NOTES ON GEOMETRIC LANGLANDS: THE EXTENDED WHITTAKER CATEGORY

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Introduction

The purpose of this series of notes is to explain the current ideas related to Geometric Langlands correspondence, which is supposed to relate the category $D(\mathrm{Bun}_G)$ to that of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$.

This installment is devoted to the study of geometries, or rather, categories, which are involved in the conjectural constructions. Most of these categories are geometric analogues of certain spaces of functions that appear in the classical theory of automorphic forms, such as principal series and the Whittaker model.

A terminological remark: by a "category" we'll mean a DG category in the conventions adopted in [GL:DG]. Limits and colimits are taken inside the $(\infty, 1)$-category DGCat_{cont} in the notation of *loc.cit.*.

For the definitions of categories of D-modules on stacks such as Bun_G or $\overline{\mathrm{Bun}}_B$, see [DrGa1].

1. The space of rational reductions to the Borel

As is well-known, the stack Bun_B is disconnected, so if we want to approximate the category $D(\mathrm{Bun}_G)$ by the pair of adjoint functors

$$\mathfrak{p}_! : D(\mathrm{Bun}_B) \rightleftarrows D(\mathrm{Bun}_G) : \mathfrak{p}^!,$$

this will lose a lot of information.

For this purpose one looks at Drinfeld's compactification $\overline{\mathrm{Bun}}_B$ and its map

$$\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G .$$

However, $\overline{\mathrm{Bun}}_B$ has another bug: it has a lot of redundancy. I.e., instead of just gluing the various connected components of Bun_B together, it glues their products by partially symmetrized powers of the curve.

The goal of this section is to define a "space" Bun_B^{rat} that will genuinely glue all the connected components of Bun_B together with no redundancy. The only problem is that the space doesn't exist. However, the category $D(\mathrm{Bun}_B^{rat})$ does exist, and this is what we are going to do. The idea is very simple: look at $D(\overline{\mathrm{Bun}}_B)$ and get rid of redundancy by imposing equivariance with respect to a suitable groupoid.

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1.1. **The "space".** Consider the stack $\overline{\mathrm{Bun}}_B$. Let $\mathcal{H}_{G,B}$ be the groupoid

$$\mathcal{H}_{G,B} \subset \overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B$$

that consists of pairs of generalized B-reductions on a given G-bundle, which agree at the generic point of the curve.

We define the category $D(\mathrm{Bun}_B^{rat})$ as the equivariant category of $D(\overline{\mathrm{Bun}}_B)$ with respect to $\mathcal{H}_{G,B}$.

Remark. One can show that Bun_B^{rat} doesn't exist as an Artin stack, i.e., a stack endowed with a map $\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_B^{rat}$ which induces the forgetful functor $D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B)$. [1] However, we'll treat it as such for notational purposes.

1.2. **Monad description.** Let $\mathbf{oblv}_B$ denote the forgetful functor

$$D(\mathrm{Bun}_B^{rat}) \to D(\overline{\mathrm{Bun}}_B).$$

The projections

$$\overline{\mathrm{Bun}}_B \overset{\overleftarrow{h}}{\longleftarrow} \mathcal{H}_{G,B} \overset{\overrightarrow{h}}{\longrightarrow} \overline{\mathrm{Bun}}_B$$

are ind-proper. Hence, the functor $\mathbf{oblv}_B$ admits a left adjoint, that we denote by $\mathbf{ind}_B$. The composition

$$\mathbf{oblv}_B \circ \mathbf{ind}_B : D(\overline{\mathrm{Bun}}_B) \to D(\overline{\mathrm{Bun}}_B)$$

identifies with $\mathrm{Av}_{\mathcal{H}_{G,B}} := \overrightarrow{h}_* \circ \overleftarrow{h}^!$. By Barr-Beck, we can think of $D(\mathrm{Bun}_B^{rat})$ as modules in $D(\overline{\mathrm{Bun}}_B)$ for the monad $\mathrm{Av}_{\mathcal{H}_{G,B}}$.

1.3. **Relation to Bun_G.** We have a pair of adjoint functors

$$\overline{\mathfrak{p}}_! : D(\overline{\mathrm{Bun}}_B) \rightleftarrows D(\mathrm{Bun}_G) : \overline{\mathfrak{p}}^!.$$

The functor $\overline{\mathfrak{p}}^!$ canonically factors as $\mathbf{oblv}_B \circ (\mathfrak{p}^{rat})^!$, where $(\mathfrak{p}^{rat})^!$ is a canonically defined functor $D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_B^{rat})$. We obtain that $(\mathfrak{p}^{rat})^!$ admits a left adjoint, denoted $(\mathfrak{p}^{rat})_!$, so that

$$\overline{\mathfrak{p}} \simeq (\mathfrak{p}^{rat})_! \circ \mathbf{ind}_B.$$

The following will be proved in [GL:contr]:

Quasi-Theorem 1.3.1. *The functor $(\mathfrak{p}^{rat})^!$ is fully faithful.*

2. Imposing $N(\mathbb{A})$-equivariance

The category $D(\mathrm{Bun}_B^{rat})$ is supposed to be a geometric analog of the space of functions on the double quotient

$$B(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

We shall now introduce a category, denoted $D(\overline{\mathrm{Bun}}_T)$, which is a geometric analog of the space of functions on the double quotient

$$T(K)N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O}).$$

Note that just as we had set-theoretically,

$$B(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \simeq B(K)\backslash B(\mathbb{A})/B(\mathbb{O}),$$

[1] The proof of non-existence follows from the non-existence of a t-structure on $D(\mathrm{Bun}_B^{rat})$ with the expected properties.

we also have

$$T(K)N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O}) \simeq T(K)\backslash T(\mathbb{A})/T(\mathbb{O}).$$

However, as the category $D(\mathrm{Bun}_B^{rat})$ was not equivalent to $D(\mathrm{Bun}_B)$, so $D(\mathrm{Bun}_T^{rat})$ won't be the same as $D(\mathrm{Bun}_T)$.

2.1. **The category.** The stack $D(\overline{\mathrm{Bun}}_B)$ carries an action of a canonical groupoid, denoted $\mathcal{H}_N$, which incarnates the action of the adelic N. We refer the reader to [Ga1] or [Ga2], where this is explained in detail.

Let $D(\overline{\mathrm{Bun}}_T)$ be the category of $\mathcal{H}_N$-equivariant objects of $D(\overline{\mathrm{Bun}}_B)$. Since $\mathcal{H}_N$ is pro-unipotent, the forgetful functor

$$D(\overline{\mathrm{Bun}}_T) \to D(\overline{\mathrm{Bun}}_B)$$

is fully faithful. We denote this functor by $\overline{\mathfrak{q}}^*$. It admits a right adjoint that we denote by $\overline{\mathfrak{q}}_*$.

Remark. Although, we cannot define $\overline{\mathrm{Bun}}_T$ as a stack beyond the case of GL_2, we'll keep treating it as such for notational purposes.

2.1.1. It is easy to see that the actions of the groupoids $\mathcal{H}_{G,B}$ and $\mathcal{H}_N$ commute. Hence, it makes sense to consider the category of $\mathcal{H}_N$-equivariant objects in $D(\mathrm{Bun}_B^{rat})$; this is a full subcategory in $D(\mathrm{Bun}_B^{rat})$, which we denote by $D(\mathrm{Bun}_T^{rat})$.

Equivalently, we note that the monad $\mathrm{Av}_{\mathcal{H}_{G,B}}$ acting on $D(\overline{\mathrm{Bun}}_B)$, sends $D(\overline{\mathrm{Bun}}_T)$ to itself. We can describe $D(\mathrm{Bun}_T^{rat})$ as the category of $\mathrm{Av}_{\mathcal{H}_{G,B}}$-modules in $D(\overline{\mathrm{Bun}}_T)$.

We denote the corresponding fully faithful embedding by

$$(\mathfrak{q}^{rat})^* : D(\mathrm{Bun}_T^{rat}) \hookrightarrow D(\mathrm{Bun}_B^{rat}),$$

and by $(\mathfrak{q}^{rat})_*$ its right adjoint.

Let us denote by

$$\mathbf{ind}_T : D(\overline{\mathrm{Bun}}_T) \rightleftarrows D(\mathrm{Bun}_T^{rat}) : \mathbf{oblv}_T$$

the resulting pair of adjoint functors.

2.2. **Relation to $D(\mathrm{Bun}_T)$.** The open embedding

$$\jmath_B : \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}}_B$$

induces an open embedding

$$\jmath_T : \mathrm{Bun}_T \hookrightarrow \overline{\mathrm{Bun}}_T,$$

and the corresponding functor

$$(\jmath_T)^! : D(\overline{\mathrm{Bun}}_T) \to D(\mathrm{Bun}_T).$$

Lemma 2.2.1. *The functor $(\jmath_T)^!$ admits a left adjoint.*

In other words, the lemma is saying that the (partially defined) left adjoint to $(\jmath_B)^!$ is defined on the full subcategory $D(\mathrm{Bun}_T) \subset D(\mathrm{Bun}_B)$. The reason that this is so is that the object $(\jmath_B)_!(\underline{k}_{\mathrm{Bun}_B}) \in D(\overline{\mathrm{Bun}}_B)$, which is defined due to holonomicity, [2] is ULA with respect to the projection $\overline{\mathfrak{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$.

Let us denote by $(\jmath_T)_!$ the left adjoint of $(\jmath_T)^!$.

[2]Here $\underline{k}_Y$ denotes the "constant sheaf" D-module on a scheme/stack Y

2.2.2. Consider the composed functor

$$(\jmath_T^{rat})_! := \mathbf{ind}_T \circ (\jmath_T)_! : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T^{rat}),$$

and its right adjoint

$$(\jmath_T^{rat})^! := \jmath_T^! \circ \mathbf{oblv}_T.$$

It is easy to see that the functor $(\jmath_T^{rat})^!$ is conservative. Hence, the category $D(\mathrm{Bun}_T^{rat})$ can be described as modules in $D(\mathrm{Bun}_T)$ with respect to the monad

$$\widetilde{\Omega} := (\jmath_T^{rat})^! \circ (\jmath_T^{rat})_! \simeq (\jmath_T)^! \circ \mathbf{oblv}_T \circ \mathbf{ind}_T \circ (\jmath_T)_!.$$

(The notation $\widetilde{\Omega}$ is due to a notational convention in [GL:Eis].)

2.3. **Description of the monad.** The monad $\widetilde{\Omega}$ can be described explicitly. For an element $\lambda \in \Lambda$, let

$$\jmath_B^\lambda : X^\lambda \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$$

be the locally closed embedding of the corresponding stratum. We shall symbolically denote by $\jmath_T^\lambda$ the "locally closed embedding" $X^\lambda \times \mathrm{Bun}_T \to \overline{\mathrm{Bun}}_T$, and all we mean by this is that there is a functor

$$(\jmath_T^\lambda)^! : D(\overline{\mathrm{Bun}}_T) \to D(X^\lambda \times \mathrm{Bun}_T).$$

As in Lemma 2.2.1, the functor $(\jmath_T^\lambda)^!$ admits a left adjoint, which we denote by $(\jmath_T^\lambda)_!$.

Consider the functor

$$\widetilde{\Omega}^\lambda : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T)$$

given by

$$\left(H(X^\lambda, -) \boxtimes \mathrm{Id}_{D(\mathrm{Bun}_T)}\right) \circ (\jmath_T^\lambda)^! \circ (\jmath_T)_!.$$

From the construction we obtain that the functor

$$\underset{\lambda \in \Lambda^{pos}}{\oplus} \widetilde{\Omega}^\lambda$$

has a natural monad structure.

Lemma 2.3.1. *The above monad* $\underset{\lambda \in \Lambda^{pos}}{\oplus} \widetilde{\Omega}^\lambda$ *is canonically isomorphic to* $\widetilde{\Omega}$.

The lemma follows from the fact the isomorphism

$$\mathrm{Bun}_B \underset{\overline{\mathrm{Bun}}_B}{\times} \mathcal{H}_{G,B} \simeq \underset{\lambda}{\sqcup} X^\lambda \times \mathrm{Bun}_B,$$

such that the "second projection" $\mathrm{Bun}_B \underset{\overline{\mathrm{Bun}}_B}{\times} \mathcal{H}_{G,B} \to \overline{\mathrm{Bun}}_B$ is given by the map $\underset{\lambda}{\sqcup} \jmath_B^\lambda$.

3. The Eisenstein series functor

Classically, Eisenstein series and consant term are operators that act between the spaces of functions on

$$G(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \text{ and } T(K)N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O})$$

by pull-push along the diagram

$$G(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \leftarrow B(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \to T(K)N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O}).$$

We'll now define a geometric analog of this procedure.

3.1. **Compactified and non-compactified geometric Eisenstein series.** Recall that the compactified Eisenstein series functor is by definition

$$\overline{\mathrm{Eis}} := \bar{\mathfrak{p}}_! \circ \bar{\mathfrak{q}}^* : D(\overline{\mathrm{Bun}}_T) \to D(\mathrm{Bun}_G).$$

Its left adjoint is the functor of "compactified constant term":

$$\overline{\mathrm{CT}} := \bar{\mathfrak{q}}_* \circ \bar{\mathfrak{p}}^! : D(\mathrm{Bun}_G) \to D(\overline{\mathrm{Bun}}_T).$$

The composed functor $\mathrm{Eis}_! := \overline{\mathrm{Eis}} \circ (\jmath_T)_!$ is the usual (non-compactified) Eisenstein series functor. Its left adjoint CT is isomorphic to $\jmath_T^* \circ \overline{\mathrm{CT}}$.

3.2. **Rational Eisenstein series.** We denote by Eis^{rat} the functor

$$\mathfrak{p}^{rat}{}_! \circ (\mathfrak{q}^{rat})^* : D(\mathrm{Bun}_T^{rat}) \to D(\mathrm{Bun}_G),$$

and by CT^{rat} its right adjoint.

We regard these functors as the geometric counterpart to the Eisenstein and constant term operators mentioned at the beginning of this section.

Tautologically, we have:

$$\overline{\mathrm{Eis}} \simeq \mathrm{Eis}^{rat} \circ \mathbf{ind}_T \text{ and } \mathrm{CT}^{rat} \simeq \mathbf{oblv}_T \circ \overline{\mathrm{CT}},$$

$$\mathrm{Eis}_! \simeq \mathrm{Eis}^{rat} \circ (\jmath_T^{rat})_! \text{ and } \mathrm{CT} \simeq (\jmath_T^{rat})^! \circ \mathcal{T}_r.$$

3.3. **Constant term and Eisenstein series.** A fundamental role in the analysis of $D(\mathrm{Bun}_G)$ is played by our ability to calculate Homs between Eisenstein series. By adjunction, we have:

$$\mathrm{Hom}_{D(\mathrm{Bun}_G)}(\mathrm{Eis}_!(\mathcal{M}_1), \mathrm{Eis}_!(\mathcal{M}_2)) \simeq \mathrm{Hom}_{D(\mathrm{Bun}_T)}(\mathcal{M}_1, \mathrm{CT} \circ \mathrm{Eis}_!(\mathcal{M}_2)).$$

I.e., we need to understand the structure of the monad

$$\mathrm{CT} \circ \mathrm{Eis}_! : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T).$$

As the functor $\mathrm{Eis}_!$ can be factored as $\mathrm{Eis}^{rat} \circ (\jmath_T^{rat})_!$, we have a canonical map of monads

$$\widetilde{\Omega} \to \mathrm{CT} \circ \mathrm{Eis}_! \,.$$

The following proposition follows from the calculation of the monad $\tilde{\Omega}$ via the Zastava spaces and Lemma 2.3.1:

Proposition 3.3.1. *The functor* $\mathrm{CT} \circ \mathrm{Eis}_!$ *has a canonical filtration indexed by the elements* $w \in W$ *of the Weyl group, such that:*

(a) *The first term* $(\mathrm{CT} \circ \mathrm{Eis}_!)_{unit}$ *has a monad structure, and as such is canonically isomorphic to* $\widetilde{\Omega}$.

(b) *The last term* $(\mathrm{CT} \circ \mathrm{Eis}_!)_{w_0}$ *is canonically isomorphic to the functor of action of* w_0 *on* $D(\mathrm{Bun}_T)$.

Remark. Point (a) of the Proposition can be interpreted as follows: the monad $\widetilde{\Omega}$ is the best approximation to the monad $\mathrm{CT} \circ \mathrm{Eis}_!$ which is local in *spectral terms* with respect to Bun_T.

Let us also note that in the classical theory of automorphic functions, the analog of the monad $\widetilde{\Omega}$ is an endomorphism of the space of functions of the adelic quotient corresponding to T, whose value on a grossen-character is the ratio of (the product over all positive roots of) the abelian L-function by its itself with the shifted argument.

4. The spectral picture

4.1. **Spectral Eisenstein series.** Consider the map of stacks

$$\mathrm{LocSys}_{\check{G}} \overset{\mathfrak{p}^{spec}}{\longleftarrow} \mathrm{LocSys}_{\check{B}} \overset{\mathfrak{q}^{spec}}{\longrightarrow} \mathrm{LocSys}_{T}\,.$$

The spectral Eisenstein series functor

$$\mathrm{Eis}^{spec}: \mathsf{QCoh}(\mathrm{LocSys}_T) \to \mathsf{QCoh}_N(\mathrm{LocSys}_{\check{G}}).$$

is defined as the composition $(\mathfrak{p}^{spec})_* \circ (\mathfrak{q}^{spec})^*$. Here $\mathsf{QCoh}_N(\mathrm{LocSys}_{\check{G}})$ is the enlargement of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$, introduced in [GL:formulation].

We'll denote by CT^{spec} the right adjoint of Eis^{spec}, and refer to it as the spectral constant term functor. [3]

On of the properties expected from the Geometric Langlands equivalence is that it intertwines the functors $\mathrm{Eis}_!$ and (the shift by $\rho(\omega_X)$ of) Eis^{spec}, i.e., the diagram of functors

$$\begin{array}{ccc} D(\mathrm{Bun}_T) & \xleftarrow{-\rho(\omega_X)\circ\Psi_T} & \mathsf{QCoh}(\mathrm{LocSys}_T) \\ \Big\downarrow{\scriptstyle \mathrm{Eis}_!} & & \Big\downarrow{\scriptstyle \mathrm{Eis}^{spec}} \\ D(\mathrm{Bun}_G) & \xleftarrow{\Psi_G} & \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \end{array} \tag{4.1}$$

must commute. (Here Ψ_G denotes the conjectural Geometric Langlands functor for G, while Ψ_T is the Fourier-Mukai transform which realizes Geometric Langlands for T.)

The goal of this section, which follows [GL:Eis], is to explain the spectral interpretation of the category $D(\mathrm{Bun}_T^{rat})$ and the functor Eis^{rat}.

4.2. **Quasi-coherent sheaves with a vertical connection.** Consider again the map

$$\mathfrak{p}^{spec}: \mathrm{LocSys}_{\check{B}} \to \mathrm{LocSys}_{\check{G}},$$

and let $T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}$ be the algebroid of vertical fields on $\mathrm{LocSys}_{\check{B}}$ with respect to $\mathfrak{p}^{spec}$.

Consider the category

$$T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r$$

of right $T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}$-modules in $\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}})$.

The universal enveloping algebra $U(T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}})$ can be regarded as a monad

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

equal to the composition of the forgetful functor

$$T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}})$$

with its left adjoint, i.e., the forgetful functor.

[3] For CT^{spec} to exist as a colimit-preserving functor, we need Eis^{spec} to send compact objects to compact ones, and this was the reason for introducing the modification $\mathsf{QCoh}(\mathrm{LocSys}_G) \mapsto \mathsf{QCoh}_N(\mathrm{LocSys}_{\check{G}})$: we needed to enlarge the collection of compact objects of $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$ which were just perfect complexes by those coherent complexes that lie in the image of Eis^{spec}.

4.3. **Refined spectral Eisenstein series.** The functor

$$\mathfrak{p}^{spec}_* : \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})$$

canonically factors as

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \overset{U(T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}})}{\longrightarrow} T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \overset{\mathfrak{p}^{spec}_{*,DR}}{\longrightarrow} \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}).$$

For the reasons that will be explained presently, we'll refer to the above functor

$$\mathfrak{p}^{spec}_{*,DR} : T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \to \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})$$

as "refined spectral Eisenstein series" and denoted it $\mathrm{Eis}^{spec,ref}$.

4.4. **Refined spectral constant term.** Consider now the right adjoint of $\mathfrak{p}^{spec}_*$, which we denote $(\mathfrak{p}^{spec}_{\mathcal{N}})^!$. Note that $\mathfrak{p}^{spec!}_{\mathcal{N}}$ is the composition

$$\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \hookrightarrow \mathsf{IndCoh}(\mathrm{LocSys}_{\check{G}}) \overset{(\mathfrak{p}^{spec})^!}{\longrightarrow} \mathsf{IndCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

where the last arrow is the right adjoint to the inclusion

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \hookrightarrow \mathsf{IndCoh}(\mathrm{LocSys}_{\check{B}}).$$

The functor $(\mathfrak{p}^{spec}_{\mathcal{N}})^!$ canonically factors as

$$\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \to T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

where the second arrow is the forgetful functor.

We'll denote the resulting functor

$$\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \to T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r$$

by $\mathrm{CT}^{spec,ref}$, and refer to it as the "refined spectral constant term" functor. The functors $(\mathrm{Eis}^{spec,ref}, \mathrm{CT}^{spec,ref})$ naturally form an adjoint pair.

4.5. **Constant term of Eisenstein series on the spectral side.** Consider now the monad:

$$(\mathfrak{p}^{spec}_{\mathcal{N}})^! \circ \mathfrak{p}^{spec}_* : \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

that controls

$$\mathrm{Hom}_{\mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}})}(\mathfrak{p}^{spec}_*(\mathcal{F}_1), \mathfrak{p}^{spec}_*(\mathcal{F}_2)),\ \mathcal{F}_1, \mathcal{F}_2 \in \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}).$$

Almost tautologically, we have:

Lemma 4.5.1. *The functor $(\mathfrak{p}^{spec}_{\mathcal{N}})^! \circ \mathfrak{p}^{spec}_*$ has a canonical filtration indexed by elements $w \in W$, such that*

(a) *The first term $((\mathfrak{p}^{spec}_{\mathcal{N}})^! \circ \mathfrak{p}^{spec}_*)_{unit}$ has a monad structure and as such is canonically isomorphic to $U(T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}})$.*

(b) *The last term $((\mathfrak{p}^{spec}_{\mathcal{N}})^! \circ \mathfrak{p}^{spec}_*)_{w_0}$ is canonically isomorphic to the composition*

$$\mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}) \overset{\iota^*}{\longrightarrow} \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \overset{w_0}{\to} \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \overset{\iota_*}{\longrightarrow} \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}}),$$

where ι is the natural map $\mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{B}})$.

4.6. **Relation between the geometric and the spectral pictures.** Recall that $\mathfrak{q}^{spec}$ denotes the projection $\mathrm{LocSys}_{\check{B}} \to \mathrm{LocSys}_{\check{T}}$. Let Ψ_T denote the Fourier-Mukai transform:

$$D(\mathrm{Bun}_T) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}).$$

The main thrust of [GL:Eis] is the following:

Quasi-Theorem 4.6.1. *There exists a canonical equivalence of categories*

$$\Psi_B : T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \to D(\mathrm{Bun}_T^{rat}),$$

which makes the following diagram commute:

$$\begin{array}{ccc} D(\mathrm{Bun}_T^{rat}) & \xleftarrow{\Psi_B} & T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \\ {\scriptstyle (\jmath_T^{rat})_!}\uparrow & & \uparrow{\scriptstyle U(T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}})\circ(\mathfrak{q}^{spec})^*} \\ D(\mathrm{Bun}_T) & \xleftarrow{-\rho(\omega_X)\circ\Psi_T} & \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}), \end{array}$$

where $\rho(\omega_X)$ is the functor of shift by the point $\rho(\omega_X) \in \mathrm{Bun}_T$.

Note that Theorem 4.6.1 can be restated as follows. Consider the monads

$$(\mathfrak{q}^{spec})_* \circ U(T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}) \circ (\mathfrak{q}^{spec})^* : \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}) \to \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}),$$

and

$$\rho(\omega_X) \circ (\jmath_T^{rat})_! \circ (\jmath_T^{rat})^! \circ -\rho(\omega_X) = \rho(\omega_X) \circ \widetilde{\Omega} \circ -\rho(\omega_X) : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_T).$$

The claim is that Ψ_T intertwines these two monads.

4.7. **Summary.** This, if we have the functor Ψ_G satisfying (4.1), we must also have the following commutative diagram:

$$\begin{array}{ccc} D(\mathrm{Bun}_T^{rat}) & \xleftarrow{-\rho(\omega_X)\circ\Psi_T} & T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r \\ {\scriptstyle \mathrm{Eis}^{rat}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{spec,ref}} \\ D(\mathrm{Bun}_G) & \xleftarrow{\Psi_G} & \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}). \end{array} \tag{4.2}$$

5. The Whittaker category

Continuing the classical analogy, we would now like to construct a category which is a geometric counterpart of the space of functions on

$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

This category would be denoted $D(\mathrm{Bun}_N^{rat})$. Furthermore, we'll define a full-subcategory

$$\mathrm{Whit}(G)_{glob} \subset D(\mathrm{Bun}_N^{rat}),$$

which is a geometric counterpart of the subspace of functions on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ obtained by imposing an equivariance condition with respect to a fixed non-degenerate character of $N(\mathbb{A})$.

The construction of these categories uses the formalism of categories over the Ran space, explained in [GL:Ran].

5.1. **The space.** Consider the following ind-stack, denoted Bun_N^{polar} that lives over Ran_X: for a finite set I we set $(\mathrm{Bun}_N^{polar})_{X^I}$ to classify quintuples $(\underline{x}, P_G, P_T, \kappa, \gamma)$, where $\underline{x} \in X^I$, (P_G, P_T, κ) is a Plücker data which is allowed to have poles at $\underline{x}$, and γ is an isomorphism of T-bundles

$$P_T \simeq \rho(\omega_X).$$

The assignment

$$I \mapsto D((\mathrm{Bun}_N^{polar})_{X^I}),$$

is naturally a category over the Ran space, which we denote $D((\mathrm{Bun}_N^{polar})_{X^{\mathrm{fSet}}})$. It is naturally defined with a unital structure and augmentation. We'll denote by $D(\mathrm{Bun}_N^{rat})$ the resulting category $D((\mathrm{Bun}_N^{polar})_{\mathrm{Ran}_X,un})$.

5.2. **The Whittaker subcategory.** For each I, there is a certain unipotent groupoid $(\mathcal{H}_N)_{X^I}$ acting on $(\mathrm{Bun}_N^{polar})_{X^I}$ (see [Ga2]; in fact $(\mathcal{H}_N)_{X^I}$ is a version of $\mathcal{H}_N$ that acted on $\overline{\mathrm{Bun}}_B$), endowed with a character $\chi : (\mathcal{H}_N)_{X^I} \to \mathbb{G}_a$.

We define $\mathrm{Whit}(G)_{X^I}$ as the corresponding equivariant category

$$\mathrm{Whit}(G)_{X^I} := D((\mathrm{Bun}_N^{polar})_{X^I})^{((\mathcal{H}_N)_{X^I},\chi)}.$$

The unipotence property of $(\mathcal{H}_N)_{X^I}$ implies that the forgetful functor

$$\mathbf{oblv}_{\mathrm{Whit},X^I} : \mathrm{Whit}(G)_{X^I} \to D((\mathrm{Bun}_N^{polar})_{X^I})$$

is fully faithful. This functor admits a right adjoint, denoted $\mathbf{ind}_{\mathrm{Whit},X^I}$.

5.2.1. We'll view the assignment $I \mapsto \mathrm{Whit}(G)_{X^I}$ as a category over the Ran space, denoted simply $\mathrm{Whit}(G)$. It is naturally unital and augmented. We'll consider the corresponding categories

$$\mathrm{Whit}(G)_{\mathrm{Ran}_X} \text{ and } \mathrm{Whit}(G)_{\mathrm{Ran}_X,un}.$$

We'll also use the notation $\mathrm{Whit}(G)_{glob} := \mathrm{Whit}(G)_{\mathrm{Ran}_X,un}$, and

$$\mathbf{ind}_{\mathrm{Whit},glob} : D(\mathrm{Bun}_N^{rat}) \rightleftarrows \mathrm{Whit}(G)_{glob} : \mathbf{oblv}_{\mathrm{Whit},glob}$$

for the corresponding pair of adjoint functors.

Note that the category $\mathrm{Whit}(G)_\emptyset$, defined as the category of equivariant objects with respect to the corresponding groupoid in $D(\mathrm{Bun}_N)$, is canonically equivalent to Vect. We shall denote the corresponding unit object of $\mathrm{Whit}(G)_\emptyset \subset \mathrm{Whit}(G)_{glob}$ by $\mathbf{1}_{\mathrm{Whit}(G)_{glob}}$.

5.3. **The Casselman-Shalika formula.** Consider the category over the Ran space built from the symmetric monoidal category $\mathrm{Rep}(\check{G})$; we'll denote it $\mathrm{Rep}(\check{G}, X^{\mathrm{fSet}})$.

5.3.1. The corresponding categories $\mathrm{Rep}(\check{G}, X^I)$ can be explicitly described as follows. Consider $\check{G}$ as a constant D-group-scheme over X, and let

$$\check{G}_{X^{\mathrm{fSet}}} : I \mapsto \check{G}_{X^I}$$

be the corresponding factorizable group-D-scheme over the Ran space.

We have that $\mathrm{Rep}(\check{G}, X^I)$ is the category of representations of $\check{G}_{X^I}$ on D-modules on X^I (compatible with the connection).

5.3.2. The following is essentially equivalent to the Casselman-Shalika formula:

Quasi-Theorem 5.3.3. *There is a natural equivalence of categories over* Ran_X*:*

$$\mathrm{Whit}(G) \simeq \mathrm{Rep}(\check{G}, X^{\mathrm{fSet}}).$$

5.3.4. From the above theorem, we deduce the equivalences

$$\mathrm{Whit}(G)_{\mathrm{Ran}_X} \simeq \mathrm{Rep}(\check{G}, \mathrm{Ran}_X) \text{ and } \mathrm{Whit}(G)_{\mathrm{Ran}_X, un} \simeq \mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un).$$

Here we view $\mathrm{Rep}(\check{G}, X^{\mathrm{fSet}})$ as unital and augmented by virtue of the co-unital structure on the factorization (group)-D-scheme $I \mapsto \check{G}_{X^I}$.

5.3.5. *Relation to* $\mathsf{QCoh}(\mathrm{LocSys}_{\check{G}})$. The relevance of the category $\mathrm{Rep}(\check{G}, \mathrm{Ran}_X)$ to us is explained by the following. Recall that there is a pair of adjoint functors

$$\mathrm{Loc}_{\check{G},spec} : \mathrm{Rep}(\check{G}, \mathrm{Ran}_X) \rightleftarrows \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) : \Gamma_{\check{G},spec}.$$

We have:

Theorem 5.3.6. *The functor* $\Gamma_{\check{G},spec}$ *is fully faithful.*

5.3.7. It is easy to see that the functor $\Gamma_{\check{G},spec}$ maps to the full subcategory

$$\mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un) \subset \mathrm{Rep}(\check{G}, \mathrm{Ran}_X).$$

Hence, the functor $\mathrm{Loc}_{\check{G},spec}$ factors through the localization

$$\mathrm{Rep}(\check{G}, \mathrm{Ran}_X) \twoheadrightarrow \mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un).$$

6. The reduced Whittaker category

From now on, we'll assume that the center of G, denoted Z_G, is connected.

In order to consider the extended Whittaker category, we'll need to replace the original $\mathrm{Whit}(G)_{glob}$ defined above by its full subcategory, which we denote $\mathrm{Whit}(G)^{red}_{glob}$.

Here is a function-theoretic analogy. As was mentioned above, $D(\mathrm{Bun}_N^{rat})$ is an analog of the space of functions on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. We are going to introduce a category, denoted $D(\mathrm{Bun}_N^{rat,red})$, which us a geometric analog of the space of functions on

$$Z_G(K)N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

The category $\mathrm{Whit}(G)^{red}_{glob}$ will be a full subcategory of $D(\mathrm{Bun}_N^{rat,red})$ obtained by imposing a Whittaker-like condition.

6.1. **The category.** Let $\overline{\mathrm{Bun}}_N^{red}$ denote the stack classifying quadruples $(P_G, P_T, \kappa, \gamma^{red})$, where (P_G, P_T, κ) is a point of $\overline{\mathrm{Bun}}_B$, and γ^{red} is a datum of a non-zero map of line bundles

$$\gamma_i^{red} : \check{\alpha}_i(P_T) \to \omega_X,$$

for every simple root $\check{\alpha}_i$ of G.

The above stack carries an action of a groupoid denoted $\mathcal{H}_{G,N}$, which is in fact pulled back from the groupoid $\mathcal{H}_{G,B}$ on $\overline{\mathrm{Bun}}_B$: it identifies two points $(P_G, P_T, \kappa, \gamma)$ and $(P_G, P_T^1, \kappa^1, \gamma^1)$ whenever there exists an isomorphism $P_T \simeq P_T^1$ defined at the generic point of the curve that intertwines κ and κ^1, and γ^{red} and γ^{red1}.

We let $D(\mathrm{Bun}_N^{rat,red})$ denote the resulting category $D(\overline{\mathrm{Bun}}_N^{red})^{\mathcal{H}_{G,N}}$.

As in the case of $D(\mathrm{Bun}_B^{rat})$, the forgetful functor $\mathbf{oblv}_N^{red} : D(\mathrm{Bun}_N^{rat,red}) \to D(\overline{\mathrm{Bun}}_N^{red})$ admits a right adjoint, denoted $\mathbf{ind}_N^{red}$, and the monad

$$\mathbf{oblv}_N^{red} \circ \mathbf{ind}_N^{red} : D(\overline{\mathrm{Bun}}_N^{red}) \to D(\overline{\mathrm{Bun}}_N^{red})$$

is given by averaging with respect to $\mathcal{H}_{G,N}$, i.e., the functor $\mathrm{Av}_{\mathcal{H}_{G,H}}$.

6.2. **The Whittaker condition.** By a slight abuse of notation we'll denote by $\mathcal{H}_N$ the pull-back of the eponymous groupoid from $\overline{\mathrm{Bun}}_B$ to $\overline{\mathrm{Bun}}_N$. Note now that the data of γ^{red} defines a canonical character $\chi : \mathcal{H}_N \to \mathbb{G}_a$. As the actions of $\mathcal{H}_N$ and $\mathcal{H}_{G,N}$ commute, we can consider the category

$$\mathrm{Whit}(G)^{red}_{glob} := D(\overline{\mathrm{Bun}}_N^{red})^{(\mathcal{H}_N,\chi),\mathcal{H}_{G,N}} = D(\mathrm{Bun}_N^{rat,red})^{(\mathcal{H}_N,\chi)}.,$$

which is a full subcategory of $D(\overline{\mathrm{Bun}}_N^{red})^{\mathcal{H}_{G,N}} =: D(\mathrm{Bun}_N^{rat,red})$.

We have a pair of adjoint functors

$$\mathbf{ind}_{\mathrm{Whit}^{red},glob} : D(\mathrm{Bun}_N^{rat,red}) \rightleftarrows \mathrm{Whit}(G)^{red}_{glob} : \mathbf{oblv}_{\mathrm{Whit}^{red},glob}.$$

6.2.1. *The canonical object.* The category $\mathrm{Whit}(G)^{red}_{glob}$ has a canonical unit object, denoted $\mathbf{1}_{\mathrm{Whit}(G)^{red}_{glob}}$: it is obtained by applying $\mathbf{ind}_N^{red}$ to a certain canonical $(\mathcal{H}_N, \chi)$-equivariant object of $D(\overline{\mathrm{Bun}}_N^{red})$.

The object in question is the extension by ! (and also by $*$) from the character sheaf on $\mathrm{Bun}_N^{red} \subset \overline{\mathrm{Bun}}_N^{red}$, where Bun_N^{red} is a locus of $\overline{\mathrm{Bun}}_N^{red}$, where the Plücker data κ has neither zeros not poles, and γ_i^{red} are isomorphisms. The character sheaf is obtained by pulling back Artin-Schreier under the residue map $\mathrm{Bun}_N^{red} \to \mathbb{G}_a$.

6.3. **Relation between $\mathrm{Whit}(G)_{glob}$ and $\mathrm{Whit}(G)^{red}_{glob}$.**

In [GL:Ran], we'll prove the following:

Proposition 6.3.1. *The category $\mathrm{Whit}(G)_{glob}$ carries a natural action of the (rigid) monoidal category $D(\mathrm{Gr}_{Z_G,\mathrm{Ran}_X,un})$, and there is a canonical equivalence*

$$\mathrm{Whit}(G)^{red}_{glob} \simeq \mathrm{Whit}(G)_{glob} \underset{D(\mathrm{Gr}_{Z_G,\mathrm{Ran}_X,un})}{\otimes} D(\mathrm{Bun}_{Z_G}).$$

In particular, there is a pair of adoint functors

$$\mathrm{Whit}(G)_{glob} \rightleftarrows \mathrm{Whit}(G)^{red}_{glob},$$

where the right adjoint is fully faithful.

Note that the image of $\mathbf{1}_{\mathrm{Whit}(G)_{glob}}$ under the above functor $\mathrm{Whit}(G)_{glob} \to \mathrm{Whit}(G)^{red}_{glob}$ is the above object $\mathbf{1}_{\mathrm{Whit}(G)^{red}_{glob}}$.

6.3.2. The above proposition, combined with Theorem 5.3.3 gives the following description of $\mathrm{Whit}(G)^{red}_{glob}$ in spectral terms:

Corollary 6.3.3. *We have:*

$$\mathrm{Whit}(G)^{red}_{glob} \simeq \mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un) \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}],\mathrm{Ran}_X,un)}{\otimes} \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}).$$

7. The extended Whittaker category

We now turn to the main character of this paper, the extended Whittaker category, denoted $\mathrm{Whit}(G)^{red,ext}_{glob}$.

The Whittaker category $\mathrm{Whit}(G)^{red}_{glob}$ is meant to receive a functor from $D(\mathrm{Bun}_G)$ that "catches" all non-degenerate Fourier coefficients of automorphic D-modules (under the assumption that G has a connected center). The extended Whittaker category $\mathrm{Whit}(G)^{red,ext}_{glob}$ is supposed to catch all Fourier coefficients.

Let us first explain the function-theoretic analogy. Let $\mathrm{Char}^{\mathrm{n.d.}}$ denote the space of non-degenerate characters on $N(\mathbb{A})$ that are trivial on $N(K)$. The adjoint action of $T(\mathbb{A})$ on $N(\mathbb{A})$ gives rise to an action of $T(K)/Z_G(K)$ on $\mathrm{Char}^{\mathrm{n.d.}}$, which is simply transitive (due to the assumption that Z_G is connected). Hence, we obtain an isomorphism:

$$Z_G(K)N(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \simeq T(K)\backslash \left(N(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \times \mathrm{Char}^{\mathrm{n.d.}}\right). \tag{7.1}$$

Moreover, on the space of functions on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \times \mathrm{Char}^{\mathrm{n.d.}}$ one can impose a natural condition of $N(\mathbb{A})$-equivariance (depending on a point of $\mathrm{Char}^{\mathrm{n.d.}}$), and this condition is invariant under the $T(K)$-action. The resulting space of functions may thus be identified with the space of Whittaker functions on $Z_G(K)N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$.

We are finally ready to explain the function-theoretic analog of $D(\mathrm{Bun}_N^{rat,red,ext})$ and its full subcategory $\mathrm{Whit}(G)^{red,ext}_{glob}$. Namely, let Char is the space of *all* characters on $N(\mathbb{A})$ that are trivial on $N(K)$. The category $D(\mathrm{Bun}_N^{rat,red,ext})$ is the geometric analog of the space of functions on

$$T(K)\backslash \left(N(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \times \mathrm{Char}\right),$$

while $\mathrm{Whit}(G)^{red,ext}_{glob}$ is the geometric analog of the subspace of functions obtained be requiring equivariance for $N(\mathbb{A})$ against the character that depends on the point of Char.

Note, in particular, that when we specialize to $\{0\} \subset \mathrm{Char}$, we recover the set of functions on

$$T(K)N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O}).$$

7.1. **The category.** The definition of the space $\overline{\mathrm{Bun}}_N^{red,ext}$ and of the category

$$D(\mathrm{Bun}_N^{rat,red,ext}) := D(\overline{\mathrm{Bun}}_N^{red,ext})^{\mathcal{H}_{G,N}}$$

follows word-for-word that of $\overline{\mathrm{Bun}}_N^{red}$ and $D(\overline{\mathrm{Bun}}_N^{rat,red})^{\mathcal{H}_{G,N}}$ with the only difference is that now the maps

$$\gamma_i^{red,ext} : \check{\alpha}_i(P_T) \to \omega_X,$$

are now allowed to vanish.

As above, we have a pair of adjoint functors

$$\mathbf{ind}_N^{red,ext} : D(\overline{\mathrm{Bun}}_N^{red,ext}) \rightleftarrows D(\mathrm{Bun}_N^{rat,red,ext}) : \mathbf{oblv}_N^{red,ext}.$$

The old discussion applies also to the groupoid $\mathcal{H}_N$, and we obtain the corresponding full subcategory

$$\mathrm{Whit}(G)^{red,ext}_{glob} := D(\overline{\mathrm{Bun}}_N^{red,ext})^{(\mathcal{H}_N,\chi),\mathcal{H}_{G,N}} \subset D(\overline{\mathrm{Bun}}_N^{red,ext})^{\mathcal{H}_{G,N}} =: D(\mathrm{Bun}_N^{rat,red,ext}).$$

We have the correponding pair of adjoint functors

$$\mathbf{ind}_{\mathrm{Whit}^{red,ext}_{glob}} : D(\mathrm{Bun}_N^{rat,red,ext}) \rightleftarrows \mathrm{Whit}(G)^{red,ext}_{glob} : \mathbf{oblv}_{\mathrm{Whit}^{red,ext}_{glob}},$$

with the right adjoint being fully faithful.

7.2. **Extended Whittaker category vs.** $D(\mathrm{Bun}_T^{rat})$**.**

Note that $\overline{\mathrm{Bun}}_B$ admits a closed embedding into $\overline{\mathrm{Bun}}_N^{red,ext}$ by setting the data of γ_i^{red} to 0. Denote this map by $\mathbf{i}$.

We'll think of $\mathbf{i}$ as inducing a "closed embedding" $\mathbf{i}^{rat} : \mathrm{Bun}_B^{rat} \hookrightarrow \mathrm{Bun}_N^{rat,red,ext}$, by which we mean that we have a pair of adjoint functors

$$(\mathbf{i}^{rat})_! =: (\mathbf{i}^{rat})_* : D(\mathrm{Bun}_B^{rat}) \rightleftarrows D(\mathrm{Bun}_N^{rat,red,ext}) : (\mathbf{i}^{rat})^!,$$

with $(\mathbf{i}^{rat})_!$ fully faithful, which make the diagram

$$\begin{array}{ccc} D(\mathrm{Bun}_B^{rat}) & \longrightarrow & D(\mathrm{Bun}_N^{rat,red,ext}) \\ \mathbf{oblv}_B \big\downarrow & & \big\downarrow \mathbf{oblv}_N^{red,ext} \\ D(\overline{\mathrm{Bun}}_B) & \longrightarrow & D(\overline{\mathrm{Bun}}_N^{red,ext}) \end{array}$$

(and three other diagrams, when we replace the corresponding functors by their adjoints) commute.

The above adjoint pair sends the corresponding $\mathcal{H}_N$-equivariant categories to each other, thereby inducing an adjoint pair:

$$(\mathbf{i}^{rat})_! =: (\mathbf{i}^{rat})_* : D(\mathrm{Bun}_T^{rat}) \rightleftarrows \mathrm{Whit}(G)^{red,ext}_{glob} : (\mathbf{i}^{rat})^!.$$

7.3. **Extended vs. non-extended categories.** We have an open embedding

$$\mathbf{j} : \overline{\mathrm{Bun}}_N^{red} \hookrightarrow \overline{\mathrm{Bun}}_N^{red,ext}.$$

We'll think of $\mathbf{j}$ as inducing an "open embedding"

$$\mathbf{j}^{rat} : \mathrm{Bun}_N^{rat,red} \hookrightarrow \mathrm{Bun}_N^{rat,red,ext},$$

corresponding to the locus where all γ_i^{red} are non-zero.

Again, rigorously, this means that we have a restriction functor

$$(\mathbf{j}^{rat})^! =: (\mathbf{j}^{rat})^* : D(\mathrm{Bun}_N^{rat,red,ext}) \to D(\mathrm{Bun}_N^{rat,red}),$$

which admits a *right* adjoint, denoted $(\mathbf{j}^{rat})_*$, which is fully faithful.

These functors give rise to four commutative diagrams with respect to the functors

$$\mathbf{ind}_{N^{red,ext}} : D(\overline{\mathrm{Bun}}_N^{red,ext}) \rightleftarrows D(\mathrm{Bun}_N^{rat,red,ext}) : \mathbf{oblv}_{N^{red,ext}}$$

and

$$\mathbf{ind}_{N^{red}} : D(\overline{\mathrm{Bun}}_N^{red}) \rightleftarrows D(\mathrm{Bun}_N^{rat,red}) : \mathbf{oblv}_{N^{red}}.$$

7.3.1. The above pair of functors sends the corresponding Whittaker subcategories to each other, so we obtain an adjoint pair:

$$(\mathbf{j}^{rat})^* : \mathrm{Whit}(G)^{red}_{glob} \rightleftarrows \mathrm{Whit}(G)^{red,ext}_{glob} : (\mathbf{j}^{rat})_*,$$

with $(\mathbf{j}^{rat})_*$ fully faithful.

We shall now prove:

Proposition 7.3.2. *The partially defined left adjoint to $(\mathbf{j}^{rat})^!$, denoted $(\mathbf{j}^{rat})_!$, is defined on the subcategory $\mathrm{Whit}(G)^{red,ext}_{glob} \subset D(\mathrm{Bun}_N^{rat,red,ext})$.*

It follows formally from the proposition that $(\mathbf{j}^{rat})_!$ sends $\mathrm{Whit}(G)^{red,ext}_{glob}$ to $\mathrm{Whit}(G)^{red}_{glob}$, and is fully faithful, and therefore is a right inverse to $(\mathbf{j}^{rat})^!$.

Proof. The Ran version of the Hecke category, perceived as $\mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un)$ acts on all the categories involved. This action commutes with the functor $(\mathbf{j}^{rat})^!$, and since $\mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un)$ is rigid, it also commutes with $(\mathbf{j}^{rat})_!$ on the subcategory on which the latter is defined.

It follows from Theorem 5.3.3 that $\mathrm{Whit}(G)^{red}_{glob}$ is generated under the Hecke action from a single object, namely, $\mathbf{1}_{\mathrm{Whit}(G)^{red}_{glob}}$. Hence, it suffices to show that

$$(\mathbf{j}^{rat})_! \left(\mathbf{1}_{\mathrm{Whit}(G)^{red}_{glob}}\right)$$

is defined. But the latter is evidently so, since $\mathbf{1}_{\mathrm{Whit}(G)^{red}_{glob}}$ is obtained by applying $\mathbf{ind}_N^{red,ext}$ to an object of $D(\overline{\mathrm{Bun}}_N^{red,ext})$ which is holonomic.

□

Quasi-Proposition 7.3.3. *The functor $(\mathbf{i}^{rat})_* : D(\mathrm{Bun}_T^{rat}) \to \mathrm{Whit}(G)^{red,ext}_{glob}$ admits a left adjoint.*

We shall now prove this proposition when G is of semi-simple rank 1:

Proof. For an object $\mathcal{F} \in \mathrm{Whit}(G)^{red,ext}_{glob}$ consider the distinguished triangle

$$(\mathbf{j}^{rat})_! \circ (\mathbf{j}^{rat})^!(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}_1,$$

where $\mathcal{F}_1$ has the property that $(\mathbf{j}^{rat})^!(\mathcal{F}_1) = 0$.

However, when G is of semi-simple rank 1, the latter means that $\mathcal{F}_1$ lies in the essential image of $(\mathbf{i}^{rat})_*$, so the functor $(\mathbf{i}^{rat})^*$ on it is well-defined.

□

7.4. **The functor of coefficient.** We have a natural forgetful map of stacks

$$\mathfrak{r} : \overline{\mathrm{Bun}}_N^{red,ext} \to \overline{\mathrm{Bun}}_B.$$

Pullback induces a functor

$$\mathfrak{r}^! : D(\overline{\mathrm{Bun}}_B) \to D(\overline{\mathrm{Bun}}_N^{red,ext}).$$

We'll think of $\mathfrak{r}$ as inducing a map of "stacks" $\mathrm{Bun}_N^{rat,red,ext} \to \mathrm{Bun}_B^{rat}$. Rigorously, we mean that there is a canonically defined functor

$$(\mathfrak{r}^{rat})^! : D(\mathrm{Bun}_B^{rat}) \to D(\mathrm{Bun}_N^{rat,red,ext}),$$

which makes the diagram

$$\begin{array}{ccc} D(\mathrm{Bun}_B^{rat}) & \xrightarrow{(\mathfrak{r}^{rat})^!} & D(\mathrm{Bun}_N^{rat,red,ext}) \\ \mathbf{oblv}_B \downarrow & & \downarrow \mathbf{oblv}_N^{red,ext} \\ D(\overline{\mathrm{Bun}}_B) & \xrightarrow{\mathfrak{r}^!} & D(\overline{\mathrm{Bun}}_N^{red,ext}) \end{array}$$

commute, as well as the other diagram with the vertical arrows given by $\mathbf{ind}_B$ and $\mathbf{ind}_N^{red,ext}$, respectively.

We define the functor

$$\mathsf{coeff}^{red,ext} : D(\mathrm{Bun}_G) \to \mathrm{Whit}(G)_{glob}^{red,ext}$$

as the composition

$$\mathsf{coeff}^{red,ext} := \mathbf{ind}_{\mathrm{Whit}^{red,ext},glob} \circ (\mathfrak{r}^{rat})^! \circ (\mathfrak{p}^{rat})^!.$$

The composition

$$\mathsf{coeff}^{red} := (\mathbf{j}^{rat})^! \circ \mathsf{coeff}^{red,ext} : D(\mathrm{Bun}_G) \to \mathrm{Whit}(G)_{glob}^{red}$$

is the usual Whittaker coefficient functor.

Note that the composition

$$(\mathbf{i}^{rat})^! \circ \mathsf{coeff}^{red,ext} : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_T^{rat})$$

coincides with the functor CT^{rat}.

Quasi-Proposition 7.4.1. *The functor* $\mathsf{coeff}^{red,ext}$ *admits a left adjoint.*

We'll denote this left adjoint functor by $\mathsf{Poinc}_!^{red,ext}$. The composition

$$\mathsf{Poinc}_!^{red} := \mathsf{Poinc}_!^{red,ext} \circ (\mathbf{j}^{rat})_! : \mathrm{Whit}(G)_{glob}^{red} \to D(\mathrm{Bun}_G),$$

which is the left adjoint of coeff^{red} is the usual functor of Poincaré series.

We'll now prove this theorem when G is of semi-simple rank 1.

Proof. As in the proof of Theorem 7.3.3, it suffices to show separately that $\mathsf{Poinc}_!^{red}$ is well-defined, and that the left adjoint to CT^{rat} is well-defined. The latter was discussed in Sect. 3.2. The former follows as in the proof of Proposition 7.3.2.

□

7.5. **Fully faithfulness.** The key of our approach to Geometric Langlands in the case of GL_n is the following statement:

Quasi-Theorem 7.5.1. *For* $G = GL_n$*, the functor* $\mathsf{coeff}^{red,ext}$ *is fully faithful.*

We do not know whether a similar statement holds for other reductive group.

7.5.2. Let's perform one step toward the proof of Theorem 7.5.1.

For a reductive group G and a parabolic P one defines the category $D(\mathrm{Bun}_P^{rat})$ along the same lines as $D(\mathrm{Bun}_B^{rat})$ (using a partial Plücker data). We have a pair of adjoint functors

$$(\mathfrak{p}_P^{rat})_! : D(\mathrm{Bun}_B^{rat}) \rightleftarrows D(\mathrm{Bun}_G) : (\mathfrak{p}_P^{rat})^!,$$

and as in Theorem 1.3.1, we have:

Quasi-Theorem 7.5.3. *The functor* $(\mathfrak{p}_P^{rat})^!$ *is fully faithful.*

We now take $G = GL_n$, and P to be the parabolic $(n-1, 1)$. The following assertion "should" be proved by the classical procedure of iterative Fourier transforms:

Quasi-Theorem 7.5.4. *There exists an equivalence of categories*

$$D(\mathrm{Bun}_P^{rat}) \simeq \mathrm{Whit}(G)_{glob}^{red,ext}.$$

The composition

$$D(\mathrm{Bun}_G) \overset{(\mathfrak{p}_P^{rat})^!}{\longrightarrow} D(\mathrm{Bun}_P^{rat}) \to \mathrm{Whit}(G)_{glob}^{red,ext}$$

is the functor $\mathsf{coeff}^{red,ext}$.

Evidently, the above two theorems imply Theorem 7.5.1.

8. Gluing the extended Whittaker, and the spectral picture

The material of this section doesn't have a classical analog.

We'd like to describe the category $\mathrm{Whit}(G)_{glob}^{red,ext}$ as glued from its subcategories such as $\mathrm{Whit}(G)_{glob}^{red}$ and $D(\mathrm{Bun}_T^{rat})$, given by the "strata" of $\mathrm{Bun}_N^{rat,red,ext}$, determined by the pattern of which of the maps γ_i^{red} vanish.

The gluing data is given by functors between these catgories, such as

$$(\mathbf{i}^{rat})^! \circ (\mathbf{j}^{rat})_! : \mathrm{Whit}(G)_{glob}^{red} \to D(\overline{\mathrm{Bun}}_T).$$

8.1. **The gluing data for semi-simple rank 1.** Let us observe that when G is of semi-simple rank 1, there are "only two strata" in $\mathrm{Bun}_N^{rat,red,ext}$, namely Bun_B^{rat} and $\mathrm{Bun}_N^{rat,red,ext}$, so the category $\mathrm{Whit}(G)_{glob}^{red,ext}$ is completely described by the functor $(\mathbf{i}^{rat})^! \circ (\mathbf{j}^{rat})_!$.

Indeed, $D(\mathrm{Bun}_N^{rat,red,ext})$ can be identified with the category, whose objects are triples $(\mathcal{F}_1, \mathcal{F}_2, \alpha)$, where $\mathcal{F}_1 \in \mathrm{Whit}(G)_{glob}^{red}$, $\mathcal{F}_2 \in D(\mathrm{Bun}_T^{rat})$,

$$\alpha \in \mathrm{Hom}_{D(\mathrm{Bun}_T^{rat})}\left((\mathbf{i}^{rat})^! \circ (\mathbf{j}^{rat})_!(\mathcal{F}_1), \mathcal{F}_2\right).$$

Morphisms are morphisms between triples in a natural sense.

For groups of higher semi-simple rank, the category $\mathrm{Whit}(G)_{glob}^{red,ext}$ is described by a diagram of such functors for all pairs of parabolics contained in one another.

8.2. **The spectral description.** According to Corollary 6.3.3, we have:

$$\mathrm{Whit}(G)^{red}_{glob} \simeq \mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un) \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}], \mathrm{Ran}_X, un)}{\otimes} \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}),$$

and according to Theorem 4.6.1, we have:

$$D(\mathrm{Bun}_T^{rat}) \simeq T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r.$$

Our current goal is to identify in these terms the functor $(\mathbf{i}^{rat})^! \circ (\mathbf{j}^{rat})_!$, i.e., as a functor

$$\mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un) \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}], \mathrm{Ran}_X, un)}{\otimes} \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}) \to T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r. \tag{8.1}$$

The following assertion is inspired by the "Gap Theorem", suggested by V. Drinfeld and proved by A. Arinkin.

Quasi-Theorem 8.2.1. *The gluing functor* (8.1) *is canonically isomorphic to the composition*

$$\mathrm{Rep}(\check{G}, \mathrm{Ran}_X, un) \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}], \mathrm{Ran}_X, un)}{\otimes} \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}) \overset{\mathrm{Loc}_{\check{G},spec}}{\longrightarrow}$$

$$\to \mathsf{QCoh}(\mathrm{LocSys}_{\check{G}}) \overset{(\mathfrak{p}^{spec})^!}{\longrightarrow} T_{\mathrm{LocSys}_{\check{B}}\,/\,\mathrm{LocSys}_{\check{G}}}\text{-mod}^r.$$

In [GL:funct] we'll show how Theorem 8.2.1 allows to construct a functor in one direction

$$\Psi_G : \mathsf{QCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{G}}) \to D(\mathrm{Bun}_G)$$

for $G = GL_2$.

References

[BD] *Chiral algebras.*
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