CHAPTER III.1. DEFORMATION THEORY

Contents

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Introduction

0.1. **What does deformation theory do?**

0.1.1. *Deformation theory via pullbacks.* 'Admitting deformation theory' refers to a certain property of a prestack. Namely, although the initial definition will be different, according to Proposition 7.2.2, a prestack $\mathcal{X}$ admits deformation theory if:

(a) It is convergent, i.e., for $S \in \mathrm{Sch}^{\mathrm{aff}}$ the map $\mathrm{Maps}(S, \mathcal{X}) \to \lim\limits_{n} \mathrm{Maps}({}^{\leq n}S, \mathcal{X})$ is an isomorphism;

(b) For a push-out diagram of objects of $\mathrm{Sch}^{\mathrm{aff}}$

$$\begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2', \end{array}$$

where the map $S_1 \to S_1'$ is a *nilpotent embedding* (i.e., the map of classical affine schemes ${}^{\mathrm{cl}}S_1 \to {}^{\mathrm{cl}}S_2$ is a closed embedding, whose ideal of definition vanishes to some power), the resulting diagram in Spc

$$\begin{array}{ccc} \mathrm{Maps}(S_1, \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2, \mathcal{X}) \\ \uparrow & & \uparrow \\ \mathrm{Maps}(S_1', \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2', \mathcal{X}) \end{array}$$

is a pullback diagram.

The reason that this notion is useful is that it allows to study the infinitesimal behavior of $\mathcal{X}$ (i.e., properties of the map $\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$ whenever $S \to S'$ is a nilpotent embedding) by using *linear* objects. Let us explain this in more detail.

0.1.2. *Pro-cotangent fibers and cotangent complex.* For $S = \text{Spec}(A)$, by considering nilpotent embeddings of the form $S \to S'$ for

$$S' = S_{\mathcal{I}} := \text{Spec}(A \oplus M), \quad M = \Gamma(S, \mathcal{I}), \quad \mathcal{I} \in \text{QCoh}(S)^{\leq 0},$$

one shows that the functor

$$\mathcal{I} \mapsto \text{Maps}_{S/}(S_{\mathcal{I}}, \mathcal{X}), \quad \text{QCoh}(S)^{\leq 0} \to \text{Spc} \tag{0.1}$$

is given by a well-defined object

$$T_x(\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^-),$$

called the *pro-cotangent space* of $\mathcal{X}$ at x. We emphasize that the fact that the functor (0.1) comes from such an object is already a non-trivial condition and amounts to this functor commuting with certain pullbacks that are among the pullbacks in Sect. 0.1.1.

Next, one shows that the assignments

$$(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}} \quad \rightsquigarrow \quad T^*_x(\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^-), \tag{0.2}$$

are compatible in the sense that for

$$f : S_1 \to S_2, \quad x_2 \in \text{Maps}(S_2, \mathcal{X}), \quad x_1 = x_2 \circ f,$$

the natural map in $\text{Pro}(\text{QCoh}(S_1)^-)$

$$T^*_{x_1}(\mathcal{X}) \to \text{Pro}(f^*)(T^*_{x_2}(\mathcal{X}))$$

is an isomorphism. This follows from the condition in Sect. 0.1.1 applied to the push-out diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ \downarrow & & \downarrow \\ (S_1)_{\mathcal{I}} & \longrightarrow & (S_2)_{f_*(\mathcal{I})}. \end{array}$$

Furthermore, one shows that for $\mathcal{X} = X \in \text{Sch}$, we have $T^*_x(X) \in \text{QCoh}(S)^{\leq 0}$, so that the assignment (0.2) comes from a well-defined object $T^*(X) \in \text{QCoh}(X)^{\leq 0}$, called the *cotangent complex* of X.

By functoriality, for $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$, we have a canonically defined map in $\text{Pro}(\text{QCoh}(S)^-)$

$$T^*_x(\mathcal{X}) \to T^*(S),$$

called the *co-differential* of x.

0.1.3. *Square-zero extensions.* Among nilpotent embeddings $S \to S'$ one singles out a particular class, called *square-zero extensions.* Namely, one shows that for every object $\mathcal{I} \in \text{QCoh}(S)^{\leq 0}$ equipped with a map

$$T^*(S) \xrightarrow{\gamma} \mathcal{I}[1]$$

one can canonically attach a nilpotent embedding

$$S \xrightarrow{i} S_{\mathcal{I},\gamma},$$

such that

$$\text{Fib}(\mathcal{O}_{S_{\mathcal{I},\gamma}} \to i_*(\mathcal{O}_S)) \simeq i_*(\mathcal{I}).$$

In fact, the pair $(i, S_{\mathcal{I},\gamma})$ can be uniquely characterized by the property that for a map $f : S \to U$ in $\mathrm{Sch}^{\mathrm{aff}}$, the data of extension of f to a map $S_{\mathcal{I},\gamma} \to U$ is equivalent to that of a *null-homotopy* of the composed map

$$f^*(T^*(U)) \to T^*(S) \overset{\gamma}{\to} \mathcal{I}[1],$$

where the map $f^*(T^*(U)) \to T^*(S)$ is the co-differential of f.

In particular, for $\gamma = 0$ we have tautologically $S_{\mathcal{I},0} = S_{\mathcal{I}}$, where $S_{\mathcal{I}}$ is as in Sect. 0.1.2.

When S is classical and $\mathcal{I} \in \mathrm{QCoh}(S)^{\heartsuit}$, one shows that the above assignment

$$(\mathcal{I}, \gamma) \rightsquigarrow S_{\mathcal{I},\gamma}$$

is an equivalence between $\mathrm{QCoh}^{\heartsuit}_{T^*(S)[-1]/}$ and the category of closed embeddings $S \to S'$ with S' being a classical scheme such that the ideal of definition of S in S' squares to 0.

0.1.4. *From square-zero extensions to all nilpotent extensions.* A fact of crucial importance is that *any* nilpotent embedding $S \to S'$ can be obtained as a composition of square-zero extensions, *up to any given truncation*. More precisely, there exists a sequence of affine schemes

$$S = S_0 \to S_1 \to S_2 \to ... \to S_n \to ... \to S'$$

such that each $S_i \to S_{i+1}$ is a square-zero extension, and for every n there exists m such that the maps

$${}^{\leq n}S_m \to {}^{\leq n}S_{m+1} \to ... \to {}^{\leq n}S'$$

are all isomorphisms.

Thus, if $\mathcal{X}$ is *convergent*, then if we can control the map

$$\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$$

when $S \to S'$ is a square-zero extension, we can control it for any nilpotent embedding.

0.1.5. *Back to deformation theory.* One shows that if $\mathcal{X}$ admits deformation theory, then, given a point $S \overset{x}{\to} \mathcal{X}$, the datum of its extension to a point $S_{\mathcal{I},\gamma} \to \mathcal{X}$ is equivalent to the datum of a null-homotopy for the map

$$T^*_x(\mathcal{X}) \to T^*(S) \overset{\gamma}{\to} \mathcal{I}[1]. \tag{0.3}$$

The latter fact, combined with Sect. 0.1.4, is the precise expression of the above-mentioned principle that extensions of a given map $S \to \mathcal{X}$ to a map $S' \to \mathcal{X}$ (for a nilpotent embedding $S \to S'$) are controlled by linear objects, the latter being the objects in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$ appearing in (0.3).

0.2. **What is done in this chapter?** This chapter splits naturally into two halves: the build-up to the formulation of what it means to admit deformation theory (Sects. 1-6) and consequences of the property of admitting deformation theory (Sects. 7-10).

0.2.1. *Push-outs.* In Sect. 1, we study the operation of push-out on affine schemes. The reason we need to do this is, as was mentioned above, we formulate the property of a prestack $\mathcal{X}$ to admit deformation theory in terms of push-outs.

Note that the operation of push-out is not so ubiquitous in algebraic geometry–we are much more used to pullbacks. In terms of rings, pullbacks are given by

$$(A_1 \leftarrow A \rightarrow A_2) \quad \mapsto \quad A_1 \underset{A}{\otimes} A_2,$$

while push-outs by

$$(A_1 \rightarrow A \leftarrow A_2) \quad \mapsto \tau^{\leq 0}(A_1 \underset{A}{\times} A_2).$$

The operation of push-out on affine schemes is not so well-behaved (for example, a push-out of affine schemes may not be a push-out in the category of all schemes). However, there is one case in which it is well-behaved: namely, when we consider push-out diagrams

$$\begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2', \end{array}$$

in which $S_1 \rightarrow S_1'$ is a nilpotent embedding.

0.2.2. *(Pro)-cotangent spaces.* In Sect. 2 we define what it means for a prestack $\mathcal{X}$ to admit a (pro)-cotangent space at a given S-point

$$S \overset{x}{\rightarrow} \mathcal{X}.$$

By definition, a (pro)-cotangent space, if it exists, is an object

$$T^*_x(\mathcal{X}) \in \operatorname{Pro}(\operatorname{QCoh}(S)^-).$$

The definition is given in terms of the notion of split-square zero extension, $S \rightarrow S_{\mathcal{F}}$, see Sect. 0.1.2.

We shall say that $\mathcal{X}$ admits a *cotangent space* at x if $T^*_x(\mathcal{X})$ actually belongs to $\operatorname{QCoh}(S)^-$.

One result in this section that goes beyond definitions is Proposition 2.5.3 that gives the expression of the (pro)-cotangent space of a prestack $\mathcal{X}$ that is itself given as a colimit of prestacks $\mathcal{X}_\alpha$. Namely, if both $\mathcal{X}$ and all $\mathcal{X}_\alpha$ admit (pro)-cotangent spaces, then the (pro)-cotangent space of $\mathcal{X}$ is the limit of the (pro)-cotangent spaces of $\mathcal{X}_\alpha$ (as is natural to expect).

0.2.3. *The tangent space.* In Sect. 3 we discuss various conditions on objects of $\operatorname{Pro}(\operatorname{QCoh}(S)^-)$ that one can impose on pro-cotangent spaces. We would like to draw the reader's attention to two of these properties: one is *convergence* and the other is *laft*-ness.

An object $\Phi \in \operatorname{Pro}(\operatorname{QCoh}(S)^-)$ is said to be *convergent* it for $\mathcal{F} \in \operatorname{QCoh}(S)^-$, the map

$$\operatorname{Maps}(\Phi, \mathcal{F}) \rightarrow \lim_n \operatorname{Maps}(\Phi, \tau^{\geq -n}(\mathcal{F}))$$

is an isomorphism.

It follows almost tautologically that if $\mathcal{X}$ is convergent (in the sense of Sect. 0.1.1), then its pro-cotangent spaces are convergent in the above sense.

An object $\Phi \in \operatorname{Pro}(\operatorname{QCoh}(S)^-)$ is said to be *laft* if it is convergent and for every m_1, m_2, the functor

$$\mathcal{F} \mapsto \operatorname{Maps}(\Phi, \mathcal{F}), \quad \mathcal{F} \in \operatorname{QCoh}(S)^{\geq -m_1, \leq m_2}$$

commutes with filtered colimits.

Again, it follows almost tautologically that if $\mathcal{X}$ is laft, then its pro-cotangent spaces are laft in the above sense.

If S is itself laft, the full subcategory

$$\text{Pro}(\text{QCoh}(S)^-)_{\text{laft}} \subset \text{Pro}(\text{QCoh}(S)^-)$$

has the following nice interpretation: Serre duality identifies it with the opposite of the category $\text{IndCoh}(S)$.

So, instead of thinking of $T^*_x(\mathcal{X})$ as an object of $\text{Pro}(\text{QCoh}(S)^-)$, we can think of its formal dual, denoted

$$T_x(\mathcal{X}) \in \text{IndCoh}(S),$$

and called the *tangent* space of $\mathcal{X}$ at x.

0.2.4. *The naive tangent space.* We want to emphasize that in our interpretation, the tangent space is *not* the *naive* dual of the (pro)-cotangent space, but rather the *Serre* dual. One can define the naive duality functor

$$(\text{Pro}(\text{QCoh}(S)^-)_{\text{laft}})^{\text{op}} \to \text{QCoh}(S) \tag{0.4}$$

by sending

$$\Phi \in \text{Pro}(\text{QCoh}(S)^-)_{\text{laft}} \quad \mapsto \quad \Phi(\mathcal{O}_S),$$

where $\Phi(\mathcal{O}_S)$ is regarded as an A-module if $S = \text{Spec}(A)$.

However, the above functor is the composition of the Serre duality equivalence

$$(\text{Pro}(\text{QCoh}(S)^-)_{\text{laft}})^{\text{op}} \to \text{IndCoh}(S),$$

followed by the functor

$$\text{IndCoh}(S) \to \text{QCoh}(S), \quad \mathcal{F} \mapsto \underline{\text{Hom}}(\omega_S, \mathcal{F}),$$

while the latter fails to be conservative (even for S eventually coconnective).

So, in general, the naive duality functor (0.4) loses information, and hence it is not a good idea to think of the tangent space as an object of $\text{QCoh}(S)$ equal to the naive dual of the (pro)-cotangent case.

In the case when S is eventually coconnective, one can explicitly describe a full subcategory inside $\text{Pro}(\text{QCoh}(S)^-)_{\text{laft}}$ on which the naive duality functor (0.4) is fully faithful:

This is the image of the fully faithful embedding

$$\text{QCoh}(S) \to (\text{Pro}(\text{QCoh}(S)^-)_{\text{laft}})^{\text{op}},$$

given by

$$\mathcal{F} \in \text{QCoh}(S) \mapsto \Phi_{\mathcal{F}}, \quad \Phi_{\mathcal{F}}(\mathcal{F}_1) = \underset{n}{\text{colim}}\, \Gamma(S, \mathcal{F} \otimes \tau^{\geq -n}(\mathcal{F}_1)), \quad \mathcal{F}_1 \in \text{QCoh}(S)^-.$$

0.2.5. *The (pro)-cotangent complex.* In Sect. 4 we impose a condition on a prestack $\mathcal{X}$ that its (pro)-cotangent spaces are compatible under pullbacks, see Sect. 0.1.2. If this condition is satisfied, we say that $\mathcal{X}$ *admits a (pro)-cotangent complex*.

If $\mathcal{X}$ is laft, we show (using Sect. 0.2.3 above) that if it admits a pro-cotangent complex, then it admits a *tangent complex*, which is an object of $\text{IndCoh}(\mathcal{X})$.

0.2.6. *Square-zero extensions.* In Sect. 5 we introduce the category of square-zero extensions of a scheme, already mentioned in Sect. 0.1.3 above.

By definition, the category of square-zero extensions of X, denoted $\mathrm{SqZ}(X)$, is

$$((\mathrm{QCoh}(X)^{\leq 0})_{T(X)[-1]/})^{\mathrm{op}}.$$

As was explained in Sect. 0.1.3 we have a functor

$$\mathrm{RealSqZ} : \mathrm{SqZ}(X) \to (\mathrm{Sch})_{X/}.$$

We note, however, that unless X is classical and we restrict ourselves to the part of $\mathrm{SqZ}(X)$ that corresponds to $((\mathrm{QCoh}(X)^{\heartsuit})_{T(X)[-1]/})^{\mathrm{op}}$, the functor RealSqZ is *not* fully faithful. I.e., being a square-zero extension is not a condition but additional structure.

We proceed to study several crucial pieces of structure pertaining to square-zero extensions:

(i) A canonical structure of square-zero extension on an $(n+1)$-coconnective scheme of square-zero extension of its n-coconnective truncation;

(ii) The approximation of any nilpotent embedding by a series of square-zero extensions, already mentioned in Sect. 0.1.4;

(iii) Functoriality of square-zero extension under push-forwards: given a square-zero extension $X_1 \to X_1'$ by means of $\mathcal{I} \in \mathrm{QCoh}(X_1)^{\leq 0}$ and an affine morphism $f : X_1 \to X_2$, we obtain a canonically defined structure of square-zero extension by means of $f_*(\mathcal{I})$ on $X_2 \to X_2' := X_1' \underset{X_1}{\sqcup} X_2$.

(iv) Functoriality of square-zero extension under pullbacks: given a square-zero extension $X_2 \to X_2'$ by means of $\mathcal{I} \in \mathrm{QCoh}(X_2)^{\leq 0}$ and a map $f' : X_1' \to X_2'$, we obtain a canonically defined structure of square-zero extension on $X_2 \underset{X_2'}{\times} X_1' =: X_1 \to X_1'$ by means of $f^*(\mathcal{I})$, where f is the resulting map $X_1 \to X_2$.

0.2.7. *Infinitesimal cohesiveness.* In Sect. 6 we define what it means for a prestack $\mathcal{X}$ to be *infinitesimally cohesive*. Namely, we say that $\mathcal{X}$ is infinitesimally cohesive if whenever $S \to S'$ is a square-zero extension of affine schemes given by

$$T^*(S) \xrightarrow{\gamma} \mathcal{I}[1], \quad \mathcal{I} \in \mathrm{QCoh}(S)^{\leq 0},$$

and $x : S \to \mathcal{X}$ is a map, the (naturally defined) map from the space of extensions of x to a map $x' : S' \to \mathcal{X}$ to the space of *null-homotopies* of the composed map

$$T^*_x(\mathcal{X}) \to T^*(S) \xrightarrow{\gamma} \mathcal{I}[1]$$

is an isomorphism.

We explain that this property can also be interpreted as the fact that $\mathcal{X}$ takes certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc.

0.2.8. *Finally: deformation theory!* In Sect. 7 we finally introduce what it means for a prestack $\mathcal{X}$ to admit deformation theory. We define it as follows: we say that $\mathcal{X}$ admits deformation theory if:

(a) It is convergent;

(b') It admits a pro-cotangent complex;

(b") It is infinitesimally cohesive.

However, as was mentioned in Sect. 0.1.1, in Proposition 7.2.2, we show that one can replace conditions (b') and (b") by just one condition (b) from Sect. 0.1.1, namely that $\mathcal{X}$ takes certain push-out to pullbacks.

We proceed to study some properties of prestacks associated with the notion of admitting deformation theory:

(i) We introduce and study the notion of *formal smoothness* of a prestack;

(ii) We show that for any integer k, prestacks that are k-Artin stacks admit deformation theory.

0.2.9. *Consequences of admitting deformation theory.* In Sect. 8 we derive some consequences of the fact that a given prestack admits deformation theory:

(i) If $\mathcal{X}_0$ is a classical prestack and $i : \mathcal{X}_0 \to {}^{\mathrm{cl}}\mathcal{X}$ is a nilpotent embedding, then if $\mathcal{X}_0$ satisfies étale descent, then so does $\mathcal{X}$;

(ii) In the above situation, if $f : \mathcal{X} \to \mathcal{X}'$ is a map where $\mathcal{X}'$ admits deformation theory such that for any classical affine scheme S and a map $x_0 : S \to \mathcal{X}_0$, the map

$$T^*_{f\circ i\circ x_0}(\mathcal{X}') \to T^*_{i\circ x_0}(\mathcal{X})$$

is an isomorphism, then f itself is an isomorphism.

0.2.10. *Deformation theory and laft-ness.* In Sect. 9 we prove two assertions related to the interaction of deformation theory with the property of a prestack to be laft (locally almost of finite type).

The first assertion, Theorem 9.1.2, gives the following infinitesimal criterion to determine whether $\mathcal{X}$ is laft. Namely, it says that a prestack $\mathcal{X}$ admitting deformation theory is laft if and only if:

(i) ${}^{\mathrm{cl}}\mathcal{X}$ is locally of finite type;

(ii) For any classical scheme of finite type S and a point $x : S \to {}^{\mathrm{cl}}\mathcal{X}$, we have $T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$,

The second assertion, Theorem 9.1.4, says that if $\mathcal{X}$ admits deformation theory, its laft-ness property implies something stronger than for arbitrary laft prestacks. Namely, it says that $\mathcal{X}$, when viewed as a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$

is the left Kan extension from

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

I.e., for any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, the space of factoring x as

$$S \to U \to \mathcal{X}, \quad U \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$$

is contractible.

Note that for arbitrary prestacks such a property holds not on all $S \in \mathrm{Sch}^{\mathrm{aff}}$ but only on truncated ones.

0.2.11. *Square-zero extensions of prestacks.* In Sect. 10 we define the notion of square-zero extension of a given prestack $\mathcal{X}$ by means of $\mathcal{I} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$. This is defined via the functoriality of square-zero extensions under pullbacks, see Sect. 0.2.6(iv).

Assuming that $\mathcal{X}$ admits deformation theory, we prove that under certain circumstances, the map from $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$ to $\mathrm{Maps}(T^*(\mathcal{X}), \mathcal{I}[1])$ is an isomorphism, and that any prestack obtained as a square-zero extension of $\mathcal{X}$ itself admits deformation theory.

1. Push-outs of schemes

In this subsection we study the operation of push-out on schemes. This operation is not so ubiqutous in algebraic geometry. However, it is crucial for deformation theory. In fact, deformation theory is defined in terms of compatibility with certain push-outs.

1.1. **Push-outs in the category of affine schemes.** In this subsection we describe what push-outs look like in the category of affine schemes.

1.1.1. Let

$$i \mapsto X_i, \quad i \in \mathbf{I}$$

be an $\mathbf{I}$-diagram in $\mathrm{Sch}^{\mathrm{aff}}$ for some $\mathbf{I} \in 1\text{-Cat}$.

Let Y denote its colimit in the category $\mathrm{Sch}^{\mathrm{aff}}$. I.e., if $X_i = \mathrm{Spec}(A_i)$, then $Y = \mathrm{Spec}(B)$, where

$$B = \lim_i A_i,$$

where the limit is taken in the category of *connective* commutative k-algebras.

Remark 1.1.2. Note that in the above formula $B = \tau^{\leq 0}(B')$, where

$$B' = \lim_i A_i,$$

the limit is taken in the category ComAlg(Vect) of all commutative k-algebras. Note also that the forgetful functor

$$\mathbf{oblv}_{\mathrm{Com}} : \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{Vect}$$

commutes with limits, so that it is easy to understand what B' looks like.

The functor $\tau^{\leq 0}$ of connective truncation used above is also very explicit. Namely, by definition, the category of connective commutative k-algebras is $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$, and the functor $\tau^{\leq 0}$ is the right adjoint to the embedding

$$\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}).$$

This functor makes the diagram

$$\begin{array}{ccc} \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) & \xleftarrow{\tau^{\leq 0}} & \mathrm{ComAlg}(\mathrm{Vect}) \\ \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Com}}} & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Com}}} \\ \mathrm{Vect}^{\leq 0} & \xleftarrow{\tau^{\leq 0}} & \mathrm{Vect} \end{array}$$

commute.

1.1.3. In particular, consider a diagram $X_1 \leftarrow X \to X_2$ in $\mathrm{Sch}^{\mathrm{aff}}$ and set $Y := X_1 \underset{X}{\sqcup} X_2$, where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$. I.e., if $X_i = \mathrm{Spec}(A_i)$ and $X = \mathrm{Spec}(A)$, then $Y = \mathrm{Spec}(B)$, where

$$B := A_1 \underset{A}{\times} A_2.$$

Note that if $X \to X_1$ is a closed embedding, then so is the map $X_2 \to Y$.

1.2. **The case of closed embeddings.** In this subsection we show that if we take push-outs with respect to maps that are closed embeddings, then this operation is well-behaved.

1.2.1. Suppose we are in the context of Sect. 1.1.3. We observe the following:

Lemma 1.2.2. *Suppose that both maps $X \to X_i$ are closed embeddings. Then:*
(a) *The Zariski topology on Y is induced by that on $X_1 \sqcup X_2$.*
(b) *For open affine subschemes $\overset{\circ}{X}_i \subset X_i$ such that $\overset{\circ}{X}_1 \cap X = \overset{\circ}{X}_2 \cap X =: \overset{\circ}{X}$, and the corresponding (by point (a)) open subscheme $\overset{\circ}{Y} \subset Y$, the map*

$$\overset{\circ}{X}_1 \underset{\overset{\circ}{X}}{\sqcup} \overset{\circ}{X}_2 \to \overset{\circ}{Y}$$

is an isomorphism, where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$.
(c) *The diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y \end{array}$$

is also a push-out diagram also in Sch.

1.2.3. From here we obtain:

Corollary 1.2.4. *Let $X_1 \leftarrow X \to X_2$ be a diagram in* Sch, *where both maps $X_i \to X$ are closed embeddings. Then:*
(a) *The push-out $Y := X_1 \underset{X}{\sqcup} X_2$ in* Sch *exists.*
(b) *The Zariski topology on Y is induced by that on $X_1 \sqcup X_2$.*
(c) *For open subschemes $\overset{\circ}{X}_i \subset X_i$ such that $\overset{\circ}{X}_1 \cap X = \overset{\circ}{X}_2 \cap X =: \overset{\circ}{X}$, and the corresponding (by point (b)) open subscheme $\overset{\circ}{Y} \subset Y$, the map*

$$\overset{\circ}{X}_1 \underset{\overset{\circ}{X}}{\sqcup} \overset{\circ}{X}_2 \to \overset{\circ}{Y}$$

is an isomorphism.

1.3. **The push-out of a closed nil-isomorphism.** The situation studied in this subsection is of crucial importance for deformation theory.

1.3.1. We recall (see [Chapter II.1, Sect. 6.1.4]) that a map $X \to Y$ is Sch is called a nil-isomorphism if it induces an isomorphism

$${}^{\mathrm{red}}X \to {}^{\mathrm{red}}Y,$$

where the notation ${}^{\mathrm{red}}X$ means the reduced classical scheme underlying ${}^{cl}X$.

We emphasize that 'nil-isomorphism' *does not* imply 'closed embedding'.

1.3.2. Let

$$X_1 \to X_1'$$

be a *closed* nil-isomorphism of affine schemes, and let $f : X_1 \to X_2$ be a map, where $X_2 \in \mathrm{Sch}^{\mathrm{aff}}$.

Let $X_2' = X_1' \underset{X_1}{\sqcup} X_2$, where the colimit is taken in $\mathrm{Sch}^{\mathrm{aff}}$. Note that the map

$$X_2 \to X_2'$$

is also a closed nil-isomorphism.

We observe:

Lemma 1.3.3. *In the above situation we have:*

(a) *For an open affine subscheme* $\overset{\circ}{X}_2 \subset X_2$, $f^{-1}(\overset{\circ}{X}_2) =: \overset{\circ}{X}_1 \subset X_1$, *and the corresponding open affine subschemes* $\overset{\circ}{X}{}'_i \subset X'_i$ *for* $i = 1, 2$, *the map*

$$\overset{\circ}{X}{}'_1 \underset{\overset{\circ}{X}_1}{\sqcup} \overset{\circ}{X}_2 \to \overset{\circ}{X}{}'_2$$

is an isomorphism, where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$.

(b) *The diagram*

$$\begin{array}{ccc} X'_1 & \longrightarrow & X'_2 \\ \uparrow & & \uparrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

is also a push-out diagram in Sch.

1.3.4. As a corollary we obtain:

Corollary 1.3.5. *Let* $X_1 \to X'_1$ *be a closed nil-isomorphism of (not necessarily affine) schemes, and let* $f : X_1 \to X_2$ *be an affine map between schemes. Then:*

(a) *The push-out* $X'_2 := X'_1 \underset{X_1}{\sqcup} X_2$ *in* Sch *exists, the map* $X_2 \to X'_2$ *is a closed nil-isomorphism (and in particular affine).*

(b) *For an open subscheme* $\overset{\circ}{X}_2 \subset X_2$, $f^{-1}(\overset{\circ}{X}_2) =: \overset{\circ}{X}_1 \subset X_1$, *and the corresponding open affine subscheme* $\overset{\circ}{X}{}'_i \subset X'_i$, *the map*

$$\overset{\circ}{X}{}'_1 \underset{\overset{\circ}{X}_1}{\sqcup} \overset{\circ}{X}_2 \to \overset{\circ}{X}{}'_2$$

is an isomorphism, where the push-out is taken in Sch.

(c) *If* f *is an open embedding, then so is the map* $X'_1 \to X'_2$.

Proof. By Lemma 1.3.3, it suffices to prove the corollary when X_2 is affine, in which case it also follows from Lemma 1.3.3.

□

Remark 1.3.6. Note that in the situation of Corollary 1.3.5(a), if the map $X_1 \to X_2$ is not affine, the map $X_2 \to X'_2$ is not necessarily a closed embedding. In fact, it can look like a "pinching":

Let $X_2 := \mathbb{A}^2$, and let X_1 be its blow-up at the origin. Let $X_1 \hookrightarrow X'_1$ be the square-zero extension supported on exceptional divisor with ideal $\mathcal{O}(-2)^{\oplus 2}$, given by the canonical element in $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-2)^{\oplus 2})$. Then X'_2 is the spectrum of the algebra consisting of polynomials with a vanishing derivative at the origin.

1.4. **Behavior of quasi-coherent sheaves.** In this subsection we will describe how the categories QCoh and IndCoh behave with respect to the operation of push-out.

1.4.1. Let $f : X_1 \to X_2$ and $g_1 : X_1 \to X'_1$ be maps of affine schemes with g_1 being a closed embedding. Let

$$X'_2 := X'_1 \underset{X_1}{\sqcup} X_2$$

be the push-out in the category $\mathrm{Sch}^{\mathrm{aff}}$.

We claim:

Proposition 1.4.2. *In the diagram*

$$\begin{array}{ccc} \mathrm{QCoh}(X_1) & \xleftarrow{g_1^*} & \mathrm{QCoh}(X_1') \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f'^*} \\ \mathrm{QCoh}(X_2) & \xleftarrow{g_2^*} & \mathrm{QCoh}(X_2'), \end{array} \tag{1.1}$$

the map

$$\mathrm{QCoh}(X_2') \to \mathrm{QCoh}(X_2) \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_1')$$

is fully faithful.

Remark 1.4.3. We note that even when the maps f and g_1 are closed embeddings, the diagram (1.1) is generally *not*[1] a pullback square in $\mathrm{DGCat}_{\mathrm{cont}}$. Indeed, this fails already for $X_1 = \mathrm{pt}$, $X_2 \simeq X_1' \simeq \mathbb{A}^1$.

That said, one can show that the diagram consisting of perfect complexes

$$\begin{array}{ccc} \mathrm{QCoh}(X_1)^{\mathrm{perf}} & \xleftarrow{g_1^*} & \mathrm{QCoh}(X_1')^{\mathrm{perf}} \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f'^*} \\ \mathrm{QCoh}(X_2)^{\mathrm{perf}} & \xleftarrow{g_2^*} & \mathrm{QCoh}(X_2')^{\mathrm{perf}}, \end{array}$$

is a pullback square, see [Chapter IV.4, Sect. A.2].

We note that there is no contradiction between the above two facts: if

$$\mathbf{C} \to \mathbf{D} \leftarrow \mathbf{E}$$

is a diagram of compactly generated categories with functors preserving compact objects, the inclusion

$$(\mathbf{C}^c \underset{\mathbf{D}^c}{\times} \mathbf{E}^c) \hookrightarrow (\mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{E})^c$$

is in general *not* an equality (although the corresponding fact is true for *filtered* limits).

Proof of Proposition 1.4.2. One readily reduces the assertion to the case when X_2 (and hence also X_1 and X_1') are affine. In the latter case, we will prove the proposition just assuming that the map g_1 is a closed embedding.

We construct a functor

$$\mathrm{QCoh}(X_1') \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_2) \to \mathrm{QCoh}(X_2'),$$

right adjoint to the tautological functor

$$(f'^* \times g_2^*) : \mathrm{QCoh}(X_2') \to \mathrm{QCoh}(X_1') \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_2),$$

by sending a datum

$$(\mathcal{F}_1' \in \mathrm{QCoh}(X_1'),\ \mathcal{F}_2 \in \mathrm{QCoh}(X_2),\ \mathcal{F}_1 \in \mathrm{QCoh}(X_1),\ g_1^*(\mathcal{F}_1') \simeq \mathcal{F}_1 \simeq f^*(\mathcal{F}_2))$$

to

$$\mathrm{Fib}\left(f'_*(\mathcal{F}_1') \oplus g_{2*}(\mathcal{F}_2) \to h_*(\mathcal{F}_1)\right),$$

where h denotes the map $X_1 \to X_2'$.

[1] We are grateful to D. Nadler who pointed this out to us, thereby correcting a mistake in the previous version.

We claim that the unit of the adjunction is an isomorphism. Indeed, we have to check that for $\mathcal{F} \in \mathrm{QCoh}(X'_2)$, the map

$$\mathcal{F} \to \mathrm{Fib}\left(f'_* \circ f'^*(\mathcal{F}) \oplus g_{2*} \circ g_2^*(\mathcal{F}) \to h_* \circ h^*(\mathcal{F})\right)$$

is an isomorphism. However, the above map is obtained by tensoring $\mathcal{F}$ with the corresponding map for $\mathcal{F} = \mathcal{O}_X$. Hence, since X is affine, it suffices to check that the map

$$\Gamma(X'_2, \mathcal{O}_{X'_2}) \to \mathrm{Fib}\left(\Gamma(X'_1, \mathcal{O}_{X'_1}) \oplus \Gamma(X_2, \mathcal{O}_{X_2}) \to \Gamma(X_1, \mathcal{O}_{X_1})\right)$$

is an isomorphism. However, the latter follows from the construction of the push-out

□

1.4.4. Assume now that in the above situation, X_1, X_2, X'_1 belong to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and the map $f : X_1 \to X_2$ is finite so that X'_2 also belongs to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

We claim:

Proposition 1.4.5. *In the diagram*

$$\begin{array}{ccc} \mathrm{IndCoh}(X_1) & \xleftarrow{g_1^!} & \mathrm{IndCoh}(X'_1) \\ f^! \uparrow & & \uparrow f'^! \\ \mathrm{IndCoh}(X_2) & \xleftarrow{g_2^!} & \mathrm{IndCoh}(X'_2), \end{array} \tag{1.2}$$

the map

$$\mathrm{IndCoh}(X'_2) \to \mathrm{IndCoh}(X_2) \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X'_1)$$

is fully faithful.

Remark 1.4.6. Unlike the situation with QCoh, in [Chapter IV.4, Sect. A.1], we will see that (1.2) *is* a pullback square[2].

Proof of Proposition 1.4.5. We construct the *left* adjoint to the

$$(f'^1 \times g_2^!) : \mathrm{IndCoh}(X'_2) \to \mathrm{IndCoh}(X'_1) \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2),$$

by sending a datum

$$(\mathcal{F}'_1 \in \mathrm{IndCoh}(X'_1),\ \mathcal{F}_2 \in \mathrm{IndCoh}(X_2),\ \mathcal{F}_1 \in \mathrm{IndCoh}(X_1),\ g_1^!(\mathcal{F}'_1) \simeq \mathcal{F}_1 \simeq f^!(\mathcal{F}_2))$$

to

$$\mathrm{coFib}\left(h_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \to (f')_*^{\mathrm{IndCoh}}(\mathcal{F}'_1) \oplus (g_2)_*^{\mathrm{IndCoh}}(\mathcal{F}_2)\right).$$

We claim that the co-unit of the adjunction is an isomorphism. I.e., we claim that for $\mathcal{F} \in \mathrm{IndCoh}(X'_2)$, the map

$$\mathrm{coFib}\left(h_*^{\mathrm{IndCoh}} \circ h^!(\mathcal{F}) \to (f')_*^{\mathrm{IndCoh}} \circ (f')^!(\mathcal{F}) \oplus (g_2)_*^{\mathrm{IndCoh}} \circ g_2^!(\mathcal{F})\right) \to \mathcal{F} \tag{1.3}$$

is an isomorphism.

Note that in order to check this, it is enough to take $\mathcal{F} \in \mathrm{Coh}(X'_2)$, in which case, both sides in (1.3) belong to $\mathrm{IndCoh}(X'_2)^+$. Hence, it is enough to show that the map (1.3) becomes an isomorphism after applying the functor

$$\Gamma^{\mathrm{IndCoh}}(X'_2, -) : \mathrm{IndCoh}(X'_2) \to \mathrm{Vect}.$$

[2] We are again grateful to D. Nadler, who pointed out a gap in the proof of this assertion in an earlier version.

Denote

$$A_i = \Gamma(X_i, \mathcal{O}_{X_i}),\ A'_i = \Gamma(X'_i, \mathcal{O}_{X'_i}), \quad i = 1, 2.$$

Denote $\mathcal{M} := \Gamma^{\mathrm{IndCoh}}(X'_2, \mathcal{F})$. After applying $\Gamma^{\mathrm{IndCoh}}(X'_2, -)$ to the left-hand side in (1.3) we obtain

$$\mathrm{coFib}\left(\mathcal{M}aps_{A'_2\text{-mod}}(A_1, M) \to \mathcal{M}aps_{A'_2\text{-mod}}(A'_1, M) \oplus \mathcal{M}aps_{A'_2\text{-mod}}(A_2, M)\right),$$

and that maps isomorphically to M, since

$$A'_2 \to \mathrm{Fib}(A'_1 \oplus A_2 \to A_1)$$

is an isomorphism.

□

2. (pro)-cotangent and tangent spaces

In this section we define what it means for a prestack to admit a *(pro)-cotangent space* at a given S-point, where $S \in \mathrm{Sch}^{\mathrm{aff}}$. The definition is given in terms of the construction known as the *split zero extension.*

2.1. **Split square-zero extensions.** In this subsection we review the construction of split zero extensions; see [Lu1, Sect. 7.3.4].

2.1.1. Let S be an object of $\mathrm{Sch}^{\mathrm{aff}}$. There is a natural functor

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}$$

that assigns to $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$ the corresponding *split square-zero extension* $\mathrm{RealSplitSqZ}(\mathcal{F})$, also denoted by $S_{\mathcal{F}}$.

Namely, if $S = \mathrm{Spec}(A)$ and $\Gamma(S, \mathcal{F}) = M \in A\text{-mod}$,

$$S_{\mathcal{F}} = \mathrm{Spec}(A \oplus M).$$

2.1.2. One can show (see [Lu1, Theorem 7.3.4.13] or [Chapter IV.2, Proposition 1.8.3]) that the functor RealSplitSqZ defines an equivalence

$$\mathrm{QCoh}(S)^{\leq 0} \simeq \mathrm{ComMonoid}(((\mathrm{Sch}^{\mathrm{aff}})_{S/\,/S})^{\mathrm{op}}),$$

where $\mathrm{ComMonoid}(-)$ denotes the category of commutative monoids in a given $(\infty, 1)$-category, see [Chapter I.1, Sect. 3.3.3].

2.1.3. The following is nearly tautological:

Lemma 2.1.4. *The functor*

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}$$

commutes with colimits.

In addition, as in Lemma 1.3.3 one shows:

Lemma 2.1.5. *The composite functor*

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/} \to \mathrm{Sch}_{S/}$$

also commutes with colimits.

2.1.6. *Terminology.* In what follows, for an affine scheme S, we will also use the notation

$$\mathrm{SplitSqZ}(S) := (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}},$$

so that RealSplitSqZ is a functor

$$\mathrm{SplitSqZ}(S) \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}.$$

2.2. **The condition of admitting a (pro)-cotangent space at a point.** The condition that a given prestack admit a (pro)-cotangent space at a point means that it is *infinitesimally linearizable*, i.e., defines an exact (=excisive) functor on split square-zero extensions.

2.2.1. Let $\mathcal{X}$ be an arbitrary object of PreStk, and let (S, x) be an object of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.

We consider the functor $\mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}$, given by

$$\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0} \mapsto \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \in \mathrm{Spc}\,. \tag{2.1}$$

2.2.2. Let $\mathcal{F}_1 \to \mathcal{F}_2$ be a map in $\mathrm{QCoh}(S)^{\leq 0}$, such that $H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2)$ is a surjection. Set

$$\mathcal{F} := 0 \underset{\mathcal{F}_2}{\times} \mathcal{F}_1.$$

By assumption, $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$.

Note that by Lemma 2.1.4,

$$S \underset{S_{\mathcal{F}_2}}{\sqcup} S_{\mathcal{F}_1} \to S_{\mathcal{F}} \tag{2.2}$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$ (and, by Lemma 2.1.5 or Lemma 1.3.3(b), also in Sch).

Consider the corresponding map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}). \tag{2.3}$$

Definition 2.2.3. *Let $\mathcal{X}$ be an object of* PreStk. *We shall say that $\mathcal{X}$ admits a* pro-cotangent *space at the point x, if the map* (2.3) *is an isomorphism for all $\mathcal{F}_1 \to \mathcal{F}_2$ as above.*

2.2.4. For example, from Lemma 1.3.3(b) we obtain:

Corollary 2.2.5. *If $\mathcal{X} = X \in$ Sch, then X admits a* pro-cotangent *space at any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/X}$.*

2.2.6. Suppose that $\mathcal{X}$ admits a pro-cotangent space at x. Note that the functor (2.1) can be extended to a functor

$$\mathrm{QCoh}(S)^{-} \to \mathrm{Spc}, \tag{2.4}$$

by sending $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq k}$ to

$$\Omega^i(\mathrm{Maps}_{S/}(S_{\mathcal{F}[i]}, \mathcal{X}))$$

for $i \geq k$. The fact that this is well-defined is guaranteed by the isomorphism (2.3).

In addition, the isomorphism (2.3) implies that the functor (2.4) is exact. Hence, it is pro-corepresentable by an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{-})$. In what follows we shall denote this object by

$$T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{-})$$

and refer to it as the *pro-cotangent space to $\mathcal{X}$ at x.*

2.2.7. Let us recall (see [Lu1, Corollary 5.3.5.4]) that for any (accessible) $(\infty,1)$-category $\mathbf{C}$ with finite limits, the category $(\mathrm{Pro}(\mathbf{C}))^{\mathrm{op}}$ is the full subcategory of $\mathrm{Funct}(\mathbf{C},\mathrm{Spc})$ that consists of (accessible) functors that preserve finite limits.

Recall also (see [Chapter I.1, Sects. 7.2.1]) that if $\mathbf{C}$ is stable, we can identify this category with that of exact functors

$$\mathbf{C}\to \mathrm{Sptr},$$

by composing with the forgetful functor $\Omega^\infty : \mathrm{Sptr}\to\mathrm{Spc}$.

Finally, if $\mathbf{C}$ is a k-linear DG category (such in our case of interest $\mathrm{QCoh}(S)^-)$), we can identify it also with that of k-linear exact functors

$$\mathbf{C}\to \mathrm{Vect},$$

by composing with the Dold-Kan functor

$$\text{Dold-Kan}^{\mathrm{Sptr}} : \mathrm{Vect}\to\mathrm{Sptr},$$

see [Chapter I.1, Sect. 10.2].)

Given an object $\Phi\in\mathrm{Pro}(\mathbf{C})$, the corresponding functor $\mathbf{C}\to\mathrm{Vect}$ is explicitly given by

$$\mathcal{F}\mapsto \mathcal{M}aps_{\mathrm{Pro}(\mathbf{C})}(\Phi,\mathcal{F}),$$

where we regard $\mathrm{Pro}(\mathbf{C})$ also as a k-linear DG category, and $\mathbf{C}$ as its full subcategory.

2.2.8. Suppose that

$$\begin{array}{ccc} \mathcal{F}'_1 & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{F}'_2 & \longrightarrow & \mathcal{F}_2 \end{array} \tag{2.5}$$

is a pullback diagram *in the category* $\mathrm{QCoh}(S)$, where all objects belong to $\mathrm{QCoh}(S)^{\leq 0}$. I.e., we have $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}'_2\in\mathrm{QCoh}(S)^{\leq 0}$, and we require that

$$\mathcal{F}'_1 := \mathcal{F}_1\underset{\mathcal{F}_2}{\times}\mathcal{F}'_2$$

also belongs to $\mathrm{QCoh}(S)^{\leq 0}$.

Note that by Lemma 2.1.4,

$$\begin{array}{ccc} S_{\mathcal{F}'_1} & \longleftarrow & S_{\mathcal{F}_1} \\ \uparrow & & \uparrow \\ S_{\mathcal{F}'_2} & \longleftarrow & S_{\mathcal{F}_2} \end{array}$$

is a push-out diagram in $(\mathrm{Sch}^{\mathrm{aff}})_{S/}$.

For a given map $x:S\to\mathcal{X}$, consider the corresponding map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}'_1},\mathcal{X})\to \mathrm{Maps}_{S/}(S_{\mathcal{F}_1},\mathcal{X})\underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2},\mathcal{X})}{\times}\mathrm{Maps}_{S/}(S_{\mathcal{F}'_2},\mathcal{X}). \tag{2.6}$$

We have:

Lemma 2.2.9. *Suppose* $\mathcal{X}$ *admits a pro-cotangent space at* x. *Then* (2.6) *is an isomorphism.*

Proof. Follows from the commutation of $\mathcal{M}aps(T^*_x(\mathcal{X}),-)$ with finite limits. $\square$

2.2.10. We end this subsection with the following definition:

Definition 2.2.11. *Let $\mathcal{X}$ be an object of* PreStk. *We shall say that $\mathcal{X}$ admits a* cotangent *space at $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ if it admits a pro-cotangent space, and $T^*_x(\mathcal{X})$ belongs to*

$$\mathrm{QCoh}(S)^- \subset \mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

2.3. The condition of admitting (pro)-cotangent spaces.

2.3.1. We give the following definition:

Definition 2.3.2. *Let $\mathcal{X}$ be an object of* PreStk.

(a) *We shall say that $\mathcal{X}$ admits* pro-cotangent spaces*, if admits a pro-cotangent space for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.*

(b) *We shall say that $\mathcal{X}$ admits* cotangent spaces*, if admits a cotangent space for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.*

2.3.3. For example, Corollary 2.2.5 can be reformulated as saying that any scheme admits pro-cotangent spaces.

Remark 2.3.4. We shall soon see that every $X \in \mathrm{Sch}$ actually admits *cotangent* spaces, see Proposition 3.2.6.

2.3.5. Zariski gluing allows us to extend the construction of split square-zero extensions to schemes that are not necessarily affine. Thus, for $Z \in \mathrm{Sch}$, we obtain a well-defined functor:

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(Z)^{\leq 0})^{\mathrm{op}} \to \mathrm{Sch}_{Z/}, \quad \mathcal{F} \mapsto Z_{\mathcal{F}}.$$

Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology.

Fix a map $x : Z \to \mathcal{X}$. It follows formally that the functor

$$\mathrm{QCoh}(Z)^{\leq 0} \to \mathrm{Spc}, \quad \mathcal{F} \mapsto \mathrm{Maps}_{Z/}(Z_{\mathcal{F}}, \mathcal{X})$$

is pro-corepresentable by an object

$$T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-).$$

2.4. The relative situation.

The same definitions apply when we work over a fixed prestack $\mathcal{X}_0$.

2.4.1. For $\mathcal{X} \in \mathrm{PreStk}_{/\mathcal{X}_0}$ and $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, we shall say that $\mathcal{X}$ admits a pro-cotangent space at x relative to $\mathcal{X}_0$ if in the situation of Sect. 2.2.2, the diagram

$$\begin{array}{ccc} \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) & \longrightarrow & * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}_0) & \longrightarrow & * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X}_0)}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}_0) \end{array}$$

is a pullback square, i.e., if the fibers of the map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X})$$

map isomorphically to the fibers of the map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}_0) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X}_0)}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}_0).$$

2.4.2. If this condition holds, we will denote by

$$T^*_x(\mathcal{X}/\mathcal{X}_0) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

the object that pro-corepresents the functor

$$\mathcal{F} \mapsto \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}_0)}{\times} *.$$

2.4.3. Note that if in the above situation $\mathcal{X}_0$ admits a pro-cotangent space at $x_0 : S \to \mathcal{X} \to \mathcal{X}_0$, then $\mathcal{X}$ admits a pro-cotangent space at x if and only if $\mathcal{X}$ admits a pro-cotangent space at x relative to $\mathcal{X}_0$ and

$$T^*_x(\mathcal{X}/\mathcal{X}_0) \simeq \mathrm{coFib}(T^*_{x_0}(\mathcal{X}_0) \to T^*_x(\mathcal{X})).$$

2.4.4. The next assertion easily results from the definitions:

Lemma 2.4.5. *A prestack $\mathcal{X}$ admits pro-cotangent spaces relative to $\mathcal{X}_0$ if and only if for every $S_0 \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$, the prestack $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits pro-cotangent spaces.*

2.5. **Describing the pro-cotangent space as a limit.** In this subsection we will study (pro)-cotangent spaces of prestacks that are presented as colimits.

2.5.1. Let $\mathcal{X}$ be an object of PreStk, written as

$$\mathcal{X} = \underset{a \in A}{\mathrm{colim}}\, \mathcal{X}_a, \tag{2.7}$$

where the colimit is taken in PreStk.

Assume that each $\mathcal{X}_a$ admits pro-cotangent spaces. We wish to express the pro-cotangent spaces of $\mathcal{X}$ (if they exist) in terms of those of $\mathcal{X}_a$.

2.5.2. For $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, let $A_{x/}$ denote the category co-fibered over A, whose fiber over a given $a \in A$ is the space of factorizations of x as

$$S \xrightarrow{x_a} \mathcal{X}_a \to \mathcal{X}.$$

We claim:

Proposition 2.5.3. *Suppose that $T^*_x(\mathcal{X})$ exists and that the category $A_{x/}$ is* sifted[3]. *Then the natural map*

$$T^*_x(\mathcal{X}) \to \underset{(a, x_a) \in (A_{x/})^{\mathrm{op}}}{\lim} T^*_{x_a}(\mathcal{X}_a), \tag{2.8}$$

where the limit is taken in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$*, is an isomorphism.*

Proof. We need to show that for $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, the map

$$\underset{(a, x_a) \in A_{x/}}{\mathrm{colim}} \mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F}) \to \mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}), \tag{2.9}$$

where the colimit is taken in Vect, is an isomorphism.

Denote $V_a := \mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})$, $V := \mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F})$. We claim that it is enough to show that for every $n \in \mathbb{N}$, the resulting composite map

$$\underset{(a, x_a) \in A_{x/}}{\mathrm{colim}} \tau^{\leq n}(V_a) \to \tau^{\leq n}(\underset{(a, x_a) \in A_{x/}}{\mathrm{colim}} V_a) \to \tau^{\leq n}(V) \tag{2.10}$$

is an isomorphism.

[3] See [Lu1, Sect. 5.5.8] for what this means.

Indeed, if (2.10) is an isomorphism, then the map

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} V_a \simeq \underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \underset{n}{\operatorname{colim}}\, \tau^{\leq n}(V_a) \simeq \underset{n}{\operatorname{colim}} \underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \tau^{\leq n}(V_a) \to \underset{n}{\operatorname{colim}}\, \tau^{\leq n}(V) \simeq V$$

is an isomorphism as well.

We note that the composite map in (2.10) can be interpreted as the shift by $[-n]$ of the map

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F}[n])\right) \to \tau^{\leq 0}\left(\mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}[n])\right) \tag{2.11}$$

in $\mathrm{Vect}^{\leq 0}$. So, it suffices to show that the map (2.11) is an isomorphism.

Now, using the assumption that $A_{x/}$ is sifted and the fact that the functor

$$\text{Dold-Kan} : \mathrm{Vect}^{\leq 0} \to \mathrm{Spc}$$

commutes with sifted colimits (see [Chapter I.1, Sect. 10.2.3]), when we apply it to (2.11), we obtain the map

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \mathrm{Maps}_{S/}(S_{\mathcal{F}[n]}, \mathcal{X}_a) \to \mathrm{Maps}_{S/}(S_{\mathcal{F}[n]}, \mathcal{X}). \tag{2.12}$$

Thus, since Dold-Kan is conservative, we obtain that it suffices to show that (2.12) is an isomorphism.

Hence, it remains to show that for a $S' \in \mathrm{Sch}^{\mathrm{aff}}_{S/}$, the map

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \mathrm{Maps}_{S/}(S', \mathcal{X}_a) \to \mathrm{Maps}_{S/}(S', \mathcal{X})$$

is an isomorphism. However, this follows from the isomorphism (2.7). □

2.5.4. We now claim:

Lemma 2.5.5. *Suppose that in the situation of* (2.7)*, the category A is filtered. Then $\mathcal{X}$ admits pro-cotangent spaces, and there is a canonical isomorphism*

$$T^*_x(\mathcal{X}) \to \underset{(a,x_a)\in (A_{x/})^{\mathrm{op}}}{\lim} T^*_{x_a}(\mathcal{X}_a).$$

Proof. As in (2.11), we have an identification

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \simeq \underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \text{Dold-Kan}\left(\tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right),$$

functorial in $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$.

First, we note that the filteredness assumption on A implies that all the categories $A_{x/}$ are filtered and in particular sifted. Hence,

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \text{Dold-Kan}\left(\tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right) \simeq$$

$$\simeq \text{Dold-Kan}\left(\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right).$$

Since $A_{x/}$ is filtered, the functor $\tau^{\leq 0} : \mathrm{Vect} \to \mathrm{Vect}^{\leq 0}$ commutes with colimits along $A_{x/}$, and we obtain:

$$\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right) \simeq \tau^{\leq 0}\left(\underset{(a,x_a)\in A_{x/}}{\operatorname{colim}} \mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right).$$

This implies the assertion of the lemma.

□

3. Properties of (pro)-cotangent spaces

By definition, the (pro)-cotangent space of a prestack at a given S-point is an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$. One can impose the condition that the (pro)-cotangent space belong to a given subcategory of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$, and obtain more restricted infinitesimal behavior. In this section, we will study various such conditions.

3.1. **Connectivity conditions.** The first type of condition is obtained by requiring that the (pro)-cotangent space be bounded above.

3.1.1. We start with the following observation: let $\mathbf{C}$ be a stable $(\infty,1)$-category, in which case the category $\mathrm{Pro}(\mathbf{C})$ is also stable [4].

Assume now that $\mathbf{C}$ is endowed with a t-structure. In this case $\mathrm{Pro}(\mathbf{C})$ also inherits a t-structure, so that its connective subcategory $\mathrm{Pro}(\mathbf{C})^{\leq 0}$ consists of those left-exact[5] functors

$$\mathbf{C} \to \mathrm{Spc}$$

that map $\mathbf{C}^{>0}$ to $* \in \mathrm{Spc}$.

Equivalently, if we interpret objects of $\mathrm{Pro}(\mathbf{C})$ as exact functors

$$\mathbf{C} \to \mathrm{Sptr},$$

the subcategory $\mathrm{Pro}(\mathbf{C})^{\leq 0}$ consists of those functors that send $\mathbf{C}^{\geq 0}$ to the subcategory $\mathrm{Sptr}^{\geq 0} \subset \mathrm{Sptr}$.

Clearly,

$$\mathbf{C} \cap \mathrm{Pro}(\mathbf{C})^{\leq n} = \mathbf{C}^{\leq n},$$

as subcategories of $\mathbf{C}$.

3.1.2. Restriction along $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$ defines a functor

$$\mathrm{Pro}(\mathbf{C})^{\leq n} \to \mathrm{Pro}(\mathbf{C}^{\leq n}), \tag{3.1}$$

Lemma 3.1.3. *The functor* (3.1) *is an equivalence.*

Similarly, for any $m \leq n$, the natural functor

$$\mathrm{Pro}(\mathbf{C})^{\geq m, \leq n} \to \mathrm{Pro}(\mathbf{C}^{\geq m, \leq n})$$

is an equivalence.

In what follows we shall denote by $\mathrm{Pro}(\mathbf{C})_{\text{event-conn}}$ the full subcategory of $\mathrm{Pro}(\mathbf{C})$ equal to $\underset{n}{\cup}\, \mathrm{Pro}(\mathbf{C})^{\leq n}$. I.e., $\mathrm{Pro}(\mathbf{C})_{\text{event-conn}}$ is the same thing as $\mathrm{Pro}(\mathbf{C})^-$.

---

[4] Note, however, that even if $\mathbf{C}$ is presentable, the category $\mathrm{Pro}(\mathbf{C})$ is not, so caution is required when applying such results as the adjoint functor theorem.

[5] We recall that a functor is said to be left-exact if it commutes with finite limits. This notion has nothing to do with t-structures.

3.1.4. We give the following definitions:

Definition 3.1.5. *Let $\mathcal{X}$ be an object of* PreStk.

(a) *We shall say that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent (resp., cotangent) space at x if it admits a pro-cotangent (resp., cotangent) space at x and $T^*_x(\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^{\leq n})$.*

(a') *We shall say that $\mathcal{X}$ admits an eventually connective pro-cotangent space at x if it admits a $(-n)$-connective pro-cotangent space at x for some n.*

(b) *We shall that $\mathcal{X}$ admits $(-n)$-connective pro-cotangent (resp., cotangent) spaces, if it admits a $(-n)$-connective pro-cotangent (resp., cotangent) space for every $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$.*

(b') *We shall that $\mathcal{X}$ admits locally eventually connective pro-cotangent spaces, if it admits an eventually connective pro-cotangent space for every $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$.*

(c) *We shall that $\mathcal{X}$ admits uniformly eventually connective pro-cotangent (resp., cotangent) spaces, if there exists an integer $n \in \mathbb{Z}$ such that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent (resp., cotangent) space for every $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$.*

3.1.6. Tautologically, if $\mathcal{X}$ admits a pro-cotangent space at x, then this pro-cotangent space is $(-n)$-connective if and only if for some/any $i \geq 0$ and $\mathcal{F} \in \text{QCoh}(S)^{\geq -i, \leq 0}$, the space

$$\text{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X})$$

is $(n + i)$-truncated.

3.1.7. Let us consider separately the case when $n = 0$ (in this case, we shall say 'connective' instead of '0-connective'). Almost tautologically, we have:

Lemma 3.1.8. *A prestack $\mathcal{X}$ admits a connective pro-cotangent space at $x : S \to \mathcal{X}$ if and only if the functor* (2.1) *commutes with finite limits (equivalently, takes pullbacks to pullbacks).*

Remark 3.1.9. The point of Lemma 3.1.8 is that the condition of admitting a connective pro-cotangent space is stronger than that of just admitting a pro-cotangent space: the former requires that functor (2.1) take any pullback square in $\text{QCoh}(S)^{\leq 0}$ to a pullback square, while the latter does so only for those pullback squares in $\text{QCoh}(S)^{\leq 0}$ that stay pullback squares in all of $\text{QCoh}(S)$.

3.1.10. From Lemma 1.3.3(b) we obtain:

Corollary 3.1.11. *Every $\mathcal{X} = X \in \text{Sch}$ admits connective pro-cotangent spaces.*

3.2. **Pro-cotangent vs cotangent.** Assume that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent space at x. We wish to give a criterion for when $\mathcal{X}$ admits a cotangent space at x. In this case, $T^*_x(\mathcal{X})$ would be an object of $\text{QCoh}(S)^{\leq n}$.

3.2.1. We have:

Lemma 3.2.2. *Let $\mathbf{C}$ be as in Sect. 3.1.1, and let $\mathbf{c}$ be an object of $\text{Pro}(\mathbf{C}^{\leq n})$. Assume that $\mathbf{C}$ contains filtered limits and retracts. Then $\mathbf{c}$ belongs to $\mathbf{C}^{\leq n}$ if and only if the corresponding functor $\mathbf{C}^{\leq n} \to \text{Spc}$ commutes with filtered limits.*

3.2.3. From the lemma we obtain:

Corollary 3.2.4. *If $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent space at x, then it admits a cotangent space at x if and only if the functor* (2.1) *commutes with filtered limits.*

3.2.5. We now claim:

Proposition 3.2.6. *Any* $\mathcal{X} = X \in \mathrm{Sch}$ *admits connective cotangent spaces.*

Proof. According to Corollaries 3.1.11, we only need to show that the composite functor

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/} \to \mathrm{Sch}_{S/} \to \mathrm{Sch}$$

commutes with filtered colimits. However, taking into account Lemma 2.1.5, this follows from the fact that the forgetful functor $\mathrm{Sch}_{S/} \to \mathrm{Sch}$ commutes with colimits *indexed by any contractible category.*

□

3.3. **The convergence condition.** The convergence condition says that the value of the (pro)-cotangent space on a given $\mathcal{F} \in \mathrm{QCoh}(S)^{-}$ is determined by the cohomological truncations $\tau^{\geq -n}(\mathcal{F})$. It is the infinitesimal version of the condition of convergence on a prestack itself.

3.3.1. For $S \in \mathrm{Sch}^{\mathrm{aff}}$, we let

$${}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^{-}) \subset \mathrm{Pro}(\mathrm{QCoh}(S)^{-})$$

denote the full subcategory spanned by objects Φ that satisfy the following convergence condition:

We require that when $\Phi \in \mathrm{Pro}(\mathrm{QCoh}(S)^{-})$ is viewed as a functor $\mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}$, then for any $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, the map

$$\Phi(\mathcal{F}) \to \lim_{n} \Phi(\tau^{\geq -n}(\mathcal{F}))$$

be an isomorphism.

We note the analogy between this definition and the notion of convergence for objects of PreStk, see [Chapter I.2, Sect. 1.4].

3.3.2. The following is nearly tautological:

Lemma 3.3.3. *Suppose that* $\mathcal{X} \in \mathrm{PreStk}$ *is convergent, and suppose that it admits a pro-cotangent space at* $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. *Then* $T^*_x(\mathcal{X})$ *belongs to* ${}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^{-})$.

In addition, we have:

Lemma 3.3.4. *Suppose that* $\mathcal{X} \in \mathrm{PreStk}$ *is convergent. Then in order to test whether* $\mathcal{X}$ *admits pro-cotangent spaces (resp.,* $(-n)$*-connective pro-cotangent spaces), it is enough to do so for* (S, x) *with* S *eventually coconnective and check that* (2.3) *is an isomorphism for* $\mathcal{F}_i \in \mathrm{QCoh}(S)^{>-\infty, \leq 0}$, $i = 1, 2$.

Similarly, we have the following extension of Lemma 2.4.5:

Lemma 3.3.5. *Let* $\pi : \mathcal{X} \to \mathcal{X}_0$ *be a morphism in* ${}^{\mathrm{conv}}\mathrm{PreStk}$. *Then in order to check that* $\mathcal{X}$ *admits pro-cotangent spaces* relative *to* $\mathcal{X}_0$, *it sufficient to check that for every* $S_0 \in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$, *the fiber product* $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ *admits pro-cotangent spaces.*

3.4. **The almost finite type condition.** In this subsection we introduce another condition on an object of $\mathrm{Pro}(\mathrm{QCoh}(X)^{-})$, namely, that it be 'almost of finite type'.

3.4.1. For a scheme X, let

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

denote the full subcategory consisting of objects Φ satisfying the following two conditions:

(1) $\Phi \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(X)^-)$;
(2) For every $m \geq 0$, the resulting functor $\Phi : \mathrm{QCoh}(X)^{\geq -m, \leq 0} \to \mathrm{Spc}$ commutes with filtered colimits.

We note the analogy between the above definition and the corresponding definition for prestacks, see [Chapter I.2, Sect. 1.7].

3.4.2. From now until the end of this subsection we will assume that $X \in \mathrm{Sch}_{\mathrm{aft}}$. In particular, we have a well-defined (non-cocomplete) DG subcategory

$$\mathrm{Coh}(X) \subset \mathrm{QCoh}(X).$$

3.4.3. Here is a more explicit interpretation of Condition (2) in Sect. 3.4.1 in the eventually connective case.

Let X be an object of $\mathrm{Sch}_{\mathrm{aft}}$, and let Φ be an object of $\mathrm{Pro}(\mathrm{QCoh}(X)^{\leq n})$ for some n. We have:

Lemma 3.4.4. *The following conditions are equivalent:*
(a) *For every $m \geq 0$, the functor $\Phi : \mathrm{QCoh}(X)^{\geq -m, \leq 0} \to \mathrm{Spc}$ commutes with filtered colimits.*
(b) *For every $m \geq 0$, the truncation $\tau^{\geq -m}(\Phi)$ belongs to the full subcategory*

$$\mathrm{Pro}(\mathrm{Coh}(X)^{\geq -m, \leq n}) \subset \mathrm{Pro}(\mathrm{QCoh}(X)^{\geq -m, \leq n}).$$

(c) *The cohomologies of Φ belong to $\mathrm{Pro}(\mathrm{Coh}(X)^{\heartsuit}) \subset \mathrm{Pro}(\mathrm{QCoh}(X)^{\heartsuit})$.*

3.4.5. Note that restriction along $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$ defines a functor

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-) \to \mathrm{Pro}(\mathrm{Coh}(X)). \tag{3.2}$$

We claim:

Proposition 3.4.6. *The functor* (3.2) *defines an equivaence*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \to \mathrm{Pro}(\mathrm{Coh}(X)).$$

Remark 3.4.7. Note the analogy between this proposition and the corresponding assertion in [Chapter I.2, Proposition 1.7.6].

Proof of Proposition 3.4.6. We construct the inverse functor as follows.

Given $\widetilde{\Phi} \in \mathrm{Pro}(\mathrm{Coh}(X))$, viewed as a functor

$$\mathrm{Coh}(X)^{\leq 0} \to \mathrm{Spc},$$

we construct a functor

$$\Phi^b : \mathrm{QCoh}(X)^{> -\infty, \leq 0} \to \mathrm{Spc},$$

as the *left Kan extension* of $\widetilde{\Phi}$ under

$$\mathrm{Coh}(X)^{\leq 0} \hookrightarrow \mathrm{QCoh}(X)^{> -\infty, \leq 0}.$$

We define the sought-for functor $\Phi : \mathrm{QCoh}(X)^{\leq 0} \to \mathrm{Spc}$ as the *right Kan extension* of Φ^b under

$$\mathrm{QCoh}(X)^{> -\infty, \leq 0} \hookrightarrow \mathrm{QCoh}(X)^{\leq 0}.$$

Explicitly,

$$\Phi(\mathcal{F}) = \lim_m \Phi^b(\tau^{\geq -m}(\mathcal{F})).$$

It is easy to check that the construction $\widetilde{\Phi} \mapsto \Phi$ is the inverse to (3.2). □

Corollary 3.4.8. *For $X \in \mathrm{Sch}_{\mathrm{aft}}$ there exists a canonical equivalence*

$$\left(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}}\right)^{\mathrm{op}} \simeq \mathrm{IndCoh}(X).$$

Proof. Follows from the canonical equivalence between $(\mathrm{Pro}(\mathrm{Coh}(X)))^{\mathrm{op}}$ and $\mathrm{IndCoh}(X)$ given by Serre duality

$$\mathbb{D}_S^{\mathrm{Serre}} : (\mathrm{Coh}(X))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Coh}(X),$$

see [Chapter II.2, Sect. 4.2.10]. □

The following results from the construction:

Lemma 3.4.9.
(a) *Under the equivalence of Corollary 3.4.8, the full subcategory of* $(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}})^{\mathrm{op}}$ *corresponding to*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \cap \mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\text{event-conn}} \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

maps onto $\mathrm{IndCoh}(X)^+ \subset \mathrm{IndCoh}(X)$.
(b) *Under the equivalence of Corollary 3.4.8, the full subcategory of* $(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}})^{\mathrm{op}}$ *corresponding to*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \cap \mathrm{QCoh}(X)^- \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

maps onto the full subcategory of $\mathrm{IndCoh}(X)^+$, *consisting of objects with* coherent *cohomologies.*

3.5. **Prestacks locally almost of finite type.** In this subsection, we will study what the 'locally almost of finite type' condition on a prestack implies about its (pro)-cotangent spaces.

3.5.1. The definition of the subcategory

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

implies:

Lemma 3.5.2. *Suppose that $\mathcal{X} \in \mathrm{PreStk}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, and suppose that it admits a pro-cotangent space at $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. Then $T_x^*(\mathcal{X})$ belongs to*

$$\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

Moreover, we have:

Lemma 3.5.3. *Suppose that $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$. Then the condition on $\mathcal{X}$ to have pro-cotangent spaces is enough to check on (S, x) with $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $\mathcal{F}_i \in \mathrm{Coh}(S)^{\leq 0}$, $i = 1, 2$.*

We also have following extension of Lemma 2.4.5:

Lemma 3.5.4. *Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in $\mathrm{PreStk}_{\mathrm{laft}}$. Then in order to check that $\mathcal{X}$ admits pro-cotangent spaces* relative *to $\mathcal{X}_0$, it sufficient to check that for every $S_0 \in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}_0}$, the fiber product $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits pro-cotangent spaces.*

3.5.5. Suppose that $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{laft}}$, and let $(x : S \to X) \in (\mathrm{Sch}^{\mathrm{aff}})_{/X}$ with $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. Consider the object

$$T^*_x(X) \in \mathrm{QCoh}(S)^{\leq 0}.$$

From Lemma 3.4.4 we obtain:

Corollary 3.5.6. *The object object $T^*_x(X)$ has coherent cohomologies.*

3.5.7. *The tangent space.* Using Corollary 3.4.8 and Lemma 3.5.2, we obtain that if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ admits a pro-cotangent space at x for $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$, then it admits a well-defined *tangent* space

$$T_x(\mathcal{X}) \in \mathrm{IndCoh}(S).$$

Namely, we let $T_x(\mathcal{X})$ be the object of $\mathrm{IndCoh}(S)$ corresponding to $T^*_x(\mathcal{X})$ via the contravariant equivalence of Corollary 3.4.8.

4. The (pro)-cotangent complex

A prestack admits a *pro-cotangent complex* if it admits pro-cotangent spaces that are compatible under the operation of pullback. We will study this notion in this section.

4.1. **Functoriality of (pro)-cotangent spaces.** In this section we define what it means for a prestack to admit a (pro)-cotangent complex. We reformulate this definition as compatibility with a certain type of push-outs.

4.1.1. Let $f : S_1 \to S_2$ be a map of affine schemes. Consider the functor

$$f^* : \mathrm{QCoh}(S_2)^- \to \mathrm{QCoh}(S_1)^-,$$

and let $\mathrm{Pro}(f^*)$ denote the resulting functor

$$\mathrm{Pro}(\mathrm{QCoh}(S_2)^-) \to \mathrm{Pro}(\mathrm{QCoh}(S_1)^-).$$

Note that we when regard $\mathrm{Pro}(\mathrm{QCoh}(S_i)^-)$ as a full subcategory of (the opposite of)

$$\mathrm{Funct}(\mathrm{QCoh}(S_i)^{\leq 0}, \mathrm{Spc}),$$

the functor $\mathrm{Pro}(f^*)$ is induced by the functor

$$\mathrm{LKE}_{f^*} : \mathrm{Funct}(\mathrm{QCoh}(S_2)^{\leq 0}, \mathrm{Spc}) \to \mathrm{Funct}(\mathrm{QCoh}(S_1)^{\leq 0}, \mathrm{Spc}).$$

Even more explicitly, for $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\leq 0}$ and $\Phi_2 \in \mathrm{Pro}(\mathrm{QCoh}(S_2)^-)$, we have:

$$((\mathrm{Pro}(f^*)(\Phi_2)))(\mathcal{F}_1) = \Phi_2(f_*(\mathcal{F}_1)). \tag{4.1}$$

4.1.2. Note also (see [Lu1, Theorem 7.3.4.18]) that we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{QCoh}(S_1)^{\leq 0} & \xrightarrow{f_*} & \mathrm{QCoh}(S_2)^{\leq 0} \\ {\scriptstyle \mathrm{RealSplitSqZ}}\downarrow & & \downarrow{\scriptstyle \mathrm{RealSplitSqZ}} \\ (\mathrm{Sch}^{\mathrm{aff}})_{S_1/} & \longrightarrow & (\mathrm{Sch}^{\mathrm{aff}})_{S_2/}, \end{array} \tag{4.2}$$

where the bottom horizontal arrow is given by push-out.

4.1.3. Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces. Let $f : S_1 \to S_2$ be a map of affine schemes. Let $x_2 : S_2 \to \mathcal{X}$ and denote $x_1 := x_2 \circ f$.

From (4.2), for $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\leq 0}$, we obtain a canonically defined map

$$\mathrm{Maps}_{S_2/}((S_2)_{f_*(\mathcal{F}_1)}, \mathcal{X}) \to \mathrm{Maps}_{S_1/}((S_1)_{\mathcal{F}_1}, \mathcal{X}), \tag{4.3}$$

which depends functorially on $\mathcal{F}_1$.

We can interpret the map (4.3) as a map

$$T^*_{x_1}(\mathcal{X}) \to \mathrm{Pro}(f^*)(T^*_{x_2}(\mathcal{X})) \tag{4.4}$$

in $\mathrm{Pro}(\mathrm{QCoh}(S_1)^-)$.

Definition 4.1.4. *We shall say that $\mathcal{X}$ admits a* pro-cotangent complex *if it admits pro-cotangent spaces* and *the map* (4.4) *is an isomorphism for any $(S_2, x_2 : S_2 \to \mathcal{X})$ and f as above.*

4.1.5. Equivalently, $\mathcal{X}$ admits a *pro-cotangent complex* if it admits pro-cotangent spaces and the map (4.3) is an isomorphism for any $(S_2, x_2 : S_2 \to \mathcal{X})$, f and $\mathcal{F}_1$ as above.

Still equivalently, from (4.2), we obtain that $\mathcal{X}$ admits a *pro-cotangent complex* it admits pro-cotangent spaces and takes push-outs of the form

$$(S_1)_{\mathcal{F}_1} \underset{S_1}{\sqcup} S_2,$$

where $(S_1)_{\mathcal{F}_1}$ is a split square-zero extension of S_1, to pullbacks in Spc.

Remark 4.1.6. Note that both the condition of admitting pro-cotangent spaces and a pro-cotangent complex are expressed as the property of taking certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc.

4.1.7. *The cotangent complex.* We give the following definition:

Definition 4.1.8. *We shall say that $\mathcal{X}$ admits a* cotangent complex *it admits cotangent spaces and a pro-cotangent complex.*

In other words, we require that for every (S, x), the object $T^*_x(\mathcal{X})$ belong to $\mathrm{QCoh}(S)^-$, and that for a map $f : S_1 \to S_2$, the resulting canonical map

$$T^*_{x_1}(\mathcal{X}) \to f^*(T^*_{x_2}(\mathcal{X}))$$

be an isomorphism in $\mathrm{QCoh}(S_1)^-$.

Thus, if $\mathcal{X}$ admits a cotangent complex, the assignment

$$(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}} \rightsquigarrow T^*_x(\mathcal{X}) \in \mathrm{QCoh}(S)$$

defines an object of $\mathrm{QCoh}(\mathcal{X})$, which we shall denote by $T^*(\mathcal{X})$ and refer to as the *cotangent complex* of $\mathcal{X}$.

4.1.9. Let $Z \in \mathrm{Sch}$, regarded as a prestack. We already know that Z admits cotangent spaces. Moreover, from Lemma 1.3.3(b), it follows that the maps (4.3) are isomorphisms. Hence, we obtain that Z admits a cotangent complex.

4.1.10. *The relative situation.* The same definitions apply in the relative situation, when we consider prestacks and affine schemes over a given prestack $\mathcal{X}_0$.

The analog of Lemma 2.4.5 holds when we replace 'cotangent spaces' by 'cotangent complex'.

4.2. **Conditions on the (pro)-cotangent complex.** In this subsection we introduce various conditions that one can impose on the (pro)-cotangent complex of a prestack.

4.2.1. *Connectivity conditions.*

Definition 4.2.2.

(a) *We shall say that $\mathcal{X}$ admits an $(-n)$-connective pro-cotangent complex (resp., cotangent complex) if it admits $(-n)$-connective pro-cotangent spaces (resp., cotangent spaces) and a pro-cotangent complex.*

(b) *We shall say that $\mathcal{X}$ admits a locally eventually connective pro-cotangent complex if it admits a pro-cotangent complex and its pro-cotangent spaces are eventually connective.*

(c) *We shall say that $\mathcal{X}$ admits a uniformly eventually connective pro-cotangent complex (resp., cotangent complex) if there exists an integer n such that $\mathcal{X}$ admits an $(-n)$-connective pro-cotangent complex (resp., cotangent complex).*

For example, we obtain that any $X \in \mathrm{Sch}$, regarded as an object of PreStk, admits a connective cotangent complex.

4.2.3. *The (pro)-cotangent complex in the convergent/finite type case.* Suppose now that $\mathcal{X}$ is convergent (resp., belongs to $\mathrm{PreStk}_{\mathrm{laft}}$). By Lemma 3.3.4 (resp., Lemma 3.5.3), the condition that $\mathcal{X}$ admit pro-cotangent spaces is sufficient to test on affine schemes that are eventually coconnective (resp., eventually coconnective and of finite type).

Similarly, we have:

Lemma 4.2.4.

(a) *Assume that $\mathcal{X}$ is convergent. Then $\mathcal{X}$ admits a pro-cotangent complex if and only if it admits pro-cotangent spaces, and the map (4.3) is an isomorphism for $S_1, S_2 \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{>-\infty,\leq 0}$.*

(b) *Assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. Then $\mathcal{X}$ admits a pro-cotangent complex if and only if it admits pro-cotangent spaces, and for any map $f : S_1 \to S_2$ in $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}$ and $\mathcal{F}_1 \in \mathrm{Coh}(S_1)^{\leq 0}$, the map*

$$\underset{\mathcal{F}_2 \in \mathrm{Coh}(S_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1}{\mathrm{colim}} \mathrm{Maps}_{S_2/}((S_2)_{\mathcal{F}_2}, \mathcal{X}) \to \mathrm{Maps}_{S_1/}((S_1)_{\mathcal{F}_1}, \mathcal{X})$$

is an isomorphism in Spc.

In addition:

Lemma 4.2.5. *Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in ${}^{\mathrm{conv}}\mathrm{PreStk}$ (resp., $\mathrm{PreStk}_{\mathrm{laft}}$). Then in order to check that $\mathcal{X}$ admits a pro-cotangent complex relative to $\mathcal{X}_0$, it is sufficient to check that for any $S_0 \in {}^{<\infty}(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$ (resp., $S_0 \in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}_0}$), the fiber product $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits a pro-cotangent complex.*

4.2.6. *Cotangent vs (pro)-cotangent.* We observe the following:

Proposition 4.2.7. *Let $\mathcal{X}$ be convergent (resp., locally almost of finite type) and admit a locally eventually connective pro-cotangent complex. Suppose that $\mathcal{X}$ admits cotangent spaces for all $S \to \mathcal{X}$ with $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ (resp., $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$). Then $\mathcal{X}$ admits a cotangent complex.*

Proof. First, we note that the assertion in the locally almost of finite type case follows formally from that in the convergent case.

To prove the latter we need to show the following. Let $\mathcal{T}$ be an object of $\text{Pro}(\text{QCoh}(S)^{\leq 0})$, such that for every truncation $i_n : {}^{\leq n}S \to S$, we have

$$(\text{Pro}(i_n^*))(\mathcal{T}) \in \text{QCoh}({}^{\leq n}S)^{\leq 0}.$$

Then $\mathcal{T} \in \text{QCoh}(S)^{\leq 0}$.

This follows from the next general observation (which is a particular case of [Chapter I.3, Proposition 3.6.10]):

Lemma 4.2.8. *The functors $\{i_n^*\}$ define an equivalence*

$$\text{QCoh}(S)^{\leq 0} \to \lim_n \text{QCoh}({}^{\leq n}S)^{\leq 0}.$$

□

4.3. **The pro-cotangent complex as an object of a category.** In this subsection we will show that for a prestack $\mathcal{X}$ locally almost of finite type that admits a (pro)-cotangent complex, there exists a *tangent* complex, which is naturally an object of $\text{IndCoh}(\mathcal{X})$.

4.3.1. Let $\mathcal{X}$ be a prestack. We define the category

$$\text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}}$$

as

$$\lim_{(S,x)\in(\text{Sch}^{\text{aff}})_{/\mathcal{X}}} \text{Pro}(\text{QCoh}(S)^-).$$

Let us emphasize that $\text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}}$ is *not* the same as $\text{Pro}(\text{QCoh}(\mathcal{X})^-)$ where the latter is the pro-completion of the category

$$\text{QCoh}(\mathcal{X})^- := \underset{n}{\cup} \text{QCoh}(\mathcal{X})^{\leq n}.$$

We have a fully faithful embedding

$$\text{QCoh}(\mathcal{X})^- \to \text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}},$$

given by $\text{QCoh}(S)^- \hookrightarrow \text{Pro}(\text{QCoh}(S)^-)$ for every $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$.

4.3.2. By definition, if $\mathcal{X}$ admits a pro-cotangent complex, then we have a well-defined object

$$T^*(\mathcal{X}) \in \text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}},$$

whose value on every $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$ is $T_x^*(\mathcal{X}) \in \text{Pro}(\text{QCoh}(S)^-)$.

4.3.3. Let

$${}^{\text{conv}}\text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}} \subset \text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}}$$

be the full subcategory equal to

$$\lim_{(S,x)\in(\text{Sch}^{\text{aff}})_{/\mathcal{X}}} {}^{\text{conv}}\text{Pro}(\text{QCoh}(S)^-).$$

We note:

Lemma 4.3.4. *Assume that $\mathcal{X}$ is convergent. Then the restriction functor*

$$ {}^{\text{conv}}\text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}} = \lim_{(S,x)\in(\text{Sch}^{\text{aff}})_{/\mathcal{X}}} {}^{\text{conv}}\text{Pro}(\text{QCoh}(S)^-) \to$$

$$\to \lim_{(S,x)\in({}^{<\infty}\text{Sch}^{\text{aff}})_{/\mathcal{X}}} {}^{\text{conv}}\text{Pro}(\text{QCoh}(S)^-)$$

is an equivalence.

4.3.5. By Lemma 3.3.3, if $\mathcal{X}$ is convergent and admits a pro-cotangent complex, we have

$$T^*(\mathcal{X}) \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}.$$

4.3.6. Assume now that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. By Lemma 4.3.4, we can rewrite

$$ {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}} \simeq \lim_{(S,x)\in({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-). \tag{4.5}$$

Let

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}} \subset {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$$

be the full subcategory equal, in terms of (4.5), to

$$\lim_{(S,x)\in({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \subset \lim_{(S,x)\in({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

By Lemma 3.5.2, obtain that if $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$ and admits a pro-cotangent complex, we have

$$T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}.$$

4.4. **The tangent complex.**

4.4.1. Assume again that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. By Corollary 3.4.8 and the convergence property of IndCoh (see [Chapter II.2, Sect. 3.4.1]), we obtain:

Corollary 4.4.2. *There exists a canonically defined equivalence*

$$\left(\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}\right)^{\mathrm{op}} \simeq \mathrm{IndCoh}(\mathcal{X}).$$

4.4.3. Assume now that admits a pro-cotangent complex. We obtain that there exists a canonically defined object

$$T(\mathcal{X}) \in \mathrm{IndCoh}(\mathcal{X}),$$

which is obtained from $T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}$ via the equivalence of Corollary 4.4.2 above.

Concretely, $T(\mathcal{X})$ is given by the assignment

$$(S,x) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}} \rightsquigarrow T_x(\mathcal{X})$$

(see Sect. 3.5.7 for the notation $T_x(\mathcal{X})$).

We shall refer to $T(\mathcal{X})$ as the *tangent complex* of $\mathcal{X}$.

4.5. **The (co)differential.** We will now introduce another basic structure associated with the pro-cotangent complex, namely, the co-differential map.

4.5.1. Let S be an object of $\mathrm{Sch}^{\mathrm{aff}}$. By the above, we have a canonical object

$$T^*(S) \in \mathrm{QCoh}(S).$$

We claim that there is a canonical map of schemes under S:

$$\mathfrak{d} : S_{T^*(S)} \to S, \tag{4.6}$$

where $T^*(S)$ is regarded as an object of $\mathrm{QCoh}(S)^{\leq 0}$.

Indeed, the map $\mathfrak{d}$ corresponds to the identity map on the left-hand side in the isomorphism

$$\mathrm{Maps}(T^*(S), T^*(S)) \simeq \mathrm{Maps}_{S/}(S_{T^*(S)}, S),$$

where we take the target prestack $\mathcal{X}$ to be S, and the map $x : S \to \mathcal{X}$ to be the identity map.

4.5.2. Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces, and let $x : S \to \mathcal{X}$ be a map. We claim that there is a canonical map in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

$$(dx)^* : T^*_x(\mathcal{X}) \to T^*(S).$$

The map $(dx)^*$ corresponds via the isomorphism

$$\mathrm{Maps}(T^*_x(\mathcal{X}), T^*(S)) \simeq \mathrm{Maps}_{S/}(S_{T^*(S)}, \mathcal{X}),$$

to the map

$$S_{T^*(S)} \xrightarrow{\mathfrak{d}} S \xrightarrow{x} \mathcal{X}.$$

We shall refer to $(dx)^*$ as the *codifferential* of x.

4.5.3. Assume for a moment that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ and $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. In this case $(dx)^*$ corresponds to a canonically defined map in $\mathrm{IndCoh}(Z)$,

$$dx : T(S) \to T_x(\mathcal{X}),$$

which we shall refer to as the *differential* of x.

4.5.4. Finally, let us note that the construction of the map $\mathfrak{d}$ is local in the Zariski topology. Hence, we obtain that it is well-defined for any $X \in \mathrm{Sch}$, which is not necessarily affine:

$$\mathfrak{d} : X_{T^*(X)} \to X,$$

4.6. **The value of the (pro)-cotangent complex on a non-affine scheme.** In this subsection we will study the pullback of the (pro)-cotangent complex of a prestack to a non-affine scheme.

4.6.1. Let Z be a scheme.

$$Z = \underset{a \in A}{\mathrm{colim}}\, U_a \tag{4.7}$$

where $U_a \in \mathrm{Sch}^{\mathrm{aff}}$, the maps $U_\alpha \to Z$ are open embeddings, and where the colimit is taken in Sch.

We have a pair of mutually adjoint functors

$$\lim_{a \in A} \mathrm{Pro}(\mathrm{QCoh}(U_a)^-) \rightleftarrows \mathrm{Pro}(\mathrm{QCoh}(Z)^-), \tag{4.8}$$

where the functor $\leftarrow$ is given by left Kan extension along each j^*_a, and the functor $\to$ sends a compatible family

$$\{\Phi_a \in \mathrm{Pro}(\mathrm{QCoh}(U_a)^-)\}$$

to the functor $\Phi : \mathrm{QCoh}(Z)^- \to \mathrm{Vect}$ given by

$$\Phi(\mathcal{F}) := \lim_{a \in A} \Phi_a(j^*_a(\mathcal{F})).$$

We have:

Lemma 4.6.2. *Let Z be quasi-compact. Then functors in* (4.8) *are mutually inverse equivalences.*

Proof. Follows easily from the fact that we can replace the limit over the category A by a finite limit. □

4.6.3. Let $\mathcal{X}$ be a prestack that admits a pro-cotangent complex. Assume that $\mathcal{X}$ is a sheaf in the Zariski topology. Let Z be a quasi-compact scheme.

Let $x : Z \to \mathcal{X}$ be a map. Recall that according to Sect. 2.3.5, we have a well-defined object

$$T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-).$$

The fact that the map (4.4) is an isomorphism implies that $T^*(\mathcal{X})$ gives rise to a well-defined object

$$\{T^*_{x|_{U_a}}(\mathcal{X})\} \in \lim_{a\in A} \mathrm{Pro}(\mathrm{QCoh}(U_a)^-). \tag{4.9}$$

By definition, we have:

Lemma 4.6.4. *The object $T^*_x(\mathcal{X})$ is canonically isomorphic to the image of $\{T^*_{x|_{U_a}}(\mathcal{X})\}$ under the functor $\to$ in* (4.8).

In particular, from Lemma 4.6.2 we obtain:

Corollary 4.6.5. *Let $f : Z_1 \to Z_2$ be a map in $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$. Then the canonical map*

$$T^*_{x_1}(\mathcal{X}) \to \mathrm{Pro}(f^*)(T^*_{x_2}(\mathcal{X}))$$

is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(Z_1)^-)$.

4.6.6. The Zariski-locality of the construction in Sect. 4.5.2 implies that that there exists a canonically defined map

$$(dx)^* : T^*_x(\mathcal{X}) \to T^*(Z).$$

Furthermore, if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ and $Z \in \mathrm{Sch}_{\mathrm{aft}}$, $(dx)^*$ corresponds to a map in $\mathrm{IndCoh}(Z)$,

$$dx : T(Z) \to T_x(\mathcal{X}).$$

5. Digression: square-zero extensions

The notion of square-zero extension is central to deformation theory. It allows to obtain nilpotent embeddings of a scheme by iterating a certain linear construction.

5.1. **The notion of square-zero extension.** In this subsection we introduce (following [Lu1, Sect. 7.4.1]) the notion of square-zero extension and study its basic properties.

5.1.1. Let X be an object of Sch. The category of *square-zero extensions* is by definition

$$\left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}}.$$

There is a naturally defined functor

$$\mathrm{RealSqZ} : \left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}} \to \mathrm{Sch}_{X/} \tag{5.1}$$

that sends

$$T^*(X) \xrightarrow{\gamma} \mathcal{F} \in (\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}$$

to

$$X' := X \underset{X_{\mathcal{F}}}{\sqcup} X,$$

where the two maps $X_{\mathcal{F}} \rightrightarrows \mathcal{F}$ are the tautological projection $X_{\mathcal{F}} \xrightarrow{pr} X$, and the map

$$X_{\mathcal{F}} \xrightarrow{\gamma} X_{T^*(X)} \xrightarrow{\mathfrak{d}} X,$$

respectively. The map $X \to X'$ corresponds to the first factor in $X \underset{X_{\mathcal{F}}}{\sqcup} X$; it is a closed nil-isomorphism.

5.1.2. Here is a functorial interpretation of the functor (5.1):

Given $X \in \mathrm{Sch}$, let $\mathrm{Sch}_{X/,\,\text{inf-closed}}$ be the full subcategory of $\mathrm{Sch}_{X/}$, spanned by those $f : X \to Y$, for which the codifferential

$$(df)^* : T^*_f(Y) = f^*(T^*(Y)) \to T^*(X)$$

induces a surjection on H^0. I.e., $T^*(X/Y) \in \mathrm{QCoh}(X)^{\leq -1}$.

We have a functor

$$\mathrm{Sch}_{X/,\,\text{inf-closed}} \to ((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/})^{\mathrm{op}}, \quad (Y, f) \mapsto T^*(X/Y). \tag{5.2}$$

Unwinding the definitions, we see that the functor RealSqZ of (5.1) is the left adjoint of (5.2).

5.1.3. The following observation may be helpful in parsing the above construction of the functor RealSqZ. Let $\mathcal{F}$ be an object of $\mathrm{QCoh}(X)^{\leq -1}$, and let γ_1, γ_2 be maps $T^*(X) \to \mathcal{F}$. We have:

Lemma 5.1.4. *There is a canonical isomorphism in* $(\mathrm{Sch})_{X/}$

$$X \underset{0, X_{\mathcal{F}}, \gamma}{\sqcup} X \simeq X \underset{\gamma_1, X_{\mathcal{F}}, \gamma_2}{\sqcup} X,$$

where X maps to both sides via the left copy of X in the push-out, and $\gamma = \gamma_1 - \gamma_2$.

Proof. By definition, the left-hand and the-right side are the co-equalizers in $\mathrm{Sch}_{X/}$ of the maps

$$X_{\mathcal{F}} \rightrightarrows X,$$

equal to $(pr, \mathfrak{d} \circ \gamma)$ and $(\mathfrak{d} \circ \gamma_1, \mathfrak{d} \circ \gamma_2)$, respectively.

Given a map $x : X \to Y$, where $Y \in \mathrm{Sch}$, in each of the two cases, the datum of a map

$$\text{co-Eq}(X_{\mathcal{F}} \rightrightarrows X) \to Y$$

in $\mathrm{Sch}_{X/}$ is equivalent to that of a map

$$T^*_x(Y) \to \mathrm{Eq}(T^*(X) \rightrightarrows \mathcal{F}),$$

where $T^*_x(Y) \to T^*(X)$ is $(dx)^*$, and the maps $T^*(X) \rightrightarrows \mathcal{F}$ are

$$(0, \gamma) \text{ and } (\gamma_1, \gamma_2),$$

respectively. This makes the assertion of the lemma manifest.

□

5.1.5. We shall denote the category

$$\left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}}$$

also by $\mathrm{SqZ}(X)$, and refer to its objects as *square-zero extensions of* X. Compare this with the notation $\mathrm{SplitSqZ}(X)$ in Sect. 2.1.6.

Thus, RealSqZ is a functor

$$\mathrm{SqZ}(X) \to \mathrm{Sch}_{X/}.$$

We shall say that $(X \hookrightarrow X') \in \mathrm{Sch}_{X/}$ has a structure of square-zero extension if it given as the image of an object of $\mathrm{SqZ}(X)$ under the functor RealSqZ.

Note, however, that in general, the functor RealSqZ is *not* fully faithful.

5.1.6. For a fixed $\mathcal{F} \in \mathrm{QCoh}(X)^{\leq -1}$, we shall refer to the category (in fact, space)

$$\mathrm{Maps}(T^*(X), \mathcal{F})$$

as that of *square-zero extensions of* X *by means of* $\mathcal{I} := \mathcal{F}[-1]$.

The reason for this terminology is the following. Let

$$(X \overset{i}{\hookrightarrow} X') = \mathrm{RealSqZ}(T^*(X) \to \mathcal{F}).$$

Then from the construction of X' as a push-out it follows that we have a fiber sequence in $\mathrm{QCoh}(X')$:

$$i_*(\mathcal{I}) \to \mathcal{O}_{X'} \to i_*(\mathcal{O}_X), \tag{5.3}$$

where i denotes the closed embedding $X \to X'$. I.e., $\mathcal{I}$ is the 'ideal' of X inside X'.

5.1.7. Finally, let us note that we have the following pullback diagram of categories:

$$\begin{array}{ccc} (\mathrm{QCoh}(X)^{\leq 0})^{\mathrm{op}} & \longrightarrow & \left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{SplitSqZ}(X) & \longrightarrow & \mathrm{SqZ}(X) \\ {\scriptstyle \mathrm{RealSplitSqZ}}\downarrow & & \downarrow{\scriptstyle \mathrm{RealSqZ}} \\ (\mathrm{Sch})_{X//X} & \longrightarrow & (\mathrm{Sch})_{X/}, \end{array}$$

where the top horizontal arrow is the functor

$$\mathcal{F} \in \mathrm{QCoh}(X)^{\leq 0} \quad \mapsto \quad (T^*(X) \overset{0}{\to} \mathcal{F}[1]) \in (\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}.$$

5.2. **Functoriality of square-zero extensions.** In this subsection we will study how square-zero extensions behave under push-outs under affine morphisms.

5.2.1. Let $f : X_1 \to X_2$ be an *affine* map in Sch. We claim that there is a canonically defined functor

$$(\mathrm{QCoh}(X_1)^{\leq -1})_{T^*(X_1)/} \to (\mathrm{QCoh}(X_2)^{\leq -1})_{T^*(X_2)/} \tag{5.4}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{SqZ}(X_1) & \longrightarrow & \mathrm{SqZ}(X_2) \\ {\scriptstyle \mathrm{RealSqZ}}\downarrow & & \downarrow{\scriptstyle \mathrm{RealSqZ}} \\ \mathrm{Sch}_{X_1/} & \longrightarrow & \mathrm{Sch}_{X_2/} \end{array}$$

commute, where the functor $\mathrm{Sch}_{X_1/} \to \mathrm{Sch}_{X_2/}$ is given by push-out:

$$(X_1 \hookrightarrow X_1') \mapsto (X_2 \hookrightarrow X_2 \underset{X_1}{\sqcup} X_1').$$

Indeed, the functor (5.4) sends $\gamma_1 : T^*(X_1) \to \mathcal{F}_1$ to

$$\gamma_2 : T^*(X_2) \to f_*(\mathcal{F}_1),$$

where γ_2 is obtained by the (f^*, f_*)-adjunction from the composition

$$f^*(T^*(X_2)) \simeq T^*_f(X_2) \overset{(df)^*}{\longrightarrow} T^*(X_1) \overset{\gamma_1}{\longrightarrow} \mathcal{F}_1.$$

Note that the assumption that f be affine was used to ensure that $f_*(\mathcal{F}_1) \in \mathrm{QCoh}(X_2)^{\leq -1}$.

Remark 5.2.2. Note that for a map $f : X_1 \to X_2$ as above, the diagram

$$\begin{array}{ccc} \text{SplitSqZ}(X_1) & \longrightarrow & \text{SplitSqZ}(X_2) \\ \downarrow & & \downarrow \\ \text{SqZ}(X_1) & \longrightarrow & \text{SqZ}(X_2). \end{array}$$

commutes, where the top horizontal arrow is

$$f_* : \text{QCoh}(X_1)^{\leq 0} \to \text{QCoh}(X_2)^{\leq 0}.$$

5.2.3. The construction in Sect. 5.2.1 makes the assignment

$$X \rightsquigarrow \text{SqZ}(X)$$

into a functor $(\text{Sch})_{\text{affine}} \to 1\text{-Cat}$, where

$$(\text{Sch})_{\text{affine}} \subset \text{Sch}$$

is the 1-full subcategory, where we restrict 1-morphisms to be affine.

Thus we obtain a co-Cartesian fibration

$$(\text{SqZ}(\text{Sch}))_{\text{affine}} \to (\text{Sch})_{\text{affine}},$$

whose fiber over $X \in \text{Sch}$ is $\text{SqZ}(X)$.

5.2.4. In particular, given an affine map $f : X_1 \to X_2$ and objects

$$(T^*(X_i) \overset{\gamma_i}{\to} \mathcal{F}_i) \in \text{SqZ}(X_i), \quad i = 1, 2$$

we obtain a well-defined notion of *map of square-zero extensions*

$$(T^*(X_1) \overset{\gamma_i}{\to} \mathcal{F}_1) \to (T^*(X_2) \overset{\gamma_i}{\to} \mathcal{F}_2),$$

extending f.

By definition, a datum of such a map amounts to a morphism $\mathcal{F}_2 \to f_*(\mathcal{F}_1)$, equipped with a datum of commutativity of the diagram

$$\begin{array}{ccc} f^*(T^*(X_2)) & \xrightarrow{(df)^*} & T^*(X_1) \\ {\scriptstyle \gamma_2}\downarrow & & \downarrow{\scriptstyle \gamma_1} \\ f^*(\mathcal{F}_2) & \longrightarrow & \mathcal{F}_1. \end{array}$$

In the above circumstances we shall say that for

$$(X_i \hookrightarrow X_i') = \text{RealSqZ}(T^*(X_i) \overset{\gamma_i}{\to} \mathcal{F}_i),$$

the resulting commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X_2 & \longrightarrow & X_2' \end{array}$$

has been given a structure of *map of sqaure-zero* extensions.

5.3. **Pull-back of square-zero extensions.** In this subsection we will show that, in addition to push-outs of square-zero extensions with respect to the source, one can also form pullbacks with respect to maps of the target.

5.3.1. Note that the category $\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}}$, introduced above, admits a forgetful functor to the category $\mathrm{Funct}([1], (\mathrm{Sch})_{\mathrm{affine}})$ of pairs of schemes $(X \to X')$ and affine maps between them.

The functor

$$\mathrm{targ} : \mathrm{Funct}([1], (\mathrm{Sch})_{\mathrm{affine}}) \to (\mathrm{Sch})_{\mathrm{affine}}, \quad (X \to X') \to X'$$

is a Cartesian fibration (via the formation of fiber products).

We claim:

Proposition 5.3.2. *The composite functor*

$$\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \to \mathrm{Funct}([1], (\mathrm{Sch})_{\mathrm{affine}}) \overset{\mathrm{targ}}{\longrightarrow} (\mathrm{Sch})_{\mathrm{affine}} \tag{5.5}$$

is a Cartesian fibration, and the forgetful functor

$$\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \to \mathrm{Funct}([1], (\mathrm{Sch})_{\mathrm{affine}})$$

sends Cartesian arrows to Cartesian arrows.

5.3.3. The concrete meaning of this proposition is that if

$$(X \hookrightarrow X') = \mathrm{RealSqZ}(T^*(X) \overset{\gamma_X}{\longrightarrow} \mathcal{F}_X),$$

then for an affine map $Y' \to X'$, the object

$$(X \underset{X'}{\times} Y' =: Y \hookrightarrow Y') \in \mathrm{Sch}_{Y'/}$$

has a canonical structure of square-zero extension; moreover as such it satisfies an appropriate universal property (for mapping into it).

5.3.4. *Proof of Proposition 5.3.2.* In the notations of Sect. 5.3.3, note that γ_X canonically factors as

$$T^*(X) \to T^*(X/X') \overset{\gamma'_X}{\longrightarrow} \mathcal{F}_X.$$

Set $\mathcal{F}_Y := f^*(\mathcal{F}_X)$. We construct the morphism

$$\gamma_Y : T^*(Y) \to \mathcal{F}_Y$$

as the composite

$$T^*(Y) \to T^*(Y/Y') \simeq f^*(T^*(X/X')) \overset{f^*(\gamma'_X)}{\longrightarrow} f^*(\mathcal{F}_X).$$

By Sect. 5.1.2, the square-zero extension of Y corresponding to γ_Y is equipped with a canonical map to Y'. This map is an isomorphism by (5.3).

The fact that this square-zero extension satisfies the required universal property is a straightforward verification.

□

5.3.5. The construction of pullback in Proposition 5.3.2 is local in the Zariski topology. This allows to extend the Cartesian fibration (5.5) to a Cartesian fibration

$$\mathrm{SqZ}(\mathrm{Sch}) \to \mathrm{Sch},$$

i.e., the formation of structure of square-zero extension on the pullback is applicable to *not necessarily affine* morphisms between schemes.

5.4. **Square-zero extensions and truncations.** In this subsection we will establish a crucial fact that a scheme can be obtained as a succession of square-zero extensions of its n-coconnective truncations.

5.4.1. We claim (which is essentially [Lu1, Theorem 7.4.1.26]):

Proposition 5.4.2.
(a) *For* $X \in {}^{\mathrm{cl}}\mathrm{Sch}$, *the category of its square-zero extensions by means of objects of* $\mathrm{QCoh}(X)^{\heartsuit}$ *is equivalent to that of closed embeddings of classical schemes* $X \hookrightarrow X'$, *where the ideal of* X *in* X' *is such that its square vanishes.*
(b) *For* $X_n \in {}^{\leq n}\mathrm{Sch}$, *the category of*

$$(X_{n+1} \in {}^{\leq n+1}\mathrm{Sch}, {}^{\leq n}X_{n+1} \simeq X_n)$$

is canonically equivalent to that of square-zero extensions of X_n *by objects of*

$$\mathrm{QCoh}(X_n)^{\heartsuit}[n+1] \subset \mathrm{QCoh}(X_n).$$

Proof. We will prove point (b), as the proof of point (a) is similar but simpler. We have the fiber sequence

$$i_*(\mathcal{F}[-1]) \to \mathcal{O}_{X_{n+1}} \to i_*(\mathcal{O}_{X_n}),$$

where $\mathcal{F} \in (\mathrm{QCoh}(X_n)^{\heartsuit})[n+2]$.

We claim that X_{n+1} has a structure of square-zero extension of X_n, corresponding to a canonically defined map $\gamma : T^*(X_n) \to \mathcal{F}$.

Indeed, consider the fiber sequence

$$T_i^*(X_{n+1}) \overset{(di)^*}{\to} T^*(X_n) \to T^*(X_n/X_{n+1}),$$

and the existence and canonicity of the required map γ follows from the next observation:

$$\begin{cases} H^k(T^*(X_n/X_{n+1})) = 0 \text{ for } k \geq -n-1 \\ H^{-n-2}(T^*(X_n/X_{n+1})) \simeq \mathcal{F}, \end{cases}$$

which in turns results from the following general assertion (see [Lu1, Theorem 7.4.3.1]):

Lemma 5.4.3. *Let* $i : X \to Y$ *be a closed embedding of schemes. Consider the corresponding fiber sequence*

$$\mathcal{I} \to \mathcal{O}_Y \to i_*(\mathcal{O}_X).$$

Then:
(a) $H^0(T^*(X/Y)) = 0$ *and*

$$H^{-1}(T^*(X/Y)) = H^0(i^*(\mathcal{I}))$$

as objects of $\mathrm{QCoh}(X)^{\heartsuit}$.
(b) *For* $n \geq 0$ *we have:*

$$\tau^{\geq -n}(\mathcal{I}) = 0 \Rightarrow \tau^{\geq -n-1}(T^*(X/Y)) = 0.$$

In the latter case ${}^{\mathrm{cl}}X \simeq {}^{\mathrm{cl}}Y$ *and*

$$H^{-n-2}(T^*(X/Y)) \simeq H^{-n-1}(\mathcal{I})$$

as objects of $\mathrm{QCoh}(X)^{\heartsuit} \simeq \mathrm{QCoh}(Y)^{\heartsuit}$.

□

5.4.4. The assertion of Proposition 5.4.2(b) in particular constructs a functor

$$ {}^{\leq n+1}\mathrm{Sch} \to \mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \underset{\mathrm{Sch}\times\mathrm{Sch}}{\times} ({}^{\leq n}\mathrm{Sch} \times {}^{\leq n+1}\mathrm{Sch}). \tag{5.6} $$

Proposition 5.4.5. *The functor* (5.6) *is the (fully faithful) right adjoint of the forgetful functor*

$$ \mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \underset{\mathrm{Sch}\times\mathrm{Sch}}{\times} ({}^{\leq n}\mathrm{Sch} \times {}^{\leq n+1}\mathrm{Sch}) \to {}^{\leq n+1}\mathrm{Sch}\,. $$

Proof. We construct the unit of the adjunction as follows. Given a square-zero extension

$$ (T^*(X) \xrightarrow{\gamma} \mathcal{F}), \quad X \in {}^{\leq n}\mathrm{Sch},\ \mathcal{F} \in \mathrm{QCoh}(X)^{\geq -n-2, \leq -1}, $$

denote

$$ (X \hookrightarrow X') := \mathrm{RealSqZ}(T^*(X) \xrightarrow{\gamma} \mathcal{F}), $$

and note that there exists a canonically defined commutative diagram of scheme[6]

$$ \begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ {}^{\leq n}X' & \longrightarrow & X'. \end{array} $$

Let $(X \hookrightarrow X')$ be given by a map $\gamma : T^*(X) \to \mathcal{F}$, where $\mathcal{F} \in \mathrm{QCoh}(X)^{\geq -n-2,\leq -1}$. We obtain a commutative diagram in $\mathrm{QCoh}(X)$:

$$ \begin{array}{ccc} f^*(T^*({}^{\leq n}X')) & \longrightarrow & f^*(T^*({}^{\leq n}X'/X')) \\ {\scriptstyle (df)^*}\downarrow & & \downarrow \\ T^*(X) & \longrightarrow & \mathcal{F}. \end{array} $$

We note that $\mathcal{F}$ lives in the cohomological degrees $\geq -n-2$, while, by Lemma 5.4.3, $T^*({}^{\leq n}X'/X')$ lives in the cohomological degrees $\leq -n-2$ with

$$ H^{-n-2}(T^*({}^{\leq n}X'/X')) \simeq H^{-n-1}(\mathcal{I}), $$

where $T^*({}^{\leq n}X') \to \mathcal{I}[1]$ is the map defining the square-zero extension ${}^{\leq n}X' \hookrightarrow X'$.

Hence, the map $f^*(T^*({}^{\leq n}X'/X')) \to \mathcal{F}$ canonically gives rise to a map $f^*(\mathcal{I}[1]) \to \mathcal{F}$, and we obtain a commutative diagram

$$ \begin{array}{ccc} f^*(T^*({}^{\leq n}X')) & \longrightarrow & f^*(\mathcal{I}[1]) \\ {\scriptstyle (df)^*}\downarrow & & \downarrow \\ T^*(X) & \longrightarrow & \mathcal{F}, \end{array} $$

which defines the sought-for unit for the adjunction. The fact that it satisfies the adjunction axioms is a straightforward check.

□

5.5. **Nilpotent embeddings.** In this subsection we will show that a nilpotent embedding of a scheme can be obtained as a (infinite) composition of square-zero extensions.

5.5.1. We shall say that a map $X \to Y$ of schemes is a *nilpotent embedding* if ${}^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$ is a closed embedding of classical schemes, such that the ideal of ${}^{\mathrm{cl}}X$ in ${}^{\mathrm{cl}}Y$ is nilpotent (i.e., there exists a power n that annihilates every section).

[6]In order to unburden the notation, for the duration of this Chapter, for a scheme Y, we will denote by ${}^{\leq n}Y$ the object of Sch that should be properly denoted by ${}^{L}\tau^{\leq n}(Y)$, see [Chapter I.2, Sect. 2.6.2]. I.e., this is the n-coconnective truncation of Y, viewed as an object of Sch, rather than ${}^{\leq n}\mathrm{Sch}$.

5.5.2. We are going to prove the following useful result:

Proposition 5.5.3. *Let $X \to Y$ be a nilpotent embedding of schemes. There exists a sequence of schemes*

$$X = X_0^0 \hookrightarrow X_0^1 \hookrightarrow ... \hookrightarrow X_0^i \hookrightarrow ... \hookrightarrow X_0^n = X_0 \hookrightarrow X_1 \hookrightarrow ... \hookrightarrow X_j \hookrightarrow ... \hookrightarrow Y,$$

such that:

- *Each of the maps $X_0^i \hookrightarrow X_0^{i+1}$, $X_0' \hookrightarrow X_0$ and $X_j \hookrightarrow X_{j+1}$ has a structure of square-zero extension;*
- *For every j, the map $g_j : X_j \to Y$ induces an isomorphism ${}^{\leq j}X_j \to {}^{\leq j}Y$.*

The rest of this subsection is devoted to the proof of Proposition 5.5.3.

5.5.4. *Step 1.* Let

$${}^{\mathrm{cl}}X = X_{\mathrm{cl},0}^0 \hookrightarrow X_{\mathrm{cl},0}^1 \hookrightarrow ... \hookrightarrow X_{\mathrm{cl},0}^k \hookrightarrow ...X_{\mathrm{cl},0}^n = {}^{\mathrm{cl}}Y$$

be a sequence of square-zero extensions of classical schemes. It exists by the assumption that the ideal of the closed embedding ${}^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$ is nilpotent. Set

$$X_0^i := X_0^0 \underset{{}^{\mathrm{cl}}X_{\mathrm{cl},0}^0}{\sqcup} X_{\mathrm{cl},0}^k.$$

By construction, $g_0 : X_0 \to Y$ induces an isomorphism ${}^{\mathrm{cl}}X_0 \to {}^{\mathrm{cl}}Y$.

5.5.5. *Step 2.* Starting from $g_0 : X_0 \to Y$, we shall construct $g_1 : X_1 \to Y$ using the following general procedure. The same procedure constructs $g_{i+1} : X_{i+1} \to Y$ starting from $g_i : X_i \to Y$.

Let $h : Z \to Y$ be a map that induces an isomorphism of the underlying classical schemes, and such that $T^*(Z/Y)$ lives in the cohomological degrees $\leq -(k+1)$ with $k \geq 0$.

We will construct a map $f : Z \hookrightarrow Z'$ with a structure of square-zero extension by an object $\mathcal{J} \in \mathrm{QCoh}(Z)^{\heartsuit}[k]$, and an extension of the map h to a map $h' : Z' \to Y$ so that h' such that $T^*(Z/Y)$ lives in the cohomological degrees $\leq -(k+2)$. (Hence, by Lemma 5.4.3, the 'ideal' of Z' in Y lives in degrees $\leq -(k+1)$, and in particular, ${}^{\leq k}Z \to {}^{\leq k}Y$ is an isomorphism.)

Namely, consider the fiber sequence

$$T_h^*(Y) \to T^*(Z) \to T^*(Z/Y),$$

and take

$$\mathcal{J} := H^{-k-1}(T^*(Z/Y))[k] = \tau^{\geq -(k+1)}(T^*(Z/Y))[-1].$$

We let the sought-for square-zero extension $Z \hookrightarrow Z'$ be given by the composite map

$$T^*(Z) \to T^*(Z/Y) \to \mathcal{J}[1].$$

The composition

$$T_h^*(Y) \to T^*(Z) \to \mathcal{J}[1]$$

acquires a canonical null-homotopy by constriction, thereby giving rise to a map $h' : Z' \to Y$.

In order to show that $T^*(Z'/Y)$ lives in $\mathrm{QCoh}(Z')^{\leq -(k+2)}$, consider the fiber sequences

$$\mathcal{I} \to \mathcal{O}_Y \to h_*(\mathcal{O}_Z) \text{ and } \mathcal{I}' \to \mathcal{O}_Y \to h'_*(\mathcal{O}_{Z'})$$

and the diagram

$$\begin{array}{ccccc} h_*(\mathcal{J}) & \longrightarrow & h'_*(\mathcal{O}_{Z'}) & \longrightarrow & h_*(\mathcal{O}_Z) \\ \uparrow & & \uparrow & & \uparrow \mathrm{Id} \\ \mathcal{I} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & h_*(\mathcal{O}_Z) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{I}' & \longrightarrow & \mathcal{I}' & \longrightarrow & 0. \end{array}$$

By Lemma 5.4.3, the map $\mathcal{I} \to h_*(\mathcal{J})$ identifies with the truncation map

$$\mathcal{I} \to \tau^{\geq -k}(\mathcal{I}).$$

Hence, $\mathcal{I}' \in \mathrm{QCoh}(Y)^{\leq -(k-1)}$. Now, this implies that $T^*(Z'/Y) \in \mathrm{QCoh}(Z')^{\leq -(k+2)}$ again by Lemma 5.4.3.

□

6. Infinitesimal cohesiveness

Infinitesimal cohesiveness is a property of a prestack that allows to describe maps into it from a square-zero extension of an affine scheme S as a data involving $\mathrm{QCoh}(S)$.

6.1. **Infinitesimal cohesiveness of a prestack.** In this subsection we introduce the notion of infinitesimal cohesiveness in terms of compatibility with certain type of push-outs.

6.1.1. Let $\mathcal{X} \in \mathrm{PreStk}$, and let (S, x) be an object of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. For

$$T^*(S) \overset{\gamma}{\to} \mathcal{F} \in \left(\mathrm{QCoh}(S)^{\leq -1})_{T^*(S)/}\right)^{\mathrm{op}} = \mathrm{SqZ}(S)$$

and the corresponding

$$(S \hookrightarrow S') := \mathrm{RealSqZ}(T^*(S) \overset{\gamma}{\to} \mathcal{F}) = S \underset{S_{\mathcal{F}}}{\sqcup} S$$

we obtain a canonically defined map

$$\mathrm{Maps}_{S/}(S', \mathcal{X}) \to * \underset{\mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})}{\times} \mathrm{Maps}(S, \mathcal{X}), \tag{6.1}$$

where $* \to \mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})$ corresponds to the composition

$$S_{\mathcal{F}} \overset{pr}{\longrightarrow} S \overset{x}{\to} \mathcal{X}.$$

Definition 6.1.2. *We shall say that* $\mathcal{X}$ *is* infinitesimally cohesive *if the map* (6.1) *is an isomorphism for all* S, x *and*

$$(T^*(S) \overset{\gamma}{\to} \mathcal{F}) \in \left((\mathrm{QCoh}(S)^{\leq -1})_{T^*(S)/}\right)^{\mathrm{op}}$$

as above.

We observe that by Lemma 1.3.3(b), any $\mathcal{X} = X \in \mathrm{Sch}$ is infinitesimally cohesive.

6.1.3. Suppose that $\mathcal{X}$ is convergent (resp., belongs to $\mathrm{PreStk}_{\mathrm{laft}}$). Then as in Lemmas 3.3.4 (resp., 3.5.3), in order to verify the condition of infinitesimal cohesiveness, it is sufficient to consider $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{>-\infty}$ (resp., $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $\mathcal{F} \in \mathrm{Coh}(S)$).

6.1.4. *The relative situation.* The notion of infinitesimal cohesiveness renders automatically to the relative situation.

We note that the analog of Lemma 2.4.5 holds when we replace 'admitting cotangent spaces' by 'infinitesimal cohesiveness'.

We also note that when $\mathcal{X}$ and $\mathcal{X}_0$ are convergent (resp., locally almost of finite type), the analog of Lemma 4.2.5 holds.

6.2. **Rewriting the condition of infinitesimal cohesiveness.** We will now rewrite the definition of infinitesimal cohesiveness in terms of QCoh.

6.2.1. Note that the space

$$* \underset{\mathrm{Maps}(S_{\mathcal{F}},\mathcal{X})}{\times} \mathrm{Maps}(S,\mathcal{X})$$

identifies with the space of homotopies between the following two points of $\mathrm{Maps}_{S/}(S_{\mathcal{F}},\mathcal{X})$: the first being

$$S_{\mathcal{F}} \overset{pr}{\longrightarrow} S \overset{x}{\to} \mathcal{X},$$

and the second being

$$S_{\mathcal{F}} \overset{\gamma}{\to} S_{T^*(S)} \overset{\mathfrak{d}}{\longrightarrow} S \overset{x}{\to} \mathcal{X}.$$

6.2.2. Assume that $\mathcal{X}$ admits a pro-cotangent space at x. We obtain that the space

$$* \underset{\mathrm{Maps}(S_{\mathcal{F}},\mathcal{X})}{\times} \mathrm{Maps}(S,\mathcal{X})$$

identifies with the space of null-homotopies of the composed map

$$T^*_x(\mathcal{X}) \overset{(dx)^*}{\longrightarrow} T^*(S) \overset{\gamma}{\to} \mathcal{F}.$$

6.2.3. Thus, we obtain that if $\mathcal{X}$ admits pro-cotangent spaces, the condition of infinitesimal cohesiveness can be formulated as saying that given (S,x), for every

$$(T^*(S) \to \mathcal{F}) \in (\mathrm{QCoh}(S)^{\leq -1})_{T^*(S)/}$$

and the corresponding

$$(S \hookrightarrow S') = \mathrm{RealSqZ}(T^*(S) \to \mathcal{F}) \in \mathrm{Sch}^{\mathrm{aff}}_{S/},$$

the canonical map of spaces

$$\mathrm{Maps}_{S/}(S',\mathcal{X}) \simeq \{\text{null homotopies of } T^*_x(\mathcal{X}) \overset{(dx)^*}{\to} T^*(S) \to \mathcal{F}\} \tag{6.2}$$

be an isomorphism.

Equivalently, this can be phrased as saying that for (S,x), the functor

$$(\mathrm{Sch}^{\mathrm{aff}})_{S//\mathcal{X}} \underset{(\mathrm{Sch}^{\mathrm{aff}})_{S/}}{\times} \mathrm{SqZ}(S) \to (\mathrm{QCoh}(S)^{\leq -1})_{\mathrm{coFib}(T^*_x(\mathcal{X}) \overset{(dx)^*}{\to} T^*(S))/}$$

is an equivalence.

6.2.4. Suppose that $\mathcal{X}$ both admits a pro-cotangent complex and is infinitesimally cohesive. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology.

Let Z be a scheme. From Lemma 4.6.2, we obtain:

Corollary 6.2.5. *For* $(T^*(Z) \to \mathcal{F}) \in (\mathrm{QCoh}(Z)^{\leq -1})_{/T^*(Z)}$ *and*

$$(Z \hookrightarrow Z') = \mathrm{RealSqZ}(T^*(Z) \to \mathcal{F}) \in \mathrm{Sch}_{Z/},$$

the map

$$\mathrm{Maps}_{Z/}(Z', \mathcal{X}) \to \{\textit{null homotopies of } T^*_x(\mathcal{X}) \overset{(dx)^*}{\to} T^*(Z) \to \mathcal{F}\}$$

is an isomorphism.

6.3. **Consequences of infinitesimal cohesiveness.** If a prestack is infinitesimally cohesive, one can deduce that it has certain properties from the fact that the underlying reduced prestack has these properties.

6.3.1. First, combining Sect. 6.2.2 and Proposition 5.4.2, we obtain:

Lemma 6.3.2. *Assume that* $\mathcal{X} \in \mathrm{PreStk}$ *admits* $(-k)$*-connective pro-cotangent spaces and is infinitesimally cohesive. Let* S *be an object of* ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$*, and let* $S_0 \subset {}^{\mathrm{cl}}S$ *be given by a nilpotent ideal. Then the fibers of the map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}(S_0, \mathcal{X})$$

are $(k+n)$*-truncated.*

In particular, we obtain that if ${}^{\mathrm{cl}}\mathcal{X}$ takes values in Sets $\subset$ Spc, then for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the space $\mathrm{Maps}(S, \mathcal{X})$ is n-truncated.

6.3.3. From Sect. 5.2.1 we obtain:

Lemma 6.3.4. *Let* $\mathcal{X}$ *be an object of* PreStk*, which both admits a pro-cotangent complex and is infinitesimally cohesive. Then if*

$$S_1' \underset{S_1}{\sqcup} S_2 \to S_2'$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$*, where* $S_1 \hookrightarrow S_1'$ *has a structure of a square-zero extension. Then*

$$\mathrm{Maps}(S_2', \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is a pullback diagram.

Iterating, from Lemma 6.3.4, we obtain:

Corollary 6.3.5. *Let* $\mathcal{X}$ *be an object of* PreStk *which both admits a pro-cotangent complex and is infinitesimally cohesive. Let*

$$S_1' \underset{S_1}{\sqcup} S_2 \to S_2'$$

be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$ *such that* S_1' *can be obtained from* S_1 *as a finite succession of square-zero extensions. Then*

$$\mathrm{Maps}(S_2', \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is a pullback diagram.

Furthermore, from Corollary 6.2.5, we obtain:

Corollary 6.3.6. *Let $\mathcal{X}$ be an object of* PreStk *which both admits a pro-cotangent complex and is infinitesimally cohesive. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology. Let*

$$Z'_1 \underset{Z_1}{\sqcup} Z_2 \to Z'_2$$

be a push-out diagram in Sch*, where the map $Z_1 \to Z_2$ is affine, and Z'_1 can be obtained from Z_1 as a finite succession of square-zero extensions. Then the map*

$$\mathrm{Maps}(Z'_2, \mathcal{X}) \to \mathrm{Maps}(Z'_1, \mathcal{X}) \underset{\mathrm{Maps}(Z_1,\mathcal{X})}{\times} \mathrm{Maps}(Z_2, \mathcal{X})$$

is an isomorphism.

7. Deformation theory

In this section we finally define what it means for a prestack to admit deformation theory, and discuss some basic consequences of this property.

7.1. **Prestacks with deformation theory.**

In this subsection we give the definition of admitting deformation theory.

7.1.1. We now give the following crucial definition:

Definition 7.1.2. *Let $\mathcal{X}$ be a prestack. We shall say that $\mathcal{X}$* admits deformation theory *(resp.,* admits corepresentable deformation theory*) if:*

- *It is convergent;*
- *It admits a pro-cotangent (resp., cotangent) complex;*
- *It is infinitesimally cohesive.*

Note that the last two conditions are of the form that the functor $\mathcal{X}$ should send certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc; see also Sect. 7.2.4.

7.1.3. In what follows we shall denote by

$$\mathrm{PreStk}_{\mathrm{def}} \subset \mathrm{PreStk} \text{ and } \mathrm{PreStk}_{\text{laft-def}} \subset \mathrm{PreStk}_{\mathrm{laft}}$$

the full subcategories spanned by objects that admit deformation theory.

It is clear that the above subcategories are closed under finite limits taken in PreStk.

7.1.4. We shall also consider the following variants:

Definition 7.1.5.

(a) *We shall say that $\mathcal{X}$ admits an $(-n)$-connective deformation theory (resp., corepresentable deformation theory) if it admits deformation theory (resp., corepresentable deformation theory) and its cotangent spaces are $(-n)$-connective.*

(b) *We shall say that $\mathcal{X}$ admits a locally eventually connective deformation theory if it admits deformation theory and its pro-cotangent spaces are locally eventually connective.*

(c) *We shall say that $\mathcal{X}$ admits a uniformly eventually connective deformation theory (resp., corepresentable deformation theory) if there exists an integer n such that $\mathcal{X}$ admits a $(-n)$-connective deformation theory (resp., corepresentable deformation theory).*

As was mentioned above, any scheme X admits a connective corepresentable deformation theory.

7.1.6. The same definitions carry over to the relative situations for $\mathcal{X} \in \mathrm{PreStk}_{/\mathcal{X}_0}$ for some fixed $\mathcal{X}_0 \in \mathrm{PreStk}$.

Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in PreStk. Replacing the words 'infinitesimal cohesiveness' by 'admitting deformation theory' we render the contents of Sect. 6.1.4 to the present context.

7.2. **Compatibility with push-outs.** In this subsection we rewrite the condition of admitting deformation theory in terms of compatibility with certain type of push-outs.

7.2.1. One of the main properties of prestacks with deformation theory is given by the following proposition:

Proposition 7.2.2. *Assume that $\mathcal{X}$ admits deformation theory, and let $S_1' \underset{S_1}{\sqcup} S_2$ be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where the map $S_1 \to S_1'$ is a nilpotent embedding. Then the map*

$$\mathrm{Maps}(S_1' \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1,\mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is an isomorphism.

Proof of Proposition 7.2.2. Follows from Corollary 6.3.5 using Proposition 5.5.3. □

Corollary 7.2.3. *Assume that $\mathcal{X}$ admits deformation theory, and is a sheaf in the Zariski topology. Let*

$$Z_1' \underset{Z_1}{\sqcup} Z_2 \to Z_2'$$

be a push-out diagram in Sch*, where $Z_1 \to Z_1'$ is a nilpotent embedding. Assume that the map $Z_1 \to Z_2$ is affine. Then the map*

$$\mathrm{Maps}(Z_2', \mathcal{X}) \to \mathrm{Maps}(Z_1', \mathcal{X}) \underset{\mathrm{Maps}(Z_1,\mathcal{X})}{\times} \mathrm{Maps}(Z_2, \mathcal{X})$$

is an isomorphism.

7.2.4. Now, we have that the following converse of Proposition 7.2.2 holds:

Proposition 7.2.5. *Let $\mathcal{X} \in \mathrm{PreStk}$ be convergent. Assume that whenever $S_1' \underset{S_1}{\sqcup} S_2$ is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where the map $S_1 \to S_1'$ is a nilpotent embedding, the map*

$$\mathrm{Maps}(S_1' \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1,\mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is an isomorphism. Then $\mathcal{X}$ admits deformation theory.

Proof. Let us first show that $\mathcal{X}$ admits pro-cotangent spaces. For $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, consider the push-out diagram

$$S \underset{S_{\mathcal{F}[1]}}{\sqcup} S \to S_{\mathcal{F}}$$

(with both maps $S_{\mathcal{F}[1]} \to S$ being pr), and the resulting map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to \Omega(\mathrm{Maps}_{S/}(S_{\mathcal{F}[1]}, \mathcal{X})). \tag{7.1}$$

Since the map $S_{\mathcal{F}[1]} \to S$ is a nilpotent embedding, by assumption, the map (7.1) is an isomorphism.

Let now $\mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2$ be a fiber sequence in $\mathrm{QCoh}(S)$ with all three terms in $\mathrm{QCoh}(S)^{\leq 0}$. Consider the push-out diagrams

$$S \underset{S_{\mathcal{F}_2}}{\sqcup} S_{\mathcal{F}_1} \to S_{\mathcal{F}}$$

and

$$S \underset{S_{\mathcal{F}_2[1]}}{\sqcup} S_{\mathcal{F}_1[1]} \to S_{\mathcal{F}[1]},$$

and the corresponding maps

$$\operatorname{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\operatorname{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \operatorname{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}) \tag{7.2}$$

and

$$\operatorname{Maps}_{S/}(S_{\mathcal{F}[1]}, \mathcal{X}) \to * \underset{\operatorname{Maps}_{S/}(S_{\mathcal{F}_2[1]}, \mathcal{X})}{\times} \operatorname{Maps}_{S/}(S_{\mathcal{F}_1[1]}, \mathcal{X}). \tag{7.3}$$

Since the map $S_{\mathcal{F}_2[1]} \to S$ is a nilpotent embedding, the map (7.3) is an isomorphism. Taking loops and using (7.1) we obtain that (7.2) is also an isomorphism.

Hence, $\mathcal{X}$ admits pro-cotangent spaces. The fact that $\mathcal{X}$ admits a pro-cotangent complex follows from the fact that $\mathcal{X}$ takes push-outs of the form

$$(S_1)_{\mathcal{F}_1} \underset{S_1}{\sqcup} S_2, \quad \mathcal{F}_1 \in \operatorname{QCoh}(S_1)^{\leq 0}$$

to pullbacks; the latter because $S_1 \to (S_1)_{\mathcal{F}_1}$ is a nilpotent embedding.

Finally, $\mathcal{X}$ is infinitesimally cohesive because it takes push-outs of the form

$$S \underset{S_{\mathcal{F}[1]}}{\sqcup} S, \quad \mathcal{F} \in \operatorname{QCoh}(S)^{\leq 0}$$

to pullbacks; the latter because $S_{\mathcal{F}[1]} \to S$ is a nilpotent embedding.

□

Combining with Proposition 5.5.3, we obtain:

Corollary 7.2.6. *Let $\mathcal{X} \in \operatorname{PreStk}$ be convergent. Assume that whenever $S'_1 \underset{S_1}{\sqcup} S_2$ is a push-out diagram in $\operatorname{Sch}^{\operatorname{aff}}$, where the map $S_1 \hookrightarrow S'_1$ has a structure of square-zero extension, the map*

$$\operatorname{Maps}(S'_1 \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \operatorname{Maps}(S'_1, \mathcal{X}) \underset{\operatorname{Maps}(S_1, \mathcal{X})}{\times} \operatorname{Maps}(S_2, \mathcal{X})$$

is an isomorphism. Then $\mathcal{X}$ admits deformation theory.

7.2.7. It is easy to see that in the circumstances of Corollary 7.2.6, it is enough to consider S_1, S_2, S'_1 that belong to ${}^{<\infty}\operatorname{Sch}^{\operatorname{aff}}$. Furthermore, if $\mathcal{X} \in \operatorname{PreStk}_{\operatorname{laft}}$, it is enough to take S_1, S_2, S'_1 that belong to ${}^{<\infty}\operatorname{Sch}^{\operatorname{aff}}_{\operatorname{ft}}$.

Hence, we obtain:

Corollary 7.2.8. *The subcategory $\operatorname{PreStk}_{\operatorname{def}} \subset {}^{\operatorname{conv}}\operatorname{PreStk}$ is closed under* filtered *colimits, and the same is true for $\operatorname{PreStk}_{\operatorname{laft\text{-}def}} \subset \operatorname{PreStk}_{\operatorname{laft}}$.*

Proof. Follows from the fact that filtered colimits commute with fiber products. □

7.3. **Formal smoothness.** In this subsection we discuss the notion of *formal smoothness* of a prestack, and rewrite it for prestacks that admit deformation theory.

7.3.1. Let $\mathcal{X}$ be an object of PreStk. We shall say that $\mathcal{X}$ is *formally smooth*, if whenever $S \to S'$ is a nilpotent embedding of affine schemes, the map

$$\operatorname{Maps}(S', \mathcal{X}) \to \operatorname{Maps}(S, \mathcal{X})$$

is surjective on π_0.

7.3.2. We have the following basic result:

Proposition 7.3.3. *Assume that $\mathcal{X}$ admits deformation theory. Then the following conditions are equivalent:*

(a) $\mathcal{X}$ *is formally smooth.*

(b) *For any $n \geq 0$, the restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\leq n}S, \mathcal{X}), \quad S \in \mathrm{Sch}^{\mathrm{aff}}$$

induces an isomorphism on π_n (equivalently, on $\pi_{n'}$ for $n' \leq n$).

(b') *The restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\mathrm{cl}}S, \mathcal{X}), \quad S \in \mathrm{Sch}^{\mathrm{aff}}$$

induces an isomorphism on π_0.

(c) *For any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$, we have*

$$\mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}) \in \mathrm{Vect}^{\leq 0}.$$

(c') *Same as* (c), but assuming that S is classical.

Proof. The implications (b) $\Rightarrow$ (b') and (c) $\Rightarrow$ (c') are tautological.

The implication (a) $\Rightarrow$ (c) is immediate: apply the definition to the nilpotent embedding $S_{\mathcal{F}[i]} \to S$. Similarly, (b') implies (c): use the fact that ${}^{\mathrm{cl}}S \simeq {}^{\mathrm{cl}}S_{\mathcal{F}[i]}$ for $i > 0$.

The implication (c) $\Rightarrow$ (b) follows from Proposition 5.4.2(b). The implication (c) $\Rightarrow$ (a) follows from Proposition 5.5.3.

The implication (c') $\Rightarrow$ (c) follows from the fact that any object of $\mathrm{QCoh}(S)^{\heartsuit}$ is the direct image under ${}^{\mathrm{cl}}S \to S$.

□

7.3.4. Now, assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. In this case we have:

Proposition 7.3.5. *Then the following conditions are equivalent:*

(i) $\mathcal{X}$ *is formally smooth.*

(i') *The condition of formal smoothness is satisfied for nilpotent embeddings $S \to S'$ with $S, S' \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.*

(ii) *For any $n \geq 0$, the restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\leq n}S, \mathcal{X}), \quad S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

induces an isomorphism on π_n (equivalently, on $\pi_{n'}$ for $n' \leq n$).

(ii') *The restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\mathrm{cl}}S, \mathcal{X}), \quad S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

induces an isomorphism on π_0.

(iii) *For any $(S, x) \in ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}$ and $\mathcal{F} \in \mathrm{Coh}(S)^{\heartsuit}$, we have*

$$\mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}) \in \mathrm{Vect}^{\leq 0}.$$

(iii') *Same as* (iii), but assuming that S is reduced.

Proof. The implications (i) $\Rightarrow$ (i'), (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') are tautological. The implications (i') $\Rightarrow$ (iii), (ii') $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) follow in the same way as in Proposition 7.3.3.

The fact that (iii') implies (iii) follows from the fact that any object in $\text{QCoh}(S)^\heartsuit$ is a finite extension of ones coming as direct image under ${}^{\text{red}}S \to S$.

It remains to show that (iii) implies (i). We will show that (iii) implies condition (c') from Proposition 7.3.3. Let (S', x') be an object of $({}^{\text{cl}}\text{Sch}^{\text{aff}})_{/\mathcal{X}}$ and $\mathcal{F}' \in \text{QCoh}(S')^\heartsuit$. Since $\mathcal{X}$ is locally almost of finite type, we can factor the map $x' : S' \to \mathcal{X}$ as

$$S' \overset{f}{\to} S \overset{x}{\to} \mathcal{X},$$

where $S \in {}^{\text{cl}}\text{Sch}^{\text{aff}}_{\text{ft}}$. Set $\mathcal{F} := f_*(\mathcal{F}')$. Since $\mathcal{X}$ admits a cotangent complex, we have

$$\text{Maps}(T^*_{x'}(\mathcal{X}), \mathcal{F}'[i]) \simeq \text{Maps}(T^*_x(\mathcal{X}), \mathcal{F}[i]).$$

Write $\mathcal{F}$ as a filtered colimit

$$\underset{\alpha}{\text{colim}}\, \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha \in \text{Coh}(S)^\heartsuit.$$

Now, since $T^*_x(\mathcal{X})$ commutes with filtered colimits in $\text{QCoh}(S)^\heartsuit$ (by Lemma 3.5.2), we have:

$$\text{Maps}(T^*_x(\mathcal{X}), \mathcal{F}_\alpha[i]) = 0 \Rightarrow \text{Maps}(T^*_x(\mathcal{X}), \mathcal{F}[i]) = 0,$$

as required.

□

7.3.6. The definition of formal smoothness and Proposition 7.3.3 can be easily extended to a relative situation.

7.4. **Artin stacks.** In this subsection we show that Artin stacks, defined as in [Chapter I.3, Sect. 4], admit deformation theory.

7.4.1. We are going to prove:

Proposition 7.4.2.

(a) *Let $\mathcal{X}$ be an n-Artin stack. Then $\mathcal{X}$ admits an $(-n)$-connective corepresentable deformation theory.*

(b) *If $\mathcal{X}$ is smooth over a scheme Z, then for $x : S \in ({}^{\text{cl}}\text{Sch}^{\text{aff}})_{/\mathcal{X}}$, the relative cotangent complex $T^*_x(\mathcal{X}/Z)$ lives in $\text{QCoh}(S)^{\geq 0, \leq n}$.*

Arguing by induction on n, the proposition follows from the next lemma:

Lemma 7.4.3. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a map in* PreStk. *Assume that:*

- *$\mathcal{X}$ satisfies étale descent;*
- *f is étale-locally surjective;*
- *$\mathcal{Y}$ admits deformation theory;*
- *$\mathcal{Y}$ admits deformation theory relative to $\mathcal{X}$;*
- *$\mathcal{Y}$ is formally smooth over $\mathcal{X}$.*

Then $\mathcal{X}$ admits deformation theory.

7.4.4. *Proof of Lemma 7.4.3.* We will show that if

$$S'_1 \underset{S_1}{\sqcup} S_2 \to S'_2$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where $S_1 \to S'_1$ has a structure of a square-zero extension, then, given a map $S_2 \to \mathcal{X}$, the map

$$\mathrm{Maps}_{S_2/}(S'_2, \mathcal{X}) \to \mathrm{Maps}_{S_1/}(S'_1, \mathcal{X})$$

is an isomorphism. The other properties are proved similarly.

By étale descent for $\mathcal{X}$, the statement is local in the étale topology on S_2. Hence, we can assume that the given map $S_2 \to \mathcal{X}$ admits a lift to a map $S_2 \to \mathcal{Y}$.

Let $\mathcal{Y}^\bullet/\mathcal{X}$ be the Čech nerve of f. We have a commutative diagram

$$\begin{array}{ccc} |\mathrm{Maps}_{S_2/}(S'_2, \mathcal{Y}^\bullet/\mathcal{X})| & \longrightarrow & \mathrm{Maps}_{S_2/}(S'_2, \mathcal{X}) \\ \downarrow & & \downarrow \\ |\mathrm{Maps}_{S_1/}(S'_1, \mathcal{Y}^\bullet/\mathcal{X})| & \longrightarrow & \mathrm{Maps}_{S_1/}(S'_1, \mathcal{X}), \end{array}$$

where the horizontal arrows are monomorphisms.

We note that the terms of $\mathcal{Y}^\bullet/\mathcal{X}$ admit deformation theory (by the deformation theory analog of Lemma 2.4.5). Hence, the left vertical arrow is an isomorphism.

Hence, it remains to show that the horizontal arrows are surjective. We claim that this follows from the last requirement on f. We claim that for any square-zero extension

$$S \hookrightarrow S',$$

a map $x' : S' \to \mathcal{X}$ and a lift of the composition

$$x : S \to S' \to \mathcal{X}$$

to a map $y : S \to \mathcal{Y}$, there *exists* a lift of x' to a map $y' : S' \to \mathcal{Y}$.

Indeed, if $S \hookrightarrow S'$ is given by a map $T^*(S) \to \mathcal{F}$, the space of lifts as above is the space of null-homotopies of the resulting map

$$T^*_y(\mathcal{Y}/\mathcal{X}) \to \mathcal{F}.$$

However, the above map admits a null-homotopy since $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq -1}$ and the assumption that $\mathcal{Y} \to \mathcal{X}$ is formally smooth.

□

8. Consequences of admitting deformation theory

In this section we discuss further properties of prestacks that admit deformation theory.

8.1. **Digression: properties of maps of prestacks.** In this subsection we define several classes of morphisms of prestacks.

8.1.1. Let ${}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$ denote the category of (classical) reduced affine schemes. For a prestack

$$\mathcal{Y} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

or an object $\mathcal{Y} \in {}^{\mathrm{cl}}\mathrm{PreStk}$, let ${}^{\mathrm{red}}\mathcal{Y}$ denote its restriction to ${}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$, which we view as a functor

$$({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

We give the following definitions:

Definition 8.1.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in* ${}^{\mathrm{cl}}\mathrm{PreStk}$.

(a) *We shall say that f is a* closed embedding *if its base change by a classical affine scheme yields a closed embedding. I.e., if for $S_2 \in ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the fiber product $S_1 := S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1$, taken in* ${}^{\mathrm{cl}}\mathrm{PreStk}$, *belongs to* ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$, *and the map $S_1 \to S_2$ is a closed embedding.*

(b) *We shall say that f is a* nil-isomorphism *if it induces an isomorphism ${}^{\mathrm{red}}\mathcal{X}_1 \to {}^{\mathrm{red}}\mathcal{X}_2$. Equivalently, if for every $S_2 \in ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the map*

$${}^{\mathrm{red}}(S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1) \to S_2$$

(*the fiber product is taken in* ${}^{\mathrm{cl}}\mathrm{PreStk}$) *is an isomorphism.*

(c) *We shall say that f is* nil-closed *if for every $S_2 \in ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the map*

$${}^{\mathrm{red}}(S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1) \to {}^{\mathrm{red}}S_2$$

(*the fiber product is taken in* ${}^{\mathrm{cl}}\mathrm{PreStk}$) *is a closed embedding.*

(d) *We shall say that f is a* nilpotent embedding *if its base change by a classical affine scheme yields a nilpotent embedding. I.e., if in the situation of* (a), *the map $S_1 \to S_2$ is a nilpotent embedding of classical schemes.*

(d') *We shall say that f is a* pseudo-nilpotent embedding *if it is a nil-isomorphism and for every $S_2 \in ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, there exists a commutative diagram*

$$\begin{array}{ccc} S_1 & \longrightarrow & \mathcal{X}_1 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & \mathcal{X}_2 \end{array}$$

with $S_1 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and $S_1 \to S_2$ a nilpotent embedding.

Definition 8.1.3. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in* PreStk. *We shall say that f is a closed embedding (resp., nil-isomorphism, nil-closed, nilpotent embedding, pseudo-nilpotent embedding), if the corresponding map ${}^{\mathrm{cl}}\mathcal{X}_1 \to {}^{\mathrm{cl}}\mathcal{X}_2$ has the corresponding property in the classical setting.*

Clearly:

'closed embedding' $\Rightarrow$ 'nil-closed';

'nilpotent embedding' $\Rightarrow$ 'closed embedding';

'nilpotent embedding' $\Rightarrow$ 'nil-isomorphism' and 'pseudo-nilpotent embedding'.

'pseudo-nilpotent embedding' $\Rightarrow$ 'nil-isomorphism'.

8.1.4. The condition of being a pseudo-nilpotent embedding may appear a little obscure, but it turns out to be useful. We note, however, that due to the next proposition, the difference between 'nil-isomorphism' and 'pseudo-nilpotent embedding' only exists when our stacks are not locally of finite type:

Lemma 8.1.5. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a nil-isomorphism in ${}^{\mathrm{cl}}\mathrm{PreStk}$. Assume that $\mathcal{X}_2 \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$. Then f is a pseudo-nilpotent embedding.*

Proof. Let $S_2 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$, and let $S_2 \to \mathcal{X}_2$ be a map. We need to find an object in the category of diagrams

$$\begin{array}{ccc} S_1 & \longrightarrow & \mathcal{X}_1 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & \mathcal{X}_2, \end{array}$$

where $S_1 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and $S_1 \to S_2$ is a nilpotent embedding.

By the assumption on $\mathcal{X}_2$, we can assume that $S_2 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. In this case the required data is supplied by taking $S_1 := {}^{\mathrm{red}}S_2$. □

8.2. **Descent properties.** In this subsection we will show that one can deduce Zariski, Nisnevich or étale descent property of a prestack from the corresponding property at the classical level.

8.2.1. We will prove:

Proposition 8.2.2. *Let $\mathcal{X} \in \mathrm{PreStk}$ admit defomation theory, and let $\mathcal{X}_{0,\mathrm{cl}} \to {}^{\mathrm{cl}}\mathcal{X}$ be a pseudo-nilpotent embedding of classical prestacks.*

(a) *Assume that $\mathcal{X}_{0,\mathrm{cl}}$ satisfies Zariski (resp., Nisnevich) descent. Then $\mathcal{X}$ also has this property.*

(b) *Assume that $\mathcal{X}_{0,\mathrm{cl}}$ satisfies étale descent. Assume also that the pro-cotangent spaces of $\mathcal{X}$ are locally eventually connective. Then $\mathcal{X}$ also satisfies étale descent.*

8.2.3. *Proof of Proposition 8.2.2.* By convergence and Proposition 5.4.2, it is enough to show that if

$$S \hookrightarrow S'$$

is a map of affine schemes that has a structure square-zero extension, $x : S \to \mathcal{X}$ is a map and $\pi : \overset{\circ}{S} \to S$ is a Zariski (resp., Nisnevich, étale) cover, then the map

$$\mathrm{Maps}_{S/}(S', \mathcal{X}) \to \mathrm{Tot}(\mathrm{Maps}_{\overset{\circ}{S}{}^\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$$

is an isomorphism, where $\pi' : \overset{\circ}{S}{}' \to S'$ is the corresponding cover, and $\overset{\circ}{S}{}^\bullet$ (resp., $\overset{\circ}{S}{}'^\bullet$) is the Čech nerve of π, (resp., π').

We rewrite $\mathrm{Maps}_{S/}(S', \mathcal{X})$ and each $\mathrm{Maps}_{\overset{\circ}{S}{}^\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$ as in (6.2). So, $\mathrm{Maps}_{S/}(S', \mathcal{X})$ identifies with the space of null-homotopies of a certain map

$$T^*_x(\mathcal{X}) \to \mathcal{F}, \quad \mathcal{F} \in \mathrm{QCoh}(S)^{>-\infty}$$

and $\mathrm{Tot}(\mathrm{Maps}_{\overset{\circ}{S}{}^\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$ identifies with the totalization of the cosimplicial space of null-homotopies of the corresponding maps

$$T^*_x(\mathcal{X}) \to \mathcal{F}^\bullet,$$

where $\mathcal{F}^\bullet$ is the Čech resolution of $\mathcal{F}$ corresponding to π.

Note, however, that in the case of Zariski and Nisnevich covers, one can replace the totalization by a limit over a finite category. Now, the required isomorphism follows from the commutation of $\mathcal{M}aps(T^*_x(\mathcal{X}), -)$ with finite limits.

For an étale cover, if $T^*_x(\mathcal{X})$ belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq n})$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{\geq -k}$, we can replace the totalization by the limit over the $n+k$-skeleton. Hence, the required isomorphism again follows from the commutation of $\mathcal{M}aps(T^*_x(\mathcal{X}), -)$ with finite limits.

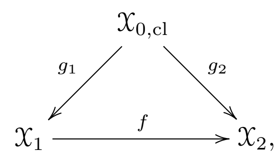
□

Remark 8.2.4. A recent result of Akhil Mathew shows that étale descent in Proposition 8.2.2 holds without the assumption of eventual connectivity.

8.3. **Isomorphism properties.** The property of having deformation theory can be used to show that certain maps between prestacks are isomorphisms.

8.3.1. We will prove:

Proposition 8.3.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map between objects of* $\mathrm{PreStk}_{\mathrm{def}}$. *Suppose that there exists a commutative diagram*



where g_1 and g_2 are pseudo-nilpotent embeddings, and $\mathcal{X}_{0,\mathrm{cl}} \in {}^{\mathrm{cl}}\mathrm{PreStk}$. Suppose also that for any $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and a map $x_0 : S \to \mathcal{X}_{0,\mathrm{cl}}$, for $x_i := g_i \circ x_0$, the induced map

$$T^*_{x_2}(\mathcal{X}_2) \to T^*_{x_1}(\mathcal{X}_1)$$

is an isomorphism. Then f is an isomorphism.

Proof. By induction and Proposition 5.4.2, we have to show that given $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $S \hookrightarrow S'$ that has a structure of square-zero extension, for a map $x_1 : S \to \mathcal{X}_1$, the space of extensions of x_1 to a map $S' \to \mathcal{X}_1$ maps isomorphically to the space of extensions of $x_2 := f \circ x_1$ to a map $S' \to \mathcal{X}_2$.

Deformation theory implies that the spaces in question are the spaces of null-homotopies of the corresponding maps

$$T^*_{x_1}(\mathcal{X}_1) \to \mathcal{F} \text{ and } T^*_{x_2}(\mathcal{X}_2) \to \mathcal{F},$$

respectively. Hence, it is enough to show that the map $T^*_{x_2}(\mathcal{X}_2) \to T^*_{x_1}(\mathcal{X}_1)$ is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

The assumption of the proposition implies that there exists a nilpotent embedding $g : \widetilde{S} \to S$, such that for $\widetilde{x}_i = x_i \circ g$, the map

$$T^*_{\widetilde{x}_2}(\mathcal{X}_2) \to T^*_{\widetilde{x}_1}(\mathcal{X}_1)$$

is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(\widetilde{S})^-)$. Therefore, it suffices to prove the following:

Lemma 8.3.3. *For a nilpotent embedding $g : \widetilde{S} \to S$, the functor*

$$\mathrm{Pro}(g^*) : \mathrm{Pro}(\mathrm{QCoh}(S)^-) \to \mathrm{Pro}(\mathrm{QCoh}(\widetilde{S})^-)$$

is conservative when restricted to ${}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

□

8.3.4. *Proof of Lemma 8.3.3.* First, we claim that if $\widetilde{S} \to S$ is a square-zero extension, then the functor

$$\text{Pro}(g^*) : \text{Pro}(\text{QCoh}(S)^-) \to \text{Pro}(\text{QCoh}(\widetilde{S})^-)$$

is conservative on all of $\text{Pro}(\text{QCoh}(S)^-)$.

Indeed, we need to show that if $\mathcal{T} \in \text{Pro}(\text{QCoh}(S)^-)$ is such that $\text{Maps}(\mathcal{T}, g_*(\widetilde{\mathcal{F}})) = 0$ for all $\widetilde{\mathcal{F}} \in \text{QCoh}(\widetilde{S})^-$, then $\text{Maps}(\mathcal{T}, \mathcal{F}) = 0$ for all $\mathcal{F} \in \text{QCoh}(S)^-$. However, this is obvious, since every object of $\text{QCoh}(S)^-$ is a two-step extension of objects in the essential image of g_*.

Hence, the functor $\text{Pro}(g^*)$ is conservative if $\widetilde{S} \to S$ can be written as a finite succession of square-zero extensions.

Using Proposition 5.5.3, we can construct a sequence of schemes

$$\widetilde{S} \to S_0 \to S_1 \to ... \to S_k \to ... \to S,$$

such that for every k, the map $\widetilde{S} \to S_k$ is a finite succession of square-zero extensions and the map $g_k : S_k \to S$ induces an isomorphism ${}^{\leq k}S_k \to {}^{\leq k}S$.

Let $\mathcal{T} \in^{\text{conv}} \text{Pro}(\text{QCoh}(S)^-)$ be in the kernel of $\text{Pro}(g^*)$. By the above, it is then in the kernel of each $\text{Pro}(g_k^*)$. Note that for $\mathcal{F} \in \text{QCoh}(S)^{\leq n}$, the map

$$\mathcal{F} \to (g_k)_* \circ (g_k)^*(\mathcal{F})$$

induces an isomorphism

$$\tau^{\geq n-k}(\mathcal{F}) \to \tau^{\geq n-k}((g_k)_* \circ (g_k)^*(\mathcal{F})).$$

Hence, by convergence, for $\mathcal{F} \in \text{QCoh}(S)^-$,

$$\text{Maps}(\mathcal{T}, \mathcal{F}) \simeq \lim_k \text{Maps}(\mathcal{T}, (g_k)_* \circ (g_k)^*(\mathcal{F})),$$

while each $\text{Maps}(\mathcal{T}, (g_k)_* \circ (g_k)^*(\mathcal{F}))$ vanishes.

□

8.3.5. From Proposition 8.3.2 we obtain:

Corollary 8.3.6. *Let*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g_1} & \mathcal{Y}_1 \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ \mathcal{X}_2 & \xrightarrow{g_2} & \mathcal{Y}_2 \end{array}$$

be a Cartesian square of objects of $\text{PreStk}_{\text{def}}$*, such that the horizontal maps are pseudo-nilpotent embeddings. Suppose that* f_X *is an isomorphism. Then* f_Y *is an isomorphism.*

9. A criterion for being locally almost of finite type

Deformation theory can be used to show that a prestack is locally almost of finite type, see Theorem 9.1.2 below.

9.1. **Statement of the result.** In this subsection we state Theorem 9.1.2 and make some initial observations.

9.1.1. The goal of this section is to prove the following:

Theorem 9.1.2. *Let $\mathcal{X}$ be an object of* $\mathrm{PreStk}_{\mathrm{def}}$*. Suppose that there exists a nilpotent embedding $\mathcal{X}_0 \to {}^{\mathrm{cl}}\mathcal{X}$, such that:*

- $\mathcal{X}_0 \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$*;*
- *For any $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $x : S \to \mathcal{X}_0$, we have $T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$.*

Then $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$.

As an immediate corollary, we obtain:

Corollary 9.1.3. *Let $\mathcal{X}$ be an object of* $\mathrm{PreStk}_{\mathrm{def}}$*. Suppose that ${}^{\mathrm{cl}}\mathcal{X} \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$, and that for any $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $x : S \to \mathcal{X}$, we have*

$$T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}.$$

Then $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$.

In addition, we will prove:

Theorem 9.1.4. *Let $\mathcal{X}$ be an object of* $\mathrm{PreStk}_{\text{laft-def}}$*. Then the fully faithful embedding functor*

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$$

is cofinal.

Remark 9.1.5. The assertion of Theorem 9.1.4 would be a tautology from the definition of $\mathrm{PreStk}_{\mathrm{laft}}$ if instead of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \subset \mathrm{Sch}^{\mathrm{aff}}$ we used ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

Remark 9.1.6. We note that the proof of Theorem 9.1.4 given in Sect. 9.6 will show that a prestack, satisfying the assumption of Corollary 9.1.3 satisfies the conclusion of Theorem 9.1.4. So, one can use the proof of Theorem 9.1.4 as an alternative (and quicker) way to prove Corollary 9.1.3.

Remark 9.1.7. The proof of Theorem 9.1.4 shows that for a not necessarily affine (but quasi-compact) sscheme Z equipped with a map to $\mathcal{X}$, the category of factorizations of this map as

$$Z \to Z' \to \mathcal{X}, \quad Z' \in \mathrm{Sch}_{\mathrm{aft}}$$

is contractible (in fact, the opposite category is filtered). Moreover, cofinal in this category is the subcategory consisting of those objects for which the map $Z \to Z'$ is affine.

9.1.8. From now until Sect. 9.6, we will be concerned with the proof of Theorem 9.1.2. We begin with the following observation:

Let $\mathcal{X}$ be any prestack, and assume that it is convergent. The condition that $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$ says that given $n \geq 0$ and an object $(S, x) \in ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, the category, denoted $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$, of factorizations of x as

$$S \to U \to \mathcal{X}, \quad U \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

is contractible.

Consider also the categories $\mathrm{Factor}(x, \mathrm{ft}, < \infty)$, $\mathrm{Factor}(x, \mathrm{aft})$ of factorizations of x as

$$S \to U \to \mathcal{X},$$

where we instead require that U belong to ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, respectively.

We have the fully faithful functors

$$\mathrm{Factor}(x, \mathrm{ft}, \leq n) \hookrightarrow \mathrm{Factor}(x, \mathrm{ft}, < \infty) \hookrightarrow \mathrm{Factor}(x, \mathrm{aft}),$$

and the map of $\text{Factor}(x, \text{ft}, \leq n)$ into both $\text{Factor}(x, \text{ft}, < \infty)$ and $\text{Factor}(x, \text{aft})$ admits a right adjoint, given by $S_0 \mapsto {}^{\leq n}S_0$.

Hence, $\text{Factor}(x, \text{ft}, \leq n)$ is contractible if and only if $\text{Factor}(x, \text{ft}, < \infty)$ is contractible and if and only if $\text{Factor}(x, \text{aft})$ is.

9.2. **Step 1.**

9.2.1. Suppose we have an object of $(S, x) \in ({}^{\leq n}\text{Sch}^{\text{aff}})_{/\mathcal{X}}$. We need to show that the category $\text{Factor}(x, \text{ft}, \leq n)$ is contractible.

Set $S_0 := {}^{\text{cl}}(S \underset{\mathcal{X}}{\times} \mathcal{X}_0)$. Let x_0 denote the resulting map $S_0 \to \mathcal{X}_0$. By assumption, the category $\text{Factor}(x_0, \text{ft}, \text{cl})$ is contractible.

We introduce the category $\mathbf{C}$ to be that of diagrams

$$\begin{array}{ccccc} S_0 & \longrightarrow & U_0 & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & U & \longrightarrow & \mathcal{X}, \end{array}$$

where $U \in {}^{\leq n}\text{Sch}^{\text{aff}}_{\text{ft}}$ and $U_0 \in {}^{\text{cl}}\text{Sch}^{\text{aff}}_{\text{ft}}$, and $U_0 \to U$ is an arbitrary map.

We have the natural forgetful functors

$$\text{Factor}(x, \text{ft}, \leq n) \leftarrow \mathbf{C} \to \text{Factor}(x_0, \text{ft}, \text{cl}).$$

We will show that both these functors are homotopy equivalences. This would imply that $\text{Factor}(x, \text{ft}, \leq n)$ is contractible.

9.2.2. The functor $\mathbf{C} \to \text{Factor}(x, \text{ft}, \leq n)$ is a co-Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, the fiber in question over a given $(S \to U \to \mathcal{X}) \in \text{Factor}(x, \text{ft}, \leq n)$ has a final object, namely, one with

$$U_0 := {}^{\text{cl}}(U \underset{\mathcal{X}}{\times} \mathcal{X}_0).$$

9.2.3. The functor $\mathbf{C} \to \text{Factor}(x_0, \text{ft}, \text{cl})$ is a Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

We note that the fiber of the above functor over a given $(S_0 \to U_0 \to \mathcal{X}_0) \in \text{Factor}(x_0, \text{ft}, \text{cl})$ can be described as follows.

Set $\widetilde{S} := S \underset{S_0}{\sqcup} U_0$. Since $\mathcal{X}$ admits deformation theory, we have a canonical map $\widetilde{x} : \widetilde{S} \to \mathcal{X}$.

The fiber in question is the category $\text{Factor}(\widetilde{x}, \text{ft}, \leq n)$ of factorizations of $\widetilde{x}$ as

$$\widetilde{S} \to U \to \mathcal{X}, \quad U \in {}^{\leq n}\text{Sch}^{\text{aff}}_{\text{ft}}.$$

9.3. **Resetting the problem.**

9.3.1. By Step 1, it suffices to prove the contractibility of the category $\operatorname{Factor}(x, \mathrm{ft}, \leq n)$ under the additional assumption that there exists a nilpotent embedding

$$S_0 \to S,$$

where $S_0 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

By Proposition 5.4.2, there exists a finite sequence of affine schemes

$$S_0 \to S_1 \to \ldots \to S_{k-1} \to S_k = S, \quad S_i \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}},$$

such that for every i, the map $S_i \hookrightarrow S_{i+1}$ has a structure of square-zero extension.

9.3.2. Repeating the manipulation of Step 1, by induction, we obtain that it suffices to prove the following: let S be an object of ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, and let $S \hookrightarrow S'$ be a square-zero extension, where $S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

Suppose we have a map $x : S' \to \mathcal{X}$. We need to show that the category $\operatorname{Factor}(x, \mathrm{ft}, \leq n)$ is contractible.

9.4. **Step 2.** Let $S \hookrightarrow S'$ be as in Sect. 9.3.2.

9.4.1. Let $\operatorname{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ be the category of factorizations of the map $x : S' \to \mathcal{X}$ as

$$S' \to \widetilde{S}' \to \mathcal{X},$$

where the composition $S \hookrightarrow \widetilde{S}'$ is equipped with a structure of square-zero extension, $S' \to \widetilde{S}'$ is equipped with a structure of map in $\mathrm{SqZ}(S)$, and where $\widetilde{S}' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Consider also the category $\mathbf{D}$ of factorizations of the map $x : S' \to \mathcal{X}$ as

$$S' \to \widetilde{S}' \to U \to \mathcal{X},$$

where the composition $S \to \widetilde{S}'$ is given a structure of square-zero extension, $S' \to \widetilde{S}'$ is given a structure of map in $\mathrm{SqZ}(S)$, and where $\widetilde{S}', U \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

We have the fogetful functors

$$\operatorname{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}} \leftarrow \mathbf{D} \to \operatorname{Factor}(x, \mathrm{ft}, \leq n).$$

We will show that both these functors are homotopy equivalences, whereas the category $\operatorname{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ is contractible. This will imply that $\operatorname{Factor}(x, \mathrm{ft}, \leq n)$ is contractible.

9.4.2. We note that the functor

$$\mathbf{D} \to \operatorname{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$$

is a Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, the fiber in question over a given $(S' \to \widetilde{S}' \to \mathcal{X}) \in \operatorname{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ has an initial point, namely, one with $U = \widetilde{S}'$.

9.4.3. The functor

$$\mathbf{D} \to \mathrm{Factor}(x, \mathrm{ft}, \leq n)$$

is a co-Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, we note that the fiber of $\mathbf{D}$ over an object $(S' \to U \to \mathcal{X}) \in \mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is the category $\mathrm{Factor}(u, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$, where u denotes the map $S' \to U$. I.e., this is a category of the same nature as $\mathrm{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$, but with $\mathcal{X}$ replaced by U.

Thus, it remains to prove the contractibilty of the category $\mathrm{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$.

9.5. **Step 3.**

9.5.1. Let the square-zero extension $(S \hookrightarrow S')$ be given by

$$T^*(S/\mathcal{X}) \xrightarrow{\gamma} \mathcal{F}, \quad \mathcal{F} \in \mathrm{QCoh}(S)^{\geq -n-1, \leq -1}.$$

The category $\mathrm{Factor}(x', \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ is that of factorizations of γ as

$$T^*(S/\mathcal{X}) \xrightarrow{\widetilde{\gamma}} \widetilde{\mathcal{F}} \to \mathcal{F},$$

where $\widetilde{\mathcal{F}} \in \mathrm{Coh}(S)^{\geq -n-1, \leq -1}$.

9.5.2. Note that $\mathcal{F}$ is isomorphic to the filtered colimit

$$\underset{\widetilde{\mathcal{F}} \in (\mathrm{Coh}(S)^{\geq -n-1, \leq -1})_{/\mathcal{F}}}{\mathrm{colim}} \widetilde{\mathcal{F}}.$$

Hence, in order to prove that $\mathrm{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ is contractible, it suffices to show that the functor

$$\mathcal{M}aps(T^*(S/\mathcal{X}), -) : \mathrm{QCoh}(S)^{\geq -n-1, \leq -1} \to \mathrm{Vect}$$

commutes with filtered colimits.

9.5.3. We have

$$T^*(S/\mathcal{X}) \simeq \mathrm{coFib}(T^*(S) \to T^*(\mathcal{X})|_S).$$

Since $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, it suffices to show that

$$T^*(\mathcal{X})|_S \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}.$$

This follows from the assumption on $\mathcal{X}$ and the next lemma:

Lemma 9.5.4. *If $i : S_0 \to S$ is a nilpotent embedding of objects of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, and $\mathcal{T}$ is an object of ${}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is such that*

$$(\mathrm{Pro}(i^*))(\mathcal{T}) \in \mathrm{Pro}(\mathrm{QCoh}(S_0)^-)_{\mathrm{laft}},$$

then $\mathcal{T} \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$.

9.5.5. *Proof of Lemma 9.5.4.* We need to show that the functor

$$\mathrm{Maps}(\mathcal{T}, -) : \mathrm{QCoh}(S)^{\heartsuit}[n] \to \mathrm{Spc}$$

commutes with filtered colimits for any n.

This allows to replace S and S_0 by ${}^{\mathrm{cl}}S$ and ${}^{\mathrm{cl}}S_0$, respectively. I.e, we can assume that S and S_0 are classical. Furthermore, by induction, we can assume that S is a classical square-zero extension of S_0. Now the required assertion follows from the fact that any $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$ can be written as an extension

$$0 \to i_*(\mathcal{F}') \to \mathcal{F} \to i_*(\mathcal{F}'') \to 0,$$

where $\mathcal{F}', \mathcal{F}'' \in \mathrm{QCoh}(S_0)^{\heartsuit}$ depend functorially on $\mathcal{F}$ (in fact, $\mathcal{F}'' := H^0(i^*(\mathcal{F}))$).

□

9.6. **Proof of Theorem 9.1.4.**

9.6.1. Suppose we have an object $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. We need to show that the category $\mathrm{Factor}(x, \mathrm{aft})$ (see Sect. 9.1.8) of factorizations

$$S \to U \to \mathcal{X}, \quad U \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$$

is contractible.

For every $n \geq 0$, consider the corresponding category $\mathrm{Factor}(x|_{{}^{\leq n}S}, \mathrm{ft}, \leq n)$ of factorizations

$${}^{\leq n}S \to U_n \to \mathcal{X}, \quad U_n \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

We note that since $\mathcal{X}$ is convergent, we have

$$\mathrm{Factor}(x, \mathrm{aft}) \simeq \lim_{n} \mathrm{Factor}(x|_{{}^{\leq n}S}, \mathrm{ft}, \leq n).$$

We will use the following observation:

Lemma 9.6.2. *Let*

$$\mathbf{C}_0 \leftarrow \mathbf{C}_1 \leftarrow \mathbf{C}_2 \leftarrow ...$$

be a sequence of $(\infty, 1)$*-categories. Assume that:*

(i) *The category* $\mathbf{C}_0$ *is filtered.*

(ii) *For every* n*, the category* $\mathbf{C}_{n+1}$ *is filtered* relative *to* $\mathbf{C}_n$.

Then the category

$$\mathbf{C} := \lim_{n} \mathbf{C}_n$$

is also filtered.

Let us recall that given a functor $\mathbf{C}' \to \mathbf{D}$, we say that $\mathbf{C}'$ is filtered *relative* to $\mathbf{D}$ if for every finite $(\infty, 1)$-category $\mathbf{K}$ and every diagram

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \mathbf{C}' \\ \downarrow & & \downarrow \\ \mathrm{Cone}(\mathbf{K}) & \longrightarrow & \mathbf{D} \end{array}$$

has a lifting property. Here $\mathrm{Cone}(\mathbf{K})$ is obtained from $\mathbf{K}$ by adjoining to it a final object. (For $\mathbf{D} = *$, we obtain the usual notion of $\mathbf{C}'$ being filtered.)

We apply the above lemma to

$$\mathbf{C}_n := (\mathrm{Factor}(x|_{{}^{\leq n}S}, \mathrm{ft}, \leq n))^{\mathrm{op}}.$$

9.6.3. To prove that the category $(\text{Factor}(x|_{^{cl}S}, \text{ft}, \text{cl}))^{\text{op}}$ is filtered we use the following lemma:

Lemma 9.6.4. *Let $\mathbf{C}' \to \mathbf{D}$ be a co-Cartesian fibration in groupoids. Suppose that $\mathbf{D}$ is filtered and $\mathbf{C}'$ is contractible. Then $\mathbf{C}'$ is also filtered.*

Proof. Let $\mathbf{C}' \to \mathbf{D}$ correspond to a functor $F : \mathbf{D} \to \text{Spc}$. Then the assumption that $\mathbf{C}'$ is contractible means that

$$\underset{\mathbf{D}}{\text{colim}}\, F \simeq *.$$

This is easily seen to imply the assertion of the lemma. □

We apply Lemma 9.6.4 to the functor

$$\text{Factor}(x|_{^{cl}S}, \text{ft}, \text{cl}) \to ({}^{cl}\text{Sch}^{\text{aff}})_{^{cl}S/}.$$

Indeed, the category $\text{Factor}(x|_{^{cl}S}, \text{ft}, \text{cl})$ is contractible because ${}^{cl}\mathcal{X}$ belongs to ${}^{cl}\text{PreStk}_{\text{lft}}$. The category (opposite) to $({}^{cl}\text{Sch}^{\text{aff}})_{/^{cl}S}$ is filtered by [Chapter I.2, Theorem 1.5.3(b)].

9.6.5. Let us now show that $(\text{Factor}(x|_{^{\leq n+1}S}, \text{ft}, \leq n+1))^{\text{op}}$ is filtered relative with respect to its projection $(\text{Factor}(x|_{^{\leq n}S}, \text{ft}, \leq n))^{\text{op}}$.

Suppose we have a functor

$$F_{n+1} : \mathbf{K} \to (\text{Factor}(x|_{^{\leq n+1}S}, \text{ft}, \leq n+1))^{\text{op}}$$

and its extension to a functor

$$\overline{F}_n : \text{Cone}(\mathbf{K}) \to (\text{Factor}(x|_{^{\leq n}S}, \text{ft}, \leq n))^{\text{op}}.$$

Let us denote by

$${}^{\leq n}S \to U_n \to \mathcal{X}$$

the object of $\text{Factor}(x|_{^{\leq n}S}, \text{ft}, \leq n)$ corresponding to the value of $\overline{F}_n$ on the final object $* \in \text{Cone}(\mathbf{K})$. For $\mathbf{k} \in \mathbf{K}$ denote also

$$U_{n+1}^{\mathbf{k}} = F_{n+1}(\mathbf{k}) \text{ and } U_n^{\mathbf{k}} = F_n(\mathbf{k}).$$

Set

$$U'_{n+1} := {}^{\leq n+1}S \underset{^{\leq n}S}{\sqcup} U_n.$$

We have ${}^{\leq n}U'_{n+1} = U_n$ and a $\mathbf{K}$-diagram of maps

$$U'_{n+1} \to U_{n+1}^{\mathbf{k}} \to \mathcal{X}. \tag{9.1}$$

We need to show that there exists $U_{n+1} \in {}^{\leq n+1}\text{Sch}^{\text{aff}}_{\text{ft}}$ equipped with a map

$$U'_{n+1} \to U_{n+1}$$

that induces an isomorphism on n-truncations, such that the $\mathbf{K}$-diagram (9.1) extends to a diagram

$$U'_{n+1} \to U_{n+1} \to U_{n+1}^{\mathbf{k}} \to \mathcal{X} \tag{9.2}$$

that induces at the level of n-truncations the diagram

$$U_n = U_n \to U_n^{\mathbf{k}} \to \mathcal{X}$$

9.6.6. Let $f_{\mathbf{k}}$ denote the map ${}^{\leq n}S \to U_n^{\mathbf{k}}$. By Proposition 5.4.2(b), the map $U_n^{\mathbf{k}} \to U_{n+1}^{\mathbf{k}}$ has a canonical structure of square-zero extension by means of some $\mathcal{J}^{\mathbf{k}} \in \mathrm{QCoh}(U_n^{\mathbf{k}})^{\heartsuit}[n+1]$. Similarly, $U_n \to U'_{n+1}$ has a canonical structure of square-zero extension by means of some $\mathcal{I}' \in \mathrm{QCoh}(U_n)^{\heartsuit}[n+1]$.

Then the datum of the diagram (9.1) is equivalent to that of the commutative diagram

$$\begin{array}{ccc} \mathrm{Pro}(f_{\mathbf{k}}^*)(T^*(U_n^{\mathbf{k}}/\mathcal{X})) & \longrightarrow & T^*(U_n/\mathcal{X}) \\ \downarrow & & \downarrow \\ f_{\mathbf{k}}^*(\mathcal{J}^{\mathbf{k}}) & \longrightarrow & \mathcal{I}' \end{array}$$

and its extension to a diagram (9.2) is equivalent to factoring the above commutative diagram as

$$\begin{array}{ccccc} \mathrm{Pro}(f_{\mathbf{k}}^*)(T^*(U_n^{\mathbf{k}}/\mathcal{X})) & \longrightarrow & T^*(U_n/\mathcal{X}) & & \\ \downarrow & & \downarrow & & \\ f_{\mathbf{k}}^*(\mathcal{J}^{\mathbf{k}}) & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{I}', \end{array}$$

where $\mathcal{I} \in \mathrm{Coh}(U_n)^{\heartsuit}[n+1]$.

The existence of such an extension follows from the combination of the following facts:

(i) The category $\mathbf{K}$ is finite;

(ii) The objects $\mathrm{Pro}(f_{\mathbf{k}}^*)(T^*(U_n^{\mathbf{k}}/\mathcal{X}))$ belong to $\mathrm{Pro}(S^-)_{\mathrm{laft}}$;

(iii) $\mathcal{I}'$ can be written as filtered colimit of $\mathcal{I}$ with $\mathcal{I} \in \mathrm{Coh}(U_n)^{\heartsuit}[n+1]$.

10. Square-zero extensions of prestacks

This section is auxilliary (it will be needed in [Chapter IV.2, Sect. 2.5]), and can be skipped on first pass. We define and (attempt to) classify square-zero extensions of a given prestack $\mathcal{X}$ by an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$.

10.1. **The notion of square-zero extension of a prestack.** We define the notion of square-zero extension of a prestack via pullback to affine schemes.

10.1.1. Let $\mathcal{X}$ be a prestack and let $\mathcal{I}$ be an object of $\mathrm{QCoh}(\mathcal{X})^{\leq 0}$ (i.e., $\mathcal{I}$ is an object of $\mathrm{QCoh}(\mathcal{X})$, whose pullback to every affine scheme is connective.)

We define the notion of *square-zero extension of* $\mathcal{X}$ *by means of* $\mathcal{I}$ to be the datum of a schematic affine map of prestacks $\mathcal{X} \hookrightarrow \mathcal{X}'$, and an assignment for every $(S', x') \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}'}$ of a structure on the map

$$S' \underset{\mathcal{X}'}{\times} \mathcal{X} =: S \hookrightarrow S'$$

of square-zero extension of S by means of $x^*(\mathcal{I})$ (where x is the resulting map $S \to \mathcal{X}$), which is functorial in (S', x') in the sense of Proposition 5.3.2.

Square-zero extensions of $\mathcal{X}$ by means of $\mathcal{I}$ form a space that we denote by $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$.

10.1.2. Let

$$(\text{PreStk}, \text{QCoh}^{\leq 0}) \to \text{PreStk}$$

denote the Cartesian fibration corresponding to the functor

$$(\text{QCoh}^{\leq 0})^*_{\text{PreStk}} : (\text{PreStk})^{\text{op}} \to 1\text{-Cat}\,.$$

The construction of Proposition 5.3.2 defines a Cartesian fibration in spaces

$$\text{SqZ}(\text{PreStk}) \to (\text{PreStk}, \text{QCoh}^{\leq 0}),$$

whose fiber over a given $(\mathcal{X}, \mathcal{I}) \in (\text{PreStk}, \text{QCoh}^{\leq 0})$ is $\text{SqZ}(\mathcal{X}, \mathcal{I})$.

In particular, given an index category I, and an I-family

$$i \mapsto (\mathcal{X}_i, \mathcal{I}_i), \quad I \to (\text{PreStk}, \text{QCoh}^{\leq 0}), \tag{10.1}$$

we have a well-defined notion of an I-family of maps $\mathcal{X}_i \hookrightarrow \mathcal{X}'_i$, equipped with a structure of square-zero extension by means of $\mathcal{I}_i$, covering (10.1).

10.2. **From square-zero extensions to maps in** QCoh**.** In this subsection we will assume that $\mathcal{X}$ admits a pro-cotangent complex. We will show that a square-zero extension of $\mathcal{X}$ gives rise to a map in $\text{QCoh}(\mathcal{X})$.

10.2.1. We claim that there is a natural map of spaces

$$\text{SqZ}(\mathcal{X}, \mathcal{I}) \to \text{Maps}(T^*(\mathcal{X}), \mathcal{I}[1]), \tag{10.2}$$

where we regard $T^*(\mathcal{X})$ and $\mathcal{I}$ as objects of $\text{Pro}(\text{QCoh}(\mathcal{X})^-)^{\text{fake}}$, see Sect. 4.3.1.

To construct (10.2), given a map $\mathcal{X} \hookrightarrow \mathcal{X}'$, equipped with a structure of square-zero extension, and $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$ we need to construct the corresponding map

$$T^*_x(\mathcal{X}) \to x^*(\mathcal{I})[1]$$

in $\text{Pro}(\text{QCoh}(S)^-)$, functorially in (S, x).

10.2.2. We will use the following lemma:

Lemma 10.2.3. *For a schematic affine map of prestacks $\mathcal{X} \to \mathcal{X}'$, the functor*

$$(\text{Sch}^{\text{aff}})_{/\mathcal{X}'} \to (\text{Sch}^{\text{aff}})_{/\mathcal{X}}, \quad S' \mapsto S' \underset{\mathcal{X}'}{\times} \mathcal{X}$$

is cofinal.

Proof. The functor in question admits a left adjoint, given by

$$(S \to \mathcal{X}) \mapsto (S \to \mathcal{X} \to \mathcal{X}').$$

□

10.2.4. Using the lemma, it suffices to construct the map

$$T^*_x(\mathcal{X}) \to x^*(\mathcal{I})[1],$$

for every $(S', x') \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}'}$, where

$$x : S = S' \underset{\mathcal{X}}{\times} \mathcal{X} \to \mathcal{X}.$$

The latter is given as the composition

$$T^*_x(\mathcal{X}) \overset{(dx)^*}{\longrightarrow} T^*(S) \to x^*(\mathcal{I})[1],$$

where the second arrow represents the structure of square-zero extension on $S \hookrightarrow S'$.

10.2.5. The following assertion results from the definitions:

Lemma 10.2.6. *Let $\mathcal{Z}$ be a prestack that admits deformation theory, and let $z : \mathcal{X} \to \mathcal{Z}$ be a map. Then for a map $\mathcal{X} \hookrightarrow \mathcal{X}'$ equipped with a structure of square-zero extension by means of $\mathcal{I} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$, the space of extensions of z to a map $z' : \mathcal{X}' \to \mathcal{Z}$ is canonically equivalent to that of null-homotopies of the composed map*

$$z^*(T^*(\mathcal{Z})) \overset{(dz)^*}{\longrightarrow} T^*(\mathcal{X}) \to \mathcal{I}[1].$$

10.3. **Classifying square-zero extensions.** In this subsection we keep the assumption that $\mathcal{X}$ admits deformation theory. We will (try to) classify square-zero extensions of $\mathcal{X}$.

10.3.1. We would like to address the following general question:

Question 10.3.2. *Is it true that the functor* $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}[1])$ *of* (10.2) *is an isomorphism of spaces?*

Unfortunately, we can't answer this question in general. In this subsection we will consider a certain particular case.

Remark 10.3.3. In [Chapter IV.4, Sect. 5.5] we will provide a far more satisfying answer under the assumption that $\mathcal{X}$ be locally almost of finite type.

10.3.4. Let Y be an object of $\mathrm{Sch}^{\mathrm{aff}}$, and let $Y \hookrightarrow Y'$ be given a structure of square-zero extension by means of $\mathcal{I}_Y \in \mathrm{QCoh}(Y)^{\leq 0}$. Let

$$\gamma_Y : T^*(Y) \to \mathcal{I}_Y[1]$$

be the corresponding map.

Fix a map $f : \mathcal{X} \to Y$, and denote $\mathcal{I}_X := f^*(\mathcal{I}_Y)$. Consider the space

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\,\mathrm{SqZ}(Y, \mathcal{I}_Y)}$$

that classifies maps $\mathcal{X} \hookrightarrow \mathcal{X}'$ equipped with a structure of square-zero extension by means of $\mathcal{I}_{\mathcal{X}}$, and a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array} \tag{10.3}$$

equipped with a structure of map of square-zero extensions that corresponds to the tautological map $f^*(\mathcal{I}_Y) \to \mathcal{I}_{\mathcal{X}}$.

Consider the space

$$\mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y}$$

that classifies maps

$$\gamma_{\mathcal{X}} : T^*(\mathcal{X}) \to \mathcal{I}_{\mathcal{X}}[1],$$

together with the data of commutativity of the diagram

$$\begin{array}{ccc} T^*(\mathcal{X}) & \xrightarrow{\gamma_{\mathcal{X}}} & \mathcal{I}_{\mathcal{X}}[1], \\ {\scriptstyle (df)^*}\uparrow & & \uparrow{\scriptstyle \sim} \\ f^*(T^*(Y)) & \xrightarrow{\gamma_Y} & f^*(\mathcal{I}_Y)[1]. \end{array}$$

As in Sect. 10.2, we have a canonically defined functor

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\,\mathrm{SqZ}(Y, \mathcal{I}_Y)} \to \mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y}. \tag{10.4}$$

We claim:

Proposition 10.3.5. *The functor* (10.4) *is an isomorphism of spaces. Furthermore, for every object of* $\mathrm{SqZ}(\mathcal{X},\mathcal{I}_{\mathcal{X}})_{/\,\mathrm{SqZ}(Y,\mathcal{I}_Y)}$ *the diagram* (10.3) *is Cartesian.*

The rest of the subsection is devoted to the proof of this proposition.

10.3.6. We construct the map

$$\mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}),\mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y} \to \mathrm{SqZ}(\mathcal{X},\mathcal{I}_{\mathcal{X}})_{/\,\mathrm{SqZ}(Y,\mathcal{I}_Y)} \tag{10.5}$$

as follows.

Given $\gamma : T^*(\mathcal{X}) \to \mathcal{I}_{\mathcal{X}}[1]$, we construct the prestack $\mathcal{X}'$ by letting for $S' \in \mathrm{Sch}^{\mathrm{aff}}$ the space $\mathrm{Maps}(S',\mathcal{X}')$ consist of the data of:

- $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$;
- A map $S \to S'$;
- A factorization of the map $x^*(\gamma) : T^*_x(\mathcal{X}) \to x^*(\mathcal{I}_{\mathcal{X}})[1]$ as
$$T^*_x(\mathcal{X}) \overset{(dx)^*}{\longrightarrow} T^*(S) \overset{\gamma_S}{\longrightarrow} x^*(\mathcal{I}_{\mathcal{X}})[1],$$
- An isomorphism $\mathrm{RealSqZ}(x^*(\mathcal{I}_{\mathcal{X}})[1],\gamma_S) \simeq (S \hookrightarrow S')$ in $\mathrm{Sch}^{\mathrm{aff}}_{S'/}$.

Note that the construction of $\mathcal{X}'$ does not appeal to the datum of the map $Y \hookrightarrow Y'$ or a structure on it of square-zero extension.

10.3.7. That above the datum of $\mathrm{Maps}(S',\mathcal{X}')$ can be rewritten as follows:

- A map $y' : S' \to Y'$ (denote $S := S' \underset{Y'}{\times} Y$, $y : S \to Y$ and $\gamma_S : T^*(S) \to y^*(\mathcal{I}_Y)[1]$);
- A factorization of y as $S \overset{x}{\longrightarrow} \mathcal{X} \overset{f}{\longrightarrow} Y$;
- A datum of homotopy between
$$x^*(T^*(\mathcal{X})) = T^*_x(\mathcal{X}) \overset{(dx)^*}{\longrightarrow} T^*(S) \overset{\gamma_S}{\longrightarrow} y^*(\mathcal{I}_Y)[1] \simeq x^*(\mathcal{I}_{\mathcal{X}})[1]$$
and $x^*(\gamma_{\mathcal{X}})$.

10.3.8. The latter description implies that the space consisting of a data of a map $x' : S' \to \mathcal{X}'$ *and* a map $S' \to S$, which is the *left* inverse of the map $S \to S'$ identifies canonically with the space $\mathrm{Maps}(S',\mathcal{X})$. Indeed, given a map $S' \to \mathcal{X}'$, both pieces of additional data amount to that of null-homotopy of the map $x^*(\gamma_{\mathcal{X}})$.

This gives rise to a canonical map $\mathcal{X} \to \mathcal{X}'$, such that for every $x' : S' \to \mathcal{X}$, the corresponding diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle x'} \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

is Cartesian.

This gives the map $\mathcal{X} \hookrightarrow \mathcal{X}'$ a structure of square-zero extension by means of $\mathcal{I}_X$, thereby providing a map in (10.5). Furthermore, the diagram (10.3) is Cartesian also by construction. □

10.4. **Deformation theory property of square-zero extensions.** In this subsection we let $\mathcal{X}$ and $Y \hookrightarrow Y'$ be as in Proposition 10.3.5. We will show that pretacks $\mathcal{X}'$ as in Proposition 10.3.5 themselves admit deformation theory.

10.4.1. Our goal is to show:

Proposition 10.4.2. *For every object of* $\mathrm{SqZ}(\mathcal{X},\mathcal{I}_\mathcal{X})_{/\,\mathrm{SqZ}(Y,\mathcal{I}_Y)}$ *we have:*
(a) *The prestack* $\mathcal{X}'$ *admits deformation theory.*
(b) *If* $Y, Y' \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ *and* $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$*, then* $\mathcal{X}' \in \mathrm{PreStk}_{\mathrm{laft}}$.

The rest of this subsection is devoted to the proof of this proposition.

10.4.3. First, we note that point (a) implies point (b):

We apply Theorem 9.1.2 to the nilpotent embedding $\mathcal{X} \hookrightarrow \mathcal{X}'$. It suffices to show that for any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$, the pullback of $T^*(\mathcal{X}/\mathcal{X}')$ under x belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$.

However, this pullback identifies with the pullback of $T^*(\mathcal{X}/\mathcal{X}')$ under $y := f \circ x$ of $T^*(Y/Y')$, and the assertion follows.

10.4.4. *Convergence.*

Using the interpretation of the space $\mathrm{Maps}(S', \mathcal{X}')$ given in Sect. 10.3.7, in order to prove that $\mathcal{X}'$ is convergent, we need to show that the space of homotopies between two fixed maps

$$x^*(T^*(\mathcal{X})) \rightrightarrows y^*(\mathcal{I}_Y)[1]$$

is mapped isomorphically to the inverse limit over n over similar spaces for

$$S_n := ({}^{\leq n}S') \underset{Y'}{\times} Y.$$

Note that for any n, we have:

$${}^{\leq n}(S_n) \simeq {}^{\leq n}S.$$

Hence, the required assertion follows from the fact that

$$x^*(T^*(\mathcal{X})) = T^*_x(\mathcal{X}) \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

(see Lemma 3.3.3).

10.4.5. *Compatibility with push-outs.* Let $\widetilde{S}'_2 := \widetilde{S}'_1 \underset{S'_1}{\sqcup} S'_2$ be a push-out in $\mathrm{Sch}^{\mathrm{aff}}$, where $S'_1 \to \widetilde{S}'_1$ is a nilpotent embedding. Let us show that the map

$$\mathrm{Maps}(\widetilde{S}'_2, \mathcal{X}') \to \mathrm{Maps}(\widetilde{S}'_1, \mathcal{X}') \underset{\mathrm{Maps}(S'_1, \mathcal{X}')}{\times} \mathrm{Maps}(S'_2, \mathcal{X}')$$

is an isomorphism.

It suffices to show that the map in question is an isomorphism over a given point of

$$\mathrm{Maps}(\widetilde{S}'_2, Y') \simeq \mathrm{Maps}(\widetilde{S}'_1, Y') \underset{\mathrm{Maps}(S'_1, Y')}{\times} \mathrm{Maps}(S'_2, Y').$$

Set

$$S_1 := S'_1 \underset{Y'}{\times} Y,\ S_2 := S'_2 \underset{Y'}{\times} Y,\ \widetilde{S}_1 := \widetilde{S}'_1 \underset{Y'}{\times} Y,\ \widetilde{S}_2 := \widetilde{S}'_2 \underset{Y'}{\times} Y.$$

It is easy to see that the map

$$\widetilde{S}_1 \underset{S_1}{\sqcup} S_2 \to \widetilde{S}_2$$

is an isomorphism.

Using the interpretation of the space $\mathrm{Maps}(-, \mathcal{X}')$ given in Sect. 10.3.7, we obtain that it suffices to show that, given a map $x : \widetilde{S}_2 \to \mathcal{X}$, the space of homotopies between two given maps

$$x^*(T^*(\mathcal{X})) \to y^*(\mathcal{I}_Y)[1]$$

maps isomorphically to the fiber product of the corresponding spaces on $\widetilde{S}_1$ and S_2 over that on S_1.

However, this follows from Proposition 1.4.2.