

## CHAPTER IV.1. FORMAL MODULI

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### INTRODUCTION

In this Chapter we prove one of the main results of this book: the existence of a well-defined procedure of taking a quotient with respect to a formal groupoid.

#### 0.1. Groupoids and quotients.

0.1.1. First off, a groupoid in  $\mathrm{Spc}$  is an object  $R^\bullet \in \mathrm{Spc}^{\Delta^{\mathrm{op}}}$  that is a Segal space such that all of its 1-morphisms are invertible. We shall say that  $R^\bullet$  acts on the space  $X = R^0$ . Sometimes we abuse the notation and instead of  $R^\bullet$  write just the space  $R := R^1$ .

In other words, a groupoid acting on  $X$  is a space  $R$ , equipped with a pair of projections

$$\begin{array}{ccc}
 & R & \\
 p_s \swarrow & & \searrow p_t \\
 X & & X,
 \end{array}$$

and a multiplication map

$$R \times_{p_t, X, p_s} R \xrightarrow{m} R$$

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over  $X \times X$ , satisfying a homotopy-coherent system of associativity conditions, and such that the map

$$R \times_{p_t, X, p_s} R \xrightarrow{m, \text{id}} R \times_{p_t, X, p_t} R$$

is an isomorphism.

0.1.2. Given a map  $X \rightarrow Y$  in  $\text{PreStk}_{\text{laft}}$ , the Čech nerve construction gives rise to a canonically defined groupoid  $R^\bullet$  acting on  $X$  with

$$R = X \times_Y X.$$

The above construction is a functor from the category of spaces under  $X$  to that of groupoids acting on  $X$ .

This functor admits a fully faithful left adjoint that sends  $R^\bullet$  to its geometric realization  $Y = |R^\bullet|$ . The image of this left adjoint is the full subcategory consisting of those  $X \rightarrow Y$  that induce a surjection on  $\pi_0$ .

0.1.3. The notion of groupoid makes sense in arbitrary  $\infty$ -category  $\mathcal{C}$  with finite limits, see [Lu1, Sect. 6.1.2].

Namely, given an object  $X \in \mathcal{C}$ , a groupoid acting on  $X$  is a simplicial object  $R^\bullet$  of  $\mathcal{C}$  with  $R^0 = X$  such that for any  $X' \in \mathcal{C}$ , the object

$$\text{Maps}_{\mathcal{C}}(X', R^\bullet) \in \text{Spc}^{\Delta^{\text{op}}}$$

is a groupoid in spaces.

As in the case of  $\mathcal{C} = \text{Spc}$ , given a map  $X \rightarrow Y$ , we canonically attach to it its Čech nerve, which is a groupoid acting on  $X$ .

However, the existence of the left adjoint can only be guaranteed if  $\mathcal{C}$  has colimits. This left adjoint will be fully faithful if geometric realizations in  $\mathcal{C}$  commute with fiber products.

0.1.4. Thus, we obtain a well-defined notion of groupoid object  $\mathcal{R}^\bullet$  in  $\text{PreStk}_{\text{laft}}$  acting on a given  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ .

Let  $\mathcal{X} \rightarrow \mathcal{Y}$  be a map in  $\text{PreStk}_{\text{laft}}$ . Taking its Čech nerve, we obtain a groupoid  $\mathcal{R}^\bullet$ . The assignment

$$\mathcal{R}^\bullet \rightarrow |\mathcal{R}^\bullet|$$

provides a fully faithful left adjoint.

## 0.2. Formal groupoids.

0.2.1. We now modify our problem: instead of the category  $\text{PreStk}_{\text{laft}}$ , we now consider the category  $\text{PreStk}_{\text{laft-def}}$ . I.e., we impose the condition that our prestacks *admit deformation theory*. In addition, we will restrict to maps between prestacks that are nil-isomorphisms.

Groupoid objects in this context will be called *formal groupoids*; for a given  $\mathcal{X}$  we denote the category of formal groupoids over  $\mathcal{X}$  by  $\text{FormGrpoid}(\mathcal{X})$ .

Starting from  $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$  and an object  $\mathcal{R}^\bullet \in \text{FormGrpoid}(\mathcal{X})$ , it is *not true* that the prestack  $|\mathcal{R}^\bullet|$  admits deformation theory. So, the existence of a fully faithful left adjoint to the Čech nerve construction is not so obvious in this case.

However, the main result of this chapter, Theorem 2.3.2 says:

**Theorem 0.2.2.** *For  $\mathcal{X} \in \text{PreStk}_{\text{lft-def}}$ , the Čech nerve construction is an equivalence between the category of  $\mathcal{Y} \in \text{PreStk}_{\text{lft-def}}$  equipped with a nil-isomorphism  $\mathcal{X} \rightarrow \mathcal{Y}$  and the category  $\text{FormGrpoid}(\mathcal{X})$ .*

In other words, this theorem says that, given a formal groupoid  $\mathcal{R}$  acting on  $\mathcal{X}$ , there is a well-defined quotient

$$B_{\mathcal{X}}(\mathcal{R}) \in (\text{PreStk}_{\text{lft}})_{\mathcal{X}/},$$

such that  $B_{\mathcal{X}}(\mathcal{R})$  admits deformation theory and the map  $\mathcal{X} \rightarrow B_{\mathcal{X}}(\mathcal{R})$  is a nil-isomorphism (it is then automatically inf-schematic).

0.2.3. As a particular case of Theorem 0.2.2 we obtain that the loop functor

$$\mathcal{Y} \mapsto \mathcal{G} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

defines an equivalence between the category of  $\mathcal{Y} \in \text{PreStk}_{\text{lft}}$ , equipped with a pair of inf-schematic nil-isomorphisms

$$\mathcal{X} \xrightarrow{i} \mathcal{Y} \xrightarrow{s} \mathcal{X}, \quad s \circ i = \text{id}$$

and that of group-objects in the category prestacks  $\mathcal{G}$  equipped with an inf-schematic nil-isomorphism  $\mathcal{G} \rightarrow \mathcal{X}$ . We denote the latter category by  $\text{Grp}(\text{Form}/_{\mathcal{X}})$ , and refer to its objects as *formal groups over  $\mathcal{X}$* .

Thus, to any  $\mathcal{G}$  as above, we can attach its classifying prestack  $B_{\mathcal{X}}(\mathcal{G})$

$$\mathcal{X} \xrightarrow{i} B_{\mathcal{X}}(\mathcal{G}) \xrightarrow{s} \mathcal{X}$$

where  $i$  and  $s$  are inf-schematic nil-isomorphisms.

**0.3. What else is done in this chapter?** Let  $X$  be an object of  $\text{Sch}_{\text{aft}}^{\text{aff}}$ . We introduce several notions of formal moduli problems associated with  $X$ , and we relate them to notions developed in [Lu6].

0.3.1. A formal moduli problem *over  $X$*  is an inf-scheme  $\mathcal{Y}$  equipped with a nil-isomorphism  $\mathcal{Y} \rightarrow X$ .

Recall that, by definition, an inf-scheme is a prestack locally almost of finite type. I.e.,  $\mathcal{Y}$  is encoded by a functor

$$((<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{/X})^{\text{op}} \rightarrow \text{Spc}.$$

In Proposition 1.2.2, we show that the data of  $\mathcal{Y}$  is completely determined by its values on a much smaller category: namely,

$$(<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{\text{nil-isom to } X} \subset (<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{/X}$$

that consists of those  $S \rightarrow X$  that are nil-isomorphisms.

Moreover, the condition that  $\mathcal{Y}$  admit deformation theory can also be expressed via the resulting functor

$$((<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{\text{nil-isom to } X})^{\text{op}} \rightarrow \text{Spc}.$$

0.3.2. Note that when  $X = \text{pt} = \text{Spec}(k)$ , the category  $(<^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}})_{\text{nil-isom to } X}$  is the one opposite to that of connective commutative DG algebras  $A$  over  $k$  that have finitely many cohomologies, and all of whose cohomologies are finite dimensional, with  $H^0(A)$  local.

So, a formal moduli problem over  $\text{pt}$  is the same as a functor on the category of such algebras, subject to some conditions that guarantee deformation theory.

0.3.3. Suppose now that  $X \in <^\infty\text{Sch}_{\text{ft}}^{\text{aff}}$ . In this case, we study the notion of formal moduli problem *under*  $X$ . By definition, this is an inf-scheme  $\mathcal{Y}$ , equipped with a nil-isomorphism

$$X \rightarrow \mathcal{Y}.$$

We show that the data of such  $\mathcal{Y}$ , viewed as a functor  $(<^\infty\text{Sch}_{\text{ft}}^{\text{aff}})^{\text{op}} \rightarrow \text{Spc}$ , is recovered from its values on a smaller category, namely the category

$$(<^\infty\text{Sch}_{\text{ft}}^{\text{aff}})_{\text{nil-isom from } X} \subset (<^\infty\text{Sch}_{\text{ft}}^{\text{aff}})_{X/},$$

consisting of those  $X \rightarrow S$  that are nil-isomorphisms.

0.3.4. Note that when  $\mathcal{X} = \text{pt}$ , for an object  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$  to be a formal moduli problem under  $\mathcal{X}$  simply means that  $\mathcal{Y}$  is an inf-scheme with  $\text{red}\mathcal{Y} = \text{pt}$ .

I.e., formal moduli problems under  $\text{pt}$  are the same as formal moduli problems over  $\text{pt}$ .

## 1. FORMAL MODULI PROBLEMS

In this section we introduce the notions of formal moduli problem *over* and *under* a given prestack  $\mathcal{X}$ .

Let us note the following substantial difference between our set-up and that of [Lu6] (in which the case  $\mathcal{X} = \text{pt}$ ):

In the context of *loc.cit.* a formal moduli problem is a functor on the category of connective finite-dimensional commutative DG algebras over  $k$ , whose 0-th cohomology is local.

By contrast, our formal moduli problems (for  $\mathcal{X} = \text{pt}$ ) are objects  $\mathcal{Y} \in \text{PreStk}_{\text{laft-def}}$  with  $\text{red}\mathcal{Y} = \text{pt}$ , so they can be evaluated on connective commutative finite-dimensional DG algebras over  $k$ . The two notions are related by the procedures of restriction and left Kan extension; the fact that these two procedures are inverses of one another is the consequence of [Chapter III.2, Corollary 4.4.6].

**1.1. Formal moduli problems over a prestack.** Unlike [Lu6], we define the category of formal moduli problems over a given  $\mathcal{X}$  to be a full subcategory in  $(\text{PreStk}_{\text{laft}})_{/\mathcal{X}}$ . The equivalence of this definition and the one in *loc. cit.* will be established in Sect. 1.2.

1.1.1. Let us fix  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ . We define

$$\text{FormMod}_{/\mathcal{X}} \subset (\text{PreStk}_{\text{laft}})_{/\mathcal{X}}$$

to be the full subcategory of spanned by those  $\mathcal{Y} \rightarrow \mathcal{X}$ , for which the above map is:

- inf-schematic (see [Chapter III.2, Definition 3.1.5] for what this means);
- a nil-isomorphism (i.e.,  $\text{red}\mathcal{Y} \rightarrow \text{red}\mathcal{X}$  is an isomorphism).

We shall refer to  $\text{FormMod}_{/\mathcal{X}}$  as the category of *formal moduli problems over*  $\mathcal{X}$ .

1.1.2. Let  $\mathcal{X}'$  be an object of  $\text{FormMod}_{/\mathcal{X}}$ . We will use the notation

$$\text{FormMod}_{\mathcal{X}'/ / \mathcal{X}}$$

to denote the category

$$(\text{FormMod}_{/\mathcal{X}})_{\mathcal{X}'/}.$$

1.1.3. The following results easily from the definitions:

**Lemma 1.1.4.** *Let  $\mathcal{Y} \rightarrow \mathcal{X}$  be a map in  $\text{PreStk}_{\text{laft}}$ . Then  $\mathcal{Y} \in \text{FormMod}_{/\mathcal{X}}$  if and only if for every  $S \in <^\infty\text{Sch}_{\text{ft}}^{\text{aff}}$ , the prestack  $S \times_{\mathcal{X}} \mathcal{Y}$  is an inf-scheme and  $\text{red}(S \times_{\mathcal{X}} \mathcal{Y}) \rightarrow \text{red}S$  is an isomorphism.*

1.1.5. The following will be useful:

**Lemma 1.1.6.** *The functor*

$$\mathrm{FormMod}/_{\mathcal{X}} \rightarrow \lim_{(Z,x) \in ((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}}} \mathrm{FormMod}/_Z$$

*is an equivalence.*

**1.2. Situation over an affine scheme.** In this subsection we will assume that that  $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$ . We will show that a formal moduli problem over  $X$ , viewed as a functor

$$((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}} \rightarrow \mathrm{Spc},$$

is completely determined by its value on those  $Z \in (<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X}$  for which  $\mathrm{red}Z \rightarrow \mathrm{red}X$  is an isomorphism.

Note that when  $X = \mathrm{pt}$ , the category  $((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}}$  is the same as that of connective finite-dimensional commutative DG algebras over  $k$ , whose 0-th cohomology is local. This brings us in contact with the definition of formal moduli problems in [Lu6].

1.2.1. We have:

**Proposition 1.2.2.**

(a) *Every  $\mathcal{Y} \in \mathrm{FormMod}/_X$ , viewed as a functor*

$$((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}} \rightarrow \mathrm{Spc},$$

*is the left Kan extension of its restriction to the full subcategory*

$$(1.1) \quad ((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil-isom\ to\ }X})^{\mathrm{op}} \subset ((<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}}.$$

(b) *Let  $\mathcal{Y}_{\mathrm{nil-isom}}$  be a presheaf on the category  $(<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil-isom\ to\ }X}$ , satisfying:*

- $\mathcal{Y}_{\mathrm{nil-isom}}(\mathrm{red}X) = *$ ;
- *For a push-out diagram  $S_1 \sqcup_S S'$  in  $(<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil-isom\ to\ }X}$ , where  $S \rightarrow S'$  has a structure of square-zero extension, the resulting map*

$$\mathcal{Y}_{\mathrm{nil-isom}}(S_1 \sqcup_S S') \rightarrow \mathcal{Y}_{\mathrm{nil-isom}}(S_1) \times_{\mathcal{Y}_{\mathrm{nil-isom}}(S)} \mathcal{Y}_{\mathrm{nil-isom}}(S')$$

*is an isomorphism.*

*Then if*

$$\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{lft}})_{/X}$$

*denotes the left Kan extension of  $\mathcal{Y}_{\mathrm{nil-isom}}$  under (1.1), then  $\mathcal{Y} \in \mathrm{FormMod}/_X$ .*

(c) *The assignments*

$$\mathcal{Y} \mapsto \mathcal{Y}|_{(<^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil-isom\ to\ }X}} \quad \text{and} \quad \mathcal{Y}_{\mathrm{nil-isom}} \mapsto \mathcal{Y}$$

*are mutually inverse equivalences of categories.*

*Proof.* Point (a) follows from [Chapter III.2, Corollary 4.3.4].

Point (b) follows from [Chapter III.2, Proposition 4.4.5].

Point (c) follows from [Chapter III.2, Corollary 4.4.6]. □

1.2.3. The following assertion will be used extensively in [Chapter IV.3]:

**Corollary 1.2.4.** *For  $\mathcal{Y} \in \text{FormMod}/X$ , the map*

$$\text{colim}_{(Z,f)} f_*^{\text{IndCoh}}(\omega_Z) \rightarrow \omega_{\mathcal{Y}}$$

*is an isomorphism, where the colimit is taken over the category*

$$((<^{\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})/_{\mathcal{Y}}.$$

*Proof.* By Proposition 1.2.2, the functor

$$((<^{\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})/_{\mathcal{Y}} \rightarrow (<^{\infty}\text{Sch}^{\text{aff}})/_{\mathcal{Y}}$$

is cofinal. Hence, the restriction functor

$$\text{IndCoh}(\mathcal{Y}) \rightarrow \lim_{(Z,f)} \text{IndCoh}(Z)$$

is an isomorphism, where the limit is taken over the category  $((<^{\infty}\text{Sch}^{\text{aff}})_{\text{nil-isom to } X})/_{\mathcal{Y}}$ .

By [Chapter III.3, Corollary 4.3.4] and [Chapter I.1, Proposition 2.5.7], we obtain that the functors  $f_*^{\text{IndCoh}} : \text{IndCoh}(Z) \rightarrow \text{IndCoh}(\mathcal{Y})$  define an equivalence

$$\text{colim}_{(Z,f)} \text{IndCoh}(Z) \rightarrow \text{IndCoh}(\mathcal{Y}),$$

where the colimit is taken with respect to the  $(\text{IndCoh}, *)$ -direct image functors.

In particular, we obtain that

$$\omega_{\mathcal{Y}} \simeq \text{colim}_{(Z,f)} f_*^{\text{IndCoh}} \circ f^!(\omega_{\mathcal{Y}}) \simeq \text{colim}_{(Z,f)} f_*^{\text{IndCoh}}(\omega_Z),$$

as required. □

**1.3. Formal moduli problems *under* a prestack.** In this subsection we consider a prestack

$$\mathcal{X} \in \text{PreStk}_{\text{lft-def}}.$$

We will consider another paradigm for formal moduli problems, by looking at prestacks *under*  $\mathcal{X}$ .

1.3.1. We define the category  $\text{FormMod}_{\mathcal{X}/}$  to be the full subcategory of  $(\text{PreStk}_{\text{lft}})_{\mathcal{X}/}$  spanned by those  $\mathcal{X} \rightarrow \mathcal{Y}$ , for which:

- $\mathcal{Y} \in \text{PreStk}_{\text{lft-def}}$ ;
- The map  $\mathcal{X} \rightarrow \mathcal{Y}$  is a nil-isomorphism.

Note that since in the above definition, the map  $\mathcal{X} \rightarrow \mathcal{Y}$  is automatically inf-schematic, and so realizes  $\mathcal{X}$  as an object of  $\text{FormMod}/_{\mathcal{Y}}$ .

1.3.2. Let  $\mathcal{X}'$  be an object of  $\text{FormMod}_{\mathcal{X}/}$ . Note that the category

$$(\text{FormMod}_{\mathcal{X}/})/_{\mathcal{X}'}$$

identifies with

$$\text{FormMod}_{\mathcal{X}/\mathcal{X}'}$$

from Sect. 1.1.2.

1.3.3. Note that when  $\mathcal{X} = \text{pt}$ , there is no difference between  $\text{FormMod}_{\mathcal{X}/}$  and  $\text{FormMod}/_{\mathcal{X}}$ .

**1.4. Formal moduli problems *under* an affine scheme.** In this subsection we specialize to the case when

$$\mathcal{X} = X \in \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle.$$

We will show that a formal moduli problem  $\mathcal{Y}$  under  $X$ , viewed as a functor

$$\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle^{\text{op}} \rightarrow \text{Spc},$$

is completely determined by its value on the category of affine schemes  $Z$ , equipped with a nil-isomorphism  $X \rightarrow Z$ .

1.4.1. Let

$$\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X} \subset \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{X/}$$

be the full subcategory formed by those  $f : X \rightarrow Z$ , for which  $f$  is a nil-isomorphism.

For a prestack  $\mathcal{Y}$  under  $X$ , consider the functor

$$\mathcal{Y}|_{\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X}} \times_{\text{Maps}(X, \mathcal{Y})} * : \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X} \rightarrow \text{Spc}$$

that sends  $X \rightarrow S$  to

$$\text{Maps}(S, \mathcal{Y}) \times_{\text{Maps}(X, \mathcal{Y})} *.$$

We claim:

**Proposition 1.4.2.**

(a) For  $\mathcal{Y} \in \text{FormMod}_{X/}$ , viewed as a functor

$$\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle^{\text{op}} \rightarrow \text{Spc},$$

the map

$$\text{LKE}_{(\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X})^{\text{op}} \rightarrow \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle^{\text{op}}} (\mathcal{Y}|_{\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X}} \times_{\text{Maps}(X, \mathcal{Y})} *) \rightarrow \mathcal{Y}$$

is an isomorphism.

(b) Let  $\mathcal{Y}_{\text{nil-isom}}$  be a presheaf on  $\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X}$ , satisfying:

- $\mathcal{Y}_{\text{nil-isom}}(X) = *$ ;
- For a push-out diagram  $S_1 \sqcup_S S'$  in  $\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X}$ , where  $S \rightarrow S'$  has a structure of square-zero extension, the resulting map

$$\mathcal{Y}_{\text{nil-isom}}(S_1 \sqcup_S S') \rightarrow \mathcal{Y}_{\text{nil-isom}}(S_1) \times_{\mathcal{Y}_{\text{nil-isom}}(S)} \mathcal{Y}_{\text{nil-isom}}(S')$$

is an isomorphism.

Then if  $\mathcal{Y} \in \text{PreStk}_{\text{left}}$  denotes the left Kan extension of  $\mathcal{Y}_{\text{nil-isom}}$  under

$$(\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X})^{\text{op}} \rightarrow \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle^{\text{op}},$$

then the canonical map  $X \rightarrow \mathcal{Y}$  makes  $\mathcal{Y}$  into an object of  $\text{FormMod}_{X/}$ .

(c) The assignments

$$\mathcal{Y} \mapsto \mathcal{Y}|_{\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X}} \times_{\text{Maps}(X, \mathcal{Y})} * \text{ and } \mathcal{Y}_{\text{nil-isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

*Proof.* Applying [Chapter III.2, Corollary 4.3.4, Proposition 4.4.5 and Corollary 4.4.6], it is enough to evaluate the functor  $\mathcal{Y}$  as in the proposition on the subcategory

$$(\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{redSch}_{\text{ft}}^{\text{aff}}} \times \{\text{red}X\}.$$

The assertion of the proposition follows now from the fact that the forgetful functor

$$(\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{nil-isom from } X} \rightarrow (\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{\text{redSch}_{\text{ft}}^{\text{aff}}} \times \{\text{red}X\})$$

admits a left adjoint, given by

$$S \mapsto S \sqcup_{\text{red}X} X.$$

□

1.4.3. As a corollary, we obtain:

**Corollary 1.4.4.** *For  $\mathcal{Y} \in \text{FormMod}_{X/}$ , the map*

$$\text{colim}_{(Z,f)} f_*^{\text{IndCoh}}(\omega_Z) \rightarrow \omega_{\mathcal{Y}}$$

*is an isomorphism, where the colimit is taken over the category*

$$((\langle \infty \text{Sch}^{\text{aff}} \rangle_{\text{nil-isom from } X})/\mathcal{Y}.$$

*Proof.* Same as that of Corollary 1.2.4. □

1.5. **The pointed case.** For  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$  we consider the category

$$\text{Ptd}(\text{FormMod}_{/\mathcal{X}}) = \text{FormMod}_{\mathcal{X}/\mathcal{X}}$$

of pointed objects in  $\text{FormMod}_{/\mathcal{X}}$ .

1.5.1. By definition,  $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$  is the category of diagrams

$$(\pi : \mathcal{Y} \rightrightarrows \mathcal{X} : s), \quad \pi \circ s = \text{id}$$

with the map  $\pi$  being an inf-schematic nil-isomorphism.

Note also that if  $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$ , then

$$\text{Ptd}(\text{FormMod}_{/\mathcal{X}}) \simeq (\text{Ptd}(\text{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}.$$

Combining Propositions 1.2.2 and 1.4.2, we obtain:

**Corollary 1.5.2.** *For  $X \in \langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle$  we have:*

(a) *Any  $\mathcal{Y} \in \text{Ptd}(\text{FormMod}_{/X})$ , viewed as a functor*

$$((\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{/X})^{\text{op}} \rightarrow \text{Spc},$$

*receives an isomorphism from the left Kan extension along*

$$(\text{Ptd}((\langle \infty \text{Sch}^{\text{aff}} \rangle_{\text{nil-isom to } X})^{\text{op}} \rightarrow ((\langle \infty \text{Sch}_{\text{ft}}^{\text{aff}} \rangle_{/X})^{\text{op}})$$

*of*

$$\mathcal{Y}_{\text{Ptd}((\langle \infty \text{Sch}^{\text{aff}} \rangle_{\text{nil-isom to } X})} \times_{\text{Maps}(X, \mathcal{Y})} *.$$

(b) *Let  $\mathcal{Y}_{\text{nil-isom}}$  be a presheaf on  $\text{Ptd}((\langle \infty \text{Sch}^{\text{aff}} \rangle_{\text{nil-isom to } X})$ , satisfying:*

- $\mathcal{Y}_{\text{nil-isom}}(X) = *;$

- For a push-out diagram  $S_1 \sqcup_S S'$  in  $\mathrm{Ptd}((\llcorner^\infty \mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil-isom} \text{ to } X})$ , where  $S \rightarrow S'$  has a structure of square-zero extension, the resulting map

$$\mathcal{Y}_{\mathrm{nil-isom}}(S_1 \sqcup_S S') \rightarrow \mathcal{Y}_{\mathrm{nil-isom}}(S_1) \times_{\mathcal{Y}_{\mathrm{nil-isom}}(S)} \mathcal{Y}_{\mathrm{nil-isom}}(S')$$

is an isomorphism.

Then if  $\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$  denotes the left Kan extension of  $\mathcal{Y}_{\mathrm{nil-isom}}$  along

$$(\mathrm{Ptd}((\llcorner^\infty \mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil-isom} \text{ to } X}))^{\mathrm{op}} \rightarrow ((\llcorner^\infty \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}},$$

then the canonical map  $X \rightarrow \mathcal{Y}$  makes  $\mathcal{Y}$  into an object of  $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ .

(c) The assignments

$$\mathcal{Y} \mapsto \mathcal{Y}|_{\mathrm{Ptd}((\llcorner^\infty \mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil-isom} \text{ to } X})} \times_{\mathrm{Maps}(X, \mathcal{Y})} * \text{ and } \mathcal{Y}_{\mathrm{nil-isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

*Remark 1.5.3.* The informal meaning of this corollary is that in order to ‘know’ an object  $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$  as a prestack, it is enough to test it on affine schemes  $S$ , equipped with a nil-isomorphism to  $X$  and a section of this nil-isomorphism.

1.5.4. As in the case of Corollary 1.2.4, from Corollary 1.5.2, we obtain:

**Corollary 1.5.5.** For  $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ , the map

$$\mathrm{colim}_{(Z, f)} f_*^{\mathrm{IndCoh}}(\omega_Z) \rightarrow \omega_{\mathcal{Y}}$$

is an isomorphism, where the colimit is taken over the category

$$(\mathrm{Ptd}((\llcorner^\infty \mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil-isom} \text{ to } X}))_{/\mathcal{Y}}.$$

**1.6. Formal groups.** In this subsection we let  $\mathcal{X}$  be an object of  $\mathrm{PreStk}_{\mathrm{laft}}$ , and we introduce formal groups over  $\mathcal{X}$  as group-objects in the category of formal moduli problems.

In [Chapter IV.3] we will show that the category of formal groups identifies with that of Lie algebras in  $\mathrm{IndCoh}(\mathcal{X})$ , thereby generalizing the main result in [Lu6].

1.6.1. Let  $\mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$  be the category of monoid-objects in  $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ , and let

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$$

be the full subcategory spanned by group-like objects.

**Lemma 1.6.2.** The inclusion  $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$  is an equivalence.

*Proof.* We need to show that for  $\mathcal{H} \in \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$  the map

$$\mathcal{H} \times_{\mathcal{X}} \mathcal{H} \rightarrow \mathcal{H} \times_{\mathcal{X}} \mathcal{H}, \quad (h_1, h_2) \mapsto (h_1, h_1 \cdot h_2)$$

is an isomorphism.

This follows from [Chapter III.1, Corollary 8.3.6], applied to the Cartesian square

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h \mapsto (1, h)} & \mathcal{H} \times_{\mathcal{X}} \mathcal{H} \\ \mathrm{id} \downarrow & & \downarrow (h_1, h_2) \mapsto (h_1, h_1 \cdot h_2) \\ \mathcal{H} & \xrightarrow{h \mapsto (1, h)} & \mathcal{H} \times_{\mathcal{X}} \mathcal{H}. \end{array}$$

□

1.6.3. We have a naturally defined functor

$$(1.2) \quad \Omega_{\mathcal{X}} : \text{Ptd}(\text{FormMod}/\mathcal{X}) \rightarrow \text{Grp}(\text{FormMod}/\mathcal{X}), \quad \mathcal{Y} \mapsto \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$

In Sect. 2.3.4 we will prove:

**Theorem 1.6.4.** *The functor  $\Omega_{\mathcal{X}}$  of (1.2) is an equivalence.*

In what follows we shall denote by  $B_{\mathcal{X}}$  the functor

$$\text{Grp}(\text{FormMod}/\mathcal{X}) \rightarrow \text{Ptd}(\text{FormMod}/\mathcal{X}),$$

inverse to  $\Omega_{\mathcal{X}}$ .

1.6.5. Combining Theorem 1.6.4 with [Chapter IV.3, Corollary 3.6.3] (which will be proved independently), we obtain:

**Corollary 1.6.6.** *The category  $\text{Ptd}(\text{FormMod}/\mathcal{X})$  contains sifted colimits, and the functor*

$$(\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{X}) \mapsto T(\mathcal{X}/\mathcal{Y}) : \text{Ptd}(\text{FormMod}/\mathcal{X}) \rightarrow \text{IndCoh}(\mathcal{X})$$

*preserves sifted colimits.*

*Remark 1.6.7.* Note, however, that the forgetful functor

$$\text{Ptd}(\text{FormMod}/\mathcal{X}) \rightarrow \text{Ptd}((\text{PreStk}_{\text{laft}})_{/\mathcal{X}})$$

does *not* preserve sifted colimits.

1.6.8. Assume for a moment that  $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$ . Consider the category  $\text{FormMod}_{\mathcal{X}/}$ . We note that even before we knew that the category  $\text{FormMod}_{\mathcal{X}/}$  contains sifted colimits, we could conclude that forgetful functor

$$(\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{X}) \mapsto (\mathcal{X} \rightarrow \mathcal{Y}) : \text{Ptd}(\text{FormMod}/\mathcal{X}) \rightarrow \text{FormMod}_{\mathcal{X}/}$$

preserves colimits. This follows from the fact that the above functor admits a right adjoint, given by

$$(\mathcal{X} \rightarrow \mathcal{Y}') \mapsto (\mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X}_{\text{aR}}} \mathcal{Y}').$$

## 2. GROUPOIDS

In this section we introduce the notion of formal groupoid over a given object  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ . We show that if  $\mathcal{X}$  admits deformation theory, then there is a well-defined procedure of taking a quotient by a formal groupoid, that produces another object of  $\text{PreStk}_{\text{laft-def}}$ .

**2.1. Digression: groupoids and groups over spaces.** In this subsection we review the definition of the notion of *groupoid* acting on a space in the category  $\text{Spc}$ .

2.1.1. Given a space  $X$ , recall the category  $\text{Grpoid}(X)$  of groupoids acting on  $X$  (see [Lu1], Sect. 6.1.2). By definition,  $\text{Grpoid}(X)$  is a full subcategory in the category of simplicial spaces, equipped with an identification of the space of 0-simplices with  $X$ . A simplicial object  $R^{\bullet}$  belongs to  $\text{Grpoid}(X)$  if the following two conditions are satisfied:

- For every  $n \geq 2$ , the map

$$R^n \rightarrow R^1 \times_X \dots \times_X R^1,$$

given by the product of the maps

$$[1] \rightarrow [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, \dots, n-1,$$

is an isomorphism.<sup>1</sup>

- The map  $R^2 \rightarrow R^1 \times_X R^1$ , given by the product of the maps  $[1] \rightarrow [2]$

$$0 \mapsto 0, 1 \mapsto 1 \text{ and } 0 \mapsto 0, 1 \mapsto 2,$$

is an isomorphism.

We shall symbolically depict a groupoid via a diagram

$$(2.1) \quad \begin{array}{ccc} & R & \\ p_s \swarrow & & \searrow p_t \\ X & & X, \end{array}$$

while properly we should be thinking about the entire simplicial object  $R^\bullet$  of  $\text{Spc}$  with  $R^0 = X$  and  $R^1 = R$ .

The category  $\text{Grpoid}(X)$  contains an initial object, the identity groupoid, where all  $R^i = X$ . We shall denote it by  $\text{diag}_X$ .

The category  $\text{Grpoid}(X)$  also contains a final object, namely  $X \times X$ .

2.1.2. The following assertion will be used repeatedly:

**Lemma 2.1.3.** *The forgetful functor  $\text{Grpoid}(X) \rightarrow \text{Spc}$  that sends a groupoid to  $R^n$  for any  $n$  preserves sifted colimits.*

*Proof.* Follows from the fact that if  $I$  is a sifted index category and

$$i \mapsto R_i^\bullet$$

is an  $I$  family of objects of  $\text{Grpoid}(X)$ , the map

$$\text{colim}_i (R_i^1 \times_X \dots \times_X R_i^1) \rightarrow (\text{colim}_i R_i^1) \times_X \dots \times_X (\text{colim}_i R_i^1)$$

is an isomorphism. □

2.1.4. Given a groupoid  $R$  acting on  $X$  we can consider the quotient space

$$X/R := |R^\bullet|,$$

which receives a natural map from  $X$ :

$$X = R^0 \rightarrow |R^\bullet| \rightarrow X/R.$$

Vice versa, given a space  $Y$  under  $X$ , we construct the groupoid over  $X$  by  $R := X \times_Y X$ , i.e.,  $R^\bullet$  is the Čech nerve of the above map  $X \rightarrow Y$ .

It is clear that the two functors

$$\text{Grpoid}(X) \rightleftarrows \text{Spc}_{X/}$$

are adjoint to one another.

We have:

**Lemma 2.1.5.** *The above two functors define equivalences between the category  $\text{Grpoid}(X)$  and the full subcategory  $\text{Spc}_{X/, \text{surj}}$  of  $\text{Spc}_{X/}$  spanned by objects  $i : X \rightarrow Y$ , for which the map  $i$  is surjective on  $\pi_0$ .*

<sup>1</sup>If we impose just this condition, the corresponding category is that of *Segal objects* (a.k.a. *category-objects*) acting on  $X$ , denoted  $\text{Seg}(X)$ .

2.1.6. Given a space  $X$ , consider the category

$$\mathrm{Grp}(\mathrm{Spc}/X),$$

of group-objects in the category of spaces over  $X$ . We have:

$$\mathrm{Grp}(\mathrm{Spc}/X) \simeq \mathrm{Grpoid}(X)_{/\mathrm{diag}_X},$$

where  $\mathrm{diag}_X \in \mathrm{Grpoid}(X)$  is the identity groupoid.

Consider also the category  $\mathrm{Ptd}(\mathrm{Spc}/X)$ , which is the same as the category of retraction diagrams

$$(2.2) \quad i : X \rightrightarrows Y : s, \quad s \circ i \simeq \mathrm{id}_X.$$

We will also use the notation

$$\mathrm{Spc}_{X//X}$$

for the above category.

We have a natural functor

$$(2.3) \quad \mathrm{Grp}(\mathrm{Spc}/X) \rightarrow \mathrm{Ptd}(\mathrm{Spc}/X),$$

given by

$$G \mapsto B_X(G),$$

where  $B_X(G)$  is the relative classifying space over  $X$ .

We also have the adjoint loop functor

$$\Omega_X : \mathrm{Ptd}(\mathrm{Spc}/X) \rightarrow \mathrm{Grp}(\mathrm{Spc}/X).$$

**Lemma 2.1.7.** *The functors  $B_X$  and  $\Omega_X$  define an equivalence between  $\mathrm{Grp}(\mathrm{Spc}/X)$  and the full subcategory  $\mathrm{Ptd}(\mathrm{Spc}/X)_{\mathrm{isom}}$  of  $\mathrm{Ptd}(\mathrm{Spc}/X)$ , spanned by those objects*

$$i : X \rightrightarrows Y : s$$

for which the map  $i$  is an isomorphism on  $\pi_0$ .

Note that the condition on an object  $(i : X \rightrightarrows Y : s) \in \mathrm{Ptd}(\mathrm{Spc}/X)$  to belong to

$$\mathrm{Ptd}(\mathrm{Spc}/X)_{\mathrm{isom}} \subset \mathrm{Ptd}(\mathrm{Spc}/X)$$

is equivalent to the map  $i$  being a surjection on  $\pi_0$ .

**2.2. Groupoids in formal geometry.** In this subsection we render the set-up of Sect. 2.1 into the context of algebraic geometry.

2.2.1. The definitions of Sect. 2.1 carry over automatically to the algebro-geometric setting. For  $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ , we consider the categories

$$(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}, \quad \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}), \quad \mathrm{Grp}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}), \quad \mathrm{Grpoid}_{\mathrm{laft}}(\mathcal{X}), \quad \text{and} \quad \mathrm{Seg}_{\mathrm{laft}}(\mathcal{X})$$

and their full subcategories

$$\mathrm{FormMod}_{/\mathcal{X}}, \quad \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}), \quad \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}), \quad \mathrm{FormGrpoid}(\mathcal{X}), \quad \text{and} \quad \mathrm{FormSeg}(\mathcal{X})$$

formed by objects that are formal as prestacks over  $\mathcal{X}$ .

Note, however, that as in Lemma 1.6.2, one shows that the inclusion

$$\mathrm{FormGrpoid}(\mathcal{X}) \hookrightarrow \mathrm{FormSeg}(\mathcal{X})$$

is an equivalence.

2.2.2. *Examples.* Let  $\mathcal{X} \rightarrow \mathcal{Y}$  be a map between objects of  $\text{PreStk}_{\text{laft}}$ . To it we attach the object of  $\text{Grpoid}_{\text{laft}}(\mathcal{X})$ , namely, the Čech nerve of this map. Thus, the corresponding

$$\mathcal{R} \rightrightarrows \mathcal{X}$$

is given by

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$

If the above map  $\mathcal{X} \rightarrow \mathcal{Y}$  is an inf-schematic nil-isomorphism, then  $\mathcal{R} \in \text{FormGrpoid}(\mathcal{X})$ .

2.2.3. As in Lemma 2.1.3, we obtain:

**Corollary 2.2.4.** *The category  $\text{FormGrpoid}(\mathcal{X})$  contains sifted colimits, and the functors*

$$\text{FormGrpoid}(\mathcal{X}) \rightarrow \text{FormMod}_{\mathcal{X}/}, \quad \text{FormGrpoid}(\mathcal{X}) \rightrightarrows \text{Ptd}(\text{FormMod}_{/\mathcal{X}})$$

and

$$\text{FormGrpoid}(\mathcal{X}) \rightarrow \text{IndCoh}(\mathcal{X})$$

that send a groupoid to

$$(\mathcal{X} \xrightarrow{\text{unit}} \mathcal{R}), \quad (\mathcal{X} \xrightleftharpoons[p_s]{\text{unit}} \mathcal{R}), \quad (\mathcal{X} \xrightleftharpoons[p_t]{\text{unit}} \mathcal{R}),$$

and  $T(\mathcal{X}/\mathcal{R})$ , respectively, commute with sifted colimits.

*Proof.* Let  $I$  be a filtered index category and

$$i \mapsto \mathcal{R}_i^\bullet$$

be an  $I$ -family of objects of  $\text{FormGrpoid}(\mathcal{X})$ . It follows from Corollary 1.6.6, Sect. 1.6.8 and [Chapter III.1, Proposition 8.3.2] that for any  $n$ , the colimit

$$\text{colim}_i (\mathcal{R}_i^1 \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{R}_i^1)$$

exists and the map

$$\text{colim}_i (\mathcal{R}_i^1 \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{R}_i^1) \rightarrow (\text{colim}_i \mathcal{R}_i^1) \times_{\mathcal{X}} \dots \times_{\mathcal{X}} (\text{colim}_i \mathcal{R}_i^1)$$

is an isomorphism. □

2.2.5. *Ind-coherent sheaves equivariant with respect to a groupoid.* Let  $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ ; let  $\mathcal{R}$  be an object of  $\text{FormGrpoid}(\mathcal{X})$ , and let  $\mathcal{R}^\bullet$  be the corresponding simplicial object of  $\text{PreStk}_{\text{laft}}$ . We define the category

$$\text{IndCoh}(\mathcal{X})^{\mathcal{R}}$$

of *ind-coherent sheaves equivariant with respect to a  $\mathcal{R}$*  to be

$$\text{Tot}(\text{IndCoh}(\mathcal{R}^\bullet)).$$

By [Chapter III.3, Proposition 3.3.3(b)], we have:

**Proposition 2.2.6.** *Let  $\mathcal{R}$  be the formal groupoid corresponding to a map  $\mathcal{X} \rightarrow \mathcal{Y}$  in  $\text{PreStk}_{\text{laft}}$ , which is an inf-schematic nil-isomorphism  $\mathcal{X} \rightarrow \mathcal{Y}$ . Then the pullback functor defines an equivalence*

$$\text{IndCoh}(\mathcal{Y}) \rightarrow \text{IndCoh}(\mathcal{X})^{\mathcal{R}}.$$

2.3. **Taking the quotient by a formal groupoid.** In this subsection we state one of the main results of this book: namely that in the world of prestacks admitting deformation theory there is a well-defined procedure of taking the quotient by a formal groupoid.

2.3.1. Assume that  $\mathcal{X} \in \text{PreStk}_{\text{lft-def}}$ . Recall the category  $\text{FormMod}_{\mathcal{X}/}$ , see Sect. 1.3.1. We have a naturally defined functor

$$(2.4) \quad \text{FormMod}_{\mathcal{X}/} \rightarrow \text{FormGrpoid}(\mathcal{X}),$$

namely,  $\mathcal{Y} \mapsto \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ , see Sect. 2.2.2.

The main result of this section is the following:

**Theorem 2.3.2.** *The functor (2.4) is an equivalence.*

2.3.3. *An example.* Let  $\mathcal{X} \rightarrow \mathcal{Y}$  be a map in  $\text{PreStk}_{\text{lft-def}}$ . Consider the formal completion  $\mathcal{Y}_{\mathcal{X}}^{\wedge}$  of  $\mathcal{Y}$  along  $\mathcal{X}$ , i.e.,

$$\mathcal{Y}_{\mathcal{X}}^{\wedge} := \mathcal{X}_{\text{dR}} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y}.$$

Then the map  $\mathcal{X} \rightarrow \mathcal{Y}_{\mathcal{X}}^{\wedge}$  defines an object of  $\text{FormMod}_{\mathcal{X}/}$ .

Consider the groupoid

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{X},$$

(see Sect. 2.2.2) and its formal completion along the diagonal map,

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})^{\wedge}.$$

It is easy to see that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})^{\wedge} \simeq \mathcal{X} \times_{\mathcal{Y}_{\mathcal{X}}^{\wedge}} \mathcal{X},$$

where the latter is an object of  $\text{FormGrpoid}(\mathcal{X})$  by Sect. 2.2.2.

Thus, the formal completion  $\mathcal{Y}_{\mathcal{X}}^{\wedge}$  can be recovered from  $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})^{\wedge}$  by taking the functor inverse to that in Theorem 2.3.2.

2.3.4. Note that Theorem 2.3.2 implies Theorem 1.6.4:

*Proof.* To prove Theorem 1.6.4 we can assume that  $\mathcal{X} \in \text{Sch}_{\text{aft}}^{\text{aff}}$ . In particular, we can assume that  $\mathcal{X} \in \text{PreStk}_{\text{lft-def}}$ . Then the required assertion follows from Theorem 2.3.2 by noting that

$$\text{Grp}(\text{FormMod}_{\mathcal{X}}) = \text{FormGrpoid}(\mathcal{X})_{/\text{diag}_{\mathcal{X}}},$$

and

$$\text{Ptd}(\text{FormMod}_{/\mathcal{X}}) = (\text{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}.$$

□

2.3.5. As another formal consequence of Theorem 2.3.2, combined with Corollary 2.2.4, we obtain:

**Corollary 2.3.6.** *The category  $\text{FormMod}_{\mathcal{X}/}$  contains sifted colimits, and the functor*

$$\text{FormMod}_{\mathcal{X}/} \rightarrow \text{IndCoh}(\mathcal{X}), \quad (\mathcal{X} \rightarrow \mathcal{Y}) \mapsto T(\mathcal{X}/\mathcal{Y})$$

*commutes with sifted colimits.*

We emphasize again that the forgetful functor

$$\text{FormMod}_{\mathcal{X}/} \rightarrow (\text{PreStk}_{\text{lft}})_{\mathcal{X}/}$$

does *not* commute with sifted colimits.

However, from [Chapter III.1, Corollary 7.2.8], we obtain:

**Corollary 2.3.7.** *The forgetful functor*

$$\mathrm{FormMod}_{\mathcal{X}/} \rightarrow (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$

*commutes with filtered colimits.*

**2.4. Constructing the classifying space of a groupoid.** In this subsection we will begin the proof of Theorem 2.3.2. In fact, we will explicitly construct the inverse functor.

2.4.1. For  $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$  we define an object  $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\mathrm{laft}}$  as follows:

For  $Z \in <^{\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ , we let  $\mathrm{Maps}(Z, B_{\mathcal{X}}(\mathcal{R}))$  be the groupoid consisting of the following data:

$$\{(\tilde{\mathcal{Z}} \rightarrow Z) \in \mathrm{FormMod}_{/Z}, \tilde{\mathcal{Z}} \rightarrow \mathcal{X}, \text{ a map of groupoids } \tilde{\mathcal{Z}} \times_Z \tilde{\mathcal{Z}} \rightarrow \mathcal{R}\},$$

where we require that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{Z}} \times_Z \tilde{\mathcal{Z}} & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{Z}} & \longrightarrow & \mathcal{X} \end{array}$$

be Cartesian, where the vertical arrows are either of the projections.

2.4.2. We have a tautological map  $\mathcal{X} \rightarrow B_{\mathcal{X}}(\mathcal{R})$  that sends  $Z \rightarrow \mathcal{X}$  to

$$\tilde{\mathcal{Z}} := Z \times_{\mathcal{X}} \mathcal{R},$$

where the fiber product  $Z \times_{\mathcal{X}} \mathcal{R}$  is formed using the map  $p_1 : \mathcal{R} \rightarrow \mathcal{X}$ , and the map  $Z \times_{\mathcal{X}} \mathcal{R} \rightarrow \mathcal{X}$  corresponds to  $p_2 : \mathcal{R} \rightarrow \mathcal{X}$ .

2.4.3. Let us show that the map  $\mathcal{X} \rightarrow B_{\mathcal{X}}(\mathcal{R})$  makes  $\mathcal{X}$  into an object of  $\mathrm{FormMod}_{/B_{\mathcal{X}}(\mathcal{R})}$ . Indeed, for a given map  $Z \rightarrow B_{\mathcal{X}}(\mathcal{R})$ , the fiber product

$$Z \times_{B_{\mathcal{X}}(\mathcal{R})} \mathcal{X}$$

identifies with  $\tilde{\mathcal{Z}}$ .

The latter observation also implies that

$$(2.5) \quad \mathcal{X} \times_{B_{\mathcal{X}}(\mathcal{R})} \mathcal{X} \simeq \mathcal{R}.$$

2.4.4. We claim that it suffices to show that the object  $B_{\mathcal{X}}(\mathcal{R})$  belongs to  $\mathrm{PreStk}_{\mathrm{laft}\text{-def}}$ . Indeed, let us assume this for a moment and conclude the proof of the theorem.

First, (2.5) implies that the construction

$$(2.6) \quad \mathcal{R} \mapsto B_{\mathcal{X}}(\mathcal{R})$$

is the right inverse to the functor (2.4).

For  $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$  we have a tautological map

$$(2.7) \quad \mathcal{Y} \mapsto B_{\mathcal{X}}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}),$$

which becomes an isomorphism after applying the functor (2.4). Hence, by [Chapter III.1, Proposition 8.3.2], the map (2.7) is an isomorphism. Hence, the construction (2.6) is also the left inverse of the functor (2.4).

We will now show that the functor (2.4) is conservative, thereby finishing the proof of Theorem 2.3.2. (The caveat here is that it is not clear a priori that the construction (2.6) is a functor<sup>2</sup>.)

2.4.5. Given  $\mathcal{Y}_1, \mathcal{Y}_2 \in \text{FormMod}_{\mathcal{X}/}$ , we will explicitly construct an inverse to the map

$$(2.8) \quad \text{Maps}_{\text{FormMod}_{\mathcal{X}/}}(\mathcal{Y}_1, \mathcal{Y}_2) \rightarrow \text{Maps}_{\text{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2),$$

where  $\mathcal{R}_i = \mathcal{X} \times_{\mathcal{Y}_i} \mathcal{X}$ . Note that we already know that  $\mathcal{Y}_i \simeq B_{\mathcal{X}}(\mathcal{R}_i)$ .

First, it follows from the construction (2.6) that for  $\mathcal{Z} \in \text{PreStk}_{\text{laft-def}}$ , the groupoid of maps  $\mathcal{Z} \rightarrow B_{\mathcal{X}}(\mathcal{R})$  admits the same description as in Sect. 2.4.1 with  $Z$  replaced by  $\mathcal{Z}$ .

Thus, given a point in  $\text{Maps}_{\text{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2)$ , we need to produce an object  $\tilde{B}_{\mathcal{X}}(\mathcal{R}_1) \in \text{FormGrpoid}/_{B_{\mathcal{X}}(\mathcal{R}_1)}$ , a map  $\tilde{B}_{\mathcal{X}}(\mathcal{R}_1) \rightarrow \mathcal{X}$  and a map of groupoids

$$\tilde{B}_{\mathcal{X}}(\mathcal{R}_1) \times_{B_{\mathcal{X}}(\mathcal{R}_1)} \tilde{B}_{\mathcal{X}}(\mathcal{R}_1) \rightarrow \mathcal{R}_2,$$

making the corresponding diagrams Cartesian.

Note, however, that  $\mathcal{R}_1 \times_{\mathcal{X}} \mathcal{R}_2$  can be viewed as a groupoid acting on  $\mathcal{R}_2$ . We set

$$\tilde{B}_{\mathcal{X}}(\mathcal{R}_1) := B_{\mathcal{R}_2}(\mathcal{R}_1 \times_{\mathcal{X}} \mathcal{R}_2).$$

It is easy to check that  $\tilde{B}_{\mathcal{X}}(\mathcal{R}_1)$  has all the required pieces of structure. Moreover, it follows from the construction that the resulting map

$$\text{Maps}_{\text{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2) \rightarrow \text{Maps}_{\text{FormMod}_{\mathcal{X}/}}(\mathcal{Y}_1, \mathcal{Y}_2)$$

is indeed the inverse of (2.8).

**2.5. Verification of deformation theory.** In this subsection we will prove that the object  $B_{\mathcal{X}}(\mathcal{R}) \in \text{PreStk}_{\text{laft}}$  constructed in Sect. 2.4.1, admits deformation theory.

2.5.1. Let  $Z$  be an object of  $Z \in <^{\infty}\text{Sch}_{\text{ft}}^{\text{aff}}$ , equipped with a map to  $B_{\mathcal{X}}(\mathcal{R})$ . We will now construct a certain object of  $\text{Pro}(\text{QCoh}(Z)^{-})_{\text{laft}}$ , which we will later identify with the pro-cotangent space to  $B_{\mathcal{X}}(\mathcal{R})$  at our given point  $Z \rightarrow B_{\mathcal{X}}(\mathcal{R})$ .

Consider the Čech nerve  $\tilde{\mathcal{Z}}^{\bullet}$  of the corresponding map  $\tilde{\mathcal{Z}} \rightarrow Z$ , and consider the resulting map of simplicial prestacks

$$\tilde{\mathcal{Z}}^{\bullet} \rightarrow \mathcal{R}^{\bullet}.$$

Let

$$T^*(\tilde{\mathcal{Z}}^{\bullet}/\mathcal{R}^{\bullet}) \in \text{Tot} \left( \text{Pro}(\text{QCoh}(\tilde{\mathcal{Z}}^{\bullet})^{-})_{\text{laft}}^{\text{fake}} \right)$$

be the corresponding relative pro-cotangent complex (see [Chapter III.1, Sect. 4.3.1]), which receives a canonically defined map from the pullback of  $T^*(Z)$ .

By nil-descent for  $\text{Pro}(\text{QCoh}(-)^{-})_{\text{laft}}^{\text{fake}}$  with respect to  $\tilde{\mathcal{Z}} \rightarrow Z$  (see [Chapter III.3, Corollary 3.3.5]), we obtain that  $T^*(\tilde{\mathcal{Z}}^{\bullet}/\mathcal{R}^{\bullet})$  gives rise to a canonically defined object, denoted,

$$'T^*(Z/B_{\mathcal{X}}(\mathcal{R})) \in \text{Pro}(\text{QCoh}(Z)^{-})_{\text{laft}},$$

which receives a map from  $T^*(Z)$ . Set

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z := \text{Fib}(T^*(Z) \rightarrow 'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))).$$

<sup>2</sup>We are grateful to Y. Zhao for pointing this out to us.

We will show that the above object

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z \in \text{Pro}(\text{QCoh}(Z)^-)_{\text{laft}},$$

identifies with the pro-cotangent space of  $B_{\mathcal{X}}(\mathcal{R})$  at the above point  $Z \rightarrow B_{\mathcal{X}}(\mathcal{R})$ .

2.5.2. We need to show that, given a square-zero extension  $Z \hookrightarrow Z'$ , corresponding to

$$\gamma : T^*(Z) \rightarrow \mathcal{J}[1], \quad \mathcal{J} \in \text{Coh}(Z)^{\leq 0},$$

the groupoid of extensions of the initial map  $Z \rightarrow B_{\mathcal{X}}(\mathcal{R})$  to a map  $Z' \rightarrow B_{\mathcal{X}}(\mathcal{R})$ , identifies canonically with groupoid of factorizations of  $\gamma$  as

$$T^*(Z) \rightarrow 'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \rightarrow \mathcal{J}[1].$$

This will show that  $B_{\mathcal{X}}(\mathcal{R})$  admits pro-cotangent spaces that are indeed identified with ones constructed in Sect. 2.5.1, and that  $B_{\mathcal{X}}(\mathcal{R})$  is infinitesimally cohesive. The fact that  $B_{\mathcal{X}}(\mathcal{R})$  admits a pro-cotangent complex (i.e., that the formation of pro-cotangent spaces is compatible with pullback) will follow from the construction in Sect. 2.5.1.

2.5.3. For  $Z \hookrightarrow Z'$  as above, by [Chapter III.1, Proposition 10.3.5], the datum of a prestack  $\tilde{\mathcal{Z}}' \rightarrow Z'$ , equipped with a Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{Z}} & \longrightarrow & \tilde{\mathcal{Z}}' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \end{array}$$

is equivalent to that of a map  $T^*(\tilde{\mathcal{Z}}) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}}[1]$  (in the category  $\text{Pro}(\text{QCoh}(\tilde{\mathcal{Z}})^-)^{\text{fake}}_{\text{laft}}$ ), and a homotopy between the composition

$$T^*(Z)|_{\tilde{\mathcal{Z}}} \rightarrow T^*(\tilde{\mathcal{Z}}) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}}[1]$$

and  $\gamma|_{\tilde{\mathcal{Z}}}$ . Moreover, by [Chapter III.1, Proposition 10.4.2], such  $\tilde{\mathcal{Z}}'$  is automatically an inf-scheme.

The same discussion applies to each term of the Čech nerve  $\tilde{\mathcal{Z}}^\bullet$ .

Furthermore, by [Chapter III.1, Proposition 10.2.6] the datum of a compatible system of maps from the Čech nerve  $\tilde{\mathcal{Z}}'^\bullet$  to  $\mathcal{R}^\bullet$ , extending the initial system  $\tilde{\mathcal{Z}}^\bullet \rightarrow \mathcal{R}^\bullet$ , is equivalent to a compatible system of factorizations of the resulting maps

$$T^*(\tilde{\mathcal{Z}}^\bullet) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}^\bullet}[1]$$

as

$$T^*(\tilde{\mathcal{Z}}^\bullet) \rightarrow T^*(\tilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}^\bullet}[1].$$

2.5.4. Hence, we obtain that the datum of extension of the initial map  $Z \rightarrow B_{\mathcal{X}}(\mathcal{R})$  to a map  $Z' \rightarrow B_{\mathcal{X}}(\mathcal{R})$  is equivalent to that of a compatible family of maps

$$T^*(\tilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}^\bullet}[1],$$

and homotopies between

$$T^*(Z)|_{\tilde{\mathcal{Z}}^\bullet} \rightarrow T^*(\tilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \rightarrow \mathcal{J}|_{\tilde{\mathcal{Z}}^\bullet}[1]$$

and  $\gamma|_{\tilde{\mathcal{Z}}^\bullet}$ .

By nil-descent for  $\text{Pro}(\text{QCoh}(-)^-)^{\text{fake}}_{\text{laft}}$  with respect to  $\tilde{\mathcal{Z}} \rightarrow Z$ , the latter datum is equivalent to that of factorizations of  $\gamma$  as

$$T^*(Z) \rightarrow 'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \rightarrow \mathcal{J}[1],$$

as desired.