $\text{D-mod}(X_1 \times X_2)$.

As we will remark in Sect. 1.7.4, from (0.1) we obtain the following isomorphism of functors

$$\text{pr}_{2!}(\text{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{Q})[-2n_1] \xrightarrow{\sim} \text{pr}_{2\bullet}(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}) = \mathsf{F}(M), \tag{0.2}$$

where the functors $\text{pr}_{2!}$ and $\overset{\bullet}{\otimes}$ a priori take values in the correponding pro-categories [1] (so, in particular, we obtain that the right-hand side in (0.2) takes values in $\text{D-mod}(X_2)$).

[1]As our D-modules are not neccesarily holonomic, for a morphism f, only the functors $f^!$ and $f_\bullet$ are defined, whereas their respective left adjoints $f_!$ and $f^\bullet$ take values in the corresponding pro-category.

0.1.5. Let us consider several most basic examples of the isomorphisms (0.1) and (0.2). In all these examples we will be assuming that X_1 is smooth of dimension n_1 and separated.

(i) Proper pushforward. Let F be the functor $f_\bullet$, where $f : X_1 \to X_2$ is a map. In this case

$$\mathcal{Q} = (\mathrm{id}_{X_1} \times f)_\bullet(\omega_{X_1}) \in \text{D-mod}(X_1 \times X_2),$$

where ω_X denotes the dualizing complex on a scheme X, and $\mathrm{id}_{X_1} \times f$ is the graph of the map f. We have $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})[2n_1] \simeq \mathcal{Q}$, so

$$\mathsf{F}_{X_2\to X_1,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})[2n_1]} \simeq f^!.$$

Assume that f is proper. In this case $f_\bullet$ preserves compactness. So (0.1) expresses the fact that for f proper, we have $f_\bullet \simeq f_!$ and hence $f^!$ is the right adjoint of $f_\bullet$.

(ii) Smooth pullback. Let F be the functor $f^!$, where $f : X_2 \to X_1$ is a *smooth* map; in particular, the functor $f^!$ preserves compactness. Note that since X_1 and f are smooth, X_2 is also smooth. We have

$$\mathcal{Q} = (f \times \mathrm{id}_{X_2})_\bullet(\omega_{X_2}) \simeq (f \times \mathrm{id}_{X_2})_!(k_{X_2})[2n_2],$$

so

$$\mathrm{pr}_{2!}(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{Q})[-2n_1] \simeq f^\bullet(\mathcal{M})[2(n_2 - n_1)].$$

Thus, (0.2) amounts to the isomorphism

$$f^!(\mathcal{M}) \simeq f^\bullet(\mathcal{M})[2(n_2 - n_1)],$$

which is valid since f is smooth.

(iii) Tensor product by a lisse D-module. Let $X_1 = X_2 = X$, and let F be the functor $\mathcal{M} \mapsto \mathcal{M}_0 \overset{!}{\otimes} \mathcal{M}$, where $\mathcal{M}_0$ is a lisse D-module on X. In this case $\mathcal{Q} = (\Delta_X)_\bullet(\mathcal{M}_0)$. The right adjoint to F is given by tensor product with the D-module $\mathbb{D}^{\mathrm{Ve}}_X(\mathcal{M}_0)[2n]$, which is the assertion of (0.1).

0.1.6. *The ULA property.* The next example may be less familiar. Let X_1 be as above, and let $f : X_2 \to X_1$ be a smooth map. Let $\mathcal{N}$ be an object in D-mod(X_2). We consider the functor

$$\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2), \quad \mathsf{F}(\mathcal{M}) = f^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{N}.$$

We shall say that $\mathcal{N}$ is *universally locally acyclic* (ULA) with respect to f if the functor F preserves compactness.

If $\mathcal{N}$ is ULA with respect to f, (0.2) says that there is a canonical isomorphism:

$$f^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{N}[-2n_1] \simeq f^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{N}.$$

0.2. **The case of Artin stacks.**

0.2.1. We now let $\mathcal{X}_1$ and $\mathcal{X}_2$ be a pair of quasi-compact Artin stacks, locally of finite type over k. We shall require that both $\mathcal{X}_1$ and $\mathcal{X}_2$ be QCA in the sense of [DrGa1]. This means that the automorphism group of any field-valued point is affine.

The category $\text{D-mod}(\mathcal{X})$ is defined for any *prestack* (see [DrGa1, Sect. 6.1]), and in particular for any Artin stack. When $\mathcal{X}$ is a QCA Artin stack, [DrGa1, Theorem 8.1.1] says that the category $\text{D-mod}(\mathcal{X})$ is compactly generated. Moreover, it is self-dual (see Sect. 6.1.2 for what this means).

This implies that the DG category of continuous functors

$$\mathsf{F} : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$$

is equivalent to $\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$. Namely, to $\mathcal{Q} \in \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$ we assign the functor $\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}}$ given by

$$\mathcal{M} \mapsto \text{pr}_{2\blacktriangle}(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}).$$

Here for a morphism f between QCA stacks we denote by $f_\blacktriangle$ the functor of *renormalized direct image*, introduced in [DrGa1, Sect. 9.3], and reviewed in Sect. 6.1.3. Here we will just say that $f_\blacktriangle$ is a colimit-preserving version of $f_\bullet$.

0.2.2. We now ask the same question as in the case of schemes: let $\mathsf{F} : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$ be a continuous functor. Assume that the right adjoint of F is also continuous (i.e., F preserves compactness). What is the relationship between the kernel F and the kernel F^R?

The answer turns out much more interesting than in the case of smooth separated schemes. To formulate it we introduce a certain endo-functor

$$\text{Ps-Id}_\mathcal{X} : \text{D-mod}(\mathcal{X}) \to \text{D-mod}(\mathcal{X})$$

defined for any QCA stack $\mathcal{X}$. Namely, $\text{Ps-Id}_\mathcal{X}$ is given by the kernel

$$(\Delta_\mathcal{X})_!(k_\mathcal{X}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X}),$$

where $\Delta_\mathcal{X}$ is the diagonal morphism for $\mathcal{X}$, and $k_\mathcal{X}$ is the "constant sheaf" on $\mathcal{X}$, i.e., the Verdier dual of the dualizing complex $\omega_\mathcal{X}$.

The main theorem for QCA stacks asserts that there is a canonical isomorphism

$$\text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R \simeq \mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}. \tag{0.3}$$

I.e., what for a smooth separated cheme X was the functor of cohomological shift by $[-2\dim(X)]$, for a QCA stack becomes the functor $\text{Ps-Id}_\mathcal{X}$.

0.2.3. The idea of considering the functor $\text{Ps-Id}_\mathcal{X}$ was suggested by V. Drinfeld.

We should also point out, that the nature of $\text{Ps-Id}_\mathcal{X}$ (and the kernel $(\Delta_\mathcal{X})_!(k_\mathcal{X})$, by which it is defined) is not a special feature of categories of D-modules. Rather, it comes from a certain general manipilation that makes sense for an arbitrary compactly generated DG category, see Sect. 5.2.

0.2.4. We would like to draw the reader's attention to the analogy between the isomorphism (0.3) and the formalism of Grothendieck-Verdier categories of [BoDr].

Namely, consider [BoDr, Example 2.2], where X scheme (or, more generally, a *safe* algebraic stack) of finite type, and Γ is the groupoid $X \times X$.

Let $\mathcal{M} := \text{D-mod}(X \times X)_{\text{hol}} \subset \text{D-mod}(X \times X)$ be the holonomic subcategory. We consider $\mathcal{M}$ as a monoidal category, endowed with the convolution product (denoted $\circ$), where, in order to be consistent with [BoDr], we now use !-pushforward and $\bullet$-pullback (instead of the $\bullet$-pushforward and !-pullback). Then $\mathcal{M}$ is a Grothendieck-Verdier category, with the dualizing object being $(\Delta_X)_\bullet(\omega_X)$.

Let $\mathcal{Q} \in \text{D-mod}(X \times X)_{\text{hol}}$ be such that the corresponding functor $\mathsf{F}_{X \to X, \mathcal{Q}}$ admits a right adjoint, given by a holonomic kernel. Denote $\mathcal{P} := \mathbb{D}^{\text{Ve}}_{X \times X}(\mathcal{Q})$. Then $\mathcal{P}$, regarded as an object of the monoidal category $\mathcal{M}$, admits a *left* rigid dual, denoted in the notation of [BoDr] by $\underline{\text{Hom}}'(\mathcal{P}, \mathbf{1})$.

The isomorphism (0.3) is equivalent to an isomorphism in $\text{D-mod}(X \times X)_{\text{hol}}$

$$(\Delta_X)_\bullet(\omega_X) \circ \underline{\text{Hom}}'(\mathcal{P}, \mathbf{1}) \simeq \mathbb{D}^{\text{Ve}}_{X \times X}(\mathcal{P}),$$

valid for any left-dualizable object in a Grothendieck-Verdier category.

However, unfortunately, we were unable to formally apply the formalism of [BoDr] to deduce our (0.3) in general.

0.2.5. Next we consider the case of non-quasi compact Artin stacks. We will require that our stacks be locally QCA and *truncatable*, see Sect. 7.1.2 for what this means. The main example of a truncatable stack that we have in mind is Bun_G–the moduli stack of G-bundles on X, where G is a reductive group and X is a smooth and complete curve.

For a truncatable stack $\mathcal{X}$ there are two categories of D-modules that one can attach to it. One is the usual category $\text{D-mod}(\mathcal{X})$, and the other is the category that we denote by $\text{D-mod}(\mathcal{X})_{\text{co}}$, whose definition uses the truncatability of $\mathcal{X}$, see Sect. 7.1.4.

To elucidate the nature of $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$ let us describe their respective categories of compact objects. An object of $\text{D-mod}(\mathcal{X})$ is compact if and only if it equals the !-extension from a compact object on a quasi-compact open substack of $\mathcal{X}$. The category of compact objects of $\text{D-mod}(\mathcal{X})_{\text{co}}$ also embeds fully faithfully into $\text{D-mod}(\mathcal{X})$, and its essential image consists of objects that are *-extensions from compact objects on quasi-compact open substacks of $\mathcal{X}$. [2]

What for a QCA stack was the Verdier duality self-equivalence of the DG category of D-modules, for a truncatable stack becomes an equivalence between the dual of $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$. This implies that for a pair of truncatable stacks $\mathcal{X}_1$ and $\mathcal{X}_2$, the DG category

$$\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$$

is equivalent to that of continuous functors

$$\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2).$$

In particular, for a QCA stack $\mathcal{X}$ there exists a canonically defined functor

$$\text{Ps-Id}_{\mathcal{X}} : \text{D-mod}(\mathcal{X})_{\text{co}} \to \text{D-mod}(\mathcal{X}),$$

given by the kernel $(\Delta_{\mathcal{X}})_! \in \text{D-mod}(\mathcal{X} \times \mathcal{X})$.

[2] One can informally think of $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$ as obtained by imposing different "growth" conditions.

The DG category of continuous functors

$$\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2)_{\text{co}},$$

which is the same as that of continuous functors $\text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1)$, is equivalent to the tensor product category

$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}.$$

Finally, we note that for a *coherent* object $\mathcal{Q} \in \text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}$ there is a well-defined Verdier dual $\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})$, which is an object of $\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)$.

0.2.6. The main theorem for non-quasi compact stacks reads as follows. Let

$$\mathsf{F} : \text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2)_{\text{co}}$$

be a continuous functor given by a coherent kernel

$$\mathcal{Q} \in \text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}.$$

Assume that F admits a continuous right adjoint (equivalently, F preserves compactness). Then we have a canonical isomorphism of functors $\text{D-mod}(\mathcal{X}_2)_{\text{co}} \to \text{D-mod}(\mathcal{X}_1)$:

$$\text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R \simeq \mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}. \tag{0.4}$$

We note that in the right-hand side, the object $\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})$ belongs to

$$\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2) = \text{D-mod}(\mathcal{X}_2 \times \mathcal{X}_1),$$

and hence the functor $\mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}$ is understood as a functor

$$\text{D-mod}(\mathcal{X}_2)_{\text{co}} \to \text{D-mod}(\mathcal{X}_1).$$

So, the initial kernel and its Verdier dual define functors between different categories, and the connection is provided by the functor $\text{Ps-Id}_{\mathcal{X}_1}$, which maps $\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_1)$.

0.3. **Contents.** We shall now review the contents of this paper section-by-section.

0.3.1. In Sect. 1 we state the main theorem pertaining to schemes, Theorem 1.3.4, which generalizes the isomorphism (0.1) to the case when the scheme X_1 is not necessarily smooth and separated. We discuss various corollaries and particular cases of Theorem 1.3.4.

We intersperse the discussion about functors between categories of D-modules with a review of some basic facts concerning DG categories.

0.3.2. In Sect. 2 we give a geometric description of a canonical natural transformation

$$\text{pr}_{2!}(\text{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \text{Ps-Id}_{X_1}(\mathcal{Q})) \to \text{pr}_{2\bullet}(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}), \tag{0.5}$$

which is an isomorphism whenever the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$, defined by $\mathcal{Q}$, preserves compactness.

When the scheme X_1 is separated, the basic ingredient of the map (0.5) is the natural transformation $f_! \to f_\bullet$ for a separated morphism f between schemes, and the natural transformation

$$g_1^\bullet \circ f_0^! \to f_1^! \circ g_0^\bullet$$

for a Cartesian diagram

$$\begin{array}{ccc} Y_{11} & \xrightarrow{f_1} & Y_{1,0} \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_0} \\ Y_{0,1} & \xrightarrow{f_0} & Y_{0,0}. \end{array}$$

0.3.3. In Sect. 3 we study the following question. Let X_i, $i = 1, 2$ be derived schemes, and let $\mathcal{Q}$ be an object of $\text{D-mod}(X_1 \times X_2)$.

Recall that for a derived scheme X there are natural forgetful functors

$$\mathbf{oblv}^{\text{left}}_X : \text{D-mod}(X) \to \text{QCoh}(X) \text{ and } \mathbf{oblv}_X : \text{D-mod}(X) \to \text{IndCoh}(X),$$

where $\text{QCoh}(X)$ is the DG category of quasi-coherent sheaves on X and $\text{IndCoh}(X)$ is its modification introduced in [Ga1]. The functors $\mathbf{oblv}^{\text{left}}_X$ and $\mathbf{oblv}_X$ are the realizations of D-modules on X as "left" and "right" D-modules, respectively (see [GR, Sect. 2.4] for more details).

We would like to know how to express the condition that the functor

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2),$$

corresponding to $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$, preserve compactness, in terms of the corresponding objects

$$\mathbf{oblv}^{\text{left}}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\mathcal{Q}) \in \text{QCoh}(X_1) \otimes \text{D-mod}(X_2)$$

and

$$\mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\mathcal{Q}) \in \text{IndCoh}(X_1) \otimes \text{D-mod}(X_2).$$

For example, we show that if the support of $\mathcal{Q}$ is proper over X_2, then $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserves compactness if and only if $\mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\mathcal{Q})$ is compact.

0.3.4. In Sect. 4 we prove the following, perhaps a little unexpected, result:

Let X_1 be quasi-projective and smooth. Then then the property of an object

$$\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$$

that the corresponding functor

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

preserve compactness is inherited by any subquotient of any cohomology of $\mathcal{Q}$ (with respect to the standard t-structure on $\text{D-mod}(X_1 \times X_2)$).

0.3.5. In Sect. 5 we prove our main result pertaining to functors between D-modules on schemes, namely, Theorem 1.3.4. In fact, we prove a more general assertion, in the general context of DG categories, namely, Theorem 5.2.3.

In more detail, Theorem 5.2.3 describes the following situation. We start with a continuous functor between DG categories

$$\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2,$$

given by a kernel $\mathcal{Q} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_2$, and we assume that the right adjoint of F is also continuous. We want to relate this right adjoint F^R to the kernel *dual* to $\mathcal{Q}$, which is an object

$$\mathcal{Q}^\vee \in \mathbf{C}_1 \otimes \mathbf{C}_2^\vee.$$

In describing the relation, we will encounter an endo-functor $\text{Ps-Id}_{\mathbf{C}}$, defined for any DG category $\mathbf{C}$. When $\mathbf{C} = \text{D-mod}(\mathcal{X})$, where $\mathcal{X}$ is a QCA stack, the corresponding endo-functor is $\text{Ps-Id}_{\mathcal{X}}$ mentioned above.

At the suggestion of Drinfeld, we also introduce the notion of Gorenstein category. Namely, this is a DG category $\mathbf{C}$ for which the functor $\text{Ps-Id}_{\mathbf{C}}$ is an equivalence. The name Gorenstein is explained by the following result: for a separated derived scheme X almost of finite type, the category $\text{QCoh}(X)$ is Gorenstein if and only if X is Gorenstein.

0.3.6. In Sect. 6 our goal is to generalize Theorem 1.3.4 to the case of QCA stacks. The generalization itself, Theorem 6.3.2, will be easy to carry out: the corresponding theorem follows from the general result about DG categories, namely, Theorem 5.2.3.

However, there are two important technical points that one needs to pay attention to in the case of Artin stacks (as opposed to schemes or Deligne-Mumford stacks).

First, for a scheme X , the subcategory $\text{D-mod}(X)^c$ of compact objects in $\text{D-mod}(X)$ is the same as $\text{D-mod}(X)_{\text{coh}}$, i.e., the subcategory spanned by cohomologically bounded objects with coherent cohomologies. This is no longer the case for stacks: for a QCA stack $\mathcal{X}$ we always have an inclusion

$$\text{D-mod}(\mathcal{X})^c \subset \text{D-mod}(\mathcal{X})_{\text{coh}},$$

which is an equality if and only if $\mathcal{X}$ is *safe*.

Second, for a non-schematic map $f : \mathcal{X}_1 \to \mathcal{X}_2$, the usual de Rham direct image functor

$$f_\bullet : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$$

may be ill-behaved (e.g., fails to be continuous). This applies in particular to the functor of de Rham cohomology $(p_\mathcal{X})_\bullet$ of a stack $\mathcal{X}$, where $p_\mathcal{X} : \mathcal{X} \to \text{pt} := \text{Spec}(k)$. To remedy this, one replaces $f_\bullet$ by its *renormalized* version, introduced in [DrGa1], and denoted $f_\blacktriangle$.

In the remainder of Sect. 6 we consider some applications of Theorem 6.3.2. For example, we consider the situation of an open embedding of stacks $j : U \hookrightarrow \mathcal{X}$, for which the functor $j_\bullet : \text{D-mod}(U) \to \text{D-mod}(\mathcal{X})$ preserves compactness (such open embeddings are in [DrGa2] called *co-truncative*), and see what Theorem 6.3.2 gives in this case.

We also consider the class of stacks $\mathcal{X}$, for which the functor of de Rham cohomology preserves compactness; we call such stacks mock-proper. For a mock-proper stack $\mathcal{X}$ we relate the functor $(p_\mathcal{X})_\blacktriangle$ (i.e., the renormalized version of de Rham cohomology) and $(p_\mathcal{X})_!$ (the functor of de Rham cohomology with compact supports).

Finally, we consider a particular example of a QCA stack, namely, $V/\mathbb{G}_m$, where V is a vector space with $\mathbb{G}_m$ acting by dilations. We show that $\text{D-mod}(V/\mathbb{G}_m)$ is Gorenstein.

0.3.7. In Sect. 7 we state and prove the theorem relating the adjoint functor to the Verdier dual kernel for locally QCA truncatable stacks.

We first review the definition of what it means for a QCA tack $\mathcal{X}$ to be truncatable, and introduce the two versions of the category of D-modules, $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$.

We proceed to stating and proving Theorem 7.4.2, which amounts to the isomorphism (0.4).

In the remainder of this section we consider applications of Theorem 7.4.2, most of which are straightforward modifications of the corresponding statements for QCA stacks, once we take into account the difference between $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$.

Finally, we consider the notion of a mock-proper truncatable stack and define the *mock-constant sheaf* on such a stack. We consider the particular case of $\mathcal{X} = \text{Bun}_G$, and show that its mock-constant sheaf $k_{\text{Bun}_G,\text{mock}}$ has some peculiar properties; this object would be invisible if one did not distinguish between $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$.

0.4. **Conventions and notation.**

0.4.1. The word "scheme" in this paper means "derived scheme almost of finite type over k, which is eventually coconnective." We refer the reader to [DrGa1, Sect. 3.1.1], where this notion is reviewed. Sometimes (more as a matter of convenience) we will use the term "prestack," by which we will always mean a prestack locally almost of finite type, see [DrGa1, Sect. 3.1].

The "good news" is that derived algebraic geometry is not needed in this paper, except in Sects. 3 and 4, in which $\mathcal{O}$-modules are discussed. So, since the material of these two sections is not used in the rest of the paper, the reader can skip them and work with ordinary schemes of finite type over k.

For a scheme/prestack $\mathcal{X}$ we denote by $p_{\mathcal{X}}$ the tautological map $\mathcal{X} \to \mathrm{pt} := \mathrm{Spec}(k)$. We let $\Delta_{\mathcal{X}}$ denote the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$.

0.4.2. Conventions and notations regarding DG categories adopted in this papar follow those reviewed in [DrGa2, Sect. 1].

In particular, we let Vect denote the category of chain complexes of k-vector spaces.

For a DG category $\mathbf{C}$ and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ we let $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Vect}$ denote the resulting chain complex of maps between them.

0.4.3. Conventions and notations regarding the category of D-modules on a scheme follow those of [DrGa1, Sect. 5], and on algebraic stack those of [DrGa1, Sect. 6]. See also [DrGa2, Sect. 2] (for a brief review), and [GR] for a systematic treatment of the foundadtions of the theory.

The only notational difference between the present paper and [DrGa1] is that the functor of de Rham direct image with respect to a morphism f is denoted here by $f_\bullet$ instead of $f_{\mathrm{dR},*}$.

For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ between prestacks, we have a tautologically defined functor

$$f^! : \text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1).$$

The symmetric monoidal structure, denoted $\overset{!}{\otimes}$, on the category of D-modules on a prestack $\mathcal{X}$ is defined by

$$\mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2 := \Delta_{\mathcal{X}}^!(\mathcal{M}_1 \boxtimes \mathcal{M}_2).$$

0.4.4. The *partially defined* left adjoint of the functor $f^!$ will be denoted by $f_!$. I.e., for $\mathcal{M}_1 \in$ D-mod$(\mathcal{X}_1)$, the object $f_!(\mathcal{M}_1) \in$ D-mod$(\mathcal{X}_1)$ is defined if and only if the functor

$$\mathcal{M}_2 \mapsto \mathcal{M}aps_{\text{D-mod}(\mathcal{X}_1)}(\mathcal{M}_1, f^!(\mathcal{M}_2)), \quad \text{D-mod}(\mathcal{X}_2) \to \mathrm{Vect}$$

is co-representable.

For a general $\mathcal{M}_1$, we can view $f_!(\mathcal{M}_1)$ as an object of Pro(D-mod$(\mathcal{X}_2)$), the pro-completion of D-mod$(\mathcal{X}_1)$. [3]

For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ between *Artin stacks* we have the functor of de Rham direct image

$$f_\bullet : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$$

see [DrGa1, Sect. 7.4.]. Its partially defined left adjoint is denoted $f^\bullet$.

We let $\overset{\bullet}{\otimes}$ denote the partially defined functor

$$\text{D-mod}(\mathcal{X}) \otimes \text{D-mod}(\mathcal{X}) \to \text{D-mod}(\mathcal{X})$$

[3] For a DG category $\mathbf{C}$, its pro-completion Pro$(\mathbf{C})$ is the category of all exact covariant functor $\mathbf{C} \to \mathrm{Vect}$ that commute with κ-filtered colimits for some sufficiently large cardinal κ.

equal to

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto (\Delta_{\mathcal{X}})^\bullet(\mathcal{M}_1 \boxtimes \mathcal{M}_2).$$

I.e., it is defined an on object of $\text{D-mod}(\mathcal{X}) \otimes \text{D-mod}(\mathcal{X})$, whenever $(\Delta_{\mathcal{X}})^\bullet$ is defined on the corresponding object of $\text{D-mod}(\mathcal{X} \times \mathcal{X})$.

As in the case of $f_!$, in general, the functors $f^\bullet$ and $\overset{\bullet}{\otimes}$ can be viewed as taking values in the pro-completion of the target category.

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are Artin stacks, the functors $f_!$, $f^\bullet$ and $\overset{\bullet}{\otimes}$ are defined on any *holonomic* object, i.e., one whose pullback to a scheme mapping smoothly to our stack has holonomic cohmologies.

0.5. **Acknowledgments.** The author would like to thank V. Drinfeld for many helpful discussions and suggestions related to this paper. The definition of the key player in the case of Artin stacks, namely, the functor $\text{Ps-Id}_{\mathcal{X}}$, is due to him.

The author is supported by NSF grant DMS-1063470.

1. Functors between categories of D-modules

In this section we state our main theorem in the case of schemes (Theorem 1.3.4) and discuss its corollaries.

1.1. **Continuous functors and kernels: recollections.**

1.1.1. Let $\mathbf{C}$ be a dualizable category, and let $\mathbf{C}^\vee$ denote its dual. We let

$$\mathbf{u}_{\mathbf{C}} \in \mathbf{C} \otimes \mathbf{C}^\vee$$

denote the object correspnding to the unit map

$$\text{Vect} \to \mathbf{C} \otimes \mathbf{C}^\vee,$$

and we let

$$\text{ev}_{\mathbf{C}} : \mathbf{C} \otimes \mathbf{C}^\vee \to \text{Vect}$$

denote the counit map.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two DG categories. Recall that an exact functor $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ is said to be *continuous* if it commutes with infinite direct sums (equivalently, all colimits). We let $\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ denote the full DG subcategory of the DG category $\text{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ of all DG functors $\mathbf{C}_1 \to \mathbf{C}_2$, spanned by continuous functors.

Assume that $\mathbf{C}_1$ dualizable. In this case, the DG category $\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ identifies with

$$\mathbf{C}_1^\vee \otimes \mathbf{C}_2.$$

Explicitly, an object $\mathcal{Q} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_2$ gives rise to the functor $\mathsf{F}_{\mathbf{C}_1 \to \mathbf{C}_2, \mathcal{Q}}$ equal to

$$\mathbf{C}_1 \overset{\text{Id}_{\mathbf{C}_1} \otimes \mathcal{Q}}{\longrightarrow} \mathbf{C}_1 \otimes \mathbf{C}_1^\vee \otimes \mathbf{C}_2 \overset{\text{ev}_{\mathbf{C}_1} \otimes \text{Id}_{\mathbf{C}_2}}{\longrightarrow} \mathbf{C}_2.$$

Vice versa, given a continuous functor $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ we construct the corresponding object $\mathcal{Q}_{\mathsf{F}} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_2$ as

$$(\text{Id}_{\mathbf{C}_1^\vee} \otimes \mathsf{F})(\mathbf{u}_{\mathbf{C}_1}).$$

In particular, $\mathbf{u}_{\mathbf{C}_1} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_1$ corresponds to the identity functor on $\mathbf{C}_1$.

We shall refer to $\mathcal{Q}_{\mathsf{F}}$ as the *kernel* of F, and to $\mathsf{F}_{\mathbf{C}_1 \to \mathbf{C}_2; \mathcal{Q}}$ as the *functor defined by* $\mathcal{Q}$.

1.1.2. Let $\mathbf{C}$ be a compactly generated category. Recall that in this case

$$\mathbf{C} \simeq \mathrm{Ind}(\mathbf{C}^c),$$

where $\mathrm{Ind}(-)$ denotes the ind-completion of a given small DG category.

Recall also that such $\mathbf{C}$ is dualizable, and we have a canonical equivalence

$$(\mathbf{C}^c)^{\mathrm{op}} \simeq (\mathbf{C}^\vee)^c, \quad \mathbf{c} \mapsto \mathbf{c}^\vee. \tag{1.1}$$

In particular,

$$\mathbf{C}^\vee \simeq \mathrm{Ind}((\mathbf{C}^c)^{\mathrm{op}}). \tag{1.2}$$

Under this identification for $\mathbf{c} \in \mathbf{C}^c$ and $\xi \in \mathbf{C}^\vee$ we have

$$\mathcal{M}aps_{\mathbf{C}^\vee}(\mathbf{c}^\vee, \xi) \simeq \mathrm{ev}_{\mathbf{C}}(\mathbf{c} \otimes \xi). \tag{1.3}$$

1.1.3. The following simple observation will be used throughout the paper (see, e.g., [DrGa2, Proposition 1.2.4] for the proof):

Lemma 1.1.4. *Let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor. If F admits a continuous right adjoint, then it preserves compactness. Vice versa, if F preserves compactness and $\mathbf{C}_1$ is compactly generated, then F admits a continuous right adjoint.*

Let us note the following consequence of Lemma 1.1.4:

Corollary 1.1.5. *Let $\mathbf{C}_1$ be compactly generated, and let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor that preserves compactness. Then for any DG category $\mathbf{C}$ the functor*

$$\mathbf{C} \otimes \mathbf{C}_1 \overset{\mathrm{Id}_{\mathbf{C}} \otimes \mathsf{F}}{\longrightarrow} \mathbf{C} \otimes \mathbf{C}_2$$

also preserves compactness.

Proof. By Lemma 1.1.4, the functor F admits a continuous right adjoint; denote it G. Hence, the functor $\mathrm{Id}_{\mathbf{C}} \otimes \mathsf{F}$ also admits a continuous right adjoint, namely, $\mathrm{Id}_{\mathbf{C}} \otimes \mathsf{G}$. Now, apply Lemma 1.1.4 again. □

1.2. **Continuous functors and kernels: the case of D-modules.**

1.2.1. Let X be a scheme of finite type over k. (We remind that in the present section, as well as elsewhere in the paper with the exception of Sects. 3 and 4, we can work within classical algebraic geometry.)

Recall (see e.g., [DrGa1, Sect. 5.3.4]) that the DG category $\text{D-mod}(X)$ canonically identifies with its own dual:

$$\mathbf{D}_X^{\mathrm{Ve}} : \text{D-mod}(X)^\vee \simeq \text{D-mod}(X),$$

where the corresponding equivalence on compact objects

$$(\text{D-mod}(X)^c)^{\mathrm{op}} = (\text{D-mod}(X)^\vee)^c \overset{\mathbf{D}_X^{\mathrm{Ve}}}{\longrightarrow} \text{D-mod}(X)^c$$

is the usual Verdier duality functor

$$\mathbb{D}_X^{\mathrm{Ve}} : (\text{D-mod}(X)^c)^{\mathrm{op}} \to \text{D-mod}(X)^c.$$

Let us also recall the corresponding evaluation and unit functors. For this we recall (see, e.g., [DrGa1, Sect. 5.1.7]) that if X_1 and X_2 are two schemes of finite type, the operation of external tensor product of D-modules defines an equivalence

$$\text{D-mod}(X_1) \otimes \text{D-mod}(X_2) \simeq \text{D-mod}(X_1 \times X_2).$$

Under the equivalence $\text{D-mod}(X)\otimes\text{D-mod}(X)\simeq\text{D-mod}(X\times X)$, the evaluation functor

$$\text{ev}_{\text{D-mod}(X)}:\text{D-mod}(X)\otimes\text{D-mod}(X)\to\text{Vect}$$

is $(p_X)_\bullet\circ(\Delta_X)^!$, where $p_X:X\to\text{pt}$ and Δ_X is the diagonal map $X\to X\times X$.

The unit object

$$\mathbf{u}_{\text{D-mod}(X)}\in\text{D-mod}(X)^\vee\otimes\text{D-mod}(X)\simeq\text{D-mod}(X)\otimes\text{D-mod}(X)\simeq\text{D-mod}(X\times X)$$

is $(\Delta_X)_\bullet(\omega_X)$, where

$$\omega_X:=p_X^!(k)$$

is the dualizing complex.

1.2.2. Let X_1 and X_2 be two schemes of finite type. By Sect. 1.1.1, the DG category

$$\text{Funct}_{\text{cont}}(\text{D-mod}(X_1),\text{D-mod}(X_2))$$

of continuous functors $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$ identifies with

$$\text{D-mod}(X_1)^\vee\otimes\text{D-mod}(X_2),$$

and further, using the equivalence $\mathbf{D}_{X_1}$, with

$$\text{D-mod}(X_1)\otimes\text{D-mod}(X_2)\simeq\text{D-mod}(X_1\times X_2).$$

I.e., continuous functors $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$ are in bijection with kernels, thought of objects of $\text{D-mod}(X_1\times X_2)$.

Explicitly, for $\mathcal{Q}\in\text{D-mod}(X_1\times X_2)$ the corresponding functor $\mathsf{F}_{X_1\to X_2;\mathcal{Q}}$ sends an object $\mathcal{M}\in\text{D-mod}(X_1)$ to

$$(\text{pr}_2)_\bullet(\text{pr}_1^!(\mathcal{M})\overset{!}{\otimes}\mathcal{Q}),$$

where $\overset{!}{\otimes}$ denotes the tensor product on the category of D-modules (see [DrGa1, Sect. 5.1.7]), and

$$\text{pr}_i:X_1\times X_2\to X_i,\quad i=1,2$$

are the two projections.

1.2.3. The question we would like to address in this section is the following: suppose that a functor $\mathsf{F}:\text{D-mod}(X_1)\to\text{D-mod}(X_2)$ admits a a *continuous* right adjoint F^R or a left adjoint F^L (the latter is automatically continuous).

We would like to relate the kernels of the functors F^R or F^L (and also of the *conjugate* functors, see Sect. 1.5.1) to that of F.

The relationship will be particularly explicit when X_1 and X_2 are separated and smooth, see Sects. 1.3.9 and 1.6.4. In the case of arbitrary schemes of finite type, the corresponding assertion is stated in Sects. 1.3.3 and 1.6.1. The situation becomes significantly more interesting when instead of schemes, we consider Artin stacks, see Sects. 6 and 7.

1.2.4. Note that from Lemma 1.1.4, we obtain:

Corollary 1.2.5. *Let* $\mathcal{Q}$ *be an object of* $\text{D-mod}(X_1\times X_2)$. *Then the functor*

$$\mathsf{F}_{X_1\to X_2,\mathcal{Q}}:\text{D-mod}(X_1)\to\text{D-mod}(X_2)$$

admits a continuous *right adjoint if and only if it preserves compactness.*

1.3. **Statement of the theorem: the case of schemes.**

1.3.1. Note that for a scheme of finite type X, the object $\omega_X \in \text{D-mod}(X)$ is compact.

We let

$$k_X := \mathbb{D}^{\text{Ve}}_X(\omega_X) \in \text{D-mod}(X)^c.$$

By definition, k_X is the D-module incarnation of the constant sheaf on X. As is well-known, if X is smooth (or rationally smooth) of dimension n, we have

$$k_X \simeq \omega_X[-2n]. \tag{1.4}$$

1.3.2. A fundamental role in this paper is played by the object

$$(\Delta_X)_!(k_X) \in \text{D-mod}(X \times X).$$

The object $(\Delta_X)_!(k_X)$ is well-defined (see Sect. 0.4.4) because k_X is holonomic. Note also that if X is separated,

$$(\Delta_X)_!(k_X) \simeq (\Delta_X)_\bullet(k_X). \tag{1.5}$$

We let

$$\text{Ps-Id}_X : \text{D-mod}(X) \to \text{D-mod}(X)$$

denote the functor, given by the kernel $(\Delta_X)_!(k_X)$.

Note that when X is separated, we have

$$\text{Ps-Id}_X(\mathcal{M}) \simeq \mathcal{M} \overset{!}{\otimes} k_X,$$

and when X is separated and smooth of dimension n, we thus have:

$$\text{Ps-Id}_X \simeq \text{Id}_{\text{D-mod}(X)}[-2n]. \tag{1.6}$$

1.3.3. We have the following theorem, which will be proved in Sect. 5, more precisely, in Sect. 5.3.4:

Theorem 1.3.4. *Let $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$ be an object such that the corresponding functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ admits a continuous right adjoint. Then:*

(a) *The object $\mathcal{Q}$ is compact.*

(b) *The functor*

$$\mathsf{F}_{X_2 \to X_1, \mathbb{D}^{\text{Ve}}_{X_1 \times X_2}(\mathcal{Q})} : \text{D-mod}(X_2) \to \text{D-mod}(X_1),$$

identifies canonically with

$$\text{D-mod}(X_2) \overset{(\mathsf{F}_{X_1 \to X_2, \mathcal{Q}})^R}{\longrightarrow} \text{D-mod}(X_1) \overset{\text{Ps-Id}_{X_1}}{\longrightarrow} \text{D-mod}(X_1).$$

Thus, informally, the functor $(\mathsf{F}_{X_1 \to X_2, \mathcal{Q}})^R$ is "almost" given by the kernel, Verdier dual to that of F, and the correction to the "almost" is given by the functor Ps-Id_{X_1}.

1.3.5. We emphasize that by Corollary 1.2.5, the condition in the theorem that the functor

$$\mathsf{F}_{X_1\to X_2,\mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

admit a continuous right adjoint is equivalent to the condition that it preserve compactness.

We note that point (a) of Theorem 1.3.4 is very simple:

Proof. The assertion follows from Corollary 1.1.5, using the fact that

$$\mathcal{Q} \simeq \left(\mathrm{Id}_{\text{D-mod}(X_1)} \otimes \mathsf{F}_{X_1\to X_2,\mathcal{Q}}\right)(\omega_{X_1}),$$

and $\omega_{X_1} \in \text{D-mod}(X_1 \times X_1)$ is compact.

□

Let us also note the following:

Proposition 1.3.6. *Let $\mathcal{Q}$ be as in Theorem 1.3.4. Let $f : X_1 \to X_1'$ (resp., $g : X_1' \to X_1$) be a smooth (resp., proper) morphism. Then the objects*

$$(f \times \mathrm{id}_{X_2})_\bullet(\mathcal{Q}) \text{ and } (g \times \mathrm{id}_{X_2})^!(\mathcal{Q})$$

of $\text{D-mod}(X_1' \times X_2)$ *also satisfy the assumption of Theorem 1.3.4; in particular, they are compact.*

Proof. Apply Theorem 1.3.4(a) to the functors $\Phi_{X_1\to X_2,\mathcal{Q}} \circ f^!$ and $\Phi_{X_1\to X_2,\mathcal{Q}} \circ g_\bullet$, respectively, and use the fact that the functors $f^!$ and $g_\bullet$ preserve compactness. □

1.3.7. Let us now swap the roles of X_1 and X_2:

Corollary 1.3.8. *Let $\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ be a continuous functor that admits a* left *adjoint, and let $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$ denote the kernel of F^L. Then:*

(a) *The object $\mathcal{Q}$ is compact.*

(b) *The functor*

$$\mathsf{F}_{X_1\to X_2,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})} : \text{D-mod}(X_1) \to \text{D-mod}(X_2),$$

identifies canonically with the composition

$$\text{D-mod}(X_1) \xrightarrow{\mathsf{F}} \text{D-mod}(X_2) \xrightarrow{\text{Ps-Id}_{X_2}} \text{D-mod}(X_2).$$

1.3.9. A particular case of Theorem 1.3.4 reads:

Corollary 1.3.10. *Let $\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ be a continuous functor, given by a kernel $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$.*

(1) *Let X_1 be smooth of dimension n_1 and separated, and suppose that F admits a continuous right adjoint. Then $\mathcal{Q}$ is compact and the functor F^R is given by the kernel $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})[2n_1]$.*

(2) *Let X_2 be smooth of dimension n_2 and separated, and suppose that F admits a left adjoint. Then $\mathcal{Q}$ is compact and the functor F^L is given by the kernel $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})[2n_2]$.*

As a particular case, we obtain:

Corollary 1.3.11. *Let X_1 and X_2 be both smooth and separated, of dimensions n_1 and n_2, respectively. Let $\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ be a continuous functor, and assume that F admits both left and continuous right adjoints. Then $\mathsf{F}^L[2(n_1 - n_2)] \simeq \mathsf{F}^R$.*

1.3.12. Finally, we have the following, perhaps a little unexpected, result that will be proved in Sect. 4:

Theorem 1.3.13. *Assume that X_1 is quasi-projective and smooth. Let $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$ be such that the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserves compactness. Then any subquotient of any of the cohomologies of $\mathcal{Q}$ with respect to the standard t-structure on $\text{D-mod}(X_1 \times X_2)$, has the same property.*

Remark 1.3.14. We are nearly sure that in Theorem 1.3.13, the assumption that X_1 be quasi-projective can be replaced by that of being separated.

1.4. **Digression: dual functors.**

1.4.1. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be dualizable DG categories. Recall that there is a canonical equivalence

$$\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \text{Funct}_{\text{cont}}(\mathbf{C}_2^\vee, \mathbf{C}_1^\vee), \tag{1.7}$$

given by the passage to the dual functor,

$$\mathsf{F} \mapsto \mathsf{F}^\vee.$$

In terms of the identification

$$\text{Funct}_{\text{cont}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \mathbf{C}_1^\vee \otimes \mathbf{C}_2 \text{ and } \text{Funct}_{\text{cont}}(\mathbf{C}_2^\vee, \mathbf{C}_1^\vee) \simeq (\mathbf{C}_2^\vee)^\vee \otimes \mathbf{C}_1^\vee,$$

the equivalence (1.7) corresponds to

$$\mathbf{C}_1^\vee \otimes \mathbf{C}_2 \simeq \mathbf{C}_2 \otimes \mathbf{C}_1^\vee \simeq (\mathbf{C}_2^\vee)^\vee \otimes \mathbf{C}_1^\vee.$$

1.4.2. Note that if

$$\mathsf{F} : \mathbf{C}_1 \rightleftarrows \mathbf{C}_2 : \mathsf{G}$$

is an adjoint pair of functors, then the pair

$$\mathsf{G}^\vee : \mathbf{C}_2^\vee \rightleftarrows \mathbf{C}_1 : \mathsf{F}^\vee$$

is also naturally adjoint.

By duality, from Lemma 1.1.4, we obtain:

Corollary 1.4.3. *Let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a continuous functor, and assume that $\mathbf{C}_2$ is compactly generated. Then F admits a* left *adjoint if and only if the functor $\mathsf{F}^\vee : \mathbf{C}_2^\vee \to \mathbf{C}_1^\vee$ preserves compactness.*

1.4.4. Let us apply the above discussion to $\mathbf{C}_i = \text{D-mod}(X_i)$, $i = 1, 2$, where X_1 and X_2 are schemes of finite type.

Thus, for

$$\mathcal{Q} \in \text{D-mod}(X_1 \times X_2) \simeq \text{D-mod}(X_2 \times X_1),$$

we have the following canonical isomorphism:

$$(\mathsf{F}_{X_1 \to X_2, \mathcal{Q}})^\vee \simeq \mathsf{F}_{X_2 \to X_1, \mathcal{Q}},$$

as functors $\text{D-mod}(X_2) \to \text{D-mod}(X_1)$, where we identify $\text{D-mod}(X_i)^\vee \simeq \text{D-mod}(X_i)$ by means of $\mathbf{D}^{\text{Ve}}_{X_i}$.

In particular, for a scheme of finite type X, we have a canonical isomorphism

$$(\text{Ps-Id}_{!,X})^\vee \simeq \text{Ps-Id}_{!,X} \, . \tag{1.8}$$

It comes from the equivariance structure on $(\Delta_X)_!(k_X)$ with respect to the flip automorphism of $X \times X$.

Applying Corollary 1.4.3 to D-modules, we obtain:

Corollary 1.4.5. *Let* $\mathcal{Q}$ *be an object of* $\text{D-mod}(X_1 \times X_2)$. *Then the functor*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

admits a left adjoint of and only if the functor

$$\mathsf{F}_{X_2 \to X_1, \mathcal{Q}} : \text{D-mod}(X_2) \to \text{D-mod}(X_1)$$

preserves compactness.

1.5. **Conjugate functors.**

1.5.1. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be compactly generated categories, and let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor that preserves compactness.

Thus, we obtain a functor

$$\mathsf{F}^c : \mathbf{C}_1^c \to \mathbf{C}_2^c,$$

and consider the corresponding functor between the opposite categories

$$(\mathsf{F}^c)^{\text{op}} : (\mathbf{C}_1^c)^{\text{op}} \to (\mathbf{C}_2^c)^{\text{op}}.$$

Hence, ind-extending $(\mathsf{F}^c)^{\text{op}}$ and using (1.2), we obtain a functor

$$\mathbf{C}_1^\vee \to \mathbf{C}_2^\vee.$$

We shall denote it by F^{op} and call it the *conjugate* functor.

1.5.2. The following is [GL:DG, Lemma 2.3.3]:

Lemma 1.5.3. *The functor* F^{op} *is the left adjoint of* $\mathsf{F}^\vee$.

Combining this with Sect. 1.4.2, we obtain:

Corollary 1.5.4. *The functor* F^{op} *is the dual of* F^R.

1.5.5. *Proof of Lemma 1.5.3.* Since all the functors in question are continuous and the categories are compactly generated, it suffices to construct a functorial equivalence

$$\mathcal{M}aps_{\mathbf{C}_2^\vee}(\mathsf{F}^{\text{op}}(\mathbf{c}_1^\vee), \mathbf{c}_2^\vee) \simeq \mathcal{M}aps_{\mathbf{C}_1^\vee}(\mathbf{c}_1^\vee, \mathsf{F}^\vee(\mathbf{c}_2^\vee)), \quad \mathbf{c}_i \in \mathbf{C}_i^c. \tag{1.9}$$

Recall (see (1.3)) that for $\xi_i \in \mathbf{C}_i^\vee$,

$$\mathcal{M}aps_{\mathbf{C}_i^\vee}(\mathbf{c}_i^\vee, \xi_i) \simeq \text{ev}_{\mathbf{C}_i}(\mathbf{c}_i \otimes \xi_i).$$

Hence, the left-hand side in (1.9) can be rewritten as

$$\text{ev}_{\mathbf{C}_2}(\mathsf{F}(\mathbf{c}_1) \otimes \mathbf{c}_2^\vee),$$

while the right-hand side as

$$\text{ev}_{\mathbf{C}_1}(\mathbf{c}_1 \otimes \mathsf{F}^\vee(\mathbf{c}_2^\vee)).$$

Finally,

$$\text{ev}_{\mathbf{C}_2}(\mathsf{F}(\mathbf{c}_1) \otimes \mathbf{c}_2^\vee) \simeq \text{ev}_{\mathbf{C}_1}(\mathbf{c}_1 \otimes \mathsf{F}^\vee(\mathbf{c}_2^\vee)),$$

by the definition of the dual functor.

□

Note that the same argument proves the following generalization of Lemma 1.5.3:

Lemma 1.5.6. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two compactly generated categories, and let* $\mathsf{G} : \mathbf{C}_2 \to \mathbf{C}_1$ *be a continuous functor; let* F *denote its partially defined left adjoint. Let* $\mathbf{c}_1 \in \mathbf{C}_1^c$ *be an object such that* $\mathsf{G}^\vee(\mathbf{c}_1^\vee) \in \mathbf{C}_2^\vee$ *is compact. Then* $\mathsf{F}(\mathbf{c}_1)$ *is defined and canonically isomorphic to*

$$\left(\mathsf{G}^\vee(\mathbf{c}_1^\vee)\right)^\vee.$$

1.6. **Back to D-modules: conjugate functors.**

1.6.1. By combining Theorem 1.3.4 with Corollary 1.5.4 and Equation (1.8), we obtain:

Corollary 1.6.2. *Under the assumptions and in the notations of of Theorem 1.3.4, the functor*

$$\mathsf{F}_{X_1\to X_2,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})}:\text{D-mod}(X_1)\to\text{D-mod}(X_2),$$

is canonically isomorphic to the composition

$$\text{D-mod}(X_1)\overset{\text{Ps-Id}_{X_1}}{\longrightarrow}\text{D-mod}(X_1)\overset{(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}}{\longrightarrow}\text{D-mod}(X_2).$$

Note that in the circumstances of Corollary 1.6.2, the functor

$$(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}:\text{D-mod}(X_1)\to\text{D-mod}(X_2)$$

also preserves compactness, by construction.

1.6.3. We emphasize that the functor $(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}$ that appears in Corollary 1.6.5 is by definition the ind-extension of the functor, defined on $\text{D-mod}(X_1)^c\subset\text{D-mod}(X_1)$ and given by

$$\mathcal{M}\mapsto\mathbb{D}^{\mathrm{Ve}}_{X_2}\circ\mathsf{F}_{X_1\to X_2,\mathcal{Q}}\circ\mathbb{D}^{\mathrm{Ve}}_{X_1}(\mathcal{M}),$$

where the right-hand side is defined, because $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}\circ\mathbb{D}^{\mathrm{Ve}}_{X_1}(\mathcal{M})\in\text{D-mod}(X_2)^c$.

In other words, $(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}|_{\text{D-mod}(X_1)^c}$ is obtained from $(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})|_{\text{D-mod}(X_1)^c}$ by conjugating by Verdier duality (hence the name "conjugate").

See Sect. 1.7.4 for an even more explicit description of $(\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}$.

1.6.4. By imposing the smoothness and separatedness hypothesis, from Corollary 1.6.2 we obtain:

Corollary 1.6.5. *Let X_1 be smooth of dimension n_1, and separated. Let*

$$\mathsf{F}:\text{D-mod}(X_1)\to\text{D-mod}(X_2)$$

be a continuous functor, given by a kernel $\mathcal{Q}\in\text{D-mod}(X_1\times X_2)$. Assume that F preserves compactness. Then the conjugate functor

$$\mathsf{F}^{\mathrm{op}}:\text{D-mod}(X_1)\to\text{D-mod}(X_2)$$

is given by the kernel $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})[2n_1]$.

Further, from Corollary 1.6.5, we deduce:

Corollary 1.6.6. *Let $\mathcal{Q}$ be a compact object of* $\text{D-mod}(X_1\times X_2)$.

(1) *Let X_1 be smooth and separated, and assume that the functor* $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$, *defined by $\mathcal{Q}$, admits a* continuous right *adjoint (i.e., preserves compactness). Then so does the functor* $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$, *defined by* $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})$.

(2) *Let X_2 be smooth and separated, and assume that the functor* $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$, *defined by $\mathcal{Q}$, admits a* left *adjoint. Then so does the functor* $\text{D-mod}(X_1)\to\text{D-mod}(X_2)$, *defined by* $\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})$.

1.7. **Another interpretation of conjugate functors.**

1.7.1. Consider the functors

$$\mathrm{pr}_1^\bullet : \text{D-mod}(X_1) \to \text{D-mod}(X_1 \times X_2);$$

$$\overset{\bullet}{\otimes} = \Delta^\bullet_{X_1 \times X_2} : \text{D-mod}((X_1 \times X_2) \times (X_1 \times X_2)) \to \mathrm{Pro}(\text{D-mod}(X_1 \times X_2)),$$

and

$$(\mathrm{pr}_2)_! : \mathrm{Pro}(\text{D-mod}(X_1 \times X_2)) \to \mathrm{Pro}(\text{D-mod}(X_2)),$$

see Sect. 0.4.4.

For an object $\mathcal{P} \in \text{D-mod}(X_1 \times X_2)$ consider the functor

$$\mathsf{F}^{\mathrm{op}}_{X_1 \to X_2; \mathcal{P}} : \text{D-mod}(X_1) \to \mathrm{Pro}(\text{D-mod}(X_2)),$$

defined by

$$\mathsf{F}^{\mathrm{op}}_{X_1 \to X_2; \mathcal{P}}(\mathcal{M}) := (\mathrm{pr}_2)_! \left(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{P}\right). \tag{1.10}$$

The assignment

$$\mathcal{P} \in \text{D-mod}(X_1 \times X_2) \rightsquigarrow \mathsf{F}^{\mathrm{op}}_{X_1 \to X_2; \mathcal{P}} : \text{D-mod}(X_1) \to \mathrm{Pro}(\text{D-mod}(X_2)),$$

is another way to construct a functor from an object on the product, using the Verdier conjugate functors, i.e., by replacing

$$p_1^! \mapsto p_1^\bullet;\ \overset{!}{\otimes} \mapsto \overset{\bullet}{\otimes},\ (\mathrm{pr}_2)_\bullet \mapsto (\mathrm{pr}_2)_!.$$

Remark 1.7.2. Let $\mathcal{M} \in \text{D-mod}(X_1)$ be such that the functors $\Delta^\bullet_{X_1 \times X_2}$ and $(\mathrm{pr}_2)_!$ are defined on the objects $\mathrm{pr}_1^\bullet(\mathcal{M}) \boxtimes \mathcal{P}$ and $\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{P}$, respectively. (E.g., this is the case when $\mathcal{P}$ and $\mathcal{M}$ are both holonomic.) Then the notation

$$(\mathrm{pr}_2)_! \left(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{P}\right) \in \text{D-mod}(X_2) \subset \mathrm{Pro}(\text{D-mod}(X_2))$$

is unambiguous.

1.7.3. Assume that $\mathcal{P} \in \text{D-mod}(X_1 \times X_2)^c$. Denote $\mathcal{Q} := \mathbb{D}^{\mathrm{Ve}}_{X_1 \times X_2}(\mathcal{P})$. Then it is easy to see that the functor

$$\mathsf{F}^{\mathrm{op}}_{X_1 \to X_2; \mathcal{P}} : \text{D-mod}(X_1) \to \mathrm{Pro}(\text{D-mod}(X_2))$$

is the left adjoint of the functor

$$\mathsf{F}_{X_2 \to X_1, \mathcal{Q}} : \text{D-mod}(X_2) \to \text{D-mod}(X_1),$$

in the sense that for $\mathcal{M}_i \in \text{D-mod}(X_i)$ we have a canonical isomorphism

$$\mathcal{M}aps_{\mathrm{Pro}(\text{D-mod}(X_2))}(\mathsf{F}^{\mathrm{op}}_{X_1 \to X_2; \mathcal{P}}(\mathcal{M}_1), \mathcal{M}_2) \simeq \mathcal{M}aps_{\text{D-mod}(X_1)}(\mathcal{M}_1, \mathsf{F}_{X_2 \to X_1, \mathcal{Q}}(\mathcal{M}_2)),$$

where the left-hand side can be also interpreted as the evaluation of an object of the pro-completion of a DG category on an object of that DG category, see Sect. 0.4.4.

1.7.4. Take now $\mathcal{P} := \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})$, where $\mathcal{Q}$ is as in Theorem 1.3.4 (i.e., the functor $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}$ preserves compactness).

By Corollary 1.4.5, the functor $\mathsf{F}_{X_2\to X_1,\mathcal{Q}} \simeq \mathsf{F}^\vee_{X_1\to X_2,\mathcal{Q}}$ admits a left adjoint. Hence, by Sect. 1.7.3, the functor $\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})}$ takes values in

$$\text{D-mod}(X_2) \subset \text{Pro}(\text{D-mod}(X_2)),$$

and provides a left adjoint to $\mathsf{F}_{X_2\to X_1,\mathcal{Q}}$. By Lemma 1.5.3, we obtain an isomorphism of functors $\text{D-mod}(X_1) \to \text{D-mod}(X_2)$:

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}}(\mathcal{Q}) \simeq (\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^{\mathrm{op}}.$$

Thus, we can interpret Corollary 1.6.2 as follows:

Corollary 1.7.5. *For $\mathcal{Q}$ as in Theorem 1.3.4 we have a canonical isomorphism*

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})} \circ \text{Ps-Id}_{X_1} \simeq \mathsf{F}_{X_1\to X_2,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})},$$

where

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})}(\mathcal{M}) = (\mathrm{pr}_2)_! \left(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} (\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q}))\right).$$

1.7.6. Combining Corollary 1.7.5 with Corollary 1.6.6(1) we obtain:

Corollary 1.7.7. *Let X_1 be smooth of dimension n_1 and separated. Let $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$ satisfy the assumption of Theorem 1.3.4. Then there is a canonical isomorphism*

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2,\mathcal{Q}} \simeq \mathsf{F}_{X_1\to X_2,\mathcal{Q}}[2n_1],$$

i.e.,

$$(\mathrm{pr}_2)_! \left(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{Q}\right) \simeq (\mathrm{pr}_2)_\bullet \left(\mathrm{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}\right)[2n_1], \quad \mathcal{M} \in \text{D-mod}(X_1);$$

in particular, the left-hand side takes values in $\text{D-mod}(X_2) \subset \text{Pro}(\text{D-mod}(X_2))$.

1.7.8. We shall now deduce a property of the functors $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}$ satisfying the assumption of Theorem 1.3.4 with respect to the standard t-structure on the category of D-modules.

In what follows, for a DG category $\mathbf{C}$, endowed with a t-structure, we let $\mathbf{C}^{\leq 0}$ (resp., $\mathbf{C}^{\geq 0}$) denote the corresponding subcategory of connective (resp., coconnective) objects. We let $\mathbf{C}^{\heartsuit} := \mathbf{C}^{\leq 0} \cap \mathbf{C}^{\geq 0}$ denote the heart of the t-structure.

Corollary 1.7.9. *Let X_1 and $\mathcal{Q}$ be as in Corollary 1.7.7. Assume in addition that the support of $\mathcal{Q}$ is affine over X_2, and that $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)^{\heartsuit}$. Then the functor*

$$\mathsf{F}_{X_1\to X_2,\mathcal{Q}}[n_1] : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

is t-exact.

Proof. The fact that $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}[n_1]$ is right t-exact is straightforward from the definition (no assumption that $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}$ preserve compactness is needed).

The fact that $\mathsf{F}_{X_1\to X_2,\mathcal{Q}}[n_1]$ is left t-exact follows from the isomorphism

$$\mathsf{F}_{X_1\to X_2,\mathcal{Q}}[n_1] \simeq \mathsf{F}^{\mathrm{op}}_{X_1\to X_2,\mathcal{Q}}[-n_1].$$

□

1.8. **An example: Fourier-Deligne transform.** Let us consider a familiar example of the situation described in Corollaries 1.7.7, 1.3.10 and 1.3.11.

1.8.1. Namely, let V be a finite-dimensional vector space, thought of a scheme over k, and let $V^\vee$ be the dual vector space. We take $X_1 = V$ and $X_2 = V^\vee$.

We take the kernel $\mathcal{Q} \in \text{D-mod}(V \times V^\vee)$ to be the pullback of exponential D-module on $\mathbb{G}_a$ under the evaluation map $V \times V^\vee \to \mathbb{G}_a$. We denote it symbolically by

$$\exp \in \text{D-mod}(V \times V^\vee),$$

and we normalize it so that it lives in cohomological degree $-\dim(V)$ with respect to the natural t-structure on the category $\text{D-mod}(V \times V^\vee)$.

The corresponding functor $\text{D-mod}(V) \to \text{D-mod}(V^\vee)$ is by definition the Fourier-Deligne transform

$$\mathsf{F}_{V\to V^\vee,\exp} = (\text{pr}_2)_\bullet \left(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \exp\right). \tag{1.11}$$

Since $\mathsf{F}_{V\to V^\vee,\exp}$ is an equivalence, it admits both left and right adjoints (which are isomorphic).

1.8.2. It is well-known that the functor $\mathsf{F}_{V\to V^\vee,\exp}$ can be rewritten as

$$\mathcal{M} \mapsto (\text{pr}_2)_! \left(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \exp\right), \tag{1.12}$$

(see Sect. 0.4.4 regarding the meaning of $(\text{pr}_2)_!$).

Now, using the fact that the map pr_1 is smooth and that the D-module exp on $V \times V^\vee$ is lisse, the expression in (1.12) can be further rewritten as

$$(\text{pr}_2)_! \left(\text{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \exp\right)[-2\dim(V)],$$

and the latter functor identifies with the functor

$$\mathsf{F}^{\text{op}}_{V\to V^\vee,\exp}[-2\dim(V)].$$

Thus, we obtain an isomorphism

$$\mathsf{F}^{\text{op}}_{V\to V^\vee,\exp}[-2\dim(V)] \simeq \mathsf{F}_{V\to V^\vee,\exp}.$$

However, we now know that the latter is not a special feature of the Fourier-Deligne trasform, but rather a particular case of Corollary 1.7.7 (for X_1 smooth and separated).

Note also that the fact that the map from (1.12) $\to$ (1.11), coming from the natural transformation $(\text{pr}_2)_! \to (\text{pr}_2)_\bullet$ is an isomorphism, follows from the description of the isomorphism of Corollary 1.6.2 in the separated case; this description will be given in the next section, specifically, Sect. 2.3.

1.8.3. The right adjoint of $\mathsf{F}_{V\to V^\vee,\exp}$, written as $\mathsf{F}^{\text{op}}_{V\to V^\vee,\exp}[-2\dim(V)]$, identifies with

$$\mathcal{M}' \mapsto (\text{pr}_1)_\bullet \left(\text{pr}_2^!(\mathcal{M}') \overset{!}{\otimes} \mathbb{D}^{\text{Ve}}_{V\times V^\vee}(\exp)\right)[2\dim(V)], \tag{1.13}$$

which in turn is the functor $\mathsf{F}_{V^\vee\to V;\text{-}\exp}$, i.e., the inverse Fourier-Deligne transform.

The isomorphism

$$(\mathsf{F}_{V\to V^\vee,\exp})^R \simeq \mathsf{F}_{V^\vee\to V;\text{-}\exp}$$

coincides with the assertion of Corollary 1.3.10(1).

1.8.4. Finally, we note that the functor $\mathsf{F}_{V\to V^\vee,\mathrm{exp}}$ admits a *left* adjoint, given by

$$\mathcal{M}' \mapsto (\mathrm{pr}_1)_! \left(\mathrm{pr}_2^\bullet(\mathcal{M}') \overset{\bullet}{\otimes} \mathbb{D}^{\mathrm{Ve}}_{V\times V^\vee}(\exp)\right),$$

i.e., $\mathsf{F}^{\mathrm{op}}_{V^\vee\to V,\text{-}\mathrm{exp}}[-2\dim(V)]$, which, by Sect. 1.8.2 with the roles of V and $V^\vee$ swapped, is well-defined and isomorphic to $\mathsf{F}_{V^\vee\to V;\text{-}\mathrm{exp}}$.

The isomorphism

$$(\mathsf{F}_{V\to V^\vee,\mathrm{exp}})^R \simeq (\mathsf{F}_{V\to V^\vee,\mathrm{exp}})^L$$

coincides with the assertion of Corollary 1.3.11.

2. The natural transformations

The goal of this section is to describe geometrically the isomorphisms of Theorem 1.3.4 and Corollary 1.7.5. This material will not be used elsewhere in the paper.

2.1. **The adjunction map.**

2.1.1. Let $\mathcal{Q}$ be as in Theorem 1.3.4. The (iso)morphism

$$\text{Ps-Id}_{X_1} \circ (\mathsf{F}_{X_1\to X_2,\mathcal{Q}})^R \to \mathsf{F}_{X_2\to X_1,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathbb{Q})} \tag{2.1}$$

of Theorem 1.3.4 gives rise to a natural transformation

$$\text{Ps-Id}_{X_1} \to \mathsf{F}_{X_2\to X_1,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})} \circ \mathsf{F}_{X_1\to X_2,\mathcal{Q}}. \tag{2.2}$$

The map (2.2) will be described exlicitly (in the context of general DG categories) in Sect. 5.3.5. We will now explain what this abstract description amounts to in the case of categories of D-modules.

2.1.2. First, we note that for a scheme Y and $\mathcal{M} \in \text{D-mod}(Y)^c$ we have a canonical map

$$\mathcal{M} \boxtimes \mathbb{D}^{\mathrm{Ve}}_Y(\mathcal{M}) \to (\Delta_Y)_\bullet(\omega_Y).$$

Applying Verdier duality, we obtain a canonical map

$$(\Delta_Y)_!(k_Y) \to \mathcal{M} \boxtimes \mathbb{D}^{\mathrm{Ve}}_Y(\mathcal{M}). \tag{2.3}$$

2.1.3. The right-hand side in (2.2) is a functor $\text{D-mod}(X_1) \to \text{D-mod}(X_1)$ given by the kernel

$$(\mathrm{id}_{X_1} \times p_{X_2} \times \mathrm{id}_{X_1})_\bullet \circ (\mathrm{id}_{X_1} \times \Delta_{X_2} \times \mathrm{id}_{X_1})^! \circ \sigma_{2,3}(\mathcal{Q} \boxtimes \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})), \tag{2.4}$$

where $\sigma_{2,3}$ is the transposition of the corresponding factors.

The datum of a map in (2.2) is equivalent to that of a map from $(\Delta_{X_1})_!(k_{X_1})$ to (2.4), and further, by the $((\Delta_{X_1})_!, \Delta^!_{X_1})$-adjunction, to a map

$$k_{X_1} \to \Delta^!_{X_1} \circ (\mathrm{id}_{X_1} \times p_{X_2} \times \mathrm{id}_{X_1})_\bullet \circ (\mathrm{id}_{X_1} \times \Delta_{X_2} \times \mathrm{id}_{X_1})^! \circ \sigma_{2,3}(\mathcal{Q} \boxtimes \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})). \tag{2.5}$$

By base change along

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{\Delta_{X_1}\times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_2 \\ {\scriptstyle \mathrm{id}_{X_1}\times p_{X_2}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times p_{X_2}} \\ X_1 & \xrightarrow{\Delta_{X_1}} & X_1 \times X_1, \end{array}$$

the right-hand side in (2.5) identifies with

$$(\mathrm{id}_{X_1} \times p_{X_2})_\bullet(\mathcal{Q} \overset{!}{\otimes} \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})).$$

2.1.4. Now, the desired map in (2.5) comes from

$$k_{X_1} \to (\mathrm{id}_{X_1} \times p_{X_2})_\bullet(k_{X_1\times X_2}) \overset{(2.3)}{\longrightarrow} (\mathrm{id}_{X_1} \times p_{X_2})_\bullet(\mathcal{Q} \overset{!}{\otimes} \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})).$$

In the above formula, the first arrow uses the canonical map (defined for any scheme Y; in our case $Y = X_2$)

$$k \to (p_Y)_\bullet(k_Y),$$

that arises from the $(p_Y^\bullet, (p_Y)_\bullet)$-adjunction.

2.2. **The map between two styles of functors.** Let $\mathcal{Q}$ be again as in Theorem 1.3.4. We shall now write down explicitly the (iso)morphism

$$(\mathsf{F}_{X_1\to X_2;\mathcal{Q}})^{\mathrm{op}} \circ \text{Ps-Id}_{X_1} \to \mathsf{F}_{X_1\to X_2,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})} \tag{2.6}$$

of Corollary 1.6.2.

2.2.1. By Sect. 1.7.4, we rewrite $(\mathsf{F}_{X_1\to X_2;\mathcal{Q}})^{\mathrm{op}}$ as

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})},$$

where $\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})}$ is as in (1.10).

Thus, we need to describe the resulting natural transformation

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})} \circ \text{Ps-Id}_{X_1} \to \mathsf{F}_{X_1\to X_2,\mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})}. \tag{2.7}$$

More generally, we will write down a natural transformation

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathcal{P}} \circ \text{Ps-Id}_{X_1} \to \mathsf{F}_{X_1\to X_2,\mathcal{P}} \tag{2.8}$$

for any $\mathcal{P} \in \text{D-mod}(X_1 \times X_2)$ (i.e., not necessarily the dual of an object defining a functor satisfying the assumption of Theorem 1.3.4).

The description of the map (2.8) occupies the rest of this subsection. The fact that (2.8), when applied to $\mathcal{P} := \mathbb{D}^{\mathrm{Ve}}_{X_1\times X_2}(\mathcal{Q})$, yields (2.7) is verified by a diagram chase, once we interpret (2.7) as obtained by passage to the dual functors in (2.1), described explicitly in Sect. 2.1.3.

We note that when the scheme X_1 is separated, the map (2.8) can be significantly simplified, see Sect. 2.3.

2.2.2. Consider the following diagram of schemes

$$\begin{array}{ccccc}
 & & & & X_1\times X_1\times X_1\times X_2 \\
 & & & & \Big\downarrow{\scriptstyle \mathrm{id}_{X_1}\times\Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2}} \\
 & & X_1\times X_1\times X_1\times X_2 & \xrightarrow{\Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2}} & X_1\times X_1\times X_1\times X_1\times X_2 \\
 & & \Big\downarrow{\scriptstyle p_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2}} & & \\
X_1\times X_2 & \xrightarrow{\Delta_{X_1}\times \mathrm{id}_{X_2}} & X_1\times X_1\times X_2 & & \\
\Big\downarrow{\scriptstyle p_{X_1}\times \mathrm{id}_{X_2}} & & & & \\
X_2 & & & &
\end{array}$$

For $\mathcal{M} \in \text{D-mod}(X_1)$ we start with the object

$$\mathcal{M} \boxtimes k_{X_1} \boxtimes \mathcal{P} \in \text{D-mod}(X_1 \times X_1 \times X_1 \times X_2). \tag{2.9}$$

The object

$$\mathsf{F}^{\mathrm{op}}_{X_1\to X_2;\mathcal{P}} \circ \text{Ps-Id}_{X_1}(\mathcal{M}) \in \text{Pro}(\text{D-mod}(X_2)),$$

i.e., the left-hand side of (2.8), applied to $\mathcal{M}$, equals the result the application to (2.9) of the following composition of functors

$$(p_{X_1}\times \mathrm{id}_{X_2})_! \circ (\Delta_{X_1}\times \mathrm{id}_{X_2})^\bullet \circ (p_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})_\bullet \circ \\ \circ (\Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1}\times \Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})_!. \tag{2.10}$$

2.2.3. Note that for a Cartesian diagram

$$\begin{array}{ccc} Y_{11} & \xrightarrow{f_1} & Y_{1,0} \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_0} \\ Y_{0,1} & \xrightarrow{f_0} & Y_{0,0} \end{array} \tag{2.11}$$

we have a canonically defined natural transformation

$$f_0^\bullet \circ (g_0)_\bullet \to (g_1)_\bullet \circ f_1^\bullet,$$

coming by adjunction from the isomorphism

$$(g_0)_\bullet \circ (f_1)_\bullet \simeq (f_0)_\bullet \circ (g_1)_\bullet.$$

Applying this to the square

$$\begin{array}{ccc} X_1\times X_1\times X_2 & \xrightarrow{\mathrm{id}_{X_1}\times\Delta_{X_1}\times\mathrm{id}_{X_2}} & X_1\times X_1\times X_1\times X_2 \\ {\scriptstyle p_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_2}}\downarrow & & \downarrow{\scriptstyle p_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_2}} \\ X_1\times X_2 & \xrightarrow{\Delta_{X_1}\times\mathrm{id}_{X_2}} & X_1\times X_1\times X_2, \end{array}$$

we obtain a natural transformation from (2.10) to

$$(p_{X_1}\times \mathrm{id}_{X_2})_! \circ (p_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})_\bullet \circ (\mathrm{id}_{X_1}\times\Delta_{X_1}\times \mathrm{id}_{X_2})^\bullet \circ \\ \circ (\Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1}\times \Delta_{X_1}\times \mathrm{id}_{X_1}\times \mathrm{id}_{X_2})_!. \tag{2.12}$$

I.e., we are now looking at the diagram

$$\begin{array}{ccccc} & & & & X_1\times X_1\times X_1\times X_2 \\ & & & & \downarrow{\scriptstyle \mathrm{id}_{X_1}\times\Delta_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_2}} \\ X_1\times X_1\times X_2 & \xrightarrow{\mathrm{id}_{X_1}\times\Delta_{X_1}\times\mathrm{id}_{X_2}} & X_1\times X_1\times X_1\times X_2 & \xrightarrow{\Delta_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_2}} & X_1\times X_1\times X_1\times X_1\times X_2 \\ {\scriptstyle p_{X_1}\times\mathrm{id}_{X_1}\times\mathrm{id}_{X_2}}\downarrow & & & & \\ X_1\times X_2 & & & & \\ {\scriptstyle p_{X_1}\times\mathrm{id}_{X_2}}\downarrow & & & & \\ X_2. & & & & \end{array}$$

2.2.4. Note also that for a Cartesian diagram (2.11) there is a canonical natural transformation

$$g_1^\bullet \circ f_0^! \to f_1^! \circ g_0^\bullet, \tag{2.13}$$

coming by adjunction from the base change isomorphism

$$f_0^! \circ (g_0)_\bullet \simeq (g_1)_\bullet \circ f_1^!.$$

Applying this to the square

$$\begin{array}{ccc}
X_1 \times X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_2 \\
{\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\
X_1 \times X_1 \times X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_1 \times X_2,
\end{array}$$

we obtain a natural transformation from (2.12) to

$$\begin{aligned}
&(p_{X_1} \times \mathrm{id}_{X_2})_! \circ (p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ (\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})^! \circ \\
&\qquad \circ (\mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})^\bullet \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})_!.
\end{aligned} \tag{2.14}$$

I.e., we are now looking at the diagram

$$\begin{array}{ccccc}
 & & & & X_1 \times X_1 \times X_1 \times X_2 \\
X_1 \times X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_2 & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} \\
{\scriptstyle p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} \downarrow & & & \searrow {\scriptstyle \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} & \\
X_1 \times X_2 & & & & X_1 \times X_1 \times X_1 \times X_1 \times X_2 \\
{\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} \downarrow & & & & \\
X_2. & & & &
\end{array}$$

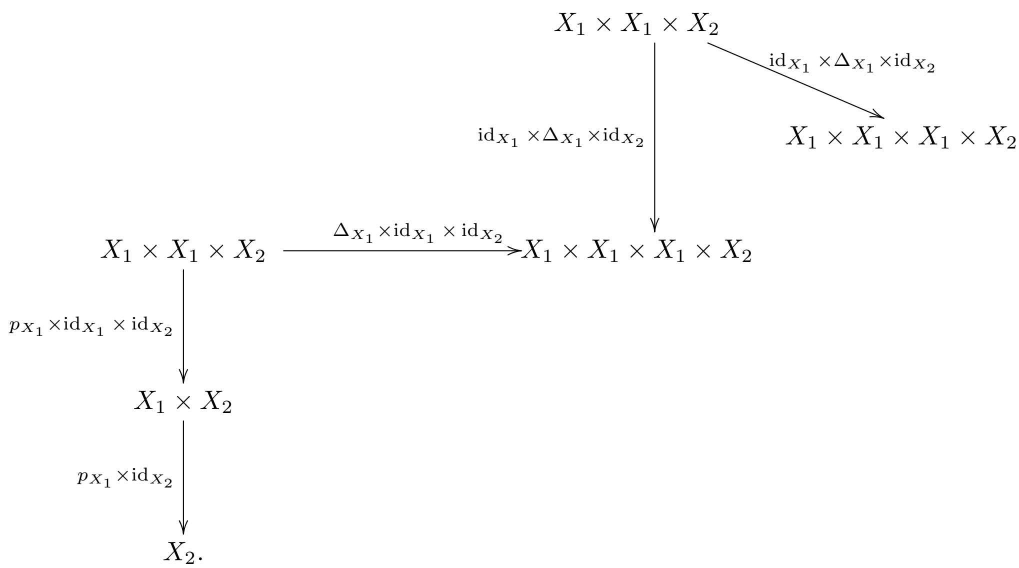
2.2.5. By base change along

$$\begin{array}{ccc}
X_1 \times X_1 \times X_2 & \xrightarrow{\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_2 \\
{\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} \\
X_1 \times X_1 \times X_1 \times X_2 & \xrightarrow[\mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}]{} & X_1 \times X_1 \times X_1 \times X_1 \times X_2,
\end{array}$$

we rewrite (2.14) as

$$\begin{aligned}
&(p_{X_1} \times \mathrm{id}_{X_2})_! \circ (p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ \\
&\qquad \circ (\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})_! \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})^\bullet.
\end{aligned} \tag{2.15}$$

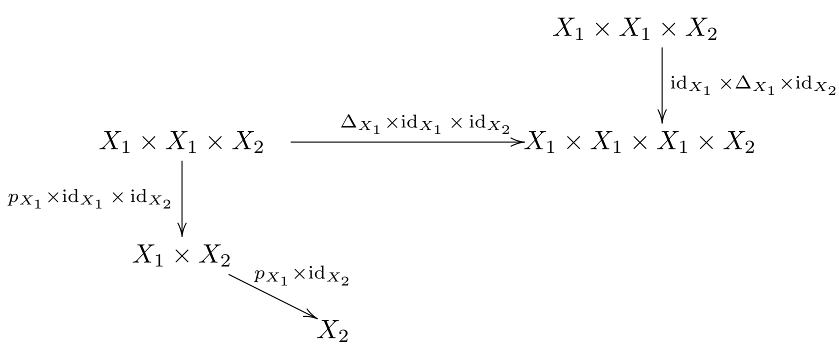
I.e., our diagram is now



2.2.6. Note now that

$$(\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})^\bullet(\mathcal{M} \boxtimes k_{X_1} \boxtimes \mathcal{P}) \simeq \mathcal{M} \boxtimes \mathcal{P}.$$

Hence, we are considering the diagram



and we need to calculate the functor

$$(p_{X_1} \times \mathrm{id}_{X_2})_! \circ (p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ (\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})_! \tag{2.16}$$

applied to $\mathcal{M} \boxtimes \mathcal{P} \in \text{D-mod}(X_1 \times X_1 \times X_2)$.

2.2.7. Consider again the Cartesian diagarm (2.11). Note that we have a canonical natural transformation

$$(f_0)_! \circ (g_1)_\bullet \to (g_0)_\bullet \circ (f_1)_! \tag{2.17}$$

that comes by adjunction from the base change isomorphism

$$(g_1)_\bullet \circ f_1^! \simeq f_0^! \circ (g_0)_\bullet.$$

Applying this to the square

$$\begin{CD}
X_1 \times X_1 \times X_2 @>{\mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}}>> X_1 \times X_2 \\
@V{p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}}VV @VV{p_{X_1} \times \mathrm{id}_{X_2}}V \\
X_1 \times X_2 @>{p_{X_1} \times \mathrm{id}_{X_2}}>> X_2,
\end{CD}$$

we obtain a natural transformation from (2.16) to the functor

$$(p_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ (\mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2})_! \circ (\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})_!. \tag{2.18}$$

I.e., we are now considering the diagram

$$\begin{array}{ccccc}
 & & & & X_1 \times X_1 \times X_2 \\
 & & & & \Big\downarrow{\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\
X_1 \times X_1 \times X_2 & & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & & X_1 \times X_1 \times X_1 \times X_2 \\
 & \searrow{\scriptstyle \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}} & & & \\
 & & X_1 \times X_2 & & \\
 & & \Big\downarrow{\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & \\
 & & X_2 & &
\end{array}$$

2.2.8. Returing again to (2.11), we have a natural transformation

$$(g_1)_! \circ f_1^! \to f_0^! \circ (g_0)_!,$$

obtained by adjunction from the isomorphism

$$f_1^! \circ g_0^! \simeq g_1^! \circ f_0^!.$$

Applying this to the square

$$\begin{CD}
X_1 \times X_1 \times X_2 @>{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}}>> X_1 \times X_1 \times X_1 \times X_2 \\
@V{\mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}}VV @VV{\mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}}V \\
X_1 \times X_2 @>{\Delta_{X_1} \times \mathrm{id}_{X_2}}>> X_1 \times X_1 \times X_2,
\end{CD}$$

we obtain a natural transformation from (2.18) to the functor

$$(p_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ (\Delta_{X_1} \times \mathrm{id}_{X_2})^! \circ (\mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2})_! \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})_!. \tag{2.19}$$

I.e., we are now looking at the diagram

$$\begin{array}{ccc}
 & & X_1 \times X_1 \times X_2 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\
 & & X_1 \times X_1 \times X_1 \times X_2 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}} \\
X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_2 \\
\Big\downarrow{\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & \\
X_2. & &
\end{array}$$

However, the composed morphism $(\mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}) \circ (\mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2})$ equals the identity, and hence, the functor in (2.19) identifies with

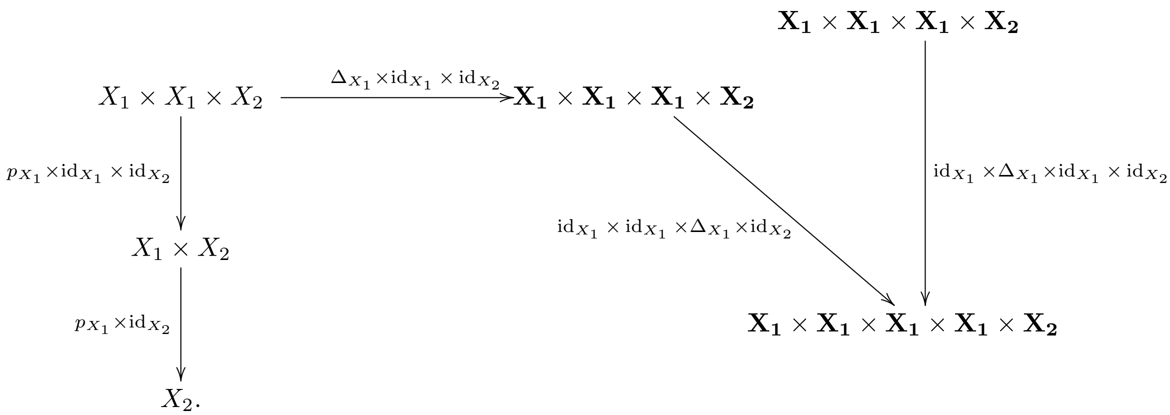
$$(p_{X_1} \times \mathrm{id}_{X_2})_\bullet \circ (\Delta_{X_1} \times \mathrm{id}_{X_2})^!.$$

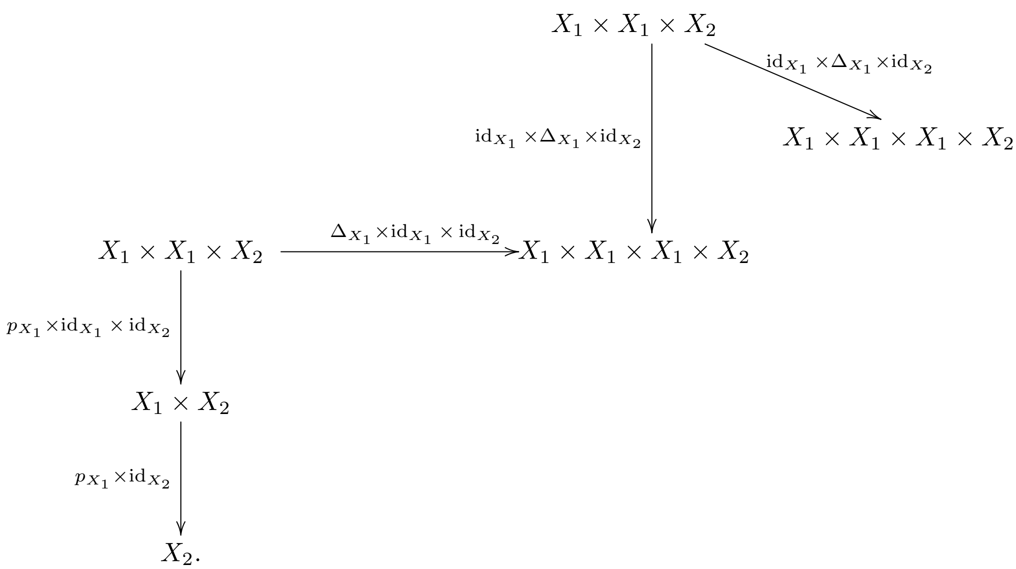
When applied to $\mathcal{M} \boxtimes \mathcal{P}$, this yields $\mathsf{F}_{X_1 \to X_2, \mathcal{P}}(\mathcal{M})$, i.e., the right-hand side in (2.8), applied to $\mathcal{M} \in \text{D-mod}(X_1)$.

2.2.9. *Summary.* Here is the picture of the evolution of diagrams (the highlighted portion is the one to undergo base change).

$$\begin{array}{ccccc}
 & & & & X_1 \times X_1 \times X_1 \times X_2 \\
 & & & & {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}}\Big\downarrow \\
 & & X_1 \times X_1 \times X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_1 \times X_2 \\
 & & \Big\downarrow{\scriptstyle p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & & \\
\mathbf{X_1} \times \mathbf{X_2} & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_2}} & \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} & & \\
\Big\downarrow{\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & & & \\
\mathbf{X_2} & & & &
\end{array}$$

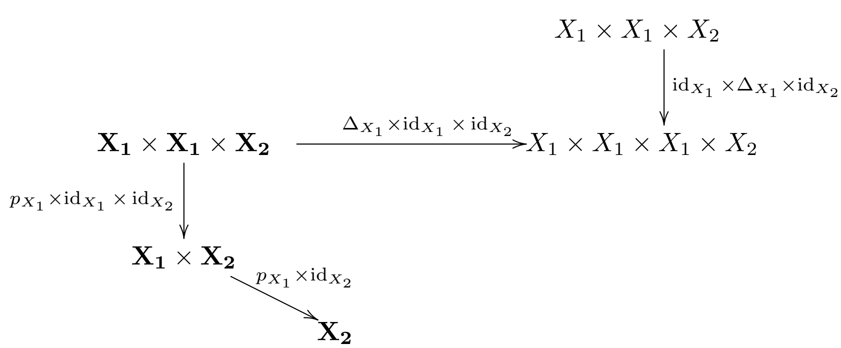
$$\begin{array}{cccc}
 & & & X_1 \times X_1 \times X_1 \times X_2 \\
\mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} & & & \\
 & \searrow{\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} & & {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}}\Big\downarrow \\
{\scriptstyle p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}}\Big\downarrow & \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} \\
X_1 \times X_2 & & & \\
{\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}}\Big\downarrow & & & \\
X_2. & & &
\end{array}$$





At this stage we note that the object of interest on $X_1 \times X_1 \times X_1 \times X_2$ comes as a pullback under $\mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}$ from $X_1 \times X_1 \times X_2$.

Hence, we resume with the next diagram:



$$
\begin{array}{ccccc}
 & & & & X_1 \times X_1 \times X_2 \\
 & & & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\
\mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} & & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & & \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_1} \times \mathbf{X_2} \\
 & \searrow {\scriptstyle \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}} & & & \\
 & & \mathbf{X_1} \times \mathbf{X_2} & & \\
 & & \downarrow {\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & \\
 & & X_2 & &
\end{array}
$$

$$
\begin{array}{ccc}
 & & X_1 \times X_1 \times X_2 \\
 & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\
 & & X_1 \times X_1 \times X_1 \times X_2 \\
 & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \mathrm{id}_{X_1} \times p_{X_1} \times \mathrm{id}_{X_2}} \\
X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_2 \\
\downarrow {\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & \\
X_2, & &
\end{array}
$$

while the latter diagram is equivalent to

$$
\begin{array}{ccc}
X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_2 \\
\downarrow {\scriptstyle p_{X_1} \times \mathrm{id}_{X_2}} & & \\
X_2. & &
\end{array}
$$

2.3. **Specializing to the separated case.** Assume now that the scheme X_1 is separated. In this case the natural transformation (2.8) can be significantly simplified.

2.3.1. First, we note if $f : Y \to Z$ is a separated morphism, there is a canonically defined natural transformation

$$f_! \to f_\bullet,$$

described as follows.

Consider the Cartesian diagram

$$
\begin{array}{ccc}
Y \underset{Z}{\times} Y & \xrightarrow{\mathrm{pr}_2} & Y \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle f} \\
Y & \xrightarrow{f} & Z.
\end{array}
$$

By (2.17), we have a natural transformation

$$f_! \circ (\mathrm{pr}_1)_\bullet \to f_\bullet \circ (\mathrm{pr}_2)_!.$$

Pre-composing with $(\Delta_{Y/Z})_\bullet \simeq (\Delta_{Y/Z})_!$, where

$$\Delta_{Y/Z} : Y \to Y \underset{Z}{\times} Y$$

(it is here that we use the assumption that $\Delta_{Y/Z}$ is a closed embedding), we obtain the desired natural transformation

$$f_! \simeq f_! \circ (\mathrm{pr}_1 \circ \Delta_{Y/Z})_\bullet \simeq f_! \circ (\mathrm{pr}_1)_\bullet \circ (\Delta_{Y/Z})_\bullet \to f_\bullet \circ (\mathrm{pr}_1)_! \circ (\Delta_{Y/Z})_! \simeq f_\bullet \circ (\mathrm{pr}_1 \circ \Delta_{Y/Z}) \simeq f_\bullet.$$

2.3.2. For X_1 separated, the morphism $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ is separated, and so the functor $\mathsf{F}^{\mathrm{op}}_{X_1 \to X_2;\mathcal{P}}$ admits a natural transformation to the functor

$$\mathcal{M} \mapsto (\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^\bullet(\mathcal{M}) \overset{\bullet}{\otimes} \mathcal{P}).$$

In this case, the natural transformation (2.8) is the composition of the above map

$$(\mathrm{pr}_2)_!(\mathrm{pr}_1^\bullet(\text{Ps-Id}_{X_1}(\mathcal{M})) \overset{\bullet}{\otimes} \mathcal{P}) \to (\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^\bullet(\text{Ps-Id}_{X_1}(\mathcal{M})) \overset{\bullet}{\otimes} \mathcal{P}),$$

and a natural transformation induced by a canonically defined map

$$\mathrm{pr}_1^\bullet(\text{Ps-Id}_{X_1}(\mathcal{M})) \overset{\bullet}{\otimes} \mathcal{P} \to \mathrm{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{P}, \tag{2.20}$$

described below.

2.3.3. Recall that for X_1 separated,

$$\text{Ps-Id}_{X_1}(\mathcal{M}) \simeq \mathcal{M} \overset{!}{\otimes} k_{X_1}.$$

The map in (2.20) comes from (2.13) applied to the Cartesian diagram

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_2 \\ {\scriptstyle \Delta_{X_1} \times \mathrm{id}_{X_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{X_1} \times \Delta_{X_1} \times \mathrm{id}_{X_2}} \\ X_1 \times X_1 \times X_2 & \xrightarrow{\Delta_{X_1} \times \mathrm{id}_{X_1} \times \mathrm{id}_{X_2}} & X_1 \times X_1 \times X_1 \times X_2, \end{array}$$

and the object

$$\mathcal{M} \boxtimes k_{X_1} \boxtimes \mathcal{P} \in \text{D-mod}(X_1 \times X_1 \times X_1 \times X_2).$$

3. Relation to $\mathcal{O}$-modules

The goal of this section is to express the condition on an object $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$ that the corresponding functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserve compactness, in terms of the underlying $\mathcal{O}$-modules. The material of this section will be used in Sect. 4, but not elsewhere in the paper.

3.1. **Recollections.** As we will be considering the forgetful functor from D-modules to $\mathcal{O}$-modules, derived algebraic geometry comes into play. Henceforth in this and the next section, by a "scheme" we will understand an *eventually coconnective DG scheme almost of finite type*, see Sect. 0.4.1.

3.1.1. For a scheme X understood as above, we will consider the categories $\text{IndCoh}(X)$ and $\text{QCoh}(X)$ (see [Ga1, Sect. 1] for the definition of the former and [GL:QCoh, Sect. 1] of the latter category).

The category $\text{IndCoh}(X)$ is compactly generated, and $\text{IndCoh}(X)^c = \text{Coh}(X)$, the latter being the full (but not cocomplete) subcategory of $\text{QCoh}(X)$ consisting of bounded complexes with coherent cohomology sheaves.

By a theorem of Thomason-Trobaugh, the category $\text{QCoh}(X)$ is also compactly generated by the subcategory $\text{QCoh}(X)^{\text{perf}}$ of perfect complexes.

The categories $\text{IndCoh}(X)$ and $\text{QCoh}(X)$ are connected by a pair of adjoint functors

$$\Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X) : \Xi_X,$$

where Ψ_X is obtained by ind-extending the tautological embedding $\text{Coh}(X) \hookrightarrow \text{QCoh}(X)$, and Ξ_X by ind-extending the tautological embedding $\text{QCoh}(X)^{\text{perf}} \hookrightarrow \text{Coh}(X) \hookrightarrow \text{IndCoh}(X)$.

The functor Ξ_X is fully faithfull by construction. The functors Ψ_X and Ξ_X are mutually inverse equivalences if and only if X is a smooth classical scheme.

3.1.2. For a pair of schemes X_1 and X_2, external tensor product defines a functor

$$\text{IndCoh}(X_1) \otimes \text{IndCoh}(X_2) \to \text{IndCoh}(X_1 \times X_2), \tag{3.1}$$

which is an equivalence by [Ga1, Proposition 6.4.2].

For a morphism $f : X_1 \to X_2$, we shall denote by f_*^{IndCoh} and $f^!$ the corresponding functors $\text{IndCoh}(X_1) \to \text{IndCoh}(X_2)$ and $\text{IndCoh}(X_2) \to \text{IndCoh}(X_1)$, respectively, see [Ga1, Sects. 3.1 and 5.2.3].

In particular, for a scheme X we have the functor

$$\overset{!}{\otimes} : \text{IndCoh}(X) \otimes \text{IndCoh}(X) \to \text{IndCoh}(X),$$

that identifies, under the equivalence $\text{IndCoh}(X) \otimes \text{IndCoh}(X) \simeq \text{IndCoh}(X \times X)$, with the functor $\Delta_X^! : \text{IndCoh}(X \times X) \to \text{IndCoh}(X)$.

We note that for $f = p_X$, the corresponding functor $(p_X)_*^{\text{IndCoh}}$ is canonically isomorphic to

$$\text{IndCoh}(X) \overset{\Psi_X}{\longrightarrow} \text{QCoh}(X) \overset{\Gamma_X}{\longrightarrow} \text{Vect},$$

where

$$\Gamma_X : \text{QCoh}(X) \to \text{Vect}$$

is the usual functor of global sections.

3.1.3. We recall (see [Ga1, Sect. 9.2.1]) that Serre duality defines a canonical equvalence

$$\mathbf{D}_X^{\text{Se}} : \text{IndCoh}(X)^\vee \simeq \text{IndCoh}(X).$$

The corresponding functor

$$(\text{IndCoh}(X)^c)^{\text{op}} = (\text{IndCoh}(X)^\vee)^c \overset{\mathbf{D}_X^{\text{Se}}}{\longrightarrow} \text{IndCoh}(X)^c$$

is the usual Serre duality functor

$$\mathbb{D}_X^{\text{Se}} : \text{Coh}(X)^{\text{op}} \to \text{Coh}(X),$$

see [Ga1, Sect. 9.5].

Under this equivalence, the unit object

$$\mathbf{u}_{\text{IndCoh}(X)} \in \text{IndCoh}(X)^\vee \otimes \text{IndCoh}(X) \simeq \text{IndCoh}(X) \otimes \text{IndCoh}(X) \simeq \text{IndCoh}(X \times X)$$

identifies with $(\Delta_X)_*^{\mathrm{IndCoh}}(\omega_X)$, where $\omega_X = p_X^!(k)$.

We note (see [Ga1, Proposition 9.6.12]) that due to the assumption that X is *eventually coconnective*, we have $\omega_X \in \mathrm{Coh}(X)$. In particular, if X is separated, the object

$$\mathbf{u}_{\mathrm{IndCoh}(X)} \in \mathrm{IndCoh}(X)^\vee \otimes \mathrm{IndCoh}(X)$$

is compact.

3.1.4. The category $\mathrm{QCoh}(X)$ is also canonically self dual: the equivalence

$$\mathbf{D}_X^{\mathrm{nv}} : \mathrm{QCoh}(X)^\vee \to \mathrm{QCoh}(X)$$

is uniquely determined by the fact that the corresponding equivalence

$$(\mathrm{QCoh}(X)^c)^{\mathrm{op}} = (\mathrm{QCoh}(X)^\vee)^c \overset{\mathbf{D}_X^{\mathrm{nv}}}{\longrightarrow} \mathrm{QCoh}(X)^c$$

is the usual duality functor

$$\mathbb{D}_X^{\mathrm{nv}} : (\mathrm{QCoh}(X)^{\mathrm{perf}})^{\mathrm{op}} \to \mathrm{QCoh}(X)^{\mathrm{perf}}, \quad \mathcal{E} \mapsto \underline{\mathrm{Hom}}_X(\mathcal{E}, \mathcal{O}_X).$$

The corresponding evaluation functor

$$\mathrm{ev}_{\mathrm{QCoh}_X} : \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \to \mathrm{Vect}$$

is

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \overset{\Delta_X^*}{\longrightarrow} \mathrm{QCoh}(X) \overset{\Gamma_X}{\to} \mathrm{Vect},$$

and the object

$$\mathbf{u}_{\mathrm{QCoh}(X)} \in \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$$

identifies with

$$(\Delta_X)_*(\mathcal{O}_X) \in \mathrm{QCoh}(X \times X) \simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X).$$

We recall (see [Ga1, Proposition 9.3.3]) that with respect to the self-dualities $\mathbf{D}_X^{\mathrm{nv}}$ and $\mathbf{D}_X^{\mathrm{Se}}$, the dual of the functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

is the functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X), \quad \mathcal{E} \mapsto \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \omega_X,$$

where $\underset{\mathcal{O}_X}{\otimes}$ is the functor

$$\mathrm{QCoh}(X) \otimes \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$$

equal to the ind-extension of the action of $\mathrm{QCoh}(X)^{\mathrm{perf}}$ on $\mathrm{Coh}(X)$ by tensor products.

3.1.5. We will consider the adjoint pair of (continuous) functors

$$\mathbf{ind}_X : \mathrm{IndCoh}(X) \rightleftarrows \text{D-mod}(X) : \mathbf{oblv}_X,$$

see [DrGa1, Sect. 5.1.5].

The functor $\mathbf{oblv}_X$ is conservative, which implies that the essential image of $\mathbf{ind}_X$ generates $\mathrm{IndCoh}(X)$. The latter, in turn, implies that the essential image of $\mathrm{IndCoh}(X)^c \simeq \mathrm{Coh}(X)$ Karoubi-generates $\text{D-mod}(X)^c$.

Consider now the functor

$$\mathbf{ind}_X^{\mathrm{left}} := \mathbf{ind}_X \circ \Upsilon_X : \mathrm{QCoh}(X) \to \text{D-mod}(X).$$

It is shown in [GR, Lemma 3.4.7] that $\mathbf{ind}_X^{\mathrm{left}}$ also admits a right adjoint, denoted $\mathbf{oblv}_X^{\mathrm{left}}$, and we have

$$\mathbf{oblv}_X \simeq \Upsilon_X \circ \mathbf{oblv}_X^{\mathrm{left}}.$$

In particular, $\mathbf{oblv}_X^{\text{left}}$ is also conservative. Hence, the essential image of $\text{QCoh}(X)^{\text{perf}}$ under the functor $\mathbf{ind}_X^{\text{left}}$ also Karoubi-generates $\text{D-mod}(X)^c$.

3.1.6. For a morphism $f : X_1 \to X_2$ we have canonical isomorphisms

$$\mathbf{ind}_{X_2} \circ f_*^{\text{IndCoh}} \simeq f_\bullet \circ \mathbf{ind}_{X_1}, \quad \text{IndCoh}(X_1) \to \text{D-mod}(X_2)$$

and

$$\mathbf{oblv}_{X_1} \circ f^! \simeq f^! \circ \mathbf{oblv}_{X_2}, \quad \text{D-mod}(X_2) \to \text{IndCoh}(X_1).$$

Finally, we recall (see [DrGa1, Sect. 5.3.4]) that with respect to the equivalences $\mathbf{D}_X^{\text{Se}}$ and $\mathbf{D}_X^{\text{Ve}}$, the functors $\mathbf{ind}_X$ and $\mathbf{oblv}_X$ satisfy

$$\mathbf{ind}_X^\vee \simeq \mathbf{oblv}_X. \tag{3.2}$$

3.2. **Criteria for preservation of compactness.** In this subsection we will give more explicit criteria for an object $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)^c$ to satisfy the assumption of Theorem 1.3.4, i.e., for the functor

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

to preserve compactness (or, equivalently, to admit a continuous right adjoint).

Remark 3.2.1. By Corollary 1.4.5, the same criterion, *with the roles of X_1 and X_2 swapped*, will tell us when $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ admits a *left* adjoint.

3.2.2. For $\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ consider the functors

$$\mathsf{F} \circ \mathbf{ind}_{X_1} : \text{IndCoh}(X_1) \to \text{D-mod}(X_2). \tag{3.3}$$

$$\mathsf{F} \circ \mathbf{ind}_{X_1}^{\text{left}} : \text{QCoh}(X_1) \to \text{D-mod}(X_2). \tag{3.4}$$

We claim:

Lemma 3.2.3. *For a functor $\mathsf{F} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$ the following conditions are equivalent:*

(a) F *preserves compactness.*

(b) $\mathsf{F} \circ \mathbf{ind}_{X_1}$ *preserves compactness.*

(c) $\mathsf{F} \circ \mathbf{ind}_{X_1}^{\text{left}}$ *preserves compactness.*

Proof. The implication (a) $\Rightarrow$ (b) (resp., (c)) follows from the fact that the functor $\mathbf{ind}_{X_1}$ (resp., $\mathbf{ind}_{X_1}^{\text{left}}$) preserves compactness, since its right adjoint, i.e., $\mathbf{oblv}_{X_1}$ (resp., $\mathbf{oblv}_{X_1}^{\text{left}}$), is continuous.

The implication (b) (resp., (c)) $\Rightarrow$ (a) follows from the fact that the image of $\text{Coh}(X_1)$ under $\mathbf{ind}_{X_1}$ (resp., $\text{QCoh}(X_1)^{\text{perf}}$ under $\mathbf{ind}_{X_1}^{\text{left}}$) Karoubi-generates $\text{D-mod}(X_1)^c$.

□

3.2.4. The usefulness of Lemma 3.2.3 lies in the fact that for $\mathcal{Q} \in \text{D-mod}(X_1 \times X_2)$, the functors $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}_{X_1}$ and $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}$ are more explicit than the original functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$.

Namely, for $\mathcal{F} \in \text{IndCoh}(X)$, the object $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}_{X_1}(\mathcal{F}) \in \text{D-mod}(X)$ is calculated as follows:

Consider the functor

$$\text{D-mod}(X_1) \overset{\mathbf{oblv}_{X_1}}{\longrightarrow} \text{IndCoh}(X_1) \overset{\mathcal{F} \overset{!}{\otimes} -}{\longrightarrow} \text{IndCoh}(X_1) \overset{(p_{X_1})^{\text{IndCoh}}_*}{\longrightarrow} \text{Vect}, \tag{3.5}$$

which is the dual of the functor $\text{Vect} \to \text{IndCoh}(X_1)$, corresponding to the object $\mathbf{ind}_{X_1}(\mathcal{F})$.

Then

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}_{X_1}(\mathcal{F}) \simeq \left((3.5) \otimes \text{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{Q}).$$

Similarly, for $\mathcal{E} \in \text{QCoh}(X)$, consider the functor

$$\text{D-mod}(X_1) \overset{\mathbf{oblv}_{X_1}}{\longrightarrow} \text{IndCoh}(X_1) \overset{\mathcal{E} \otimes -}{\longrightarrow} \text{IndCoh}(X_1) \overset{(p_{X_1})^{\text{IndCoh}}_*}{\longrightarrow} \text{Vect}, \tag{3.6}$$

or which is the same

$$\text{D-mod}(X_1) \overset{\mathbf{oblv}_{X_1}}{\longrightarrow} \text{IndCoh}(X_1) \overset{\Psi_{X_1}}{\longrightarrow} \text{QCoh}(X_1) \overset{\mathcal{E} \otimes -}{\longrightarrow} \text{QCoh}(X_1) \overset{\Gamma_{X_1}}{\longrightarrow} \text{Vect}\,.$$

Then

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}(\mathcal{E}) \simeq \left((3.6) \otimes \text{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{Q}).$$

In other words, the point is that the functors (3.3) and (3.4) only involve the operation of direct image

$$(p_{X_1})^{\text{IndCoh}}_* : \text{IndCoh}(X_1) \to \text{Vect} \text{ and } \Gamma_{X_1} : \text{QCoh}(X_1) \to \text{Vect},$$

rather than the more complicated functor of de Rham cohomology

$$(p_{X_1})_\bullet : \text{D-mod}(X_1) \to \text{Vect}\,.$$

3.2.5. From Lemma 3.2.3 we obtain:

Corollary 3.2.6. *Assume that X_1 is quasi-projective with an ample line bundle $\mathcal{L}$. Let $\mathcal{Q}$ be an object* $\text{D-mod}(X_1 \times X_2)$. *Then the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserves compactness if and only if the following equivalent conditions hold:*

(i) *For any $i \in \mathbb{Z}$, the object*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}(\mathcal{L}^{\otimes i}) \in \text{D-mod}(X_2)$$

is compact.

(ii) *There exists an integer i_0 such that the objects*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}(\mathcal{L}^{\otimes i}) \in \text{D-mod}(X_2)$$

are compact for all $i \geq i_0$.

(iii) *There exists an integer i_0 such that the objects*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}(\mathcal{L}^{\otimes i}) \in \text{D-mod}(X_2)$$

are compact for all $i \leq i_0$.

(iv) *For some specific interval $[i_1, i_2]$ that only depends on X_1, the objects*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}(\mathcal{L}^{\otimes i}) \in \text{D-mod}(X_2)$$

are compact for all $i_1 \leq i \leq i_2$.

Proof. By Lemma 3.2.3, we need to check when the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}^{\text{left}}_{X_1}$ preserves compactness. The statement of the corollary follows from the fact that the objects $\mathcal{L}^{\otimes i}$ in all of the four cases Karoubi-generate $\text{QCoh}(X)^{\text{perf}}$.

□

In particular, we obtain:

Corollary 3.2.7. *Assume that X_1 is affine. Then Let $\mathcal{Q}$ be an object* D-mod$(X_1 \times X_2)$. *Then the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserves compactness if and only if*

$$\left((\Gamma_{X_1} \circ \Psi_{X_1} \circ \mathbf{oblv}_{X_1}) \otimes \text{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{Q}) \in \text{D-mod}(X_2) \tag{3.7}$$

is compact.

3.2.8. Let us note that Corollary 3.2.7 implies the assertion of Theorem 1.3.13 in the particular case when X_1 is affine:

Let us recall that for a scheme X the category IndCoh(X) carries a canonical t-structure, see [Ga1, Sect. 1.2]. Its basic feature is that the functor $\Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X)$ is t-exact.

Note that since X_1 is smooth, the functor $\mathbf{oblv}_{X_1}$ is t-exact (see [GR, Proposition 4.2.11(a)]). Since X_1 is affine, we obtain that the composed functor

$$\Gamma_{X_1} \circ \Psi_{X_1} \circ \mathbf{oblv}_{X_1} : \text{D-mod}(X_1) \to \text{Vect}$$

is t-exact.

Hence, the same is true for the functor (3.7) (see Sect. 4.1, where the general statement along these lines is explained).

Now, the assertion of the theorem follows from the fact that if an object of D-mod(X_2) is compact, then the same is true for any subquotient of any of its cohomologies.

□

3.3. Preservation of compactness and compactness of the kernel.

3.3.1. Consider the category

$$\text{IndCoh}(X_1) \otimes \text{D-mod}(X_2),$$

which is endowed with a forgetful functor

$$\text{D-mod}(X_1 \times X_2) \simeq \text{D-mod}(X_1) \otimes \text{D-mod}(X_2) \overset{\mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}}{\longrightarrow} \to \text{IndCoh}(X_1) \otimes \text{D-mod}(X_2). \tag{3.8}$$

We claim:

Proposition 3.3.2. *Assume that X_1 is separated. Let $\mathcal{Q}$ be an object of* D-mod$(X_1 \times X_2)$*, such that the functor*

$$\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

preserves compactness. Then the image of $\mathcal{Q}$ under the functor (3.8) *is compact in*

$$\text{IndCoh}(X_1) \otimes \text{D-mod}(X_2).$$

Proof. If $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$ preserves compactness, then so does the functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}_{X_1}$. Hence, by Corollary 1.1.5, the same is true for the functor

$$\text{Id}_{\mathbf{C}} \otimes (\mathsf{F}_{X_1 \to X_2, \mathcal{Q}} \circ \mathbf{ind}_{X_1}) : \mathbf{C} \otimes \text{IndCoh}(X_1) \to \mathbf{C} \otimes \text{D-mod}(X_2) \tag{3.9}$$

for any DG category $\mathbf{C}$.

Note that the functor $\mathsf{F}_{\mathcal{Q},X_1\to X_2}\circ \mathbf{ind}_{X_1}$ is defined by the kernel

$$\left(\mathrm{Id}_{\mathrm{IndCoh}(X_1)^\vee}\otimes(\mathsf{F}_{X_1\to X_2,\mathcal{Q}}\circ \mathbf{ind}_{X_1})\right)(\mathbf{u}_{\mathrm{IndCoh}(X_1)})\in \mathrm{IndCoh}(X_1)^\vee\otimes \text{D-mod}(X_2). \tag{3.10}$$

By Sect. 3.1.3, the assumption that X_1 be separated implies that the object

$$\mathbf{u}_{\mathrm{IndCoh}(X_1)}\in \mathrm{IndCoh}(X_1)^\vee\otimes \mathrm{IndCoh}(X_1)\simeq \mathrm{IndCoh}(X_1)\otimes \mathrm{IndCoh}(X_1)\simeq \mathrm{IndCoh}(X_1\times X_1)$$

is compact. Hence, taking in (3.9) $\mathbf{C}:=\mathrm{IndCoh}(X_1)^\vee$, we obtain that the object in (3.10) is compact.

Finally, we observe that in terms of the identification

$$\mathrm{IndCoh}(X_1)^\vee\otimes \text{D-mod}(X_2)\overset{\mathbf{D}^{\mathrm{Se}}_{X_1}\otimes \mathrm{Id}_{\text{D-mod}(X_2)}}{\longrightarrow}\mathrm{IndCoh}(X_1)\otimes \text{D-mod}(X_2),$$

and using (3.2), the kernel of the functor $\mathsf{F}_{\mathcal{Q},X_1\to X_2}\circ \mathbf{ind}_{X_1}$ identifies with

$$(\mathbf{oblv}_{X_1}\otimes \mathrm{Id}_{\text{D-mod}(X_2)})(\mathcal{Q}).$$

□

3.3.3. We shall now prove:

Theorem 3.3.4. *Assume that the support of $\mathcal{Q}$ is* proper *over X_2. Then the assertion of Proposition 3.3.2 is "if and only if."*

Proof. Set

$$\mathcal{K}:=\left(\mathbf{oblv}_{X_1}\otimes \mathrm{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{Q})\in \mathrm{IndCoh}(X_1)\otimes \text{D-mod}(X_2).$$

Let $X_1\overset{j}{\hookrightarrow}\overline{X}_1$ be a compactification of X_1. Consider the object

$$\overline{\mathcal{K}}:=\left(j_*^{\mathrm{IndCoh}}\otimes \mathrm{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{K})\in \mathrm{IndCoh}(\overline{X}_1)\otimes \text{D-mod}(X_2).$$

We claim that $\overline{\mathcal{K}}$ is compact. Let us assume this and finish the proof of the theorem.

By Lemma 3.2.3 and Sect. 3.2.4, it suffices to show that for any $\mathcal{E}\in \mathrm{QCoh}(\overline{X}_1)^{\mathrm{perf}}$, we have

$$((p_{X_1})_*^{\mathrm{IndCoh}}\otimes \mathrm{Id}_{\text{D-mod}(X_2)})(\mathcal{E}|_{X_1}\underset{\mathcal{O}_{X_1}}{\otimes}\mathcal{K})\in \text{D-mod}(X_2)^c.$$

However,

$$((p_{X_1})_*^{\mathrm{IndCoh}}\otimes \mathrm{Id}_{\text{D-mod}(X_2)})(\mathcal{E}|_{X_1}\underset{\mathcal{O}_{X_1}}{\otimes}\mathcal{K})\simeq ((p_{\overline{X}_1})_*^{\mathrm{IndCoh}}\otimes \mathrm{Id}_{\text{D-mod}(X_2)})(\mathcal{E}\underset{\mathcal{O}_{\overline{X}_1}}{\otimes}\overline{\mathcal{K}}).$$

Note that the functor

$$\mathcal{E}\underset{\mathcal{O}_{\overline{X}_1}}{\otimes}-:\mathrm{IndCoh}(\overline{X}_1)\otimes \text{D-mod}(X_2)\to \mathrm{IndCoh}(\overline{X}_1)\otimes \text{D-mod}(X_2)$$

preserves compactness. Indeed, it admits a continuous right adjoint, given by $\mathcal{E}^\vee\underset{\mathcal{O}_{\overline{X}_1}}{\otimes}-$.

Now, the required assertion follows from the fact that the functor

$$(p_{\overline{X}_1})_*^{\mathrm{IndCoh}}\otimes \mathrm{Id}_{\text{D-mod}(X_2)}:\mathrm{IndCoh}(\overline{X}_1)\otimes \text{D-mod}(X_2)\to \text{D-mod}(X_2)$$

preserves compactness, which follows from the corresponding fact (Serre's theorem) for

$$(p_{\overline{X}_1})_*:\mathrm{IndCoh}(\overline{X}_1)\to \mathrm{Vect}.$$

To prove that $\overline{\mathcal{K}}$ is compact we proceed as follows.

By [Ga1, Corollary 10.3.6], we interpret the category $\operatorname{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2)$ as the category IndCoh of the prestack $\overline{X}_1 \times (X_2)_{\text{dR}}$ (see [GR, Sect. 1.1.1] for the definition of the de Rham prestack). Recall also that the assignment

$$\mathcal{X} \rightsquigarrow \operatorname{IndCoh}(\mathcal{X}), \quad \mathcal{X} \in \operatorname{PreStk}_{\text{laft}}$$

satisfies Zariski descent (see [Ga1, Sect. 10.4.2]).

Note that the Zariski site of $\overline{X}_1 \times (X_2)_{\text{dR}}$ is in bijection with that of $\overline{X}_1 \times X_2$. Set

$$U := X_1 \times X_2 \text{ and } V := \overline{X}_1 \times X_2 - S,$$

where S is the support of $\mathcal{Q}$, which is closed in $\overline{X}_1 \times X_2$, by assumption. By Zariski descent, the category $\operatorname{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2)$ identifies with

$$\left(\operatorname{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2)\right)_U \underset{\left(\operatorname{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2)\right)_{U \cap V}}{\times} \left(\operatorname{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2)\right)_V.$$

Hence, it suffices to show that the restriction of $\overline{\mathcal{K}}$ to both U and V is compact. However, the former yields $\mathcal{K}$, and the latter zero.

□

3.4. **The ULA property.**

3.4.1. Let X_1 and X_2 be smooth classical schemes, and let $f : X_2 \to X_1$ be a smooth morphism.

Definition 3.4.2. *We say that* $\mathcal{M} \in \text{D-mod}(X_2)$ *is ULA with respect to* f *if the functor*

$$\mathcal{N} \mapsto \mathcal{M} \overset{!}{\otimes} f^!(\mathcal{N}), \quad \text{D-mod}(X_1) \to \text{D-mod}(X_2) \tag{3.11}$$

preserves compactness.

Note that the question of being ULA is Zariski-local on X_2, and hence also on X_1. So, with no restriction of generailty we can assume that X_1 and X_2 are affine.

3.4.3. For $\mathcal{M}$ as above take

$$\mathcal{Q} := (f \times \operatorname{id}_{X_2})_\bullet(\mathcal{M}) \in \text{D-mod}(X_1 \times X_2),$$

where by a slight abuse of notation we denote by $f \times \operatorname{id}_{X_2} : X_2 \to X_1 \times X_2$ the graph of the map f.

Then the functor (3.11) is the same as the corresponding functor $\mathsf{F}_{X_1 \to X_2, \mathcal{Q}}$, so the above analysis applies.

In particular, we obtain:

Corollary 3.4.4. *If* $\mathcal{M}$ *is ULA with respect to* f*, then the same is true for any subquotient of any of its cohomologies.*

This follows immediately from Theorem 1.3.13 in the affine case, established in Sect. 3.2.8. Another proof follows from Proposition 3.4.10 below.

3.4.5. Applying Corollary 1.6.6, we obtain:

Corollary 3.4.6. *If* $\mathcal{M} \in \text{D-mod}(X_2)$ *is ULA with respect to* f*, then it is compact, and* $\mathbb{D}^{\text{Ve}}_{X_2}(\mathcal{M})$ *is also ULA.*

Finally, from Corollary 1.7.7 and Sect. 2.3 we obtain:

Corollary 3.4.7. *Let* $\mathcal{M} \in \text{D-mod}(X_2)$ *be ULA with respect to* f*. Then the functor*

$$\mathcal{N} \in \text{D-mod}(X_1) \rightsquigarrow \mathcal{M} \overset{\bullet}{\otimes} f^\bullet(\mathcal{N})$$

takes values in $\text{D-mod}(X_2) \subset \text{Pro}(\text{D-mod}(X_2))$*, and the natural map*

$$\mathcal{M} \overset{\bullet}{\otimes} f^\bullet(\mathcal{N}) \to \mathcal{M} \overset{!}{\otimes} f^!(\mathcal{N})[2\dim(X_1)]$$

coming from (2.13) *for the commutative diagram*

$$\begin{array}{ccc} X_2 & \xrightarrow{f\times \text{id}_{X_2}} & X_1 \times X_2 \\ {\scriptstyle f\times\text{id}_{X_2}}\downarrow & & \downarrow{\scriptstyle \Delta_{X_1}\times\text{id}_{X_2}} \\ X_1 \times X_2 & \xrightarrow{\text{id}_{X_1}\times(f\times\text{id}_{X_2})} & X_1\times X_1\times X_2 \end{array}$$

and the object

$$\mathcal{N} \boxtimes k_{X_1} \boxtimes \mathcal{M} \in \text{D-mod}(X_1 \times X_1 \times X_2),$$

is an isomorphism.

3.4.8. Let D_{X_2/X_1} be the sheaf of vertical differential operators on X_2 with respect to f. I.e., this is the subsheaf of rings in D_{X_2} generated by all functions and $T_{X_2/X_1} \subset T_{X_2}$. Still equivalently, D_{X_2/X_1} is the centralizer of $f^\cdot(\mathcal{O}_{X_1})$ in D_{X_2}.

We consider the corresponding DG category $\text{D-mod}_{\text{rel}}(X_2)$ (see, e.g., [DrGa1, Sect. 6.3]). By definition, in the affine situation, the category $\text{D-mod}_{\text{rel}}(X_2)$ is compactly generated by the object D_{X_2/X_1}. The category $\text{D-mod}_{\text{rel}}(X_2)$ is endowed with continuous conservative functors

$$\text{D-mod}(X_2) \overset{\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}}{\longrightarrow} \text{D-mod}_{\text{rel}}(X_2) \overset{\mathbf{oblv}_{\text{rel},X_2}}{\longrightarrow} \text{IndCoh}(X_2),$$

whose composition is the functor $\mathbf{oblv}_{X_2}$. The functors $\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}$ and $\mathbf{oblv}_{\text{rel},X_2}$ admit left adjoints, denoted $\mathbf{ind}_{\text{rel}\to\text{abs},X_2}$ and $\mathbf{ind}_{\text{rel},X_2}$, respectively.

In addition, the category $\text{D-mod}_{\text{rel}}(X_2)$ carries a t-structure in which the functor

$$\mathbf{oblv}_{\text{rel},X_2} : \text{D-mod}_{\text{rel}}(X_2) \to \text{IndCoh}(X_2)$$

is t-exact. This property determines the above t-structure uniquely.

Finally, it is easy to see that an object of $\text{D-mod}_{\text{rel}}(X_2)$ is compact if and only if it is cohomologically bounded, and its cohomologies are finitely generated as D_{X_2/X_1}-modules.

3.4.9. We claim:

Proposition 3.4.10. *An object* $\mathcal{M} \in \text{D-mod}(X_2)$ *is ULA with respect to* f *if and only if its image under the forgetful functor*

$$\mathbf{oblv}_{\text{abs}\to\text{rel},X_2} : \text{D-mod}(X_2) \to \text{D-mod}_{\text{rel}}(X_2)$$

is compact.

Proof. With no restriction of generality, we can assume that X_1 and X_2 are affine. Then the functor (3.11) preserves compactness if and only if it sends D_{X_1} to a compact object of $\text{D-mod}(X_2)$. Since X_1 is smooth, instead of D_{X_1} we can take $\mathbf{ind}_{X_1}(\omega_{X_1})$; it will still be a generator of $\text{D-mod}(X_1)$.

Thus, we need to show that the object

$$\mathcal{M}\overset{!}{\otimes} f^!(\mathbf{ind}_{X_1}(\omega_{X_1}))\in \text{D-mod}(X_2)$$

is compact if and only if $\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}(\mathcal{M})$ is compact.

Now, recall (see, e.g., [DrGa1, Sect. 6.3.4]) that for $\mathcal{F}\in\text{IndCoh}(X_1)$, the object

$$f^!(\mathcal{F})\in\text{IndCoh}(X_2)$$

has a natural structure of object of $\text{D-mod}_{\text{rel}}(X_2)$, i.e., is the image under $\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}$ of the same-named object of $\text{D-mod}_{\text{rel}}(X_2)$. Furthermore, by [DrGa1, Lemma 6.3.15]

$$f^!(\mathbf{ind}_{X_1}(\mathcal{F}))\simeq \mathbf{ind}_{\text{rel}\to\text{abs},X_2}(f^!(\mathcal{F})).$$

Combining this with the projection formula of [DrGa1, Proposition 6.3.12(b')], for $\mathcal{M}\in\text{D-mod}(X_2)$ we obtain a canonical isomorphism

$$\mathcal{M}\overset{!}{\otimes} f^!(\mathbf{ind}_{X_1}(\mathcal{F}))\simeq \mathbf{ind}_{\text{rel}\to\text{abs},X_2}(\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}(\mathcal{M})\overset{!}{\otimes} f^!(\mathcal{F})).$$

Hence, we obtain that $\mathcal{M}$ is ULA if and only if the object

$$\mathbf{ind}_{\text{rel}\to\text{abs},X_2}(\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}(\mathcal{M})\overset{!}{\otimes} f^!(\omega_{X_1}))\simeq \mathbf{ind}_{\text{rel}\to\text{abs},X_2}(\mathbf{oblv}_{\text{abs}\to\text{rel},X_2}(\mathcal{M}))\in\text{D-mod}(X_2)$$

is compact.

However, it is easy to see that an object $\mathcal{M}'\in\text{D-mod}_{\text{rel}}(X_2)$ is compact if and only if

$$\mathbf{ind}_{\text{rel}\to\text{abs},X_2}(\mathcal{M}')\in\text{D-mod}(X_2)$$

is compact.

□

4. Proof of the subquotient theorem

The goal of this section is to prove Theorem 1.3.13. The results of this section will not be used elsewhere in the paper.

4.1. **The tensor product t-structure.** Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two DG categories, each endowed with a t-structure. Consider the DG category $\mathbf{C}_1\otimes\mathbf{C}_2$. It inherits a t-structure where we set $(\mathbf{C}_1\otimes\mathbf{C}_2)^{>0}$ to be the full subcategory spanned by objects $\mathbf{c}$ that satisfy

$$\text{Maps}(\mathbf{c}_1\otimes\mathbf{c}_2,\mathbf{c})=0,\ \forall \mathbf{c}_1\in\mathbf{C}_1^{\leq 0},\mathbf{c}_2\in\mathbf{C}_2^{\leq 0}.$$

Equivalently, the subcategory $(\mathbf{C}_1\otimes\mathbf{C}_2)^{\leq 0}$ is generated under colimits by objects of the form $\mathbf{c}_1\otimes\mathbf{c}_2$ with $\mathbf{c}_i\in\mathbf{C}_i^{\leq 0}$.

4.1.1. Let us recall that a t-structure on a DG category $\mathbf{C}$ is said to be *compactly generated* if the category $\mathbf{C}^{\leq 0}$ is generated under colimits by the subcategory $\mathbf{C}^{\leq 0}\cap\mathbf{C}^c$. Equivalently, if

$$\mathbf{c}\in\mathbf{C}^{>0}\ \Leftrightarrow\ \text{Maps}(\mathbf{c}',\mathbf{c})=0,\ \forall\,\mathbf{c}'\in\mathbf{C}^{\leq 0}\cap\mathbf{C}^c.$$

E.g., this is the case for the standard t-structures on $\text{QCoh}(X)$, $\text{IndCoh}(X)$ and $\text{D-mod}(X)$ for a scheme X.

4.1.2. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be DG categories, both equipped with t-structures. Note that, by construction, if the t-structures on $\mathbf{C}_i$ are compactly generated, the same will be true for one on $\mathbf{C}_1 \otimes \mathbf{C}_2$.

We will use the following assertion:

Lemma 4.1.3. *Let $\mathbf{C}_1, \mathbf{C}_2, \widetilde{\mathbf{C}}_2$ be DG categories, each endowed with a t-structure, and let $\mathsf{F} : \mathbf{C}_2 \to \widetilde{\mathbf{C}}_2$ be a continuous functor. Consider the functor*

$$(\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F}) : \mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{C}_1 \otimes \widetilde{\mathbf{C}}_2.$$

(i) *If the functor F is right t-exact, then so is $\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F}$.*

(ii) *If the functor F is left t-exact, and the t-structure on $\mathbf{C}_1$ is compactly generated, then the functor $\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F}$ is also left t-exact.*

Remark 4.1.4. We do not know whether in point (ii) one can get rid of the assumption that the t-structure on $\mathbf{C}_1$ be compactly generated.

Proof. Point (i) is tautological. For point (ii), by the assumption on $\mathbf{C}_1$, it suffices to show that for $\mathbf{c} \in (\mathbf{C}_1 \otimes \mathbf{C}_2)^{>0}$ and for $\mathbf{c}_1 \in \mathbf{C}_1^{\leq 0} \cap \mathbf{C}_1^c$ and $\widetilde{\mathbf{c}}_2 \in \widetilde{\mathbf{C}}_2^{\leq 0}$, the object

$$\mathcal{M}aps_{\mathbf{C}_1 \otimes \widetilde{\mathbf{C}}_2}\Big(\mathbf{c}_1 \otimes \widetilde{\mathbf{c}}_2, (\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F})(\mathbf{c})\Big) \in \mathrm{Vect} \tag{4.1}$$

belongs to $\mathrm{Vect}^{>0}$.

Note that for a pair of DG categories $\mathbf{C}_1$ and $\widetilde{\mathbf{C}}_2$, and objects $\mathbf{c}_1 \in \mathbf{C}_1^c$, $\widetilde{\mathbf{c}}_2 \in \widetilde{\mathbf{C}}_2$ and $\mathbf{c}' \in \mathbf{C}_1 \otimes \widetilde{\mathbf{C}}_2$, we have a canonical isomorphism

$$\mathcal{M}aps_{\mathbf{C}_1 \otimes \widetilde{\mathbf{C}}_2}(\mathbf{c}_1 \otimes \widetilde{\mathbf{c}}_2, \mathbf{c}') \simeq \mathcal{M}aps_{\widetilde{\mathbf{C}}_2}\left(\widetilde{\mathbf{c}}_2, (\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\widetilde{\mathbf{C}}_2})(\mathbf{c}_1^\vee \otimes \mathbf{c}')\right),$$

where $\mathbf{c}_1^\vee$ is the object of $(\mathbf{C}_1^\vee)^c \simeq (\mathbf{C}_1^c)^{\mathrm{op}}$ corresponding to $\mathbf{c}_1 \in \mathbf{C}_1^c$ and where

$$\mathrm{ev}_{\mathbf{C}_1} : \mathbf{C}_1^\vee \otimes \mathbf{C}_1 \to \mathrm{Vect}$$

is the canonical evaluation functor.

Hence, we can rewrite (4.1) as

$$\mathcal{M}aps_{\widetilde{\mathbf{C}}_2}\left(\widetilde{\mathbf{c}}_2, (\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\widetilde{\mathbf{C}}_2})\left(\mathbf{c}_1^\vee \otimes (\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F})(\mathbf{c})\right)\right). \tag{4.2}$$

We have

$$(\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\widetilde{\mathbf{C}}_2})\left(\mathbf{c}_1^\vee \otimes (\mathrm{Id}_{\mathbf{C}_1} \otimes \mathsf{F})(\mathbf{c})\right) \simeq \mathsf{F} \circ (\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\mathbf{C}_2})(\mathbf{c}_1^\vee \otimes \mathbf{c}).$$

Now, since $\mathbf{c}_1 \in \mathbf{C}_1^{\leq 0}$ and $\mathbf{c} \in (\mathbf{C}_1 \otimes \mathbf{C}_2)^{>0}$, we have

$$(\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\mathbf{C}_2})(\mathbf{c}_1^\vee \otimes \mathbf{c}) \in \mathbf{C}_2^{>0}.$$

Hence, $\mathsf{F} \circ (\mathrm{ev}_{\mathbf{C}_1} \otimes \mathrm{Id}_{\mathbf{C}_2})(\mathbf{c}_1^\vee \otimes \mathbf{c}) \in \widetilde{\mathbf{C}}_2^{>0}$, since F is left t-exact. Hence, the expression in (4.2) belongs to $\mathrm{Vect}^{>0}$ since $\widetilde{\mathbf{c}}_2 \in \widetilde{\mathbf{C}}_2^{\leq 0}$.

□

4.2. **The t-structure on $(\mathcal{O}, \mathrm{D})$-bimodules.**

4.2.1. For a pair of schemes X_1 and X_2 consider the DG category

$$\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2),$$

endowed with the t-structure, induced by the t-structures on $\mathrm{IndCoh}(X_1)$ and $\text{D-mod}(X_2)$, respectively.

The goal of this subsection is to prove the following assertion:

Proposition 4.2.2. *Let $\mathcal{K}$ be a compact object in $\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2)$. Then any subquotient of any of its cohomologies (with respect to the above t-structure) is compact.*

The rest of this subsection is devoted to the proof of this proposition; so, the reader, who is willing to take the assertion of Proposition 4.2.2 on faith, can skip it.

4.2.3. Consider the DG category

$$\mathrm{QCoh}(X_1) \otimes \text{D-mod}(X_2),$$

endowed with the t-structure induced by the t-structures on $\mathrm{QCoh}(X_1)$ and $\text{D-mod}(X_2)$, respectively.

By Lemma 4.1.3, the functor

$$\Psi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)} : \mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2) \to \mathrm{QCoh}(X_1) \otimes \text{D-mod}(X_2)$$

is t-exact.

Lemma 4.2.4. *The functor $\Psi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}$ induces an equivalence*

$$(\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2))^{\geq 0} \to (\mathrm{QCoh}(X_1) \otimes \text{D-mod}(X_2))^{\geq 0}.$$

Proof. The functor in question admits a left adjoint, which is also a right inverse, given by

$$\mathcal{M} \mapsto \tau^{\geq 0}(\Xi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}(\mathcal{M})).$$

Hence, it remains to check that $\Psi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}$ is conservative when restricted to the subcategory $(\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2))^{\geq 0}$. Note that

$$\ker(\Psi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}) = \ker(\Psi_{X_1}) \otimes \text{D-mod}(X_2).$$

So, we need to show that the essential image of the fully faithful embedding

$$\ker(\Psi_{X_1}) \otimes \text{D-mod}(X_2) \hookrightarrow \mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2)$$

has a zero intersection with $(\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2))^{\geq 0}$.

Note (see [Ga1, Sect. 1.2.7]) that the essential image of $\ker(\Psi_{X_1})$ in $\mathrm{IndCoh}(X_1)$ belongs to $\mathrm{IndCoh}(X_1)^{<0}$ (in fact, to $\mathrm{IndCoh}(X_1)^{<-n}$ for any n). Hence, the desired assertion follows from Lemma 4.1.3(i).

□

Corollary 4.2.5. *The functor $\Psi_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}$ has the following properties:*

(i) *It is fully faithful when restricted to $(\mathrm{IndCoh}_{X_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)})^c$.*

(ii) *It induces an equivalence*

$$(\mathrm{IndCoh}(X_1) \otimes \text{D-mod}(X_2))^{\heartsuit} \to (\mathrm{QCoh}(X_1) \otimes \text{D-mod}(X_2))^{\heartsuit}.$$

4.2.6. We claim that the abelian category

$$(\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\heartsuit}$$

is the usual category of quasi-coherent sheaves of $(\mathcal{O}_{X_1}, \mathrm{D}_{X_2})$-modules on $X_1 \times X_2$.

Indeed, it is easy to see that the assertion is local, so we can assume that X_1 and X_2 are affine. In this case $(\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\heartsuit}$ admits a projective generator, namely $\tau^{\geq 0}(\mathcal{O}_{X_1}) \boxtimes \mathrm{D}_{X_2}$, where $\tau^{\geq 0}$ is the truncation functor.

4.2.7. Let

$$(\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}} \subset \mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2)$$

be the full subcategory spanned by cohomologically bounded objects with finitely generated cohomologies. As in [Ga1, Proposition 1.2.4] one shows that the category

$$(\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}$$

has a unique t-structure such that the functor

$$\mathrm{Ind}\left((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}\right) \to \mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2), \tag{4.3}$$

obtained by ind-extending the tautological embedding, is t-exact and induces an equivalence

$$\mathrm{Ind}\left((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}\right)^{\geq 0} \simeq (\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\geq 0}. \tag{4.4}$$

The Noetherianness of the sheaf of rings $\mathcal{O}_{X_1} \otimes \mathrm{D}_{X_2}$ implies that the functor $\Psi_{X_1} \otimes \mathrm{Id}_{\mathrm{D\text{-}mod}(X_2)}$ sends the compact generators of $\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2)$ to $(\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}$. Hence, we obtain that the functor $\Psi_{X_1} \otimes \mathrm{Id}_{\mathrm{D\text{-}mod}(X_2)}$ factors as a composition of a canonically defined functor

$$\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2) \to \mathrm{Ind}((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}), \tag{4.5}$$

followed by (4.3).

Lemma 4.2.8. *The functor* (4.5) *is an equivalence and is t-exact.*

Proof. The functor (4.5) is right t-exact by construction. We construct a functor

$$\mathrm{Ind}((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}}) \to \mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2), \tag{4.6}$$

right adjoint to (4.5) by ind-extending

$$(\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}} \hookrightarrow (\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{+} \simeq$$
$$\simeq (\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{+},$$

where the last equivalence is given by Lemma 4.2.4.

Being the right adjoint to a right t-exact functor, the functor (4.6) is left t-exact. Consider the composition

$$\mathrm{Ind}((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}})^{+} \overset{(4.6)}{\longrightarrow}$$
$$\to (\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{+} \to (\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{+}.$$

By Lemma 4.2.4 and (4.4), we obtain that the functor (4.6) is t-exact and induces an equivalence

$$\mathrm{Ind}((\mathrm{QCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{\mathrm{f.g.}})^{+} \to (\mathrm{IndCoh}(X_1) \otimes \mathrm{D\text{-}mod}(X_2))^{+}.$$

Since the compact objects of both

$$\text{Ind}((\text{QCoh}(X_1) \otimes \text{D-mod}(X_2))^{\text{f.g.}}) \text{ and } \text{IndCoh}(X_1) \otimes \text{D-mod}(X_2)$$

are contained in their eventually coconnective parts, we obtain that (4.6) is an equivalence.

□

4.2.9. *Proof of Proposition 4.2.2.* Follows from the fact that the sheaf of rings $\mathcal{O}_{X_1} \otimes \text{D}_{X_2}$ is Noetherian, combined with Lemma 4.2.8.

□

4.3. **Proof of Theorem 1.3.13.**

4.3.1. *Step 1.* Set

$$\mathcal{K} := \mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\mathcal{Q}) \in \text{IndCoh}(X_1) \otimes \text{D-mod}(X_2).$$

The functor $\mathbf{oblv}_{X_1}$ is t-exact because X_1 is smooth. Hence, the functor $\mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}$ is also t-exact by Lemma 4.1.3.

Therefore, if $\mathcal{Q}'$ is a subquotient of the n-th cohomology of $\mathcal{Q}$, we obtain that

$$\mathcal{K}' := \mathbf{oblv}_{X_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\mathcal{Q}') \in (\text{IndCoh}(X_1) \otimes \text{D-mod}(X_2))^{\heartsuit}$$

is a subquotient of the n-th cohomology of $\mathcal{K}$.

4.3.2. *Step 2.* Choose an affine open embedding $X_1 \overset{j}{\hookrightarrow} \overline{X}_1$, where $\overline{X}_1$ is projective, but not necessarily smooth (for aesthetic reasons we do not want to use desingularization; the latter allows to choose $\overline{X}_1$ smooth as well).

Set $\overline{\mathcal{Q}} := (j \times \text{id}_{X_2})_{\bullet}(\mathcal{Q})$, and

$$\overline{\mathcal{K}} := \mathbf{oblv}_{\overline{X}_1} \otimes \text{Id}_{\text{D-mod}(X_2)}(\overline{\mathcal{Q}}) \simeq (j_*^{\text{IndCoh}} \otimes \text{Id}_{\text{D-mod}(X_2)})(\mathcal{K}).$$

Since j is affine, the functor $j_*^{\text{IndCoh}} : \text{IndCoh}(X_1) \to \text{IndCoh}(\overline{X}_1)$ is t-exact. Hence, by Lemma 4.1.3, so is the functor

$$(j_*^{\text{IndCoh}} \otimes \text{Id}_{\text{D-mod}(X_2)}) : \text{IndCoh}(X_1) \otimes \text{D-mod}(X_2) \to \text{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2).$$

Hence, if $\mathcal{K}'$ is a subquotient of the n-th cohomology of $\mathcal{K}$, we obtain that

$$\overline{\mathcal{K}}' := (j_*^{\text{IndCoh}} \otimes \text{Id}_{\text{D-mod}(X_2)})(\mathcal{K}') \in (\text{IndCoh}(\overline{X}_1) \otimes \text{D-mod}(X_2))^{\heartsuit}$$

is a subquotient of the n-th cohomology of $\overline{\mathcal{K}}$.

By Proposition 1.3.6, the object $\overline{\mathcal{Q}}$ is compact. Hence, $\overline{\mathcal{K}}$ is compact by Proposition 3.3.2. By Proposition 4.2.2, we obtain that the object $\overline{\mathcal{K}}'$ is compact as well.

4.3.3. *Step 3.* We have have the following assertion, proved in Sect. 4.3.5 below:

Lemma 4.3.4. *Let Y_1 be projective with ample line bundle $\mathcal{L}$. Then for*

$$\mathcal{T} \in (\text{IndCoh}(Y_1) \otimes \text{D-mod}(Y_2))^c \cap (\text{IndCoh}(Y_1) \otimes \text{D-mod}(Y_2))^{\heartsuit},$$

there exists an integer i_0 such that for all $i \geq i_0$, the non-zero cohomologies of

$$\left((p_{Y_1})_*^{\text{IndCoh}} \otimes \text{Id}_{\text{D-mod}(Y_2)}\right)(\mathcal{L}^i \underset{\mathcal{O}_{Y_1}}{\otimes} \mathcal{T})$$

vanish.

Let $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}'$ be as in Step 2. By Lemma 4.3.4, we obtain that there exists an integer i_0 such that for $i \geq i_0$, the object

$$\left(\Gamma_{\overline{X}_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{L}^i \underset{\mathcal{O}_{\overline{X}_1}}{\otimes} \overline{\mathcal{K}}')$$

is acyclic off cohomological degree n, and appears as a subquotient of the n-th cohomology of

$$\left(\Gamma_{\overline{X}_1} \otimes \mathrm{Id}_{\text{D-mod}(X_2)}\right)(\mathcal{L}^i \underset{\mathcal{O}_{\overline{X}_1}}{\otimes} \overline{\mathcal{K}}).$$

Hence, we obtain that the assertion of the theorem follows from Corollary 3.2.6(ii), combined with the fact for a scheme X, if an object $\mathcal{M} \in \text{D-mod}(X)$ is compact, then the same is true for any subquotient of any cohomology of $\mathcal{M}$.

□

4.3.5. *Proof of Lemma 4.3.4.* The functor

$$(p_{Y_1})_*^{\mathrm{IndCoh}} \otimes \mathrm{Id}_{\text{D-mod}(Y_2)} : \mathrm{IndCoh}(Y_1) \otimes \text{D-mod}(Y_2) \to \text{D-mod}(Y_2)$$

is left t-exact by Lemma 4.1.3(ii).

Hence, for any

$$\mathcal{T} \in (\mathrm{IndCoh}(Y_1) \otimes \text{D-mod}(Y_2))^{\heartsuit}$$

the object

$$\left((p_{Y_1})_*^{\mathrm{IndCoh}} \otimes \mathrm{Id}_{\text{D-mod}(Y_2)}\right)(\mathcal{K}) \in \text{D-mod}(Y_2) \tag{4.7}$$

lives in $\text{D-mod}(Y_2)^{\geq 0}$.

Now, any $\mathcal{T}$ as above admits a *left* resolution $\mathcal{T}_\bullet$ whose terms $\mathcal{T}_n$ are of the form

$$\mathcal{F}_n \boxtimes \mathcal{M}_n, \quad \mathcal{F}_n \in \mathrm{Coh}(Y_1)^{\heartsuit},\ \mathcal{M}_n \in \text{D-mod}(Y_2)^c \cap \text{D-mod}(Y_2)^{\heartsuit}.$$

Note that the functor $(p_{Y_1})_*^{\mathrm{IndCoh}} \otimes \mathrm{Id}_{\text{D-mod}(Y_2)}$ has cohomological amplitude bounded *on the right* by $\dim(Y_1)$, because this is trie for $(p_{Y_1})_*^{\mathrm{IndCoh}}$. Hence, it is enough to show that for $n = 0, ..., \dim(Y_1)$ and $i \gg 0$, the higher cohomologies of

$$\left((p_{Y_1})_*^{\mathrm{IndCoh}} \otimes \mathrm{Id}_{\text{D-mod}(Y_2)}\right)(\mathcal{L}^i \underset{\mathcal{O}_{Y_1}}{\otimes} \mathcal{T}_n) \tag{4.8}$$

vanish. However, the expression in (4.8) is isomorphic to

$$\Gamma(Y_1, \mathcal{L}^{\otimes i} \otimes \mathcal{F}_n) \otimes \mathcal{M}_n,$$

and the assertion follows from the correponding fact for $\mathcal{F}_n$.

□

5. Proof of the main theorem for schemes, and generalizations

In this section we will prove Theorem 1.3.4 by establishing a general result along the same lines for arbitrary DG categories.

5.1. **Duality in a compactly generated category.**

5.1.1. Let $\mathbf{C}$ be a compactly generated category. Recall that we have a natural equivalence

$$(\mathbf{C}^c)^{\mathrm{op}} \simeq (\mathbf{C}^\vee)^c, \quad \mathbf{c} \mapsto \mathbf{c}^\vee.$$

We shall now extend the above assignment to a (non-continuous) functor

$$\mathbf{C}^{\mathrm{op}} \to \mathbf{C}^\vee. \tag{5.1}$$

Namely, for $\mathbf{c} \in \mathbf{C}$ we let $\mathbf{c}^\vee$ be the object of $\mathbf{C}^\vee$ characterized by the property that

$$\mathrm{Hom}_{\mathbf{C}^\vee}(\mathbf{c}_1^\vee, \mathbf{c}^\vee) := \mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, \mathbf{c}_1) \text{ for } \mathbf{c}_1 \in \mathbf{C}^c.$$

5.1.2. Explicitly, if $\mathbf{c} = \underset{i}{colim}\, \mathbf{c}_i$ with $\mathbf{c}_i \in \mathbf{C}^c$, then

$$\mathbf{c}^\vee = \underset{i}{lim}\, \mathbf{c}_i^\vee. \tag{5.2}$$

By construction, the assignment $\mathbf{c} \mapsto \mathbf{c}^\vee$ sends colimits to limits. In general, it is very ill-behaved.

5.1.3. From (5.2) we obtain:

Lemma 5.1.4. *The functor* (5.1) *is the* right Kan extension *of its restriction to* $(\mathbf{C}^c)^{\mathrm{op}}$.

Proof. This is the property of any functor from $\mathbf{C}^{\mathrm{op}}$ that commutes with limits. □

5.1.5. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be two objects of $\mathbf{C}$. We claim that there is a canonical map

$$\mathrm{ev}_{\mathbf{C}}(\mathbf{c}_1^\vee \otimes \mathbf{c}_2) \to \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2). \tag{5.3}$$

Indeed, for $\mathbf{c}_2$ compact, the map (5.3) is the isomorphism resulting from the tautological isomorphism

$$\mathrm{ev}_{\mathbf{C}}(- \otimes \mathbf{c}_2) \simeq \mathcal{M}aps_{\mathbf{C}^\vee}(\mathbf{c}_2^\vee, -).$$

In general, the map (5.3) results from the fact that the left-hand side is continuous as a functor of $\mathbf{c}_2$, and hence is the left Kan extension from its restriction to $\mathbf{C}^c$.

5.1.6. Note that for $\mathbf{c} \in \mathbf{C}$ we have a canonical map

$$\mathbf{c} \to (\mathbf{c}^\vee)^\vee. \tag{5.4}$$

We shall say that $\mathbf{c}$ is *reflexive* if the map (5.4) is an isomorphism.

It is clear that every compact object is reflexive. But the converse is obviously false.

5.1.7. *Interaction with functors.* Let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor that sends compact objects to compact ones. Consider the conjugate functor $\mathsf{F}^{\mathrm{op}} : \mathbf{C}_1^\vee \to \mathbf{C}_2^\vee$, see Sect. 1.5.1.

We claim that for $\mathbf{c}_1 \in \mathbf{C}_1$ we have a canonical map

$$\mathsf{F}^{\mathrm{op}}(\mathbf{c}_1^\vee) \to (\mathsf{F}(\mathbf{c}_1))^\vee, \tag{5.5}$$

that extends the tautological isomorphism for $\mathbf{c}_1 \in \mathbf{C}_1^c$.

The natural transformation (5.5) follows by adjunction from the fact that the functor

$$\mathbf{c}_1 \mapsto (\mathsf{F}(\mathbf{c}_1))^\vee, \quad (\mathbf{C}_1)^{\mathrm{op}} \to \mathbf{C}_2$$

is the right Kan extension of its restriction to $(\mathbf{C}_1^c)^{\mathrm{op}}$ (i.e., takes colimits in $\mathbf{C}_1$ to limits in $\mathbf{C}_2^\vee$).

5.2. **A general framework for Theorem 1.3.4.**

5.2.1. Let $\mathbf{C}$ be a compactly generated DG category. Recall that $\mathbf{u_C} \in \mathbf{C} \otimes \mathbf{C}^\vee$ denotes the object that defines the identity functor.

We consider the object

$$(\mathbf{u_C})^\vee \in (\mathbf{C} \otimes \mathbf{C}^\vee)^\vee = \mathbf{C}^\vee \otimes \mathbf{C}.$$

We let $\text{Ps-Id}_\mathbf{C}$ be the functor $\mathbf{C} \to \mathbf{C}$ defined by the kernel $(\mathbf{u_C})^\vee$. I.e., in the notations of Sect. 1.1.1,

$$\text{Ps-Id}_\mathbf{C} := \mathsf{F}_{\mathbf{C}\to\mathbf{C},(\mathbf{u_C})^\vee}.$$

Note that by construction

$$(\text{Ps-Id}_\mathbf{C})^\vee \simeq \text{Ps-Id}_{\mathbf{C}^\vee}, \quad \mathbf{C}^\vee \to \mathbf{C}^\vee. \tag{5.6}$$

5.2.2. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two compactly generated categories, and let $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between them that preserves compactness.

Let $\mathcal{Q} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_2$ be kernel of F i.e.,

$$\mathcal{Q} = (\text{Id}_{\mathbf{C}_1^\vee} \otimes \mathsf{F})(\mathbf{u}_{\mathbf{C}_1}).$$

Consider the functor

$$(\text{Id}_{\mathbf{C}_1^\vee} \otimes \mathsf{F}) : \mathbf{C}_1^\vee \otimes \mathbf{C}_1 \to \mathbf{C}_1^\vee \otimes \mathbf{C}_2.$$

By Corollary 1.1.5, this functor still preserves compactness. Applying (5.5) to this functor and the object $\mathbf{u}_{\mathbf{C}_1} \in \mathbf{C}_1^\vee \otimes \mathbf{C}_1$, we obtain a map

$$(\text{Id}_{\mathbf{C}_1} \otimes \mathsf{F}^{\text{op}})((\mathbf{u}_{\mathbf{C}_1})^\vee) \to ((\text{Id}_{\mathbf{C}_1^\vee} \otimes \mathsf{F})(\mathbf{u}_{\mathbf{C}_1}))^\vee, \tag{5.7}$$

where we note that

$$((\text{Id}_{\mathbf{C}_1^\vee} \otimes \mathsf{F})(\mathbf{u}_{\mathbf{C}_1}))^\vee \simeq \mathcal{Q}^\vee. \tag{5.8}$$

Theorem 5.2.3. *Assume that the map* (5.7) *is an isomorphism. Then the composed functor*

$$\mathbf{C}_2 \xrightarrow{\mathsf{F}^R} \mathbf{C}_1 \xrightarrow{\text{Ps-Id}_{\mathbf{C}_1}} \mathbf{C}_1$$

is given by the kernel

$$\mathcal{Q}^\vee \in \mathbf{C}_1 \otimes \mathbf{C}_2^\vee \simeq \mathbf{C}_2^\vee \otimes \mathbf{C}_1.$$

Proof. This is a tautology:

The kernel of the composition

$$\mathbf{C}_2 \xrightarrow{\mathsf{F}^R} \mathbf{C}_1 \xrightarrow{\text{Ps-Id}_{\mathbf{C}_1}} \mathbf{C}_1,$$

viewed as an object of $\mathbf{C}_2^\vee \otimes \mathbf{C}_1$, is obtained from the kernel of $\text{Ps-Id}_{\mathbf{C}_1}$, viewed as an object of $\mathbf{C}_1^\vee \otimes \mathbf{C}_1$, by applying the functor

$$(\mathsf{F}^R)^\vee \otimes \text{Id}_{\mathbf{C}_1} : \mathbf{C}_1^\vee \otimes \mathbf{C}_1 \to \mathbf{C}_2^\vee \otimes \mathbf{C}_1.$$

By Lemma 1.5.3, the latter is the same as

$$(\mathsf{F}^{\text{op}} \otimes \text{Id}_{\mathbf{C}_1})((\mathbf{u}_{\mathbf{C}_1})^\vee),$$

which identifies with $\mathcal{Q}$ by (5.8) and the assumption of the theorem.

□

5.3. **The smooth case.**

5.3.1. Recall that a DG category $\mathbf{C}$ is called *smooth* if the object

$$\mathbf{u}_{\mathbf{C}} \in \mathbf{C} \otimes \mathbf{C}^\vee$$

is compact.

Remark 5.3.2. The terminology "smooth" originates in the fact that for a separated scheme X, the DG category $\operatorname{QCoh}(X)$ is smooth if and only if X is a smooth classical scheme (see Sect. 3.1 for our conventions regarding schemes).

5.3.3. Note that the assumption of Theorem 5.2.3 is trivially satisfied when $\mathbf{C}_1$ is *smooth*. Indeed, the map (5.5) is by definition an isomorphism for $\mathbf{c}_1$ compact.

5.3.4. *Proof of Theorem 1.3.4.* This is follows immediately from Theorem 5.2.3 and Sect. 5.3.3, using the fact that

$$\mathbf{u}_{\text{D-mod}(X)} = (\Delta_X)_\bullet(\omega_X) \in \text{D-mod}(X \times X),$$

being a bounded holonomic complex, is compact.

$\square$

5.3.5. *The natural transformation.* Let us continue to assume that $\mathbf{C}_1$ is smooth, and let us be in the situation of Theorem 5.2.3.

The (iso)morphism of functors

$$\text{Ps-Id}_{\mathbf{C}_1} \circ (\mathsf{F}_{\mathbf{C}_1 \to \mathbf{C}_2, \mathcal{Q}})^R \to \mathsf{F}_{\mathbf{C}_2 \to \mathbf{C}_1, \mathcal{Q}^\vee}$$

gives rise to (and is determined by) the natural transformation

$$\text{Ps-Id}_{\mathbf{C}_1} \to \mathsf{F}_{\mathbf{C}_2 \to \mathbf{C}_1, \mathcal{Q}^\vee} \circ \mathsf{F}_{\mathbf{C}_1 \to \mathbf{C}_2, \mathcal{Q}}. \tag{5.9}$$

Let us write down the natural transformation (5.9) explicitly:

The datum of a map

$$\text{Ps-Id}_{\mathbf{C}_1} \to \mathsf{F}_{\mathbf{C}_2 \to \mathbf{C}_1, \mathcal{Q}^\vee} \circ \mathsf{F}_{\mathbf{C}_1 \to \mathbf{C}_2, \mathcal{Q}}$$

is equivalent to that of a map between the corresponding kernels, i.e.,

$$(\mathbf{u}_{\mathbf{C}_1})^\vee \to (\operatorname{Id}_{\mathbf{C}_1^\vee} \otimes \operatorname{ev}_{\mathbf{C}_2} \otimes \operatorname{Id}_{\mathbf{C}_1})(\mathcal{Q} \otimes \mathcal{Q}^\vee),$$

and the latter is the same as a datum of a vector in

$$\operatorname{ev}_{\mathbf{C}_1^\vee}\left((\operatorname{Id}_{\mathbf{C}_1^\vee} \otimes \operatorname{ev}_{\mathbf{C}_2} \otimes \operatorname{Id}_{\mathbf{C}_1})(\mathcal{Q} \otimes \mathcal{Q}^\vee)\right) \simeq \operatorname{ev}_{\mathbf{C}_1^\vee \otimes \mathbf{C}_2}(\mathcal{Q} \otimes \mathcal{Q}^\vee).$$

Now vector corresponding to (5.9) is the canonical vector in

$$\operatorname{ev}_{\mathbf{C}}(\mathbf{c} \otimes \mathbf{c}^\vee) \simeq \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, \mathbf{c}),$$

defined for any DG category $\mathbf{C}$ and $\mathbf{c} \in \mathbf{C}^c$ (where we take $\mathbf{C} = \mathbf{C}_1^\vee \otimes \mathbf{C}_2$ and $\mathbf{c} = \mathcal{Q}$).

5.4. **Gorenstein categories.**

5.4.1. Following Drinfeld, we shall say that a compactly generated category $\mathbf{C}$ is *Gorenstein* if the functor

$$\text{Ps-Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$$

is an equivalence.

For example, the category $\text{D-mod}(X)$ on a smooth separated scheme X is Gorenstein.

5.4.2. The origin of the name is explained by the following assertion. Recall (see [Ga1, Sect. 7.3.3]) that a scheme X is said to be Gorenstein if ω_X, regarded as an object of $\operatorname{Coh}(X)$, is invertible (i.e., a cohomologically shifted line bundle).

Proposition 5.4.3. *For a separated scheme X the following assertions are equivalent:*

(a) *The scheme X is Gorenstein;*

(b) *The category $\operatorname{QCoh}(X)$ is Gorenstein;*

(c) *The category $\operatorname{IndCoh}(X)$ is Gorenstein.*

Proof. First we note that for a separated scheme X the object

$$(\mathbf{u}_{\operatorname{QCoh}(X)})^\vee \in \operatorname{QCoh}(X)\otimes\operatorname{QCoh}(X)\simeq\operatorname{QCoh}(X\times X)$$

identifies with

$$\underline{\operatorname{Hom}}_{X\times X}((\Delta_X)_*(\mathcal{O}_X),\mathcal{O}_{X\times X})\simeq(\Delta_X)_*(\Delta_X^!(\mathcal{O}_{X\times X})). \tag{5.10}$$

and the object

$$(\mathbf{u}_{\operatorname{IndCoh}(X)})^\vee \in \operatorname{IndCoh}(X)\otimes\operatorname{IndCoh}(X)\simeq\operatorname{IndCoh}(X\times X)$$

identifies with

$$\mathbb{D}^{\mathrm{Se}}_{X\times X}((\Delta_X)_*^{\operatorname{IndCoh}}(\omega_X))\simeq(\Delta_X)_*^{\operatorname{IndCoh}}(\mathcal{O}_X). \tag{5.11}$$

Assume first that X is Gorenstein, i.e., $\omega_X\simeq\mathcal{L}$, where $\mathcal{L}$ is a cohomologically shifted line.

In this case $\Delta_X^!(\mathcal{O}_{X\times X})$ identifies with $\omega_X\otimes\mathcal{L}^{\otimes -2}\simeq\mathcal{L}^{\otimes -1}$. I.e., the functor $\text{Ps-Id}_{\operatorname{QCoh}(X)}$ is given by tensor product by $\mathcal{L}^{\otimes -1}$, and thus is an equivalence.

Similarly, $\text{Ps-Id}_{\operatorname{IndCoh}(X)}$ is also given by the action of $\mathcal{L}^{\otimes -1}$, in the sense of the monoidal action of $\operatorname{QCoh}(X)$ on $\operatorname{IndCoh}(X)$, and hence is also an equivalence.

Vice versa, assume that $\text{Ps-Id}_{\operatorname{QCoh}(X)}$ is an equivalence. It suffices to show that for every k-point $i_x:\mathrm{pt}\to X$, the object

$$i_x^*(\omega_X)\in\operatorname{Vect}$$

is invertible. By duality (in Vect) it suffices to show that $(i_x^*(\omega_X))^\vee$ is invertible. However,

$$(i_x^*(\omega_X))^\vee=\mathcal{M}aps_{\operatorname{Vect}}(i_x^*(\omega_X),k)\simeq\mathcal{M}aps_{\operatorname{Coh}(X)}(\omega_X,(i_x)_*(k)),$$

which by Serre duality identifies with $(i_x)^!(\mathcal{O}_X)$.

By (5.10), the assumption that $\text{Ps-Id}_{\operatorname{QCoh}(X)}$ is an equivalence means that that the object

$$\Delta_X^!(\mathcal{O}_{X\times X})\in\operatorname{QCoh}(X)$$

has the property that the functor of tensoring by it is an equivalence. Hence, $\Delta_X^!(\mathcal{O}_{X\times X})$ is a cohomologically shifted line bundle; denote it by $\mathcal{L}'$. Hence, for i_x as above,

$$i_x^!(\mathcal{O}_X)\otimes i_x^!(\mathcal{O}_X)\simeq(i_x\times i_x)^!(\mathcal{O}_{X\times X})\simeq i_x^!\circ\Delta_X^!(\mathcal{O}_{X\times X})\simeq i_x^!(\mathcal{L}')\simeq i_x^!(\mathcal{O}_X)\otimes i_x^*(\mathcal{L}'),$$

from which it follows that $i_x^!(\mathcal{O}_X)$ is invertible, as required.

Assume now that $\text{Ps-Id}_{\operatorname{IndCoh}(X)}$ is an equivalence. By (5.11), this implies that $\mathcal{O}_X$, regarded as object of $\operatorname{IndCoh}(X)$, is invertible with respect to the $\overset{!}{\otimes}$ symmetric monoidal structure on $\operatorname{IndCoh}(X)$. In particular, for every $i_x:\mathrm{pt}\to X$ as above, $(i_x)^!(\mathcal{O}_X)$ is invertible in Vect. By the above, this implies that $i_x^*(\omega_X)$ is invertible, as required.

□

Remark 5.4.4. The following observation is due to A. Arinkin: the same proof as above shows that the functor

$$\mathcal{E} \mapsto \Upsilon(\mathcal{E}) := \mathcal{E} \otimes \omega_X$$

establishes an equivalence between $\mathrm{QCoh}(X)^{\mathrm{perf}}$ and the category of *dualizable* objects of IndCoh with respect to the $\overset{!}{\otimes}$ symmetric monoidal structure.

5.4.5. We shall now give a criterion for a compactly generated DG category $\mathbf{C}$ to be Gorenstein.

Proposition 5.4.6. *Suppose that the functors* $\text{Ps-Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ *and* $\text{Ps-Id}_{\mathbf{C}^\vee} : \mathbf{C}^\vee \to \mathbf{C}^\vee$ *both satisfy the assumption of Theorem 5.2.3. Suppose also that* $\mathbf{u} \in \mathbf{C} \otimes \mathbf{C}^\vee$ *is reflexive. Then* $\mathbf{C}$ *is Gorenstein.*

Remark 5.4.7. Note that Proposition 5.4.6 has the following flavor: *certain finiteness properties of a functor imply that this functor is an equivalence.*

Proof. We apply Theorem 5.2.3 to $\mathsf{F} = \text{Ps-Id}_{\mathbf{C}}$. Combining with the assumtion that

$$((\mathbf{u}_{\mathbf{C}})^\vee)^\vee \simeq \mathbf{u}_{\mathbf{C}},$$

we obtain

$$\text{Ps-Id}_{\mathbf{C}} \circ \mathsf{F}^R \simeq \text{Id}_{\mathbf{C}}. \tag{5.12}$$

I.e., we obtain that $\text{Ps-Id}_{\mathbf{C}}$ admits a right inverse. Passing to the dual functors in (5.12) for $\mathbf{C}^\vee$, and using the fact that $(\text{Ps-Id}_{\mathbf{C}})^\vee \simeq \text{Ps-Id}_{\mathbf{C}^\vee}$, we obtain that $\text{Ps-Id}_{\mathbf{C}}$ also has a left inverse. Hence, it is an equivalence.

□

6. Generalization to Artin stacks: quasi-compact case

6.1. QCA stacks: recollections.

6.1.1. In this section all algebraic stacks will be assumed QCA. Recall (see [DrGa1, Definition 1.1.8]) that an algebraic stack $\mathcal{X}$ is said to be QCA if it is quasi-compact and the automorphism group of every field-valued point is *affine*.

We recall (see [DrGa1, Theorem 8.1.1]) that for a QCA stack the category $\text{D-mod}(X)$ is compactly generated. Furthermore, by [DrGa1, Corollary 8.3.4], for *any* prestack $\mathcal{X}'$, the operation of external tensor product defines an equivalence

$$\text{D-mod}(\mathcal{X}) \otimes \text{D-mod}(\mathcal{X}') \to \text{D-mod}(\mathcal{X} \times \mathcal{X}').$$

We let $\text{D-mod}(\mathcal{X})_{\text{coh}} \subset \text{D-mod}(\mathcal{X})$ be the full subcategory of *coherent* D-modules. We remind that an object of $\text{D-mod}(\mathcal{X})$ is called coherent if its pullback to any scheme, mapping smoothly to $\mathcal{X}$, is compact (see [DrGa1, Sect. 7.3.1]).

We always have

$$\text{D-mod}(\mathcal{X})^c \subset \text{D-mod}(\mathcal{X})_{\text{coh}},$$

and the containment is an equality if and only if $\mathcal{X}$ is *safe*, which means that the automorphism group of every field-valued point is such that its neutral connected component is unipotent ([DrGa1, Corollary 10.2.7]). For example, any Deligne-Mumford stack (and, in particular, any algebraic space) is safe.

The category $\text{D-mod}(\mathcal{X})_{\text{coh}}$ carries a canonical Verdier duality anti-involution

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}} : (\text{D-mod}(\mathcal{X})_{\text{coh}})^{\text{op}} \to \text{D-mod}(\mathcal{X})_{\text{coh}}.$$

6.1.2. The basic property of the functor $\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}$ is that it preserves the subcategory $\text{D-mod}(\mathcal{X})^c$, thereby inducing an equivalence

$$\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}} : (\text{D-mod}(\mathcal{X})^c)^{\mathrm{op}} \to \text{D-mod}(\mathcal{X})^c$$

(see [DrGa1, Corollary 8.4.2]).

Hence, it induces an equivalence $\text{D-mod}(\mathcal{X})^\vee \to \text{D-mod}(\mathcal{X})$ that we denote by $\mathbf{D}^{\mathrm{Ve}}_{\mathcal{X}}$. The unit and counit corresponding to the identification $\mathbf{D}^{\mathrm{Ve}}_{\mathcal{X}}$ are described below, see Sect. 6.1.4.

6.1.3. For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ we have the functor $f^! : \text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1)$. The usual de Rham direct image functor (defined as in [DrGa1, Sect. 7.4.1])

$$f_\bullet : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$$

is in general non-continuous.

In fact, $f_\bullet$ is continuous if and only if f is *safe* (i.e., its geometric fibers are safe algebraic stack). E.g., any schematic or representable morphism is safe.

In [DrGa1, Sect. 9.3] another functor

$$f_\blacktriangle : \text{D-mod}(\mathcal{X}_1) \to \text{D-mod}(\mathcal{X}_2)$$

is introduced, which is by definition the ind-extension of the restriction of the functor $f_\bullet$ to $\text{D-mod}(\mathcal{X}_1)^c$. I.e., $f_\blacktriangle$ is the unique continuous functor which equals $f_\bullet$ when restricted to $\text{D-mod}(\mathcal{X}_1)^c$.

We have a natural transformation

$$f_\blacktriangle \to f_\bullet, \tag{6.1}$$

which is an isomorphism if f is safe. For any f, (6.1) is an isomorphism when evaluated on compact objects.

6.1.4. We can now describe explicitly the unit and the counit of the identification $\mathbf{D}^{\mathrm{Ve}}_{\mathcal{X}}$. Namely, the unit is given by the object

$$(\Delta_{\mathcal{X}})_\bullet(\omega_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X}) \simeq \text{D-mod}(\mathcal{X}) \otimes \text{D-mod}(\mathcal{X}) \simeq \text{D-mod}(\mathcal{X})^\vee \otimes \text{D-mod}(\mathcal{X}),$$

where $(\Delta_{\mathcal{X}})_\bullet \simeq (\Delta_{\mathcal{X}})_\blacktriangle$ since the morphism $\Delta_{\mathcal{X}}$ is representable and hence safe. The object $\omega_{\mathcal{X}}$ is, as in the case of scheme, $(p_{\mathcal{X}})^!(k)$, where $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{pt}$.

The counit corresponds to the functor

$$\text{D-mod}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\Delta^!_{\mathcal{X}}} \text{D-mod}(\mathcal{X}) \xrightarrow{(p_{\mathcal{X}})_\blacktriangle} \mathrm{Vect}\,.$$

6.1.5. For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$, with respect to the equivalences

$$\mathbf{D}^{\mathrm{Ve}}_{\mathcal{X}_i} : \text{D-mod}(\mathcal{X}_i)^\vee \to \text{D-mod}(\mathcal{X}_i),$$

we have

$$(f_\blacktriangle)^\vee \simeq f^!.$$

For a pair of QCA algebraic stacks, we have an equivalence of DG categories

$$\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2) \simeq \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{X}_1), \text{D-mod}(\mathcal{X}_2)),$$

$$\mathcal{Q} \mapsto \mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}}, \quad \mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}}(\mathcal{M}) = (\mathrm{pr}_2)_\blacktriangle(\mathrm{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}).$$

and

$$\mathsf{F} \mapsto \mathcal{Q}_{\mathsf{F}} := (\mathrm{Id}_{\text{D-mod}(\mathcal{X}_1)} \otimes \mathsf{F})((\Delta_{\mathcal{X}_1})_\bullet(\omega_{\mathcal{X}_1})).$$

6.1.6. We have the following assertion to be used in the sequel:

Lemma 6.1.7. *The restriction of the functor* $\mathcal{F} \mapsto \mathcal{F}^\vee$ *to* $\text{D-mod}(\mathcal{X})_{\text{coh}}$ *identifies canonically with* $\mathbb{D}^{\text{Ve}}_{\mathcal{X}}$.

Proof. We need to show that for $\mathcal{F} \in \text{D-mod}(\mathcal{X})_{\text{coh}}$ and $\mathcal{F}_1 \in \text{D-mod}(\mathcal{X})^c$ there exists a canonical isomorphism

$$\mathcal{M}aps(\mathcal{F}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F})) \simeq \mathcal{M}aps(\mathcal{F}, \mathcal{F}_1^\vee).$$

However, this follows from the fact that $\mathcal{F}_1^\vee = \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}_1)$ and the fact that $\mathbb{D}^{\text{Ve}}_{\mathcal{X}}$ is an anti-self equivalence on $\text{D-mod}(\mathcal{X})_{\text{coh}}$.

□

Corollary 6.1.8. *Every object of* $\text{D-mod}(\mathcal{X})_{\text{coh}} \subset \text{D-mod}(\mathcal{X})$ *is reflexive.*

We will need the following generalization of Corollary 6.1.8:

Proposition 6.1.9. *Every object of* $\text{D-mod}(\mathcal{X})$ *with coherent cohomologies is reflexive. The functor* $\mathcal{F} \mapsto \mathcal{F}^\vee$*, restricted to the full subcategory of* $\text{D-mod}(\mathcal{X})$ *spanned by objects with coherent cohomologies, is an involutive anti-self equivalence and is of bounded cohomological amplitude.*

Proof. Note that the functor $\mathbb{D}^{\text{Ve}}_{\mathcal{X}}$ has a bounded cohomological amplitude, say by k. We claim that for $\mathcal{F} \in \text{D-mod}(\mathcal{X})$ with coherent cohomologies we have

$$\tau^{\geq -n, \leq n}(\mathcal{F}^\vee) \simeq \tau^{\geq -n, \leq n}\left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\tau^{\geq -n-k, \leq n+k}(\mathcal{F}))\right), \quad \forall n \geq 0. \tag{6.2}$$

This would prove the assertion of the proposition.

To prove (6.2), we note that since the t-structure on $\text{D-mod}(\mathcal{X})$ is left and right complete, for $\mathcal{F}$ with coherent cohomologies there is a canonically defined object $\widetilde{\mathcal{F}} \in \text{D-mod}(\mathcal{X})$ such that

$$\tau^{\geq -n, \leq n}(\widetilde{\mathcal{F}}) = \tau^{\geq -n, \leq n}\left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\tau^{\geq -n-k, \leq n+k}(\mathcal{F}))\right), \quad \forall n \geq 0.$$

We have to show that for $\mathcal{F}_1 \in \text{D-mod}(\mathcal{X})^c$, there is a canonical isomorphism

$$\mathcal{M}aps(\mathcal{F}_1, \widetilde{\mathcal{F}}) \simeq \mathcal{M}aps(\mathcal{F}, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}_1)).$$

We shall do it separately in the cases $\mathcal{F} \in \text{D-mod}(\mathcal{X})^-$ and $\mathcal{F} \in \text{D-mod}(\mathcal{X})^+$ in such a way that the two isomorphisms coincide for $\mathcal{F} \in \text{D-mod}(\mathcal{X})^- \cap \text{D-mod}(\mathcal{X})^+$, i.e., when $\mathcal{F}$ belongs $\text{D-mod}(\mathcal{X})_{\text{coh}}$.

For $\mathcal{F} \in \text{D-mod}(\mathcal{X})^-$, we have

$$\widetilde{\mathcal{F}} \simeq \underset{n}{colim}\, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\tau^{\geq -n}(\mathcal{F})).$$

Hence, since $\mathcal{F}_1$ is compact,

$$\mathcal{M}aps(\mathcal{F}_1, \widetilde{\mathcal{F}}) \simeq \underset{n}{colim}\, \mathcal{M}aps\left(\mathcal{F}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\tau^{\geq -n}(\mathcal{F}))\right) \simeq \underset{n}{colim}\, \mathcal{M}aps(\tau^{\geq -n}(\mathcal{F}), \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}_1)).$$

However, since $\mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}_1)$ is in $\text{D-mod}(\mathcal{X})^+$, the last colimit stabilizes to

$$\mathcal{M}aps(\mathcal{F}, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}_1)),$$

as required.

For $\mathcal{F} \in \text{D-mod}(\mathcal{X})^+$, we have

$$\widetilde{\mathcal{F}} \simeq \underset{n}{lim}\, \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\tau^{\leq n}(\mathcal{F})).$$

Hence,

$$\mathcal{M}aps(\mathcal{F}_1, \widetilde{\mathcal{F}}) \simeq \underset{n}{lim}\, \mathcal{M}aps\left(\mathcal{F}_1, \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}(\tau^{\leq n}(\mathcal{F}))\right) \simeq$$
$$\simeq \underset{n}{lim}\, \mathcal{M}aps(\tau^{\leq n}(\mathcal{F}), \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}(\mathcal{F}_1)) \simeq \mathcal{M}aps(\mathcal{F}, \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}(\mathcal{F}_1)).$$

□

In what follows, for $\mathcal{F} \in \text{D-mod}(\mathcal{X})$ with coherent cohomologies we shall denote

$$\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}(\mathcal{F}) := \mathcal{F}^{\vee}.$$

6.2. **Direct image with compact supports.**

6.2.1. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism between QCA stacks. Let $f_!$ denote the partially defined left adjoint to the functor $f^! : \text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1)$.

The following is a particular case of Lemma 1.5.6:

Lemma 6.2.2. *Let $\mathcal{F}_1$ be an object of* $\text{D-mod}(\mathcal{X}_1)^c$ *for which the object*

$$f_\blacktriangle(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1)) \in \text{D-mod}(\mathcal{X}_2)$$

belongs to $\text{D-mod}(\mathcal{X}_2)^c$*. Then the functor $f_!$ is defined on $\mathcal{F}_1$ and we have a canonical isomorphism*

$$f_!(\mathcal{F}_1) \simeq \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_2}\left(f_\blacktriangle(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right).$$

We shall now prove its generalization where instead of compact objects we consider coherent ones:

Proposition 6.2.3. *Let $\mathcal{F}_1$ be an object of* $\text{D-mod}(\mathcal{X}_1)_{\mathrm{coh}}$ *for which the object*

$$f_\bullet(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1)) \in \text{D-mod}(\mathcal{X}_2)$$

belongs to $\text{D-mod}(\mathcal{X}_2)_{\mathrm{coh}}$*. Then $f_!(\mathcal{F}_1)$ is well-defined and we have a canonical isomorphism*

$$f_!(\mathcal{F}_1) \simeq \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_2}\left(f_\bullet(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right).$$

Remark 6.2.4. Note that in Proposition 6.2.3 we use the functor $f_\bullet$ rather than $f_\blacktriangle$. This does not contradict Lemma 6.2.2 since the two functors coincide on compact objects. We also remind that the two functors coincide when f is safe (e.g., schematic or representable).

Proof. We need to establish a functorial isomorphism

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{X}_2)}\left(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_2}\left(f_\bullet(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right), \mathcal{F}_2\right) \simeq \mathcal{M}aps_{\text{D-mod}(\mathcal{X}_1)}(\mathcal{F}_1, f^!(\mathcal{F}_2)), \quad \mathcal{F}_2 \in \text{D-mod}(\mathcal{X}_2).$$

Since both $\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_2}\left(f_\bullet(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right)$ and $\mathcal{F}_1$ are coherent, and the functor $f^!$ has a bounded cohomological amplitude, we can assume that $\mathcal{F}_2 \in \text{D-mod}(\mathcal{X}_2)^-$. Furthermore, since both $\text{D-mod}(\mathcal{X}_1)$ and $\text{D-mod}(\mathcal{X}_2)$ are left complete in their respective t-structures, and we can moreover assume that $\mathcal{F}_2 \in \text{D-mod}(\mathcal{X}_2)^b$.

Note that for a QCA stack $\mathcal{X}$ and $\mathcal{F} \in \text{D-mod}(\mathcal{X})_{\mathrm{coh}}$, the functor $\mathcal{M}aps_{\text{D-mod}(\mathcal{X})}(\mathcal{F}, -)$ commutes with colimits *taken in* $\text{D-mod}(\mathcal{X})^{\geq -n}$, for any fixed n.

This allows to assume that $\mathcal{F}_2 \in \text{D-mod}(\mathcal{X}_2)_{\mathrm{coh}}$. Hence, we need to establish an isomorphism

$$(6.3)\quad \mathcal{M}aps_{\text{D-mod}(\mathcal{X}_2)}\left(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_2}(\mathcal{F}_2), f_\bullet(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right) \simeq \mathcal{M}aps_{\text{D-mod}(\mathcal{X}_1)}(\mathcal{F}_1, f^!(\mathcal{F}_2)), \quad \text{D-mod}(\mathcal{X}_2)_{\mathrm{coh}}.$$

We claim that the latter isomorphism holds for any $\mathcal{F}_i \in \text{D-mod}(\mathcal{X}_i)_{\mathrm{coh}}$, $i = 1, 2$.

Indeed, the definition of $f_\bullet$ (see [DrGa1, Sect. 7.4.1]) allows to reduce the proof of (6.3) to the case when $\mathcal{X}_1$ is a scheme. Thus, we can assume that $\mathcal{F}_1 \in \text{D-mod}(\mathcal{X}_1)^c$ and that f is safe, so $f_\bullet = f_\blacktriangle$. In this case, the right-hand side of (6.3) identifies with

$$\text{ev}_{\text{D-mod}(\mathcal{X}_1)}\left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1) \otimes f^!(\mathcal{F}_2)\right) \simeq \text{ev}_{\text{D-mod}(\mathcal{X}_2)}\left(\mathcal{F}_2 \otimes f_\blacktriangle(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))\right).$$

Moreover, by [DrGa1, Lemma 10.4.2(a)], the object $f_\blacktriangle(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1)) \in \text{D-mod}(\mathcal{X}_2)$ is *safe*.

Let $\mathcal{X}$ be any QCA stack, and $\mathcal{F} \in \text{D-mod}(\mathcal{X})_{\text{coh}}$, $\mathcal{F}' \in \text{D-mod}(\mathcal{X})$. The morphism (5.3) gives rise to a map

$$\text{ev}_{\text{D-mod}(\mathcal{X})}(\mathcal{F} \otimes \mathcal{F}') \to \mathcal{M}aps_{\text{D-mod}(\mathcal{X})}\left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}), \mathcal{F}'\right). \tag{6.4}$$

The map $\leftarrow$ in (6.3) will be the map (6.4) for $\mathcal{X} := \mathcal{X}_2$, $\mathcal{F} := \mathcal{F}_2$, $\mathcal{F}' = f_\blacktriangle(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1}(\mathcal{F}_1))$. Hence, it remains to show that the map (6.4) is an isomorphism whenever $\mathcal{F}'$ is safe.

We have:

$$\text{ev}_{\text{D-mod}(\mathcal{X})}(\mathcal{F} \otimes \mathcal{F}') \simeq (p_{\mathcal{X}})_\blacktriangle(\mathcal{F} \overset{!}{\otimes} \mathcal{F}'),$$

and by [DrGa1, Lemma 7.3.5],

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{X})}\left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\mathcal{F}), \mathcal{F}'\right) = (p_{\mathcal{X}})_\bullet(\mathcal{F} \overset{!}{\otimes} \mathcal{F}'),$$

and the map (6.4) comes from the natural transformation $(p_{\mathcal{X}})_\blacktriangle \to (p_{\mathcal{X}})_\bullet$.

Finally, if $\mathcal{F}'$ is safe, then so is $\mathcal{F} \overset{!}{\otimes} \mathcal{F}'$, and hence the map

$$(p_{\mathcal{X}})_\blacktriangle(\mathcal{F} \overset{!}{\otimes} \mathcal{F}') \to (p_{\mathcal{X}})_\bullet(\mathcal{F} \overset{!}{\otimes} \mathcal{F}')$$

is an isomorphism by [DrGa1, Proposition 9.2.9].

□

6.2.5. For a QCA algebraic stack, we consider the object

$$k_{\mathcal{X}} := \mathbb{D}^{\text{Ve}}_{\mathcal{X}}(\omega_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X})_{\text{coh}}.$$

By Proposition 6.2.3, the object

$$(\Delta_{\mathcal{X}})_!(k_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X})_{\text{coh}}$$

is well-defined and is isomorphic to

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}\times\mathcal{X}}\left((\Delta_{\mathcal{X}})_\blacktriangle(\omega_{\mathcal{X}})\right),$$

where we recall that $(\Delta_{\mathcal{X}})_\blacktriangle \simeq (\Delta_{\mathcal{X}})_\bullet$, since $\Delta_{\mathcal{X}}$ is representable and hence safe.

Note, however, that neither $(\Delta_{\mathcal{X}})_\blacktriangle(\omega_{\mathcal{X}})$ nor $(\Delta_{\mathcal{X}})_!(k_{\mathcal{X}})$ are in general compact.

We define the functor

$$\text{Ps-Id}_{\mathcal{X}} : \text{D-mod}(\mathcal{X}) \to \text{D-mod}(\mathcal{X})$$

to be given by the kernel $(\Delta_{\mathcal{X}})_!(k_{\mathcal{X}})$ in the sense of Sect. 6.1.4

6.3. **The theorem for stacks.**

6.3.1. We have the following analog of Theorem 1.3.4 for QCA stacks.

Theorem 6.3.2. *Let* $\mathcal{Q}$ *be an object of* $\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)_{\text{coh}}$. *Assume that the corresponding functor*

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

admits a continuous right adjoint. Then the functor

$$\mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})} : \text{D-mod}(X_2) \to \text{D-mod}(X_1)$$

identifies canonically with

$$\text{D-mod}(\mathcal{X}_2) \overset{(\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R}{\longrightarrow} \text{D-mod}(\mathcal{X}_1) \overset{\text{Ps-Id}_{\mathcal{X}_1}}{\longrightarrow} \text{D-mod}(\mathcal{X}_1).$$

6.3.3. Using Lemma 1.5.3, from Theorem 6.3.2 we obtain:

Corollary 6.3.4. *Let* $\mathcal{Q}$ *be an object of* $\text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)_{\text{coh}}$. *Assume that the corresponding functor*

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

admits a continuous right adjoint. Then the functor

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})} : \text{D-mod}(X_1) \to \text{D-mod}(X_2)$$

identifies canonically with

$$\text{D-mod}(\mathcal{X}_1) \overset{\text{Ps-Id}_{\mathcal{X}_1}}{\longrightarrow} \text{D-mod}(\mathcal{X}_1) \overset{(\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^{\text{op}}}{\longrightarrow} \text{D-mod}(\mathcal{X}_2).$$

6.4. **Proof of Theorem 6.3.2.**

6.4.1. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be QCA stacks, let $\mathcal{M}$ be an object of $\text{D-mod}(\mathcal{Y}_1)_{\text{coh}}$, and let

$$\mathsf{G} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2),$$

given by a kernel $\mathcal{P} \in \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)_{\text{coh}}$. Assume that G preserves compactness.

We wish to know when the map

$$\mathsf{G}^{\text{op}}(\mathbb{D}^{\text{Ve}}_{\mathcal{Y}_1}(\mathcal{M})) = \mathsf{G}^{\text{op}}(\mathcal{M}^\vee) \to \mathsf{G}(\mathcal{M})^\vee \tag{6.5}$$

of (5.5) is an isomorphism.

Consider the map

$$\mathsf{G}(\mathcal{M}) = (\text{pr}_2)_\blacktriangle(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{P}) \to (\text{pr}_2)_\bullet(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{P}) \tag{6.6}$$

of (6.1).

Lemma 6.4.2. *If* (6.6) *is an isomorphism, then so is* (6.5).

Remark 6.4.3. The proof of Lemma 6.4.2 will show that if (6.6) is an isomorphism, then $\mathsf{G}(\mathcal{M})$ has coherent cohomologies and hence $\mathsf{G}(\mathcal{M})^\vee$ is the same as $\mathbb{D}^{\text{Ve}}_{\mathcal{Y}_2}(\mathsf{G}(\mathcal{M}))$.

6.4.4. Let us assume Lemma 6.4.2 and finish the proof of the theorem. We need to show that the functor $\mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}}$ satisfies the condition of Theorem 5.2.3.

We will apply Lemma 6.4.2 in the following situation. We take

$$\mathcal{Y}_1=\mathcal{X}_1\times\mathcal{X}_1,\ \mathcal{Y}_2=\mathcal{X}_1\times\mathcal{X}_2,\ \mathcal{M}=(\Delta_{\mathcal{X}_1})_{\blacktriangle}(\omega_{\mathcal{X}_1}),\ \mathsf{G}=\mathrm{Id}_{\text{D-mod}(\mathcal{X}_1)}\otimes\mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}},$$

so that

$$\mathcal{P}\in\text{D-mod}(\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_2)$$

is

$$\sigma_{2,3}\left((\Delta_{\mathcal{X}_1})_{\blacktriangle}(\omega_{\mathcal{X}_1})\boxtimes\mathcal{Q}\right),$$

where $\sigma_{2,3}$ is the transposition of the corresponding factors.

Base change for the $\blacktriangle$-pushforward and !-pullback for the Cartesian diagram

$$\begin{array}{ccc}
\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_2 & \xrightarrow{\sigma_{4,5}(\Delta_{\mathcal{X}_1\times\mathcal{X}_1}\times\mathrm{id}_{\mathcal{X}_1\times\mathcal{X}_2})} & \mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_2 \\
{\scriptstyle \Delta^2_{\mathcal{X}_1}\times\mathrm{Id}_{\mathcal{X}_2}}\uparrow & & \uparrow{\scriptstyle \sigma_{2,3}(\Delta_{\mathcal{X}_1\times\mathcal{X}_1}\times\mathrm{id}_{\mathcal{X}_1\times\mathcal{X}_2})} \\
\mathcal{X}_1\times\mathcal{X}_2 & \xrightarrow{\Delta^2_{\mathcal{X}_1}\times\mathrm{Id}_{\mathcal{X}_2}} & \mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_2
\end{array}$$

(here $\Delta^2_{\mathcal{X}_2}$ denotes the diagonal morphism $\mathcal{X}_1\to\mathcal{X}_1\times\mathcal{X}_1\times\mathcal{X}_1$) implies that in our case the left-hand side in (6.6) is canonically isomorphic to

$$\mathcal{Q}\in\text{D-mod}(\mathcal{X}_1\times\mathcal{X}_2).$$

Now, the base change morphism for the $\bullet$-pushforward and !-pullback is not always an isomorphism, but by [DrGa1, Proposition 7.6.8] it is an isomorphism for eventually coconnective objects. Hence, the right-hand side in (6.6) identifies with

$$(p_{\mathcal{X}_1\times\mathcal{X}_1}\times\mathrm{id}_{\mathcal{X}_1\times\mathcal{X}_2})_{\bullet}\circ(\Delta^2_{\mathcal{X}_1}\times\mathrm{Id}_{\mathcal{X}_2})_{\bullet}(\mathcal{Q}). \tag{6.7}$$

Again, the $\bullet$-pushforward is not always functorial with respect to compositions of morphisms (see [DrGa1, Sect. 7.8.7]), but it is functorial when evaluated on eventually coconnective objects by [DrGa1, Sect. 7.8.6(iii)]. Hence, (6.7) is isomorphic to $\mathcal{Q}$, as required.

□[Theorem 6.3.2]

6.4.5. *Proof of Lemma 6.4.2.* By [DrGa1, Lemma 9.4.7(b)], we can find an inverse system of objects $\mathcal{M}_n\in\text{D-mod}(\mathcal{Y}_1)^c$, equipped with a compatible system of maps

$$\mathcal{M}\to\mathcal{M}_n,$$

such that $\mathrm{Cone}(\mathcal{M}\to\mathcal{M}_n)\in\text{D-mod}(\mathcal{Y}_1)^{\geq n}$. Then

$$\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_1}(\mathcal{M})\simeq\underset{n}{colim}\,\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_1}(\mathcal{M}_n),$$

since the functor $\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_1}$ has a bounded cohomological amplitude.

Hence, the left-hand side in (6.5) is given by

$$\underset{n}{colim}\,\mathsf{G}^{\mathrm{op}}(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_1}(\mathcal{M}_n))\simeq\underset{n}{colim}\,\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_2}\left(\mathsf{G}(\mathcal{M}_n)\right).$$

By Proposition 6.1.9, in order to prove that (6.5) is an isomorphism, it suffices to show that for every integer k there exists n_0 such that the map

$$\mathsf{G}(\mathcal{M})\to\mathsf{G}(\mathcal{M}_n)$$

induces an isomorphism in cohomological degrees $\leq k$ for $n\geq n_0$.

Consider the commutative diagram

$$
\begin{array}{ccccc}
\mathsf{G}(\mathcal{M}_n) & \xrightarrow{=} & (\mathrm{pr}_2)_\blacktriangle(\mathrm{pr}_1^!(\mathcal{M}_n)\overset{!}{\otimes}\mathcal{P}) & \longrightarrow & (\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{M}_n)\overset{!}{\otimes}\mathcal{P}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{G}(\mathcal{M}) & \xrightarrow{=} & (\mathrm{pr}_2)_\blacktriangle(\mathrm{pr}_1^!(\mathcal{M})\overset{!}{\otimes}\mathcal{P}) & \longrightarrow & (\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{M})\overset{!}{\otimes}\mathcal{P}).
\end{array}
\tag{6.8}
$$

By assumption, the bottom horizontal arrows in (6.8) are isomorphisms. We have the following assertion, proved below:

Lemma 6.4.6. *For any $\mathcal{N}\in \text{D-mod}(\mathcal{Y}_1)^c$ and $\mathcal{P}\in\text{D-mod}(\mathcal{Y}_1\times\mathcal{Y}_2)$, the map*

$$(\mathrm{pr}_2)_\blacktriangle(\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P})\to(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P})$$

is an isomorphism.

Assuming Lemma 6.4.6, we obtain that the top horizontal arrows in (6.8) are also isomorphisms. Hence, it is sufficient to show that for every integer k there exists n_0 such that the map

$$(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{M})\overset{!}{\otimes}\mathcal{P})\to(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{M}_n)\overset{!}{\otimes}\mathcal{P})$$

induces an isomorphism in cohomological degrees $\leq k$ for $n\geq n_0$.

However, this follows from the fact that the functor $\overset{!}{\otimes}$ has a bounded cohomological amplitude, and the functor of $\bullet$-direct image is left t-exact up to a cohomological shift.

$\square$[Lemma 6.4.2]

6.4.7. *Proof of Lemma 6.4.6.* First, by [DrGa1, Proposition 9.3.7] the map

$$(\mathrm{pr}_2)_\blacktriangle(\mathrm{pr}_1^!(\mathcal{N}')\overset{!}{\otimes}\mathcal{P}')\to(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{N}')\overset{!}{\otimes}\mathcal{P}')$$

is an isomorphism for any $\mathcal{N}'\in\text{D-mod}(\mathcal{Y}_1)$ and $\mathcal{P}'\in\text{D-mod}(\mathcal{Y}_1\times\mathcal{Y}_2)^c$. Hence, it suffices to show that for $\mathcal{N}\in\text{D-mod}(\mathcal{Y}_1)^c$, the functor

$$\mathcal{P}\mapsto(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P})$$

is continuous.

This is equivalent to showing that for any fixed $\mathcal{M}\in\text{D-mod}(\mathcal{Y}_2)^c$, the functor

$$\mathcal{P}\mapsto\mathcal{M}aps_{\text{D-mod}(\mathcal{Y}_2)}\left(\mathcal{M},(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P})\right)$$

is continuous. We have:

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y}_2)}\left(\mathcal{M},(\mathrm{pr}_2)_\bullet(\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P})\right)\simeq\mathcal{M}aps_{\text{D-mod}(\mathcal{Y}_1\times\mathcal{Y}_2)}(k_{\mathcal{Y}_1}\boxtimes\mathcal{M},\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P}),$$

which by [DrGa1, Lemma 7.3.5] can be rewritten as

$$(p_{\mathcal{Y}_1\times\mathcal{Y}_2})_\bullet\left(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_1\times\mathcal{Y}_2}(k_{\mathcal{Y}_1}\boxtimes\mathcal{M})\overset{!}{\otimes}\mathrm{pr}_1^!(\mathcal{N})\overset{!}{\otimes}\mathcal{P}\right)\simeq(p_{\mathcal{Y}_1\times\mathcal{Y}_2})_\bullet\left((\mathcal{N}\boxtimes\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_2}(\mathcal{M}))\overset{!}{\otimes}\mathcal{P}\right).$$

Now, the object

$$\mathcal{N}\boxtimes\mathbb{D}^{\mathrm{Ve}}_{\mathcal{Y}_2}(\mathcal{M})\in\text{D-mod}(\mathcal{Y}_1\times\mathcal{Y}_2)$$

is compact, and hence, by [DrGa1, Proposition 9.2.3], safe. This implies the assertion of the lemma, by the definition of safety.

$\square$[Lemma 6.4.6]

6.5. **Mock-proper stacks.** We shall now discuss some applications of Theorem 6.3.2.

6.5.1. Let us call a QCA stack $\mathcal{X}$ *mock-proper* if the functor $(p_{\mathcal{X}})_\blacktriangle$ preserves compactness. (Recall that $(p_{\mathcal{X}})_\blacktriangle|_{\text{D-mod}(\mathcal{X})^c} = (p_{\mathcal{X}})_\bullet|_{\text{D-mod}(\mathcal{X})^c}$, so the above condition is equivalent to $(p_{\mathcal{X}})_\bullet$ preserving compactness.)

An example of a mock-proper stack will be given in Sect. 6.8. Another set of examples is supplied by Corollary 7.6.10.

Note that from Corollary 1.4.3, we obtain that $\mathcal{X}$ is mock-proper if and only if the functor $(p_{\mathcal{X}})_!$, left adjoint to $p^!_{\mathcal{X}}$, is defined.

Proposition 6.5.2. *Let $\mathcal{X}$ be mock-proper and smooth of dimension n. Then we have a canonical isomorphism of functors*

$$(p_{\mathcal{X}})_\blacktriangle \simeq (p_{\mathcal{X}})_! \circ \text{Ps-Id}_{\mathcal{X}}[2n].$$

Proof. We apply Corollary 6.3.4 to the functor $(p_{\mathcal{X}})_\blacktriangle$. The functor in question is given by the kernel $\omega_{\mathcal{X}} \in \text{D-mod}(\mathcal{X})$. Since $(p_{\mathcal{X}})_! \simeq ((p_{\mathcal{X}})_\blacktriangle)^{\text{op}}$, we obtain that the functor $(p_{\mathcal{X}})_! \circ \text{Ps-Id}_{\mathcal{X}}$ is given by the kernel $k_{\mathcal{X}}$. Since $\mathcal{X}$ is smooth of dimension n, we obtain that $(p_{\mathcal{X}})_! \circ \text{Ps-Id}_{\mathcal{X}}[2n]$ is given by the kernel $\omega_{\mathcal{X}}$, i.e., the same as $(p_{\mathcal{X}})_\blacktriangle$.

□

Remark 6.5.3. Retracing the proof of Theorem 6.3.2 one can prove the following generalization of Proposition 6.5.2. Let $\mathcal{X}$ be mock-proper, but not necessarily smooth. Then there is a canonical isomorphism

$$(p_{\mathcal{X}})_\blacktriangle \simeq (p_{\mathcal{X}})_! \circ \mathsf{F}_{\mathcal{X}\to\mathcal{X},(\Delta_{\mathcal{X}})_!(\omega_{\mathcal{X}})}.$$

6.5.4. Passing to dual functors in Proposition 6.5.2, and using Lemma 1.5.3, we obtain:

Corollary 6.5.5. *Let $\mathcal{X}$ be mock-proper and smooth of dimension n. Then we have a canonical isomorphism of functors*

$$p^!_{\mathcal{X}} \simeq \text{Ps-Id}_{\mathcal{X}} \circ ((p_{\mathcal{X}})_\blacktriangle)^R[2n].$$

For a mock-proper stack, we shall denote by $\omega_{\mathcal{X},\text{mock}}$ the object

$$((p_{\mathcal{X}})_\blacktriangle)^R(k) \in \text{D-mod}(\mathcal{X}).$$

We note that when $\mathcal{X}$ is a proper scheme, $\omega_{\mathcal{X},\text{mock}} = \omega_{\mathcal{X}}$.

Note that Corollary 6.5.5 can be reformulated as saying that for $\mathcal{X}$ smooth of dimension n we have:

$$\text{Ps-Id}_{\mathcal{X}}(\omega_{\mathcal{X},\text{mock}})[2n] \simeq \omega_{\mathcal{X}}.$$

Remark 6.5.6. Again, if $\mathcal{X}$ is mock-proper, but not necessarily smooth, we have

$$\mathsf{F}_{\mathcal{X}\to\mathcal{X},(\Delta_{\mathcal{X}})_!(\omega_{\mathcal{X}})}(\omega_{\mathcal{X},\text{mock}}) \simeq \omega_{\mathcal{X}}.$$

6.6. **Truncative and co-truncative substacks.**

6.6.1. *Co-truncative substacks.* Let $j : \mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ be an open embedding of QCA stacks. Recall that according to [DrGa2, Definition 3.1.5], j is said to be *co-truncative* if the partially defined left adjoint to $j^!$, i.e., the functor $j_!$, is defined on all $\text{D-mod}(\mathcal{X}_1)$.

According to Corollary 1.4.3, this condition is equivalent to the functor $j_\bullet$ (which is the same as $j_\blacktriangle$) preserving compactness.

A typical example of a co-truncative open embedding will be considered in Sect. 6.8. Another series of examples is supplied in [DrGa2], where it is shown that the moduli stack Bun_G of G-bundles on X (here G is a reductive group and X a smooth complete curve) can be written as a union of quasi-compact substacks under co-truncative open embeddings.

Proposition 6.6.2 (Drinfeld). *Let j be co-truncative. Then there is a canonical isomorphism of functors*

$$\text{Ps-Id}_{\mathcal{X}_2} \circ j_\bullet \simeq j_! \circ \text{Ps-Id}_{\mathcal{X}_1}.$$

Just as an illustration, we will give a proof of Proposition 6.6.2 using Theorem 6.3.2. However, one can give a more direct proof, see Lemma 7.5.3.

Proof. Consider the functor $j_\bullet$. It is given by the kernel

$$\mathcal{Q} := (\text{id}_{\mathcal{X}_1} \times j)_\bullet(\omega_{\mathcal{X}_1}) \in \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2), \tag{6.9}$$

where by a slight abuse of notation we denote by $\text{id}_{\mathcal{X}_1} \times j$ the graph of the map j.

Note that $j_! \simeq (j_\bullet)^{\text{op}}$ by Lemma 1.5.3. Hence, by Corollary 6.3.4 applied to $j_\bullet$, the functor $j_! \circ \text{Ps-Id}_{\mathcal{X}_1}$ is given by the kernel

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_1}(\mathcal{Q}).$$

Consider now the functor $j^!$. It is also given by the kernel (6.9). Since j is an open embedding, we have $j^! \simeq j^\bullet$, and hence $(j^!)^R \simeq j_\bullet$. Hence, by Theorem 6.3.2 applied to $j^!$, we obtain that $\text{Ps-Id}_{\mathcal{X}_2} \circ j_\bullet$ is also given by

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_1}(\mathcal{Q}),$$

as required. [4]

□

Passing to the dual functors, we obtain:

Corollary 6.6.3. *There is a canonical isomorphism of functors*

$$\text{Ps-Id}_{\mathcal{X}_1} \circ j^? \simeq j^\bullet \circ \text{Ps-Id}_{\mathcal{X}_2},$$

where $j^?$ denotes the (continuous!) right adjoint of $j_\bullet$.

6.6.4. *Truncative substacks.* Let $i : \mathcal{X}_1 \to \mathcal{X}_2$ be a closed embedding. Recall (see [DrGa2, Definition 3.1.5]) that i is said to be *truncative* if the partially defined left adjoint to $i_\bullet$, i.e., the functor $i^\bullet$, is defined on all of $\text{D-mod}(\mathcal{X}_2)$.

According to Corollary 1.4.3, this is equivalent to the functor $i^!$ preserving compactness. Still, equivalently, i is truncative if and only if the complementary open embedding is co-truncative; see [DrGa2, Sects. 3.1-3.3] for a detailed discussion of the properties of truncativeness and co-truncativeness.

As in Proposition 6.6.2 we show:

Proposition 6.6.5. *Let $i : \mathcal{X}_1 \to \mathcal{X}_2$ be truncative. Then we have a canonical isomorphism of functors*

$$i^\bullet \circ \text{Ps-Id}_{\mathcal{X}_2} \simeq \text{Ps-Id}_{\mathcal{X}_1} \circ i^!.$$

Passing to dual functors, one obtains:

Corollary 6.6.6. *There is a canonical isomorphism of functors*

$$\text{Ps-Id}_{\mathcal{X}_2} \circ i_? \simeq i_\bullet \circ \text{Ps-Id}_{\mathcal{X}_1},$$

where $i_?$ is the (continuous!) right adjoint to $i^!$.

6.7. **Miraculous stacks.**

[4] Note that the above kernel is isomorphic to $(\text{id}_{\mathcal{X}_1} \times j)_!(k_{\mathcal{X}_1})$.

6.7.1. Following [DrGa2, Definition 4.5.2], we shall say that a QCA stack $\mathcal{X}$ is *miraculous* if the category D-mod($\mathcal{X}$) is Gorenstein (see Sect. 5.4), i.e., if the functor $\text{Ps-Id}_{\mathcal{X}}$ is an equivalence.

From Proposition 5.4.6 and Theorem 6.3.2, we obtain:

Corollary 6.7.2. *Let $\mathcal{X}$ be a QCA stack for which the functor* $\text{Ps-Id}_{\mathcal{X}}$ *preserves compactness. Then $\mathcal{X}$ is miraculous.*

6.7.3. *Classifying space of a group.* Let G be an affine algebraic group, and consider the stack $\mathcal{X} := \text{pt}/G$. We claim that it is both mock-proper and miraculous.

Indeed, the category D-mod(pt $/G$) is compactly generated by one object, namely, $\pi_\bullet(k)$, where $\pi : \text{pt} \to \text{pt}/G$. Now,

$$(p_{\text{pt}/G})_\bullet(\pi_\bullet(k)) \simeq k \in \text{Vect}^c .$$

Hence pt $/G$ is mock-proper.

Similarly, it is easy to see that

$$(\Delta_{\text{pt}/G})_!(k_{\text{pt}/G}) \simeq (\Delta_{\text{pt}/G})_\bullet(k_{\text{pt}/G})[-d_G] \simeq (\Delta_{\text{pt}/G})_\bullet(\omega_{\text{pt}/G})[-d_G + 2\dim(G)],$$

where

$$d_G = \begin{cases} 2\dim(G) \text{ if } G \text{ is unipotent;} \\ \dim(G) \text{ if } G \text{ is reductive.} \end{cases}$$

Hence,

$$\text{Ps-Id}_{\text{pt}/G} \simeq \text{Id}_{\text{D-mod}(\text{pt}/G)}[-d_G + 2\dim(G)]].$$

6.8. **An example of a miraculous stack.** The results of this and the next subsection were obtained jointly with A. Beilinson and V. Drinfeld.

6.8.1. Let V be a vector space, considered as a scheme, and consider the stack $V/\mathbb{G}_m$. We will prove:

Proposition 6.8.2. *The stack $V/\mathbb{G}_m$ is miraculous and mock-proper.*

6.8.3. Let i denote the closed embedding pt $/\mathbb{G}_m \to V/\mathbb{G}_m$, and let j denote the complementary open embedding

$$\mathbb{P}(V) \simeq (V - 0)/\mathbb{G}_m \hookrightarrow V/\mathbb{G}_m.$$

Let, in addition, π denote the projection map $V/\mathbb{G}_m \to \text{pt}/\mathbb{G}_m$.

According to [DrGa2, Sect. 3.2.2], the closed embedding i (resp., open embedding j) is truncative (resp., co-truncative). Moreover, by [DrGa2, Sect. 5.3], we have canonical isomorphisms of functors

$$i^\bullet \simeq \pi_\bullet, \quad i^! \simeq \pi_!,$$

and hence

$$i_? \simeq \pi^!. \tag{6.10}$$

6.8.4. The fact that $V/\mathbb{G}_m$ is mock-proper follows from the fact that the functor $\pi_\bullet$ preserves compactness (being the left adjoint of $i_\bullet$), combined with the fact that pt $/\mathbb{G}_m$ is mock-proper.

6.8.5. Let us write down the isomorphisms of Propositions 6.6.2 and 6.6.5 and Corollaries 6.6.3 and 6.6.6 in our case.

For that we note that the functor $\text{Ps-Id}_{\mathbb{P}(V)}$ identifies with $\text{Id}_{\text{D-mod}(\mathbb{P}(V))}[-2(\dim(V)-1)]$, since $\mathbb{P}(V)$ is a smooth separated scheme of dimension $\dim(V)-1$. By Sect. 6.7.3, the functor $\text{Ps-Id}_{\text{pt}\,/\mathbb{G}_m}$ identifies with $\text{Id}_{\text{D-mod}(\text{pt}\,/\mathbb{G}_m)}[1]$.

From Proposition 6.6.2, we obtain:

$$\text{Ps-Id}_{V/\mathbb{G}_m}\circ j_\bullet \simeq j_![-2(\dim(V)-1)]. \tag{6.11}$$

From Corollary 6.6.6 and (6.10), we obtain:

$$\text{Ps-Id}_{V/\mathbb{G}_m}\circ \pi^! \simeq i_\bullet[1]. \tag{6.12}$$

From Proposition 6.6.5 we obtain

$$i^\bullet \circ \text{Ps-Id}_{V/\mathbb{G}_m} \simeq i^![1]$$

and from Corollary 6.6.3:

$$j^\bullet \circ \text{Ps-Id}_{V/\mathbb{G}_m} \simeq j^?[-2(\dim(V)-1)].$$

6.8.6. In order to show that $V/\mathbb{G}_m$ is miraculous, by Corollary 6.7.2, it is sufficient to show that the functor $\text{Ps-Id}_{V/\mathbb{G}_m}$ preserves compactness.

The category $\text{D-mod}(V/\mathbb{G}_m)^c$ is generated by the essential images of $\text{D-mod}((V-0)/\mathbb{G}_m)^c$ and $\text{D-mod}(\text{pt}\,/\mathbb{G}_m)^c$ under the functors $j_\bullet$ and $\pi^!$, respectively. Hence, it is sufficient to show that the functors

$$\text{Ps-Id}_{V/\mathbb{G}_m}\circ j_\bullet \text{ and } \text{Ps-Id}_{V/\mathbb{G}_m}\circ \pi^!$$

preserve compactness.

However, this follows from (6.11) and (6.12), respectively.

Remark 6.8.7. To complete the picture, one can show that there is a canonical isomorphism of functors

$$\text{Ps-Id}_{V/\mathbb{G}_m}\circ i_\bullet \simeq \pi^![-2(\dim(V))+1].$$

In particular, if $\dim(V)>1$, it is *not* true that $\text{Ps-Id}_{V/\mathbb{G}_m}$ is an involution.

However, one can show that $\text{Ps-Id}_{V/\mathbb{G}_m}$ is an involution if $\dim(V)=1$. Indeed, for $V=k$, one can show that $\text{Ps-Id}_{V/\mathbb{G}_m}$ is isomorphic to the functor of Fourier-Deligne transform.

6.9. **A non-example.**

6.9.1. Consider now the following stack $X:=(\mathbb{A}^2-0)/\mathbb{G}_m$, where we consider the *hyperbolic* action of $\mathbb{G}_m$ on $\mathbb{A}^2$,

$$\lambda(x_1,x_2)=(\lambda\cdot x_1,\lambda^{-1}\cdot x_2).$$

In fact X is a non-separated scheme, namely, $\widetilde{\mathbb{A}}^1$, i.e., $\mathbb{A}^1$ with a double point. Let i_1 and i_2 denote the corresponding two closed embeddings $\text{pt}\to X$.

We claim that X is *not* miraculous. We will show that the functor Ps-Id_X fails to preserve compactness.

6.9.2. Consider the canonical map

$$(\Delta_X)_!(k_X) \to (\Delta_X)_\bullet(k_X) \simeq (\Delta_X)_\bullet(\omega_X)[-2]. \tag{6.13}$$

Lemma 6.9.3. *The cone of the map* (6.13) *is isomorphic to the direct sum*

$$(i_1 \times i_2)_\bullet(k \oplus k[-1]) \oplus (i_2 \times i_1)_\bullet(k \oplus k[-1]).$$

Proof. Let U_1 and U_2 be the two open charts of X, each isomorphic to $\mathbb{A}^1$, so that

$$U_1 \cap U_2 \to U_1 \times U_2$$

is the map

$$\mathbb{A}^1 - 0 \hookrightarrow \mathbb{A}^1 \xrightarrow{\Delta} \mathbb{A}^1 \times \mathbb{A}^1.$$

The assertion of the lemma follows by calculating the map (6.13) on the charts $U_1 \times U_1$ and $U_2 \times U_2$ (where it also an isomorphism), and $U_1 \times U_2$ and $U_2 \times U_1$, each of which contributes the corresponding direct summand.

□

6.9.4. By Lemma 6.9.3, it is sufficient to show that the functor $\text{D-mod}(X) \to \text{D-mod}(X)$, given by the kernel $(i_1 \times i_2)_\bullet(k)$ does not preserve compactness.

However, the letter functor identifies with $(i_2)_\bullet \circ (i_1)^!$, which sends D_X to a non-compact object.

7. Artin stacks: the non-quasi compact case

7.1. **Truncatable stacks.**

7.1.1. Let $\mathcal{X}$ be an algebraic stack, which is locally QCA, i.e., one that can be covered by quasi-compact algebraic stacks that are QCA.

Recall (see [DrGa2, Lemma 2.3.2]) that the category $\text{D-mod}(\mathcal{X})$ is equivalent to

$$\underset{U \in \text{Open-qc}(\mathcal{X})^{\text{op}}}{lim} \text{D-mod}(U),$$

where $\text{Open-qc}(\mathcal{X})$ is the poset of quasi-compact open substacks of $\mathcal{X}$.

7.1.2. We shall say that an open substack $U \subset \mathcal{X}$ is co-truncative if for any quasi-compact open substack $U' \subset \mathcal{X}$, the inclusion

$$U \cap U' \hookrightarrow U'$$

is co-truncative.

Recall (see [DrGa2, Definition 4.1.1]) that $\mathcal{X}$ is said to be *truncatable* if $\mathcal{X}$ can be covered by its quasi-compact open co-truncative substacks.

Our main example of a truncatable stack is Bun_G, the moduli stack of G-bundles on a smooth complete curve X, where G is a reductie group. This fact is proved in [DrGa2, Theorem 4.1.8].

7.1.3. We let $\mathrm{Ctrnk}(\mathcal{X})$ denote the poset of quasi-compact open co-truncative subsets of $\mathcal{X}$. Note that according to [DrGa2, Lemma 3.8.4], the union of co-truncative subsets is co-truncative. Hence, $\mathrm{Ctrnk}(\mathcal{X})$ is filtered.

Furthermore, the condition of being truncatable is equivalent to the map of posets

$$\mathrm{Ctrnk}(\mathcal{X}) \to \text{Open-qc}(\mathcal{X})$$

being co-final. I.e., $\mathcal{X}$ is truncatable if and only if every quasi-compact open substack of $\mathcal{X}$ is contained in one which is co-truncative.

Hence, for $\mathcal{X}$ truncatable, we have

$$\text{D-mod}(\mathcal{X}) \simeq \underset{U\in \mathrm{Ctrnk}(\mathcal{X})^{\mathrm{op}}}{lim} \text{D-mod}(U).$$

7.1.4. From now on we will assume that all our algebraic stacks are locally QCA and truncatable.

According to [DrGa2, Proposition 4.1.6], the category $\text{D-mod}(\mathcal{X})$ is compactly generated. The set of compact generators is provided by the objects

$$j_!(\mathcal{F}), \quad j: U \hookrightarrow \mathcal{X}, \quad U \in \mathrm{Ctrnk}(\mathcal{X}), \quad \mathcal{F} \in \text{D-mod}(U)^c.$$

We introduce the DG category $\text{D-mod}(\mathcal{X})_{\mathrm{co}}$ as

$$\underset{\mathrm{Ctrnk}(\mathcal{X})^{\mathrm{op}}}{lim} \text{D-mod}^?,$$

where the functor $\text{D-mod}^? : \mathrm{Ctrnk}(\mathcal{X})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ sends

$$U \rightsquigarrow \text{D-mod}(U) \text{ and } (U_1 \overset{j_{1,2}}{\hookrightarrow} U_2) \rightsquigarrow j_{1,2}^?$$

(the functor $j_{1,2}^?$ is the *right* adjoint of $(j_{1,2})_\bullet$, see Sect. 6.6, and also Sect. 7.1.6 below).

For $(U \overset{j}{\hookrightarrow} \mathcal{X}) \in \mathrm{Ctrnk}(\mathcal{X})$, the tautological evaluation functor

$$j^? : \text{D-mod}(\mathcal{X})_{\mathrm{co}} \to \text{D-mod}(U)$$

admits a *left* adjoint, denoted $j_{\mathrm{co},\bullet}$. The category $\text{D-mod}(\mathcal{X})_{\mathrm{co}}$ is compactly generated by objects

$$j_{\mathrm{co},\bullet}(\mathcal{F}), \quad (j: U \hookrightarrow \mathcal{X}) \in \mathrm{Ctrnk}(\mathcal{X}), \quad \mathcal{F} \in \text{D-mod}(U)^c.$$

For $U \overset{j}{\hookrightarrow} \mathcal{X}$ as above, the functor $j_{\mathrm{co},\bullet}$ also admits a left adjoint, denoted $j^\bullet_{\mathrm{co}}$.

7.1.5. By [DrGa2, Corollaries 4.3.2 and 4.3.5], there is a canonically defined equivaence

$$\mathbf{D}^{\mathrm{Ve}}_{\mathcal{X}} : \text{D-mod}(\mathcal{X})^\vee \to \text{D-mod}(\mathcal{X})_{\mathrm{co}}.$$

It is characterized by the property that the corresponding functor

$$\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}} : (\text{D-mod}(\mathcal{X})^c)^{\mathrm{op}} \to (\text{D-mod}(\mathcal{X})_{\mathrm{co}})^c$$

acts as follows

$$\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}}(j_!(\mathcal{F})) = j_{\mathrm{co},\bullet}(\mathbb{D}^{\mathrm{Ve}}_U(\mathcal{F})), \quad \mathcal{F} \in \text{D-mod}(U)^c, \quad (U \overset{j}{\hookrightarrow} \mathcal{X}) \in \mathrm{Ctrnk}(\mathcal{X}).$$

7.1.6. For a co-truncative quasi-compact $U \overset{j}{\hookrightarrow} \mathcal{X}$ we have the following isomorphisms:

$$(j_!)^{\mathrm{op}} \simeq j_{\mathrm{co},\bullet}, \quad (j_!)^\vee \simeq j^?, \quad (j^\bullet)^\vee \simeq j_{\mathrm{co},\bullet},$$

from which, using Lemma 1.5.3, we obtain

$$(j_\bullet)^\vee \simeq j^\bullet_{\mathrm{co}}, \quad (j^\bullet)^{\mathrm{op}} \simeq j^\bullet_{\mathrm{co}}.$$

7.2. **Additional properties of** $\text{D-mod}(\mathcal{X})_{\text{co}}$**.** The following several additional pieces of information regarding the categories $\text{D-mod}(\mathcal{X})$ and $\text{D-mod}(\mathcal{X})_{\text{co}}$ will be used in the sequel.

7.2.1. According to [DrGa2, Corollaries 4.3.2 and 4.3.5], the functors

$$j_{\text{co},\bullet} : \text{D-mod}(U) \to \text{D-mod}(\mathcal{X})_{\text{co}}, \quad (j : U \hookrightarrow \mathcal{X}) \in \text{Ctrnk}(\mathcal{X})$$

have the property that the induced functor

$$\underset{\text{Ctrnk}(\mathcal{X})}{colim}\ \text{D-mod}_\bullet \to \text{D-mod}(\mathcal{X})_{\text{co}} \tag{7.1}$$

is an equivalence, where the functor

$$\text{D-mod}_\bullet : \text{Ctrnk}(\mathcal{X}) \to \text{DGCat}_{\text{cont}}$$

sends

$$U \rightsquigarrow \text{D-mod}(U) \text{ and } (U_1 \overset{j_{12}}{\hookrightarrow} U_2) \rightsquigarrow (j_{12})_\bullet.$$

7.2.2. For a truncatable stack $\mathcal{X}$ we define a continuous functor

$$(p_{\mathcal{X}})_\blacktriangle : \text{D-mod}(\mathcal{X})_{\text{co}} \to \text{Vect}$$

to be the dual of the functor

$$(p_{\mathcal{X}})^! : \text{Vect} \to \text{D-mod}(\mathcal{X}).$$

In terms of the equivalence

$$\underset{U \in \text{Ctrnk}(\mathcal{X})}{colim}\ \text{D-mod}(U) \to \text{D-mod}(\mathcal{X})$$

of (7.1), the funtcor $(p_{\mathcal{X}})_\blacktriangle$ corresponds to the family of functors $\text{D-mod}(U) \to \text{Vect}$, given by $(p_U)_\blacktriangle$, which are naturally compatible under

$$(p_{U_1})_\blacktriangle \circ (j_{1,2})_\bullet \simeq (p_{U_2})_\blacktriangle, \quad U_1 \overset{j_{1,2}}{\hookrightarrow} U_2 \overset{j_2}{\hookrightarrow} \mathcal{X}.$$

7.2.3. Next, we claim that the $\overset{!}{\otimes}$ operation defines a canonical action of the monoidal category $\text{D-mod}(\mathcal{X})$ on $\text{D-mod}(\mathcal{X})_{\text{co}}$. In terms of the equivalence (7.1), for $\mathcal{F} \in \text{D-mod}(\mathcal{X})$ and

$$\mathcal{F}_U \in \text{D-mod}(U), \quad (U \overset{j}{\hookrightarrow} \mathcal{X}) \in \text{Ctrnk}(\mathcal{X}),$$

we have

$$\mathcal{F} \overset{!}{\otimes} j_{\text{co},\bullet}(\mathcal{F}_U) := j_{\text{co},\bullet}(j^\bullet(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}_U).$$

The following assertion will be used in the sequel:

Lemma 7.2.4. *For* $\mathcal{F} \in (\text{D-mod}(\mathcal{X})_{\text{co}})^c$ *and* $\mathcal{F}_1 \in \text{D-mod}(\mathcal{X})_{\text{co}}$*, there is a canonical isomorphism*

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{X})_{\text{co}}}(\mathcal{F}, \mathcal{F}') \simeq (p_{\mathcal{X}})_\blacktriangle(\mathbb{D}^{\text{Ve}}_{\text{Bun}(G)}(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}'),$$

where $\mathcal{F} \mapsto \mathbb{D}^{\text{Ve}}_{\text{Bun}(G)}(\mathcal{F})$ *is the equivalence*

$$((\text{D-mod}(\mathcal{X})_{\text{co}})^c)^{\text{op}} \to \text{D-mod}(\mathcal{X})^c,$$

induced by $\mathbf{D}^{\text{Ve}}_{\mathcal{X}}$*.*

7.3. **Kernels in the non-quasi compact situation.**

7.3.1. For a pair of truncatable stacks $\mathcal{X}_1$ and $\mathcal{X}_2$, let $\mathcal{Q}$ be an object of the category

$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}.$$

We shall say that $\mathcal{Q}$ is *coherent* if for any pair of quasi-compact open co-truncative substacks $U_1 \overset{j_1}{\hookrightarrow} \mathcal{X}_1$ and $U_2 \overset{j_2}{\hookrightarrow} \mathcal{X}_2$ we have

$$((j_1)^\bullet \otimes (j_2)^\bullet_{\text{co}})(\mathcal{Q}) \in \text{D-mod}(U_1) \otimes \text{D-mod}(U_2) \simeq \text{D-mod}(U_1 \times U_2)$$

is coherent.

We claim that for any $\mathcal{Q}$ which is coherent, there is a well-defined object, denoted

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q}) \in \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)_{\text{coh}}$$

(Note that the notion of coherence for an object of D-mod($\mathcal{X}$) makes sense for not necessarily quasi-compact algebraic stacks.)

Namely, we define $\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})$ be requiring that for any quasi-compact open co-truncative $U_1 \overset{j_1}{\hookrightarrow} \mathcal{X}_1$ and $U_2 \overset{j_2}{\hookrightarrow} \mathcal{X}_2$, we have:

$$(j_1 \times j_2)^\bullet \left(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})\right) \simeq \mathbb{D}^{\text{Ve}}_{U_1 \times U_2}\left(((j_1)^\bullet \otimes (j_2)^\bullet_{\text{co}})(\mathcal{Q})\right).$$

7.3.2. Let us note that by Sect. 1.1.1, the category

$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}.$$

is equivalent to that of continuous functors

$$\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2)_{\text{co}}.$$

The category

$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2) \simeq \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$$

is equivalent to that of continuous functors

$$\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2).$$

In both cases, we will denote this assignment by

$$\mathcal{Q} \rightsquigarrow \mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}}.$$

7.3.3. Note now that for a stack $\mathcal{X}$, the object

$$(\Delta_{\mathcal{X}})_!(k_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X})$$

is well-defined.

It has the property that for every quasi-compact open $j : U \hookrightarrow \mathcal{X}$, we have

$$(j \times j)^\bullet\left((\Delta_{\mathcal{X}})_!(k_{\mathcal{X}})\right) \simeq (\Delta_U)_!(k_U).$$

Indeed, the functor $(j \times j)^\bullet \circ (\Delta_{\mathcal{X}})_!$ is the partially defined left adjoint to

$$\Delta^!_{\mathcal{X}} \circ (j \times j)_\bullet \simeq j_\bullet \circ \Delta^!_U,$$

as is $(\Delta_U)_! \circ j^\bullet$.

7.3.4. We define the functor

$$\text{Ps-Id}_{\mathcal{X}} : \text{D-mod}(\mathcal{X})_{\text{op}} \to \text{D-mod}(\mathcal{X})$$

to be given by the kernel

$$(\Delta_{\mathcal{X}})_!(k_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X}) \simeq \text{D-mod}(\mathcal{X}) \otimes \text{D-mod}(\mathcal{X}) \simeq (\text{D-mod}(\mathcal{X})_{\text{op}})^\vee \otimes \text{D-mod}(\mathcal{X}).$$

7.3.5. Finally, in addition to the functor $\text{Ps-Id}_{\mathcal{X}} : \text{D-mod}(\mathcal{X})_{\text{co}} \to \text{D-mod}(\mathcal{X})$, introduced above, there is another functor, denoted

$$\text{Ps-Id}^{\text{naive}}_{\mathcal{X}} : \text{D-mod}(\mathcal{X})_{\text{co}} \to \text{D-mod}(\mathcal{X}).$$

It is given by the kernel

$$(\Delta_{\mathcal{X}})_\bullet(\omega_{\mathcal{X}}) \in \text{D-mod}(\mathcal{X} \times \mathcal{X}).$$

In terms of the equivalence (7.1), the functor $\text{Ps-Id}^{\text{naive}}_{\mathcal{X}}$ corresponds to the family of functors

$$\text{D-mod}(U) \to \text{D-mod}(\mathcal{X}), \quad (U \overset{j}{\hookrightarrow} \mathcal{X}) \in \text{Ctrnk}(\mathcal{X}),$$

given by $j_\bullet$, that are compatible under the isomorphisms

$$(j_1)_\bullet \simeq (j_{12})_\bullet \circ (j_2)_\bullet, \quad U_1 \overset{j_{12}}{\hookrightarrow} U_2 \overset{j_2}{\hookrightarrow} \mathcal{X}.$$

This functor is not an equivalence, unless the closure of any quasi-compact open substack of $\mathcal{X}$ is quasi-compact, see [DrGa2, Proposition 4.4.5].

In Sect. 7.7 we will describe a particular object in the kernel of this functor for $\mathcal{X} = \text{Bun}_G$.

7.4. **The theorem for truncatable stacks.**

7.4.1. The following is an extension of Theorem 6.3.2 to the case of truncatable (but not necessarily quasi-compact) stacks:

Theorem 7.4.2. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be truncatable stacks, and let*

$$\mathcal{Q} \in \text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}$$

be coherent. Assume that the corresponding functor

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} : \text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2)_{\text{co}}$$

preserves compactness. Then we have a canonical isomorphism

$$\text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R \simeq \mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}$$

as functors

$$\text{D-mod}(\mathcal{X}_2)_{\text{co}} \to \text{D-mod}(\mathcal{X}_1).$$

7.4.3. By passing to the dual functors, we obtain:

Corollary 7.4.4. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be truncatable stacks, and let*

$$\mathcal{Q} \in \text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\text{co}}$$

be coherent. Assume that the corresponding functor

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} : \text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2)_{\text{co}}$$

preserves compactness. Then we have a canonical isomorphism

$$(\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^{\text{op}} \circ \text{Ps-Id}_{\mathcal{X}_1} \simeq \mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathbb{D}^{\text{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}$$

as functors

$$\text{D-mod}(\mathcal{X}_1)_{\text{co}} \to \text{D-mod}(\mathcal{X}_2).$$

Remark 7.4.5. Note that Theorem 7.4.2 *does not* fit into the paradigm of Theorem 5.2.3. Indeed, we start with $\mathbf{C}_i = \text{D-mod}(\mathcal{X}_i)_{\text{co}}$, $i = 1, 2$ and a functor $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$, and while Theorem 5.2.3 talks about an isomorphism between two functors $\mathbf{C}_2 \to \mathbf{C}_1$, in Theorem 7.4.2, the target category is no longer $\mathbf{C}_2 = \text{D-mod}(\mathcal{X}_2)_{\text{co}}$, but rather $\text{D-mod}(\mathcal{X}_2)$.

7.5. **Proof of Theorem 7.4.2.**

7.5.1. We shall first consider the case when $\mathcal{X}_1$ is quasi-compact. In this case we will not distinguish between $\text{D-mod}(\mathcal{X}_1)$ and $\text{D-mod}(\mathcal{X}_1)_{\text{co}}$.

Using the equivalence (7.1), in order to prove the theorem, it suffices to construct a compatible family of isomorphisms of functors

$$\text{(7.2)} \quad \text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}})^R \circ (j_2)_{\text{co},\bullet} \simeq \\ \simeq \mathsf{F}_{\mathcal{X}_2\to\mathcal{X}_1,\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}(\mathcal{Q})} \circ (j_2)_{\text{co},\bullet}, \quad (j_2 : U_2 \hookrightarrow \mathcal{X}_2) \in \text{Ctrnk}(\mathcal{X}_2).$$

We have:

$$(\mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}})^R \circ (j_2)_{\text{co},\bullet} \simeq ((j_2)^\bullet_{\text{co}} \circ \mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}})^R \simeq (\mathsf{F}_{\mathcal{X}_1\to U_2,\mathcal{Q}_U})^R,$$

where

$$\mathcal{Q}_U := \left(\text{Id}_{\text{D-mod}(\mathcal{X}_1)} \otimes (j_2)^\bullet_{\text{co}}\right)(\mathcal{Q}).$$

The functor

$$\mathsf{F}_{\mathcal{X}_2\to\mathcal{X}_1,\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}(\mathcal{Q})} \circ (j_2)_{\text{co},\bullet}$$

is given by the kernel

$$(\text{Id}_{\text{D-mod}(\mathcal{X}_1)} \otimes ((j_2)_{\text{co},\bullet})^\vee)(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}(\mathcal{Q})) \simeq (\text{Id}_{\text{D-mod}(\mathcal{X}_1)} \otimes j_2^\bullet)(\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}(\mathcal{Q})),$$

which by the definition of $\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}$ identifies with

$$\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times U_2}(\mathcal{Q}_U).$$

Hence, both sides of (7.2) identify with the corresponding functors when we replace $\mathcal{X}_2$ by U_2 and $\mathcal{Q}$ by $\mathcal{Q}_U$. In this case, the required isomorhism for (7.2) follows from Theorem 6.3.2. Furthermore, this system of isomorphisms is compatible under the restrictions for $U_2' \hookrightarrow U_2$.

This establishes the isomorphism of the theorem in the case when $\mathcal{X}_1$ is quasi-compact.

7.5.2. Let now $\mathcal{X}_1$ be general truncatable. By the definition of the category $\text{D-mod}(\mathcal{X}_1)$, it is enough to show that for every quasi-compact open co-truncative $U_1 \overset{j_1}{\hookrightarrow} \mathcal{X}_1$, there exists a canonical isomorphism of functors

$$\text{(7.3)} \qquad j_1^\bullet \circ \text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1\to\mathcal{X}_2,\mathcal{Q}})^R \simeq j_1^\bullet \circ \mathsf{F}_{\mathcal{X}_2\to\mathcal{X}_1,\mathbb{D}^{\text{Ve}}_{\mathcal{X}_1\times\mathcal{X}_2}(\mathcal{Q})},$$

compatible with the restriction maps under $U_1' \hookrightarrow U_1''$.

Lemma 7.5.3. *For a truncatable stack $\mathcal{X}$ and $(U \overset{j}{\hookrightarrow} \mathcal{X}) \in \text{Ctrnk}$ there is a canonical isomorphism of functors*

$$j^\bullet \circ \text{Ps-Id}_{\mathcal{X}} \simeq \text{Ps-Id}_U \circ j^?, \quad \text{D-mod}(\mathcal{X})_{\text{co}} \to \text{D-mod}(U).$$

(Note that if $\mathcal{X}$ were quasi-compact, the assertion of the lemma is a particular case of Corollary 6.6.3.)

Proof. The functor $j^\bullet \circ \text{Ps-Id}_{\mathcal{X}}$ is given by the kernel

$$(j \times \text{id}_{\mathcal{X}})^\bullet \left((\Delta_{\mathcal{X}})_!(k_{\mathcal{X}})\right),$$

which by base change identifies with

$$((\text{id}_U \times j) \circ \Delta_U)_!(k_U),$$

i.e., $(\text{id}_U \times j)_! \circ (\Delta_U)_!(k_U)$.

The functor $\text{Ps-Id}_U \circ j^?$ is given by the kernel

$$\left(\text{Id}_{\text{D-mod}(U)} \otimes (j^?)^\vee\right)\left((\Delta_U)_!(k_U)\right) \simeq \left(\text{Id}_{\text{D-mod}(U)} \otimes j_!\right)\left((\Delta_U)_!(k_U)\right).$$

We note that the functor $\mathrm{Id}_{\text{D-mod}(U)} \otimes j_!$ is the left adjoint of $\mathrm{Id}_{\text{D-mod}(U)} \otimes j^\bullet$, and hence identifies with $(\mathrm{id}_U \times j)_!$.

□

7.5.4. Hence, we obtain that the left-hand side in (7.3) identifies canonically with

$$\text{Ps-Id}_{U_1} \circ j_1^? \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R,$$

which we further rewrite as

$$\text{Ps-Id}_{U_1} \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} \circ (j_1)_{\mathrm{co},\bullet})^R.$$

Note that the functor

$$\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}} \circ (j_1)_{\mathrm{co},\bullet} : \text{D-mod}(U_1) \to \text{D-mod}(\mathcal{X}_2)_{\mathrm{co}}$$

preserves compactness since $(j_1)_{\mathrm{co},\bullet}$ does. The above functor is given by the kernel

$$\mathcal{Q}_U := (((j_1)_{\mathrm{co},\bullet})^\vee \otimes \mathrm{Id}_{\text{D-mod}(\mathcal{X}_2)_{\mathrm{co}}})(\mathcal{Q}) \simeq (j_1^\bullet \otimes \mathrm{Id}_{\text{D-mod}(\mathcal{X}_2)_{\mathrm{co}}})(\mathcal{Q}) \in \text{D-mod}(U_1) \otimes \text{D-mod}(\mathcal{X}_2)_{\mathrm{co}}.$$

Now, the functor

$$j_1^\bullet \circ \mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})},$$

appearing in the right-hand side of (7.3), is given by the kernel

$$(j_1^\bullet \otimes \mathrm{Id}_{\text{D-mod}(\mathcal{X}_2)}) \left(\mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})\right) \simeq \mathbb{D}^{\mathrm{Ve}}_{U_1 \times \mathcal{X}_2}(\mathcal{Q}_U).$$

Hence, both sides in (7.3) identify with the corresponding functors when we replace $\mathcal{X}_1$ by U_1 and $\mathcal{Q}$ by $\mathcal{Q}_U$. We define the isomorphism in (7.3) to be the isomorphism of Sect. 7.5.1 for the stack $U_1 \times \mathcal{X}_2$. These identifications are compatible under further restrictions for $U_1' \hookrightarrow U_1''$.

This establishes the required isomorphism

$$\text{Ps-Id}_{\mathcal{X}_1} \circ (\mathsf{F}_{\mathcal{X}_1 \to \mathcal{X}_2, \mathcal{Q}})^R \simeq \mathsf{F}_{\mathcal{X}_2 \to \mathcal{X}_1, \mathbb{D}^{\mathrm{Ve}}_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{Q})}.$$

□

7.6. **Applications.**

7.6.1. First, passing to dual functors in Lemma 7.5.3, we obtain:

Corollary 7.6.2. *For* $(U \overset{j}{\hookrightarrow} \mathcal{X}) \in \mathrm{Ctrnk}(\mathcal{X})$ *there is a canonical isomorphism of functors*

$$\text{Ps-Id}_{\mathcal{X}} \circ j_{\mathrm{co},\bullet} \simeq j_! \circ \text{Ps-Id}_U, \quad \text{D-mod}(U) \to \text{D-mod}(\mathcal{X}).$$

7.6.3. Let $\mathcal{X}$ be a truncatable stack. We shall say that $\mathcal{X}$ is miraculous if the functor $\text{Ps-Id}_{\mathcal{X}}$ is an equivalence.

Proposition 7.6.4. *For a stack* $\mathcal{X}$ *the following conditions are equivalent:*

(a) *The* $\mathcal{X}$ *is miraculous.*

(b) *Every co-truncatable quasi-compact open substack of* $\mathcal{X}$ *is miraculous.*

(c) *There is a cofinal family in* $\mathrm{Ctrnk}(\mathcal{X})$ *consisting of miraculous stacks.*

Proof. We reproduce the proof from [DrGa2, Lemma 4.5.7]. Assume that $\mathcal{X}$ is miraculous, and let $(j : U \to \mathcal{X}) \in \mathrm{Ctrnk}(\mathcal{X})$.

Let us first show that the functor Ps-Id_U has a left inverse. For this it is enough to show that the composition $j_! \circ \text{Ps-Id}_U$ has a left inverse. Taking into account the isomorphism of Corollary 7.6.2, it suffices to show that each of the functors $\text{Ps-Id}_{\mathcal{X}}$ and $j_{\mathrm{co},\bullet}$ admits a left inverse. For $\text{Ps-Id}_{\mathcal{X}}$ this follows from the assumption that $\mathcal{X}$ is miraculous. For $j_{\mathrm{co},\bullet}$, the left inverse is $j_{\mathrm{co}}^\bullet$.

Now, if $(\text{Ps-Id}_U)^{-1,L}$ is the left inverse of Ps-Id_U, passing to dual functors in

$$(\text{Ps-Id}_U)^{-1,L} \circ \text{Ps-Id}_U \simeq \text{Id}_{\text{D-mod}(U)},$$

we obtain that $((\text{Ps-Id}_U)^{-1,L})^\vee$ is the right inverse of Ps-Id_U. Hence, Ps-Id_U is an equivalence.

The implication (b) $\Rightarrow$ (c) is tautological.

The implication (c) $\Rightarrow$ (a) follows from Lemma 7.5.3, since the functors Ps-Id_U define an equivalence between the limits

$$\underset{\text{Ctrnk}^{\text{op}}}{lim} \text{D-mod}^? \to \underset{\text{Ctrnk}^{\text{op}}}{lim} \text{D-mod}^\bullet.$$

□

The proof of the following result is given in [Ga3]:

Theorem 7.6.5. *The stack* Bun_G *of principal* G*-bundles on a complete smooth curve* X*, where* G *is a reductive group, is miraculous.*

7.6.6. We shall say that a truncatable stack $\mathcal{X}$ is mock-proper if the functor $(p_\mathcal{X})_\blacktriangle$ (defined in Sect. 7.2.2) preserves compactness.

By Lemma 1.5.3, $\mathcal{X}$ mock-proper if and only if the functor

$$(p_\mathcal{X})_! : \text{D-mod}(\mathcal{X}) \to \text{pt},$$

right adjoint to $p^!_\mathcal{X}$, is defined.

The following assertion is proved in [Ga2, Corollary 4.3.2]:

Proposition 7.6.7. *The stack* Bun_G *is mock-proper.*

7.6.8. The following is immediate from the definitions:

Lemma 7.6.9. *For a stack* $\mathcal{X}$ *the following conditions are equivalent:*

(a) *The* $\mathcal{X}$ *is mock-proper.*

(b) *Every quasi-compact open co-truncative substack of* $\mathcal{X}$ *is mock-proper.*

(c) *There is a cofinal family in* $\text{Ctrnk}(\mathcal{X})$ *consisting of mock-proper stacks.*

Hence, we obtain:

Corollary 7.6.10. *Every quasi-compact open co-truncative substack of* Bun_G *is mock-proper.*

7.6.11. The next assertion is proved in the same way as Proposition 6.5.2:

Proposition 7.6.12. *Let* $\mathcal{X}$ *be mock-proper and smooth of dimension* n*. Then there exists a canonical isomorphism of functors*

$$(p_\mathcal{X})_\blacktriangle \simeq (p_\mathcal{X})_! \circ \text{Ps-Id}_\mathcal{X}[2n].$$

Passing to dual functors, and using Lemma 1.5.3, we obtain:

Corollary 7.6.13. *Let* $\mathcal{X}$ *be mock-proper and smooth of dimension* n*. Then we have a canonical isomorphism of functors*

$$p^!_\mathcal{X} \simeq \text{Ps-Id}_\mathcal{X} \circ ((p_\mathcal{X})_\blacktriangle)^R[2n].$$

For a mock-proper stack, we shall denote by $\omega_{\mathcal{X},\text{mock}}$ the object

$$((p_\mathcal{X})_\blacktriangle)^R(k) \in \text{D-mod}(X)_{\text{co}}.$$

Hence, Corollary 7.6.13 can be reformulated as saying that for $\mathcal{X}$ smooth of dimension n we have:

$$\text{Ps-Id}_\mathcal{X}(\omega_{\mathcal{X},\text{mock}})[2n] \simeq \omega_\mathcal{X}.$$

7.7. **A bizarre object in** $\text{D-mod}(\text{Bun}_G)_{\text{co}}$.

7.7.1. We consider the object

$$\omega_{\text{Bun}_G,\text{mock}} \in \text{D-mod}(\text{Bun}_G)_{\text{co}}.$$

The goal of this subsection and the next is to prove the following assertion:

Theorem 7.7.2. *Let G be a reductive group with a non-trivial semi-simple part (i.e., G is not a torus). Then the object $\omega_{\text{Bun}_G,\text{mock}}$ belongs to the kernel of the functor*

$$\text{Ps-Id}^{\text{naive}}_{\text{Bun}_G} : \text{D-mod}(\text{Bun}_G)_{\text{co}} \to \text{D-mod}(\text{Bun}_G).$$

(See Sect. 7.3.5 where the functor $\text{Ps-Id}^{\text{naive}}_{\mathcal{X}}$ is introduced.)

In order to prove this theorem we will use a description of the object $\omega_{\text{Bun}_G,\text{mock}}$, which is valid for any reductive group.

7.7.3. Let us recall the setting of [Ga2, Sect. 4.1.1]. We let $\text{Gr}_{G,\text{Ran}(X)}$ denote the prestack, which is the Ran version of affine Grassmannian for G. Let π denote the canonical map

$$\text{Gr}_{G,\text{Ran}(X)} \to \text{Bun}_G .$$

The following is [Ga2, Theorem 4.1.6]:

Theorem 7.7.4. *The functor $\pi^! : \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})$ is fully faithful.*

7.7.5. We recall that the pre-stack $\text{Gr}_{G,\text{Ran}(X)}$ is by definition the colimit

$$\underset{i\in I}{colim}\, Z_i, \tag{7.4}$$

where Z_i are proper schemes, and I is some index category. In particular, for $\alpha : i \to j$, the corresponding map $f_\alpha : Z_i \to Z_j$ is proper. We let f_i denote the corresponding map $Z_i \to \text{Gr}_{G,\text{Ran}(X)}$.

The category $\text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})$ is the limit

$$\underset{i\in I^{\text{op}}}{lim}\ \text{D-mod}(Z_i), \tag{7.5}$$

where for $(\alpha : i \to j) \in I$, the functor $\text{D-mod}(Z_j) \to \text{D-mod}(Z_i)$ is $f^!_\alpha$. The corresponding evaluation functor $\text{D-mod}(\text{Gr}_{G,\text{Ran}(X)}) \to \text{D-mod}(Z_i)$ is $f^!_i$.

Hence, by [DrGa2, Proposition 1.7.5], we have a canonical equivalence

$$\text{D-mod}(\text{Gr}_{G,\text{Ran}(X)}) \simeq \underset{i\in I}{colim}\ \text{D-mod}(Z_i), \tag{7.6}$$

where for $(\alpha : i \to j) \in I$, the functor $\text{D-mod}(Z_i) \to \text{D-mod}(Z_j)$ is $(f_\alpha)_\bullet$. For $i \in I$, the corresponding functor $\text{D-mod}(Z_i) \to \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})$ will be denoted $(f_i)_\bullet$, and it is the left adjoint of $f^!_i$.

In particular, by [DrGa2, Proposition 1.8.3], the Verdier duality equivalences

$$\mathbf{D}^{\text{Ve}}_{Z_i} : \text{D-mod}(Z_i)^\vee \to \text{D-mod}(Z_i)$$

give rise to an equivalence

$$\mathbf{D}^{\text{Ve}}_{\text{Gr}_{G,\text{Ran}(X)}} : \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})^\vee \simeq \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)});$$

under which we have:

$$(f^!_i)^\vee \simeq (f_i)_\bullet.$$

7.7.6. Let

$$\pi_\bullet : \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)}) \to \text{D-mod}(\text{Bun}_G)_{\text{co}}$$

denote the functor dual to $\pi^!$ under the identifications $\mathbf{D}^{\text{Ve}}_{\text{Gr}_{G,\text{Ran}(X)}}$ and $\mathbf{D}^{\text{Ve}}_{\text{Bun}_G}$.

The functor $\pi_\bullet$ can be described more explicity as follows. By (7.6), the datum of $\pi_\bullet$ is equivalent to a compatible collection of functors

$$(\pi \circ f_i)_\bullet : \text{D-mod}(Z_i) \to \text{D-mod}(\text{Bun}_G)_{\text{co}}.$$

For each i, the category of factorizations of the map $\pi \circ f_i$ as

$$Z_i \overset{f_{i,U}}{\longrightarrow} U \overset{j}{\hookrightarrow} \text{Bun}_G, \quad U \in \text{Ctrnk}(\text{Bun}_G). \tag{7.7}$$

is cofinal in Ctrnk(Bun_G), and hence, is contractible.

The sought-for functor $(\pi \circ f_i)_\bullet$ is

$$j_{\text{co},\bullet} \circ (f_{i,U})_\bullet$$

for some/any factorization (7.7).

In the sequel, we will use the following version of the projection formula, which follows immediately from the defintions:

Lemma 7.7.7. *For* $\mathcal{F} \in \text{D-mod}(\text{Bun}_G)$ *and* $\mathcal{F}' \in \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})$ *there is a canonical isomorphism*

$$\mathcal{F} \overset{!}{\otimes} \pi_\bullet(\mathcal{F}') \simeq \pi_\bullet(\pi^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{F}'),$$

where $\overset{!}{\otimes}$ *in the left-hand side is understood in the sense of Sect. 7.2.3.*

7.7.8. We claim:

Theorem 7.7.9. *There exists a canonical isomorphism*

$$\omega_{\text{Bun}_G,\text{mock}} \simeq \pi_\bullet(\omega_{\text{Gr}_{G,\text{Ran}(X)}}).$$

Proof. We need to establish a functorial isomorphism

$$\mathcal{M}aps_{\text{D-mod}(\text{Bun}_G)_{\text{co}}}(\mathcal{F}, \omega_{\text{Bun}_G,\text{mock}}) \simeq \mathcal{M}aps_{\text{D-mod}(\text{Bun}_G)_{\text{co}}}\left(\mathcal{F}, \pi_\bullet(\omega_{\text{Gr}_{G,\text{Ran}(X)}})\right). \tag{7.8}$$

for $\mathcal{F} \in (\text{D-mod}(\text{Bun}_G)_{\text{co}})^c$.

By definition, the left-hand side in (7.8) can be rewritten as

$$\mathcal{M}aps_{\text{Vect}}\left((p_{\text{Bun}_G})_\blacktriangle(\mathcal{F}), k\right),$$

and further, by Lemma 1.5.3, as

$$(p_{\text{Bun}_G})_!(\mathbb{D}^{\text{Ve}}_{\text{Bun}_G}(\mathcal{F})), \tag{7.9}$$

where

$$\mathbb{D}^{\text{Ve}}_{\text{Bun}_G} : (\text{D-mod}(\text{Bun}_G)^c_{\text{co}})^{\text{op}} \simeq \text{D-mod}(\text{Bun}_G)^c$$

is the equivalence indiced by

$$\mathbf{D}^{\text{Ve}}_{\text{Bun}_G} : (\text{D-mod}(\text{Bun}_G)_{\text{co}})^\vee \simeq \text{D-mod}(\text{Bun}_G).$$

We rewrite the right-hand side of (7.8) using Lemma 7.2.4 as

$$(p_{\text{Bun}_G})_\blacktriangle\left(\mathbb{D}^{\text{Ve}}_{\text{Bun}_G}(\mathcal{F}) \overset{!}{\otimes} \pi_\bullet(\omega_{\text{Gr}_{G,\text{Ran}(X)}})\right).$$

Using Lemma 7.2.3, we further rewrite it as

$$(p_{\mathrm{Bun}_G})_{\blacktriangle} \circ \pi_\bullet(\pi^!(\mathbb{D}^{\mathrm{Ve}}_{\mathrm{Bun}_G}(\mathcal{F})) \overset{!}{\otimes} \omega_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}}) \simeq (p_{\mathrm{Bun}_G})_{\blacktriangle} \circ \pi_\bullet(\pi^!(\mathbb{D}^{\mathrm{Ve}}_{\mathrm{Bun}_G}(\mathcal{F}))),$$

and hence as

$$(p_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}})_\bullet(\pi^!(\mathbb{D}^{\mathrm{Ve}}_{\mathrm{Bun}_G}(\mathcal{F}))). \tag{7.10}$$

Comparing (7.9) and (7.10), the assertion of the theorem follows from the next lemma:

Lemma 7.7.10. *For $\mathcal{F}' \in \text{D-mod}(\mathrm{Bun}_G)$ there is a canonical isomorphism*

$$(p_{\mathrm{Bun}_G})_!(\mathcal{F}') \simeq (p_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}})_\bullet(\pi^!(\mathcal{F}')).$$

□

Proof of Lemma 7.7.10. It is enough to establish the isomorphism in question in the case when $\mathcal{F}' \in \text{D-mod}(\mathrm{Bun}_G)^c$. We will show that

$$\mathcal{M}aps_{\mathrm{Vect}}((p_{\mathrm{Bun}_G})_!(\mathcal{F}'), V) \simeq \mathcal{M}aps_{\mathrm{Vect}}((p_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}})_\bullet(\pi^!(\mathcal{F}')), V), \quad V \in \mathrm{Vect}.$$

We rewrite the left-hand side and the right-hand side as

$$\mathcal{M}aps_{\text{D-mod}(\mathrm{Bun}_G)}(\mathcal{F}', p^!_{\mathrm{Bun}_G}(V)) \text{ and } \mathcal{M}aps_{\text{D-mod}(\mathrm{Gr}_{G,\mathrm{Ran}(X)})}(\pi^!(\mathcal{F}'), p^!_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}}(V)),$$

respectively, and the required assertion follows from Theorem 7.7.4.

□

7.8. **Proof of Theorem 7.7.2.**

7.8.1. Taking into account Theorem 7.7.9, we need to show that the object

$$\text{Ps-Id}^{\mathrm{naive}}_{\mathrm{Bun}_G} \circ \pi_\bullet(\omega_{\mathrm{Gr}_{G,\mathrm{Ran}(X)}}) \in \text{D-mod}(\mathrm{Bun}_G)$$

is zero.

First, we recall that the prestack $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ is the colimit of ind-schemes, denoted Gr_{G,X^n}, see [Ga2, Sect. 4.1.1]. We will show that for every n

$$\text{Ps-Id}^{\mathrm{naive}}_{\mathrm{Bun}_G} \circ (\pi_n)_\bullet(\omega_{\mathrm{Gr}_{G,X^n}}) = 0, \tag{7.11}$$

where π_n denotes the map $\mathrm{Gr}_{G,X^n} \to \mathrm{Bun}_G$.

7.8.2. Recall that for an ind-scheme (of ind-finite type) $\mathcal{X}$ the category D-mod($\mathcal{X}$) carries a canonical t-structure, see [GR, Sect. 4.3]. It is characterized by the property that if

$$\mathcal{X} \simeq \underset{i\in I}{colim}\, X_i,$$

where $f_i : X_i \to \mathcal{X}$ are closed subschemes of $\mathcal{X}$, the category $\text{D-mod}(\mathcal{X})^{\leq 0}$ is generated under colimits by the essential images of the categories D-mod(X_i) under the functors $(f_i)_\bullet$.

The assertion of (7.11) follows from the combination of the following two statements:

Proposition 7.8.3. *For a reductive group G, the functor*

$$\text{Ps-Id}^{\mathrm{naive}}_{\mathrm{Bun}_G} \circ (\pi_n)_\bullet : \text{D-mod}(\mathrm{Gr}_{G,X^n}) \to \text{D-mod}(\mathrm{Bun}_G)$$

has cohomological amplitude bounded on the right by n.

Proposition 7.8.4. *If the semi-simple compnent of G is non-trivial, the object*

$$\omega_{\mathrm{Gr}_{G,X^n}} \in \text{D-mod}(\mathrm{Gr}_{G,X^n})$$

is infinitely connective, i.e., belongs to $\text{D-mod}(\mathrm{Gr}_{G,X^n})^{\leq -n}$ *for any n.*

7.8.5. *Proof of Proposition 7.8.3.* Let us write Gr_{G,X^n} as

$$\underset{i\in I}{colim}\, Z_i,$$

where Z_i's are closed subschemes of Gr_{G,X^n}.

By the definition of the t-structure on $\text{D-mod}(\mathrm{Gr}_{G,X^n})$, it is enough to show that each of functors

$$\text{Ps-Id}^{\text{naive}}_{\mathrm{Bun}_G}\circ(\pi_n)_\bullet\circ(f_i)_\bullet:\text{D-mod}(Z_i)\to\text{D-mod}(\mathrm{Bun}_G)$$

has cohomological amplitude bounded on the right by n.

However, it follows from the definitions, that the above composed functor is the usual direct image functor for the map

$$(\pi_n\circ f_i):Z_i\to\mathrm{Bun}_G\,.$$

We factor the above map as a composition

$$Z_i\overset{(s_n\times f_i)\times(\pi_n\circ f_i)}{\longrightarrow}X^n\times\mathrm{Bun}_G\to\mathrm{Bun}_G,$$

where s_n is the natural projection $\mathrm{Gr}_{G,X^n}\to X^n$.

The required assertion follows from the fact that the map

$$(s_n\times f_i)\times(\pi_n\circ f_i):Z_i\to X^n\times\mathrm{Bun}_G$$

is schematic and affine. The latter follows from the fact that the map

$$s_n\times\pi:\mathrm{Gr}_{G,X^n}\to X^n\times\mathrm{Bun}_G$$

is ind-affine.

□

7.8.6. *Proof of Proposition 7.8.4.* Consider the diagonal stratification of X^n. It is easy to see that it is sufficient to show that the !-restriction of $\omega_{\mathrm{Gr}_{G,X^n}}$ to the preimage of each stratum is infinitely connective.

Using the factorization property of Gr_{G,X^n} over X^n, the assertion is further reduced to the case when instead of Gr_{G,X^n} we consider $\mathrm{Gr}_{G,x}$, i.e., its local version at some point $x\in X$.

In the latter case we can assume that $X=\mathbb{P}^1$ and $x=\infty\in\mathbb{P}^1$. Denote the corresponding ind-scheme simply by Gr_G. We need to show that ω_{Gr_G} is infinitely connective.

We have the following lemma, proved below:

Lemma 7.8.7. *For an ind-scheme $\mathcal{X}$, the t-structure on* $\text{D-mod}(\mathcal{X})$ *is local in the Zariski topology, i.e., if $\mathcal{X}=\underset{i}{\cup}U_i$, where $U_i\subset\mathcal{X}$ are Zariski open subschemes, then an object $\mathcal{F}\in$* $\text{D-mod}(\mathcal{X})$ *is connective/coconnective if and only if its restrictions to U_i have this property.*

Let

$$\mathrm{Gr}^0_G\subset\mathrm{Gr}_G$$

be the open Bruhat cell, i.e., the preimage of $\mathrm{pt}/G\subset\mathrm{Bun}_G$ under the map π. The entire ind-scheme Gr_G can be covered by translates of Gr^0_G by means of the loop group. Hence, by Lemma 7.8.7, it is sufficient to show that $\omega_{\mathrm{Gr}^0_G}$ is infinitely connective.

However, it is known that for a reductive group with a nontrivial semi-simple part, the ind-scheme Gr^0_G is isomorphic to

$$\mathbb{A}^\infty\simeq\underset{k\geq 0}{colim}\,\mathbb{A}^k.$$

Now, for any n, we can write

$$\omega_{\mathbb{A}^\infty} \simeq \underset{m\geq n}{colim}\, (i_m)_\bullet(\omega_{\mathbb{A}^m}),$$

where $i_m : \mathbb{A}^m \to \mathbb{A}^\infty$. The functors $(i_m)_\bullet$ are t-exact, and

$$\omega_{\mathbb{A}^m} \in \text{D-mod}(\mathbb{A}^m)^\heartsuit[m] \subset \text{D-mod}(\mathbb{A}^m)^{\leq -n},$$

since $m \geq n$. Hence,

$$\omega_{\mathbb{A}^\infty} \in \text{D-mod}(\mathbb{A}^\infty)^{\leq -n},$$

as required.

□

7.8.8. *Proof of Lemma 7.8.7.* First, we note that the functor of restriction

$$\text{D-mod}(\mathcal{X}) \to \text{D-mod}(U)$$

for an open embedding $U \hookrightarrow \mathcal{X}$ is t-exact.

Let us show that the property of being coconnective is local in the Zariski topology. I.e., let $\mathcal{F} \in \text{D-mod}(\mathcal{X})$ be such that $\mathcal{F}|_{U_i} \in \text{D-mod}(U_i)^{>0}$, and we need to show that $\mathcal{F} \in \text{D-mod}(\mathcal{X})^{>0}$.

I.e., we need to show that for $\mathcal{F}' \in \text{D-mod}(\mathcal{X})^{\leq 0}$, we have

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{X})}(\mathcal{F}', \mathcal{F}) \in \text{Vect}^{>0}.$$

Let $U^\bullet$ be the Čech nerve of the cover $\underset{i}{\cup} U_i \to \mathcal{X}$.

The category $\text{D-mod}(\mathcal{X})$ satisfies Zariski descent. Hence, $\mathcal{M}aps_{\text{D-mod}(\mathcal{X})}(\mathcal{F}', \mathcal{F})$ is the totalization of a co-simplicial object of Vect whose n-th term is

$$\mathcal{M}aps_{\text{D-mod}(U^n)}(\mathcal{F}'|_{U^n}, \mathcal{F}|_{U^n}).$$

However, $\mathcal{F}'|_{U^n} \in \text{D-mod}(U^n)^{\leq 0}$ and $\mathcal{F}|_{U^n} \in \text{D-mod}(U^n)^{>0}$, and the assertion follows, as the functor of totalization is left t-exact.

The proof in the connective case is similar.

□

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