CHAPTER II.2. IND-COHERENT SHEAVES AS A FUNCTOR OUT OF THE CATEGORY OF CORRESPONDENCES

Contents

Introduction

0.1. The !-pullback and base change.

0.1.1. In [Chapter II.1] we constructed the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad X \rightsquigarrow \mathrm{IndCoh}(X), \quad (X \xrightarrow{f} Y) \rightsquigarrow f_*^{\mathrm{IndCoh}}. \tag{0.1}$$

In addition, we constructed the functors

$$\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} : ((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{0.2}$$

$$X \rightsquigarrow \mathrm{IndCoh}(X), \quad (X \xrightarrow{f} Y) \rightsquigarrow f^!$$

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and

$$\text{IndCoh}^*_{(\text{Sch}_{\text{aft}})_{\text{open}}} : ((\text{Sch}_{\text{aft}})_{\text{open}})^{\text{op}} \to \text{DGCat}_{\text{cont}}, \quad X \rightsquigarrow \text{IndCoh}(X), \quad (X \overset{f}{\to} Y) \rightsquigarrow f^{\text{IndCoh},*}, \tag{0.3}$$

where (0.2) is obtained from (0.1) by passing to *right* adjoints along proper maps, and (0.3) is obtained from (0.1) by passing to *left* adjoints along open embeddings.

The goal of the present chapter is to combine the above functors to a single piece of structure.

0.1.2. It is easy to phrase (but not to prove!) what it means to combine the functors (0.2) and (0.3): we will have a single functor

$$\text{IndCoh}^!_{\text{Sch}_{\text{aft}}} : (\text{Sch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}, \quad X \rightsquigarrow \text{IndCoh}(X), \quad (X \overset{f}{\to} Y) \rightsquigarrow f^!. \tag{0.4}$$

It is trickier to say what kind of structure encodes both (0.1) and (0.4). The idea that we want to express that these two functors are compatible via base change. I.e., for a Cartesian diagram in Sch

$$\begin{array}{ccc} X' & \xrightarrow{g_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g_Y} & Y \end{array} \tag{0.5}$$

we want to be *given* an isomorphism of functors

$$g^!_Y \circ f^{\text{IndCoh}}_* \simeq (f')^{\text{IndCoh}}_* \circ g^!_X. \tag{0.6}$$

The problem is that for a general diagram (0.5), there is no adjunction that gives rise to a map in (0.6) in either direction. Namely, if g_Y is proper, the natural map points $\leftarrow$, and when g_Y is an open embedding, the natural map points $\to$.

So, in general, the isomorphism (0.6) is really an additional piece of data, and once we want to say that these isomorphisms are compatible with the compositions of f's a and g's, we need to specify what we mean by that, i.e., what a *homotopy-compatible* system of isomorphisms means in this case.

0.1.3. Here enters the idea of the *category of correspondences*, suggested to us by J. Lurie, and developed in [Chapter V.1]. This is the category, denoted $\text{Corr}(\text{Sch}_{\text{aft}})$, whose objects are $X \in \text{Sch}_{\text{aft}}$, and whose 1-morphisms are diagrams

$$\begin{array}{ccc} X_{0,1} & \xrightarrow{g} & X_0 \\ {\scriptstyle f}\downarrow & & \\ X_1. & & \end{array} \tag{0.7}$$

Compositions of 1-morphisms are given by fiber products: the composition of (0.7) with the 1-morphism

$$\begin{array}{ccc} X_{1,2} & \longrightarrow & X_1 \\ \downarrow & & \\ X_2 & & \end{array}$$

is given by the diagram

$$\begin{array}{ccc} X_{0,2} & \longrightarrow & X_0 \\ \big\downarrow & & \\ X_2, & & \end{array}$$

where $X_{0,2} := X_{1,2} \underset{X_1}{\times} X_{0,1}$. We refer the reader to [Chapter V.1], where it is explained how to define $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})$ as an ∞-category.

0.1.4. The main goal of this chapter is to define IndCoh as a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{DGCat}_{\mathrm{cont}} \tag{0.8}$$

that, at the level of objects sends $X \rightsquigarrow \mathrm{IndCoh}(X)$, and at the level of 1-morphisms sends the diagram (0.7) to $f_*^{\mathrm{IndCoh}} \circ g^!$.

The functor in (0.8) will encode the initial functor (0.1) by restricting to the 1-full subcategory of $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})$, where we only allow 1-morphisms (0.7) with α being an isomorphism (this subcategory is tautologically equivalent to $\mathrm{Sch}_{\mathrm{aft}}$).

The functor in (0.8) will encode the initial functor (0.4) by restricting to the 1-full subcategory of $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})$, where we only allow 1-morphisms (0.7) with β being an isomorphism (this subcategory is tautologically equivalent to $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$).

0.1.5. In order to construct the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})}$ we will apply the machinery developed in [Chapter V.1]. It turns out that the data of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})}$ is *uniquely recovered* from the data of the functor $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$ of (0.1).

However, there is one caveat: in order for the uniqueness statement mentioned above to hold, and in order to perform the construction of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})}$, one needs to work not with $(\infty,1)$-category $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})$, but with the $(\infty,2)$-category $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}$.

The latter $(\infty,2)$-category is one where we allow non-invertible 2-morphisms of the following kind: a 2-morphism from the 1-morphism (0.7) to the 1-morphism

$$\begin{array}{ccc} X'_{0,1} & \xrightarrow{g'} & X'_0 \\ {\scriptstyle f'}\big\downarrow & & \\ X'_1 & & \end{array}$$

is a commutative diagram

$$\begin{array}{ccccc} X_{0,1} & & \xrightarrow{\quad f \quad} & & X_0 \\ & \searrow^{h} & & \nearrow_{f'} & \\ & & X'_{0,1} & & \\ & {\scriptstyle g}\searrow & \big\downarrow{\scriptstyle g'} & & \\ & & X_1, & & \end{array} \tag{0.9}$$

where h is proper.

What we will actually construct is the functor

$$\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}} : \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}} . \tag{0.10}$$

The additional piece of data that is contained in $\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}}$ as compared to $\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})}$ is that of *adjunction* between $f^!$ and f^{IndCoh}_* for a proper morphism f.

0.2. **The !-pullback and** IndCoh **on prestacks.** Having constructed the functor

$$\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}} : \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}} \to \text{DGCat}_{\text{cont}},$$

we restrict it to $(\text{Sch}_{\text{aft}})^{\text{op}} \subset \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}$ and obtain the functor

$$\text{IndCoh}^!_{\text{Sch}_{\text{aft}}} : (\text{Sch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

of (0.4).

Remark 0.2.1. My emphasize that even if one is only interested in the functor (0.4), one has to employ the machinery of $(\infty, 2)$-categories of correspondences in order to construct it.

0.2.2. Starting with the functor $\text{IndCoh}^!_{\text{Sch}_{\text{aft}}}$, we will *right-Kan-extend* it to a functor

$$\text{IndCoh}^!_{\text{PreStk}_{\text{aft}}} : (\text{PreStk}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}} .$$

I.e., we now have a well-defined category $\text{IndCoh}(\mathcal{X})$ for $\mathcal{X} \in \text{PreStk}_{\text{aft}}$.

0.2.3. The assignment

$$\mathcal{X} \rightsquigarrow \text{IndCoh}(\mathcal{X})$$

provides a theory of $\mathcal{O}$-modules on prestacks, that exists alongside of QCoh; the former is functorial with respect to the !-pullback, while the assignment

$$\mathcal{X} \rightsquigarrow \text{QCoh}(\mathcal{X})$$

is functorial with respect to the *-pullback.

In [Chapter II.3, Sect. 3.3] we will see that the categories $\text{QCoh}(\mathcal{X})$ and $\text{QCoh}(\mathcal{X})$ are related by a functor

$$\Upsilon_{\mathcal{X}} : \text{QCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \omega_{\mathcal{X}}, \tag{0.11}$$

where $\omega_{\mathcal{X}} \in \text{IndCoh}(\mathcal{X})$ is the dualizing object, and $\otimes$ is the action of $\text{QCoh}(\mathcal{X})$ on $\text{IndCoh}(\mathcal{X})$, defined in [Chapter II.3, Sect. 3.3].

However, in general, the functor $\Upsilon_{\mathcal{X}}$ is far from being an equivalence.

Remark 0.2.4. Recall that when $\mathcal{X} = X$ is a scheme, we have a different functor relating $\text{IndCoh}(X)$ and $\text{QCoh}(X)$, namely

$$\Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X)$$

(this functor was instrumental of getting the theory of IndCoh off the ground; we used it in order to defined the *-push forward functors for IndCoh).

In [Chapter II.3, Sect. 4.4] we will see that the functors

$$\Psi_X : \text{IndCoh}(X) \to \text{QCoh}(X) \text{ and } \Upsilon_X : \text{IndCoh}(X) \to \text{QCoh}(X)$$

are naturally *duals* of one another.

However, for a general prestack $\mathcal{X}$, only the functor $\Upsilon_{\mathcal{X}}$ makes sense; the functor $\Psi_{\mathcal{X}}$ is a feature of schemes (or, more generally, Artin stacks).

0.2.5. Here is a typical manifestation of the usefulness of the category $\text{IndCoh}(\mathcal{X})$ for a prestack $\mathcal{X}$.

Let us take $\mathcal{X}$ to be an *ind-scheme*. In this case, $\text{IndCoh}(\mathcal{X})$ is compactly generated by the direct images of $\text{Coh}(X)$ for closed subschemes $X \to \mathcal{X}$.

This is while it is not clear (and probably not true) that $\text{QCoh}(\mathcal{X})$ is compactly generated.

Note, however, that in [GaRo1, Theorem 10.1.1] it is proved that if $\mathcal{X}$ is a *formally smooth*, ind-scheme then the functor $\Upsilon_{\mathcal{X}}$ of (0.11) is an equivalence.

0.2.6. Let us now take $\mathcal{X} = X_{\text{dR}}$, where $X \in \text{Sch}_{\text{aft}}$. It is shown in [GaRo2, Proposition 2.4.4] that in this case the functor

$$\Upsilon_{X_{\text{dR}}} : \text{QCoh}(X_{\text{dR}}) \to \text{IndCoh}(X_{\text{dR}})$$

is an equivalence.

We can view

$$\text{QCoh}(X_{\text{dR}}) =: \text{D-mod}(X) := \text{IndCoh}(X_{\text{dR}})$$

as the two incarnations of the category of D-modules on X: as 'left D-modules' and as 'right D-modules'. Correspondingly, we have the two forgetful functors

$$\text{D-mod}(X) = \text{QCoh}(X_{\text{dR}}) \to \text{QCoh}(X) \text{ and } \text{D-mod}(X) = \text{IndCoh}(X_{\text{dR}}) \to \text{IndCoh}(X),$$

corresponding to the *- and !-pullback, respectively, with respect to the map $X \to X_{\text{dR}}$.

In [GaRo2, Sects. 2-4] is it shown that the above 'right' forgetful functor $\text{D-mod}(X) \to \text{IndCoh}(X)$ has much better properties than the the 'left' forgetful functor $\text{D-mod}(X) \to \text{IndCoh}(X)$.

This is closely related to the fact that the category $\text{IndCoh}(X^\wedge)$ is better behaved (see Sect. 0.2.5 above) than the category $\text{QCoh}(X^\wedge)$, where $X^\wedge$ is the ind-scheme

$$X^\wedge := X \underset{X_{\text{dR}}}{\times} X,$$

i.e., the formal completion of X in $X \times X$.

0.3. **What is done in this chapter?**

0.3.1. In Sect. 1 we collect some geometric preliminaries needed for the proof of the main theorem (Theorem 2.1.4) in Sect. 2.

Namely, we show that the operation of the *closure of the image of a morphism* is well-behaved in the context of derived algebraic geometry. Specifically, for a morphism $X \overset{f}{\to} Y$, its closure is the initial object in the category of factorizations of f as

$$X \to X' \overset{f'}{\to} Y,$$

where f' is a closed embedding (i.e., the corresponding map of classical schemes ${}^{\text{cl}}X' \to {}^{\text{cl}}Y$ is a closed embedding).

The main result of this section is Proposition 1.3.2, which establishes the *transitivity property* of the operation of closure of the image of a morphism. Namely, it says that for a composition of morphisms

$$X \overset{f}{\to} Y \overset{g}{\to} Z,$$

if Y' denotes the closure of the image of f, and $g' := g|_{Y'}$, then the canonical map from the closure of the image of $g \circ f$ maps isomorphically to the closure of the image of g'.

0.3.2. The central section of this Chapter is Sect. 2, where we construct the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}}$$

of (0.10), starting from the functor $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$ of (0.1). This is done by applying [Chapter V.1, Theorem 5.2.4].

In order to apply this theorem, we need to check one condition of geometric nature. Namely, we need to establish a derived version of the Nagata compactification theorem: given a morphism between schemes $X \xrightarrow{f} Y$, the *category of its factorizations* as

$$X \xrightarrow{j} Z \xrightarrow{g} Y$$

with j an open embedding and g is proper is *contractible*.

0.3.3. In Sect. 3 we study the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

of (0.4) that is obtained from $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}}$ by restriction to the 1-full subcategory

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}},$$

see Sect. 0.1.4.

The main point of this section is that, having the functor $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ at our disposal, we can extend it to a functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

The latter extension procedure is simply the right Kan extension along the embedding

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

In other words, for a prestack $\mathcal{Y}$, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$ is a compatible family of objects

$$\mathcal{F}_{X,y} \in \mathrm{IndCoh}(X), \quad (X \xrightarrow{y} \mathcal{Y}) \in \mathrm{Sch}_{/\mathcal{Y}},$$

where the compatibility is understood in the sense of the !-pullback functor.

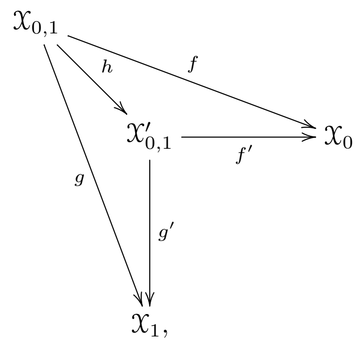
Furthermore, we can canonically extend the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}}$ to a functor

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

where $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}}$ is an $(\infty, 2)$-category, whose objects are $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, 1-morphisms are diagrams

$$\begin{array}{ccc} \mathcal{X}_{0,1} & \xrightarrow{g} & \mathcal{X}_0 \\ {\scriptstyle f}\downarrow & & \\ \mathcal{X}_1, & & \end{array} \tag{0.12}$$

with g arbitrary and f *schematic quasi-compact*, and 2-morphisms are diagrams



with h *schematic and proper.*

0.3.4. In Sect. 4 we show that the functor $\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}}$ of (0.10) has a natural symmetric monoidal structure, where $\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}$ acquires a structure of symmetric monoidal $(\infty, 2)$-category from the operation of Cartesian product on Sch_{aft}.

We show that the symmetric monoidal structure on $\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})}$ gives rise to the *Serre duality equivalence*

$$\mathbf{D}_X^{\text{Serre}} : \text{IndCoh}(X)^\vee \simeq \text{IndCoh}(X), \quad X \in \text{Sch}_{\text{aft}} . \tag{0.13}$$

At the level of the subcategories of compact objects,

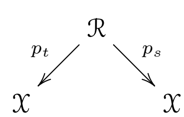
$$\text{Coh}(X) \simeq \text{IndCoh}(X)^c,$$

the functor $\text{Coh}(X)^{\text{op}} \to \text{Coh}(X)$, corresponding to $\mathbf{D}_X^{\text{Serre}}$, is the usual (contravariant) Serre duality auto-equivalence

$$\mathbb{D}_X^{\text{Serre}} : \text{Coh}(X)^{\text{op}} \simeq \text{Coh}(X).$$

Under the equivalence (0.10), for a morphism $X \xrightarrow{f} Y$, the functor $f^!$ identifies with the *dual* of the functor f_*^{IndCoh}.

0.3.5. In Sect. 5 we apply the theory developed in the preceding sections show that if



is a groupoid-object in $\text{PreStk}_{\text{laft}}$, where the maps p_s and p_t are schematic, then the category $\text{IndCoh}(\mathcal{R})$ has a natural monoidal structure, and as such it acts on $\text{IndCoh}(\mathcal{X})$.

We show, moreover, that if p_s and p_t are proper, then the dualizing object

$$\omega_{\mathcal{R}} \in \text{IndCoh}(\mathcal{R})$$

acquires a natural structure of associative algebra in $\text{IndCoh}(\mathcal{R})$.

1. Factorizations of morphisms of DG schemes

In this section we will study what happens to the notion of the closure of the image of a morphism between schemes in derived algebraic geometry. The upshot is that there is essentially 'nothing new' as compared to the classical case.

Remark 1.0.1. Added in November 2021: this section was intended to be used for the proof of Proposition 2.1.6. However, that proof turned out to be erroneous, and was replaced by a different one upon revision.

1.1. **Colimits of closed embeddings.** In this subsection we will show that colimits exist and are well-behaved in the category of closed subschemes of a given ambient scheme.

1.1.1. Recall that a map $X \to Y$ in Sch is called a closed embedding if the map

$$^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$$

is a closed embedding of classical schemes.

1.1.2. Let $f : X \to Y$ be a morphism in Sch. We let

$$\mathrm{Sch}_{X/,\,\text{closed in } Y}$$

denote the full subcategory of $\mathrm{Sch}_{X/\,/Y}$ consisting of diagrams

$$X \to X' \overset{f'}{\to} Y,$$

where the map f' is a closed embedding.

We claim:

Proposition 1.1.3.
(a) *The category* $\mathrm{Sch}_{X/,\,\text{closed in } Y}$ *has finite colimits (including the initial object).*
(b) *The formation of colimits commutes with Zariski localization on* Y.

Proof.

Step 1. Assume first that Y is affine, $Y = \mathrm{Spec}(A)$. Let

$$i \rightsquigarrow (X \to X'_i \overset{f'_i}{\to} Y), \tag{1.1}$$

be a finite diagram in $\mathrm{Sch}_{X/,\,\text{closed in } Y}$.

Set $B := \Gamma(X, \mathcal{O}_X)$. This is a (not necessarily connective) commutative k-algebra. Set also $X'_i = \mathrm{Spec}(B'_i)$. Consider the corresponding diagram

$$i \rightsquigarrow (A \to B'_i \to B) \tag{1.2}$$

in $\mathrm{ComAlg}_{A/\,/B}$.

Set

$$(\widetilde{B}' \to B) := \lim_i \, (B'_i \to B),$$

where the limit taken in $\mathrm{ComAlg}_{/B}$. Note that we have a canonical map $A \to \widetilde{B}'$, and

$$(A \to \widetilde{B}' \to B) \in \mathrm{ComAlg}_{A/\,/B}$$

maps isomorphically to the limit of (1.2) taken in, $\mathrm{ComAlg}_{A/\,/B}$.

Set

$$B' := \tau^{\leq 0}(\widetilde{B}') \underset{H^0(\widetilde{B}')}{\times} \mathrm{Im}\left(H^0(A) \to H^0(\widetilde{B}')\right),$$

where the fiber product is taken in the category of *connective* commutative algebras (i.e., it is $\tau^{\leq 0}$ of the fiber product taken in the category of all commutative algebras).

We still have the canonical maps

$$A \to B' \to B,$$

and it is easy to see that for $X' := \operatorname{Spec}(B')$, the object

$$(X \to X' \to Y) \in \operatorname{Sch}_{X/,\,\text{closed in } Y}$$

is the colimit of (1.1).

Step 2. To treat the general case it it suffices to show that the formation of colimits in the affine case commutes with Zariski localization. I.e., we need to show that if Y is affine, $\overset{\circ}{Y} \subset Y$ is a basic open, then for $\overset{\circ}{X} := f^{-1}(\overset{\circ}{Y})$, $\overset{\circ}{X}{}'_i := (\phi'_i)^{-1}(\overset{\circ}{Y})$, $\overset{\circ}{X}{}' := (f')^{-1}(\overset{\circ}{Y})$, then the map

$$\underset{i}{\operatorname{colim}}\, \overset{\circ}{X}{}'_i \to \overset{\circ}{X}{}',$$

is an isomorphism, where the colimit is taken in $\operatorname{Sch}_{\overset{\circ}{X}/,\,\text{closed } \overset{\circ}{Y}}$.

However, the required isomorphism follows from the description of the colimit in Step 1.

□

1.1.4. We note the following property of colimits in the situation of Proposition 1.1.3.

Let $g : Y \to \widetilde{Y}$ be a closed embedding. Set

$$(X \to X' \to Y) = \underset{i}{\operatorname{colim}}\, (X \to X'_i \to Y) \text{ and } (X \to \widetilde{X}' \to \widetilde{Y}) = \underset{i}{\operatorname{colim}}\, (X \to X'_i \to \widetilde{Y}),$$

where the colimits are taken in $\operatorname{Sch}_{X/,\,\text{closed in } Y}$ and $\operatorname{Sch}_{X/,\,\text{closed } \widetilde{Y}}$, respectively.

Consider the composition

$$X \to X' \to Y \to \widetilde{Y},$$

and the corresponding object

$$(X \to X' \to \widetilde{Y}) \in \operatorname{Sch}_{X/,\,\text{closed } \widetilde{Y}}.$$

It is endowed with a compatible family of maps

$$(X \to X'_i \to \widetilde{Y}) \to (X \to X' \to \widetilde{Y}).$$

Hence, by the universal property of $(X \to \widetilde{X}' \to \widetilde{Y})$, we obtain a canonically defined map

$$\widetilde{X}' \to X'. \tag{1.3}$$

We claim:

Lemma 1.1.5. *The map* (1.3) *is an isomorphism.*

Proof. We construct the inverse map as follows. We note that by the universal property of $(X \to \widetilde{X}' \to \widetilde{Y})$, we have a canonical map

$$(X \to \widetilde{X}' \to \widetilde{Y}) \to (X \to Y \to \widetilde{Y}).$$

This produces a compatible family of maps

$$(X \to X'_i \to Y) \to (X \to \widetilde{X}' \to Y),$$

and hence the desired map

$$X' \to \widetilde{X}'.$$

□

1.1.6. In the situation of Proposition 1.1.3 let us consider the case of $X = \emptyset$. We shall denote the resulting category by $\mathrm{Sch}_{\text{closed in } Y}$. Thus, Proposition 1.1.3 guarantees the existence and compatibility with Zariski localization of finite colimits in $\mathrm{Sch}_{\text{closed in } Y}$.

Explicitly, if $Y = \mathrm{Spec}(A)$ is affine and

$$i \rightsquigarrow Y'_i \subset Y$$

is a diagram of closed subschemes, $Y'_i = \mathrm{Spec}(A'_i)$, then

$$\underset{i}{\mathrm{colim}}\, Y'_i = Y',$$

where

$$Y' = \mathrm{Spec}(A'), \quad A' := \tau^{\leq 0}(\widetilde{A}') \underset{H^0(\widetilde{A}')}{\times} \mathrm{Im}\left(H^0(A) \to H^0(\widetilde{A}')\right), \quad \widetilde{A}' := \underset{i}{\lim}\, A'_i.$$

1.2. **The closure.** In this subsection we will define the notion of closure of the image of a morphism of schemes.

1.2.1. In what follows, in the situation of Proposition 1.1.3, we shall refer to the initial object in the category $\mathrm{Sch}_{X/, \text{closed in } Y}$ as the *closure* of X and Y, and denote it by $\overline{f(X)}$.

Explicitly, if $Y = \mathrm{Spec}(A)$ is affine, we have:

$$\overline{f(X)} = \mathrm{Spec}(A'), \quad A' = \tau^{\leq 0}(\Gamma(X, \mathcal{O}_X)) \underset{H^0(\Gamma(X,\mathcal{O}_X))}{\times} \mathrm{Im}\left(H^0(A) \to H^0(\Gamma(X, \mathcal{O}_X))\right). \tag{1.4}$$

A particular case of Lemma 1.1.5 says:

Corollary 1.2.2. *If $f : X \to Y$ is a closed embedding, then $X \to \overline{f(X)}$ is an isomorphism.*

1.2.3. The following property of the operation of taking the closure will be used in the sequel. Let us be in the situation of Proposition 1.1.3,

$$X = X_1 \cup X_2,$$

where $X_i \subset X$ are open and set $X_{12} = X_1 \cap X_2$. Denote $f_i := f|_{X_i}$.

We have a canonical map

$$\overline{f_1(X_1)} \underset{\overline{f_{12}(X_{12})}}{\sqcup} \overline{f_2(X_2)} \to \overline{f(X)}, \tag{1.5}$$

where the colimit is taken in $\mathrm{Sch}_{\text{closed in } Y}$.

Lemma 1.2.4. *The map* (1.5) *is an isomorphism.*

Proof. Follows by reducing to the case when Y is affine, and in the latter case by (1.4). □

1.2.5. We give the following definition:

Definition 1.2.6. *A map $f : X \to Y$ is said to be a* locally closed embedding *if Y contains an open $\overset{\circ}{Y} \subset Y$, such that f defines a closed embedding $X \to \overset{\circ}{Y}$.*

We have:

Lemma 1.2.7. *Suppose that f is a locally closed embedding. Then f defines an open embedding of X into $\overline{f(X)}$.*

Proof. Follows by combining Corollary 1.2.2 and Proposition 1.1.3(b). □

1.3. **Transitivity of closure.** The basic fact established in this subsection, Proposition 1.3.2, will be of crucial technical importance for the proof of Theorem 2.1.4.

1.3.1. Consider a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Set $Y' := \overline{f(X)}$ and $g' := g|_{Y'}$. By the universal property of closure, we have a canonical map

$$\overline{g \circ f(X)} \to \overline{g'(Y')}. \tag{1.6}$$

We claim:

Proposition 1.3.2. *The map* (1.6) *is an isomorphism.*

The rest of this subsection is devoted to the proof of this proposition.

1.3.3. *Step 1.* As in the proof of Proposition 1.1.3, the assertion reduces to the case when $Z = \operatorname{Spec}(A)$ is affine. Assume first that Y is affine as well, $Y = \operatorname{Spec}(B)$. Then we have the following descriptions of the two sides in (1.6).

Set $C := \Gamma(X, \mathcal{O}_X)$. We have $Y' := \operatorname{Spec}(B')$, where

$$B' = \tau^{\leq 0}(C) \underset{H^0(C)}{\times} \operatorname{Im}\left(H^0(B) \to H^0(C)\right),$$

where here and below the fiber product is taken in the category of connective commutative algebras.

Furthermore, $\overline{g \circ f(X)} = \operatorname{Spec}(A')$, where

$$A' = \tau^{\leq 0}(C) \underset{H^0(C)}{\times} \operatorname{Im}\left(H^0(A) \to H^0(C)\right).$$

Finally, $\overline{g'(Y')} = \operatorname{Spec}(A'')$, where

$$A'' = B' \underset{H^0(B')}{\times} \operatorname{Im}\left(H^0(A) \to H^0(B')\right).$$

Note that

$$H^0(B') = \operatorname{Im}\left(H^0(B) \to H^0(C)\right) \text{ and } \operatorname{Im}\left(H^0(A) \to H^0(B')\right) = \operatorname{Im}(H^0(A) \to H^0(C)).$$

The map (1.6) corresponds to the homomorphism

$$A'' = B' \underset{H^0(B')}{\times} \operatorname{Im}(H^0(A) \to H^0(B')) =$$

$$= \left(\tau^{\leq 0}(C) \underset{H^0(C)}{\times} \operatorname{Im}(H^0(B) \to H^0(C))\right) \underset{\operatorname{Im}(H^0(B) \to H^0(C))}{\times} \operatorname{Im}(H^0(A) \to H^0(C)) \simeq$$
$$\simeq \tau^{\leq 0}(C) \underset{H^0(C)}{\times} \operatorname{Im}(H^0(A) \to H^0(C)) \simeq A',$$

which is an isomorphism, as required.

1.3.4. *Step 2.* Let Y be arbitrary. Choose an open affine cover $Y = \underset{i}{\cup} Y_i$ and set $X_i = f^{-1}(Y_i)$. Then the assertion of the proposition follows from Step 1 using Lemma 1.2.4.

□

2. IndCoh as a functor from the category of correspondences

This section realizes one of the main goal of our book, namely, the construction of IndCoh as a functor out of the category of correspondences.

It will turn out that IndCoh, equipped with the operation of direct image, and left and right adjoints, corresponding to open embeddings and proper morphisms, respectively will uniquely extend to the sought-for formalism of correspondences.

2.1. The category of correspondences.

In this subsection we introduce the category of correspondences on schemes and state our main theorem.

2.1.1. We consider the category $\mathrm{Sch}_{\mathrm{aft}}$ equipped with the following classes of morphisms:

$$vert = \text{all},\ horiz = \text{all},\ adm = \text{proper},$$

and consider the corresponding category

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}},$$

see [Chapter V.1, Sect. 1].

Our goal in this section is to extend the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

of [Chapter II.1, Sect. 2.2] to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}.$$

We shall do so in several stages.

2.1.2. We start with the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and consider the class of morphisms

$$\text{open} \subset \text{all}.$$

By [Chapter II.1, Proposition 3.2.2], the functor $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$, viewed as a functor

$$\mathrm{Sch}_{\mathrm{aft}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}},$$

satisfies the *left Beck-Chevalley condition* with respect to the class[1] open ⊂ all.

Applying [Chapter V.1, Theorem 3.2.2(a)], we extend $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$ to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}} \to \left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}.$$

We restrict the latter functor to

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}},$$

and denote the resulting functor by $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}}$, viewed as a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

We note that due to the uniqueness assertion in [Chapter V.1, Theorem 4.1.3], the restriction procedure

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}}} \rightsquigarrow \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}}$$

---

[1] We note that the left Beck-Chevalley condition is intrinsic to the target $(\infty, 2)$-category, in our case $\left(\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}\right)^{2\text{-op}}$.

loses no information. I.e., the datum of the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}}$ is *equivalent* to that of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{all;open}}}$.

2.1.3. The main result of this section reads:

Theorem 2.1.4. *There exists a unque extension of the functor* $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}}$ *to a functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

We will deduce this theorem from [Chapter V.1, Theorem 5.2.4]. We refer the reader to [Chapter V.1, Sects. 5.1 and 5.2] where the notations involved in this theorem are introduced.

2.1.5. *Proof of Theorem 2.1.4.* We start with the following three classes of morphisms

$$\mathit{vert} = \mathrm{all},\ \mathit{horiz} = \mathrm{all},\ \mathit{co}\text{-}\mathit{adm} = \mathrm{open},\ \mathit{adm} = \mathrm{proper}.$$

We note that the class *open* $\cap$ *proper* is that of embeddings of a connected component. This implies that the condition of [Chapter V.1, Sect. 5.1.2] holds.

The fact that

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;open}}}|_{\mathrm{Sch}_{\mathrm{aft}}} = \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{\mathrm{Sch}_{\mathrm{aft}}}$$

satisfies the *left Beck-Chevalley condition* with respect to the class of proper maps is the content of [Chapter II.1, Proposition 5.2.1].

Finally, the fact that the condition of [Chapter V.1, Sect. 5.2.2] holds is the content of [Chapter II.1, Proposition 5.3.4].

Hence, in order to deduce Theorem 2.1.4 from [Chapter V.1, Theorem 5.2.4], it remains to verify that the factorization condition of [Chapter V.1, Sect. 5.1.3]. I.e. we need to prove the following derived version of the Nagata compactification theorem:

Proposition 2.1.6. *For a morphism $f : X \to Y$ in $\mathrm{Sch}_{\mathrm{aft}}$, the category $\mathrm{Factor}(f)$ of factorizations of f as*

$$X \overset{j}{\to} Z \overset{g}{\to} Y, \tag{2.1}$$

where j is an open embedding, and g is proper, is contractible.

□

2.2. Proof of Proposition 2.1.6.

2.2.1. First, we claim that it is enough to show that the category $\mathrm{Factor}(f)$ is non-empty (for any f). Indeed, suppose that this is the case. We claim that this implies that it is co-filtered, and hence contractible.

Let us be given a finite category I and a functor

$$Z_I : I \to \mathrm{Factor}(f).$$

Let $I^{\triangleleft}$ be a left cone over I. We need to show that Z_I can be extended to a functor

$$Z_{I^{\triangleleft}} : I^{\triangleleft} \to \mathrm{Factor}(f).$$

Let us compose Z_I with the forgetful functor

$$\mathrm{Factor}(f) \to (\mathrm{Sch}_{\mathrm{aft}})_{/Y},$$

and let Y' denote its limit.

Then Y' is a scheme, endowed with a proper map $h : Y' \to Y$, and equipped with a map $f' : X \to Y'$ so that $h \circ f' = f$. By assumption, we can factor the morphism f' as

$$X \overset{j'}{\to} Z \overset{g'}{\to} Y',$$

where g' is proper and h' is an open embedding.

Then

$$X \overset{j'}{\to} Z \overset{h \circ g'}{\to} Y,$$

is an object of $\text{Factor}(f)$ that supplies an extension of Z_I to a functor $Z_{I^{\triangleleft}}$.

2.2.2. We now prove the existence of a factorization. We first prove the assertion when X is a classical scheme.

We start with the classical Nagata's theorem, which says that the map

$$^{\text{red}}X \to Y$$

can be factored as

$$^{\text{red}}X \overset{j}{\to} Z \overset{g}{\to} Y,$$

where Z is a scheme of finite type, j is an open embedding and g is proper.

Blowing up Z along any closed complement of the image of $^{\text{red}}X$, we can assume that the morphism j is affine.

2.2.3. The scheme X can be obtained from $^{\text{red}}X$ as finite sequence of square-zero extensions (see [Chapter III.1, Sect. 5.1] for what this means).

Hence, by induction, it suffices to show that if $X \hookrightarrow X'$ is a square-zero extension, and we are given a map $f' : X' \to Y$ and a factorization of $f := f'|_X$ as

$$X \overset{j}{\to} Z \overset{g}{\to} Y,$$

with Z of finite type, j an affine open embedding and g proper, then we can extend it to a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{j'} & Z' & \xrightarrow{g'} & Y \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ X & \xrightarrow{j} & Z & \xrightarrow{g} & Y, \end{array} \tag{2.2}$$

where Z' is also of finite type, $Z \to Z'$ is a square-zero extension, j' is an open embedding, g' is proper and $g' \circ j' = f'$. (Note that in this situation, the map j' is also affine because it is such at the reduced level.)

We will carry out the construction of (2.2) is the derived context, i.e., when X and X' are derived, but *eventually coconnective* schemes.

2.2.4. Set

$$\widetilde{Z}' := X' \underset{X}{\sqcup} Z, \tag{2.3}$$

see [Chapter III.1, Corollary 1.3.5] for the existence and the properties of push-out in this situation. The map $\widetilde{j}' : X' \to \widetilde{Z}'$ is an open embedding, and the map g extends to a map $\widetilde{g}' : \widetilde{Z}' \to Y$.

However, $\widetilde{Z}'$ is not necessarily of finite type. In the next few subsections, we will show how to replace $\widetilde{Z}'$ by a scheme of finite type, and thereby construct the diagram (2.2).

2.2.5. The square-zero extension $X \hookrightarrow X'$ is classified by an object $\mathcal{I}_X \in \mathrm{Coh}(X)^{\leq 0}$ and a map

$$\gamma_X : T^*(X) \to \mathcal{I}_X[1].$$

Note that the assumption that X and X' are eventually coconnective implies that $\mathcal{I}_X$ is *eventually coconnective.*

Set

$$\widetilde{\mathcal{I}}_Z := j_*(\mathcal{I}_X) \in \mathrm{QCoh}(Z).$$

Note that since j was assumed affine, the object $\widetilde{\mathcal{I}}_Z$ is connective (i.e., belongs to $\mathrm{QCoh}(Z)^{\leq 0}$).

Consider the map

$$T^*(Z) \to j_*(T^*(X)) \overset{\gamma_X}{\to} \widetilde{\mathcal{I}}_Z[1], \tag{2.4}$$

where the first arrow is obtained by adjunction from the co-differential

$$(dj)^* : j^*(T^*(Z)) \to T^*(X).$$

Denote the map (2.4) by $\widetilde{\gamma}_Z$.

The map $\widetilde{\gamma}_Z$ defines a square-zero extension of Z with ideal $\widetilde{\mathcal{I}}_Z$. Unwinding the definitions, we obtain that the scheme underlying this square-zero extension is

$$Z \to \widetilde{Z}'.$$

2.2.6. The extension of the map $f : X \to Y$ to a map $f' : X' \to Y$ corresponds to a null-homotopy of the map

$$f^*(T^*(Y)) \overset{(df)^*}{\to} T^*(X) \overset{\gamma_X}{\to} \mathcal{I}_X[1], \tag{2.5}$$

(see [Chapter III.1, Corollary 6.2.5], applied to the target Y).

The extension of the map $g : Z \to Y$ to the map $\widetilde{g}' : \widetilde{Z}' \to Y$ corresponds to a null-homotopy of the composition

$$g^*(T^*(Y)) \overset{(dg)^*}{\longrightarrow} T^*(Z) \overset{\widetilde{\gamma}_Z}{\to} \widetilde{\mathcal{I}}_Z[1], \tag{2.6}$$

which is from the null-homotopy of (2.5) by adjunction.

2.2.7. The object $\widetilde{\mathcal{I}}_Z \in \mathrm{QCoh}(Z)^{\leq 0}$ is eventually coconnective, but not necessarily coherent, which corresponds to the fact that the scheme $\widetilde{Z}'$ is not necessarily of finite type.

However, since $T^*(Z)$ and $g^*(T^*(Y))$ are coherent, we can find a *coherent* object $\mathcal{I}_Z \in \mathrm{QCoh}(Z)^{\leq 0}$, equipped with a map $\mathcal{I}_Z \to \widetilde{\mathcal{I}}_Z$, such that:

- The map $\mathcal{I}_Z \to \widetilde{\mathcal{I}}_Z$ is an isomorphism over (the open subscheme) $X \subset Z$;
- The map $\widetilde{\gamma}_Z$ factors via a map

$$\gamma_Z : T^*(Z) \to \mathcal{I}_Z[1]; \tag{2.7}$$

- The null-homotopy of (2.6) is induced by a null-homotopy

$$g^*(T^*(Y)) \overset{(dg)^*}{\longrightarrow} T^*(Z) \overset{\gamma_Z}{\to} \mathcal{I}_Z[1]. \tag{2.8}$$

The map (2.7) defines a square-zero extension

$$Z \to Z'.$$

Moreover, the resulting map $\widetilde{Z}' \to Z'$ is an isomorphism over (the open subscheme) $X \subset Z$.

The fact that $\mathcal{I}_Z$ is coherent implies that Z if almost of finite type.

The null-homotopy of (2.8) defines an extension of g to a map $g' : Z' \to Y$. The compatibility of this null-homotopy with the null-homotopy of (2.6) implies the commutativity of the diagram (2.2).

2.2.8. We now consider the case of a general derived X. Assume by induction that we have found a factorization of $f^i := f|_{\leq i X}$

$$^{\leq i}X \overset{j^i}{\to} Z^i \overset{g^i}{\to} Y$$

with Z^i almost of finite type, j^i an affine open embedding and g^i proper.

It suffices to show that in this case we can find a factorization of f^{i+1} that fits into a commutative diagram

$$\begin{array}{ccccc} ^{\leq i+1}X & \xrightarrow{j^{i+1}} & Z^{i+1} & \xrightarrow{g^{i+1}} & Y \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ ^{\leq i}X & \xrightarrow{j^i} & Z^i & \xrightarrow{g^i} & Y, \end{array} \tag{2.9}$$

such that the maps $Z^i \to Z^{i+1}$ induce isomorphisms

$$^{\leq i}(Z^i) \to {}^{\leq i}(Z^{i+1}).$$

Indeed, in this case, we will have a well-defined derived scheme Z with

$$^{\leq i}Z := {}^{\leq i}(Z^m), \quad m \geq i,$$

and the maps j^i, g^i extend to maps

$$X \overset{j}{\to} Z \overset{g}{\to} Y,$$

as desired.

2.2.9. Now, the existence of (2.9) follows the existence of (2.2):

Indeed, by [Chapter III.1, Proposition 5.4.2(b)], the map $^{\leq i}X \to {}^{\leq i+1}X$ has a structure of square-zero extension with ideal

$$\mathcal{I}_X \in \operatorname{Coh}(^{\mathrm{cl}}X)^{\heartsuit}[i+1].$$

Let us show that Z^{i+1} can be chosen so that the map $Z^i \to Z^{i+1}$ is an isomorphism on the i-th truncation. For that we revisit the construction of the scheme Z' in Sect. 2.2.7.

The map $Z \to Z'$ has a structure of square-zero extension with ideal denoted $\mathcal{I}_Z$. Note, however, that if $\mathcal{I}_X \in \operatorname{Coh}(X)$ belongs to $\operatorname{Coh}(X)^{<-m}$ for some m, then $\widetilde{\mathcal{I}}_Z := j_*(\mathcal{I}_X)$ belongs to $\operatorname{QCoh}(Z)^{<-m,>-\infty}$, and we can choose $\mathcal{I}_Z$ in $\operatorname{Coh}(Z)^{<-m}$. In this case, the map $Z \to Z'$ induces an isomorphism on the m-th truncations.

This applies to our situation with $m = i$.

$\square$[Proposition 2.1.6]

3. The functor of !-pullback

Having defined IndCoh as a functor out of the category of correspondences, restricting to 'horizontal morphisms', we in particular obtain the functor of !-pullback, which is now defined on all morphisms.

In this section we study the basic properties of this functor.

3.1. **Definition of the functor.** In this subsection we summarize the basic properties of the !-pullback that follow formally from Theorem 2.1.4.

3.1.1. We let $\text{IndCoh}^!_{\text{Sch}_{\text{aft}}}$ denote the restriction of the functor $\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{all;all}}}$ to

$$(\text{Sch}_{\text{aft}})^{\text{op}} \to \text{Corr}(\text{Sch}_{\text{aft}})^{\text{proper}}_{\text{all;all}}.$$

In particular, for a morphism $f : X \to Y$, we let $f^! : \text{IndCoh}(Y) \to \text{IndCoh}(X)$ the resulting morphism.

The functor $\text{IndCoh}^!_{\text{Sch}_{\text{aft}}}$ is essentially defined by the following two properties:

- The restriction $\text{IndCoh}^!_{\text{Sch}_{\text{aft}}}|_{((\text{Sch}_{\text{aft}})_{\text{proper}})^{\text{op}}}$ identifies with $\text{IndCoh}^!_{(\text{Sch}_{\text{aft}})_{\text{proper}}}$.
- The restriction $\text{IndCoh}^!_{\text{Sch}_{\text{aft}}}|_{((\text{Sch}_{\text{aft}})_{\text{open}})^{\text{op}}}$ identifies with $\text{IndCoh}^*_{(\text{Sch}_{\text{aft}})_{\text{open}}}$.

In the above formula,

$$\text{IndCoh}^*_{(\text{Sch}_{\text{aft}})_{\text{open}}} := \text{IndCoh}^*_{(\text{Sch}_{\text{aft}})_{\text{event-coconn}}}|_{((\text{Sch}_{\text{aft}})_{\text{open}})^{\text{op}}},$$

see [Chapter II.1, Corollary 3.1.10], where the functor

$$\text{IndCoh}^*_{(\text{Sch}_{\text{aft}})_{\text{event-coconn}}} : ((\text{Sch}_{\text{aft}})_{\text{event-coconn}})^{\text{op}} \to \text{DGCat}_{\text{cont}}$$

is introduced.

3.1.2. In what follows we shall denote by $\omega_X \in \text{IndCoh}(X)$ the canonical object equal to

$$p^!_X(k),$$

where $p_X : X \to \text{pt}$.

3.1.3. *Base change.* Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g_X} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_Y} & Y_2 \end{array}$$

be a Cartesian diagram in Sch_{aft}. As the main corollary of Theorem 2.1.4 we obtain:

Corollary 3.1.4. *There exists a canonical isomorphism of functors*

$$g^!_Y \circ (f_2)^{\text{IndCoh}}_* \simeq (f_1)^{\text{IndCoh}}_* \circ g^!_X, \tag{3.1}$$

compatible with compositions of vertical and horizontal morphisms in the natural sense. Furthermore:

(a) *Suppose that g_Y (and hence g_X) is proper. Then the morphism $\leftarrow$ in* (3.1) *comes by the $(g^{\text{IndCoh}}_*, g^!)$-adjunction from the isomorphism*

$$(f_2)^{\text{IndCoh}}_* \circ (g_X)^{\text{IndCoh}}_* \simeq (g_Y)^{\text{IndCoh}}_* \circ (f_1)^{\text{IndCoh}}_*.$$

(b) *Suppose that f_2 (and hence f_1) is proper. Then the morphism $\leftarrow$ in* (3.1) *comes by the* $(f_*^{\mathrm{IndCoh}}, f^!)$*-adjunction from the isomorphism*

$$f_1^! \circ g_Y^! \simeq g_X^! \circ f_2^!.$$

(c) *Suppose that g_Y (and hence g_X) is an open embedding. Then the morphism $\to$ in* (3.1) *comes by the* $(g^!, g_*^{\mathrm{IndCoh}})$*-adjunction from the isomorphism*

$$(f_2)_*^{\mathrm{IndCoh}} \circ (g_X)_*^{\mathrm{IndCoh}} \simeq (g_Y)_*^{\mathrm{IndCoh}} \circ (f_1)_*^{\mathrm{IndCoh}}.$$

(d) *Suppose that f_2 (and hence f_1) is an open embedding. Then the morphism $\to$ in* (3.1) *comes by the* $(f^!, f_*^{\mathrm{IndCoh}})$*-adjunction from the isomorphism*

$$f_1^! \circ g_Y^! \simeq g_X^! \circ f_2^!.$$

Remark 3.1.5. The real content of Theorem 2.1.4 is that there exists a uniquely defined family of functors $f^!$, that satisfies the properties listed in Corollary 3.1.4 and those of Sect. 3.1.1.

3.2. **Some properties.**

3.2.1. Let $\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}}$ denote the restriction

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}}.$$

We claim:

Lemma 3.2.2. *The functor*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{RKE}_{(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}}) \to \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$$

is an isomorphism.

Proof. Follows from the fact that $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ satisfies Zariski descent ([Chapter II.1, Proposition 4.2.2]), combined with the fact that affine schemes form a basis for the Zariski topology:

For a given $X \in \mathrm{Sch}_{\mathrm{aft}}$, we need to show that the functor

$$\mathrm{IndCoh}(X) \to \lim_{S \to X} \mathrm{IndCoh}(S) \tag{3.2}$$

is an equivalence, where the limit is taken over the index category $((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/X})^{\mathrm{op}}$.

Choose a Zariski cover $U \to X$ with $U \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, and let $U^\bullet$ be its Čech nerve. We extend (3.2) to a string of functors

$$\mathrm{IndCoh}(X) \to \lim_{S \to X} \mathrm{IndCoh}(S) \to \mathrm{Tot}(\mathrm{IndCoh}(U^\bullet)) \to \lim_{S \to X} \left(\mathrm{Tot}(\mathrm{IndCoh}(S \underset{X}{\times} U^\bullet) \right).$$

Now, Zariski descent for IndCoh implies that the two composites

$$\mathrm{IndCoh}(X) \to \lim_{S \to X} \mathrm{IndCoh}(S) \to \mathrm{Tot}(\mathrm{IndCoh}(U^\bullet))$$

and

$$\lim_{S \to X} \mathrm{IndCoh}(S) \to \mathrm{Tot}(\mathrm{IndCoh}(U^\bullet)) \to \lim_{S \to X} \left(\mathrm{Tot}(\mathrm{IndCoh}(S \underset{X}{\times} U^\bullet) \right)$$

are both equivalences.

□

3.2.3. *Convergence.* Let

$$\mathrm{IndCoh}^!_{<\infty\mathrm{Sch}_{\mathrm{ft}}} \text{ and } \mathrm{IndCoh}^!_{<\infty\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}$$

denote the restrictions of $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ to the corresponding subcategories.

We claim:

Lemma 3.2.4. *The functors*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} \to \mathrm{RKE}_{({<\infty}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \to (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{<\infty\mathrm{Sch}_{\mathrm{ft}}})$$

and

$$\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}} \to \mathrm{RKE}_{({<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}}(\mathrm{IndCoh}^!_{<\infty\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}})$$

are isomorphisms.

Proof. Both statements are equivalent to the assertion that for $X \in \mathrm{Sch}_{\mathrm{aft}}$, the functor

$$\mathrm{IndCoh}(X) \to \lim_{n} \mathrm{IndCoh}({}^{\leq n}X)$$

is an equivalence.

The latter assertion is the content of [Chapter II.1, Proposition 6.4.3].

□

3.3. **h-descent.**

We will now use proper descent for IndCoh to show that it in fact has h-descent.

3.3.1. Let $\mathbf{C}$ be a category with Cartesian products, and let α be an isomorphism class of 1-morphisms, closed under base change.

We define the Grothendieck topology generated by α to be the minimal Grothendieck topology that contains all morphisms from α and has the following "2-out-of-3" property:

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps in $\mathbf{C}$ such that f and $g \circ f$ are coverings, then so is g.

The following is well-known:

Lemma 3.3.2. *Let $\mathcal{F}$ be a presheaf on $\mathbf{C}$ that satisfies descent with respect to morphisms from the class α. Then $\mathcal{F}$ is a sheaf with respect to the Grothendieck topology generated by α.*

3.3.3. We recall that the h-topology on $\mathrm{Sch}_{\mathrm{aft}}$ is the one generated by the class of proper surjective maps and Zariski covers.

From Lemma 3.3.2, combined with [Chapter II.1, Propositions 4.2.2 and 7.2.2] we obtain:

Corollary 3.3.4. *The functor*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{op}}$$

satisfies h-descent.

3.3.5. We have:

Lemma 3.3.6. *Any ppf covering is an h-covering.*

Proof. Let $f : X \to Y$ be an ppf covering. Consider the Cartesian square

$$\begin{array}{ccc} {}^{\mathrm{cl}}Y \underset{Y}{\times} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ {}^{\mathrm{cl}}Y & \longrightarrow & Y. \end{array}$$

It suffices to show that ${}^{\mathrm{cl}}Y \underset{Y}{\times} X \to {}^{\mathrm{cl}}Y$ is an h-covering. By flatness, ${}^{\mathrm{cl}}Y \underset{Y}{\times} X$ is classical. Hence, we are reduced to an assertion at the classical level, in which case it is well-known. □

Hence, combining, we obtain:

Corollary 3.3.7. *The functor*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{op}}$$

satisfies ppf-descent.

3.4. **Extension to prestacks.** The functor of !-pullback for arbitrary morphisms of schemes allows to define the category IndCoh on arbitrary prestacks (locally almost of finite type).

3.4.1. We consider the category $\mathrm{PreStk}_{\mathrm{laft}}$ and define the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

as the right Kan extension of $\mathrm{IndCoh}^!_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}}$ along the Yoneda functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

According to Lemmas 3.2.2 and 3.2.4, we can equivalently define $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ as the right Kan extension of

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}},\ \mathrm{IndCoh}^!_{<\infty\mathrm{Sch}_{\mathrm{ft}}} \text{ or } \mathrm{IndCoh}^!_{<\infty\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}$$

from the corresponding subcategories.

For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ we let $\mathrm{IndCoh}(\mathcal{X})$ denote the value of $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ on it. For a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ we let

$$f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1)$$

denote the corresponding functor.

For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, we let $\omega_X \in \mathrm{IndCoh}(X)$ denote the canonical object equal to $p^!_{\mathcal{X}}(k)$, where $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{pt}$.

3.4.2. We now consider the category

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}},$$

where

$$\mathrm{sch} \text{ and } \mathrm{sch}\,\&\,\mathrm{proper}$$

signify the classes of schematic and quasi-compact (resp., schematic and proper) morphisms between prestacks.

We claim:

Theorem 3.4.3. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}} \to \mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}},$$

equipped with isomorphisms

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}}}|_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}} \simeq \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$$

and

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}}}|_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{corr:all;all}}} \simeq \mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{corr:all;all}}},$$

where the latter two isomorphisms are compatible in a natural sense.

Proof. Follows from [Chapter V.2, Theorem 6.1.5]. □

3.4.4. The actual content of Theorem 3.4.3 can be summarized as follows:

First, for any *schematic quasi-compact* morphism $f : \mathcal{X} \to \mathcal{Y}$ we have a well-defined functor

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y}).$$

Furthermore, if $f : \mathcal{X} \to \mathcal{Y}$ is schematic and proper, the functor f^{IndCoh}_* is the left adjoint of $f^!$.

When $\mathcal{Y}$ is a scheme (and hence $\mathcal{X}$ is one as well), the above functor f^{IndCoh}_* is the usual f^{IndCoh}_* defined in this case, and similarly for the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction.

Second, let

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g_X} & \mathcal{X}_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathcal{Y}_1 & \xrightarrow{g_Y} & \mathcal{Y}_2 \end{array} \tag{3.3}$$

be a Cartesian diagram in $\mathrm{PreStk}_{\mathrm{laft}}$, where the vertical maps are schematic. Then we have a canonical isomorphism of functors

$$g^!_Y \circ (f_2)^{\mathrm{IndCoh}}_* \simeq (f_1)^{\mathrm{IndCoh}}_* \circ g^!_X, \tag{3.4}$$

compatible with compositions. Furthermore, if the vertical (resp., horizontal) morphisms are proper (resp., schematic and proper), the map $\leftarrow$ in (3.4) comes by adjunction in a way similar to Corollary 3.1.4(a) (resp., Corollary 3.1.4(b)).

3.4.5. In [Chapter III.3, Proposition 5.3.6], we will show that for a morphism $f : \mathcal{X} \to \mathcal{Y}$, which is an open embedding, the functor f^{IndCoh}_* is the *right* adjoint of $f^!$.

Furthermore, if in the Cartesian diagram (3.3) the vertical (resp., horizontal) morphisms are open embeddings, the map $\to$ in (3.4) comes by adjunction in a way similar to Corollary 3.1.4(c) (resp., Corollary 3.1.4(d)).

3.4.6. For future use, we note that the statement and proof of [Chapter II.1, Proposition 7.2.2] remains valid for groupoid objects in $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch}\,\&\,\mathrm{proper}}$.

4. Multiplicative structure and duality

In this section we will show that the functor IndCoh, when viewed as a functor out of the category of correspondences, and equipped with a natural symmetric monoidal structure, encodes Serre duality.

4.1. IndCoh as a symmetric monoidal functor.

In this subsection we show that the functor IndCoh possesses a natural symmetric monoidal structure.

4.1.1. We recall that by [Chapter II.1, Proposition 6.3.6], the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

carries a natural symmetric monoidal structure.

Applying [Chapter V.3, Proposition 3.1.5], we obtain:

Theorem 4.1.2. *The functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}$$

carries a canonical symmetric monoidal structure that extends one on $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$.

In particular, we obtain that the functors

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

both carry natural symmetric monoidal structures.

4.1.3. Note that the symmetric monoidal structure on $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ automatically upgrades the functor $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ to a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}, \tag{4.1}$$

due to the fact that the identity functor on $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$ naturally lifts to a functor

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{ComAlg}\left((\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}\right)$$

via the diagonal map.

Explicitly, the monoidal operation on $\mathrm{IndCoh}(X)$ is given by

$$\mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X) \overset{\boxtimes}{\to} \mathrm{IndCoh}(X \times X) \xrightarrow{\Delta^!_X} \mathrm{IndCoh}(X).$$

We shall denote the above monoidal operation by $\overset{!}{\otimes}$:

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{IndCoh}(X) \mapsto \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2 \in \mathrm{IndCoh}(X).$$

The unit for this symmetric monoidal structure is given by $\omega_X \in \mathrm{IndCoh}(X)$.

4.1.4. Applying the functor of right Kan extension along

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}$$

of the functor (4.1), we obtain that the functor

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}} \tag{4.2}$$

naturally upgrades to a functor

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}. \tag{4.3}$$

The functor (4.3) is tautologically right-lax symmetric monoidal with respect to the *coCartesian* symmetric monoidal structures on the source and the target. Since the forgetful functor

$$\mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is symmetric monoidal when viewed with respect to the coCartesian symmetric monoidal structures on the source and the Lurie tensor product on the target (see [Chapter I.1, Sect. 3.3.6]), we obtain that the functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ of (4.2) acquires a natural right-lax symmetric monoidal structure.

4.1.5. The above right-lax symmetric monoidal structure on $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ can be enhanced:

Indeed, applying [Chapter V.3, Proposition 3.2.4], we obtain that the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\,\&\,proper}}_{\mathrm{sch\text{-}qc;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\,\&\,proper}}_{\mathrm{sch\text{-}qc;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}$$

carries a canonical right-lax symmetric monoidal structure.

4.2. **Duality.** In this subsection we will formally deduce Serre duality for schemes from the symmetric monoidal structure on IndCoh.

4.2.1. Let $\mathbf{O}$ be a symmetric monoidal category, and let $\mathbf{O}^{\mathrm{dualizable}} \subset \mathbf{O}$ be the full subcategory spanned by dualizable objects. This subcategory carries a canonical symmetric monoidal anti-involution

$$\left(\mathbf{O}^{\mathrm{dualizable}}\right)^{\mathrm{op}} \overset{\mathrm{dualization}}{\longrightarrow} \mathbf{O}^{\mathrm{dualizable}},$$

given by the passage to the dual object, see [Chapter I.1, Sect. 4.1.4]:

$$\mathbf{o} \mapsto \mathbf{o}^{\vee}.$$

Note that that if

$$F : \mathbf{O}_1 \to \mathbf{O}_2$$

is a symmetric monoidal functor between symmetric monoidal categories, then it maps

$$\mathbf{O}_1^{\mathrm{dualizable}} \to \mathbf{O}_2^{\mathrm{dualizable}},$$

and the following diagram commutes

$$\begin{array}{ccc} (\mathbf{O}_1^{\mathrm{dualizable}})^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & (\mathbf{O}_2^{\mathrm{dualizable}})^{\mathrm{op}} \\ {\scriptstyle \mathrm{dualization}}\downarrow & & \downarrow{\scriptstyle \mathrm{dualization}} \\ \mathbf{O}_1^{\mathrm{dualizable}} & \xrightarrow{F} & \mathbf{O}_2^{\mathrm{dualizable}}. \end{array}$$

4.2.2. Recall now that by [Chapter V.3, Sect. 2.2] the category $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}$ carries a canonical anti-involution ϖ, which is the identity on objects, and at the level of 1-morphisms is maps a 1-morphism

$$\begin{array}{ccc} X_{12} & \xrightarrow{f} & X_1 \\ {\scriptstyle g}\big\downarrow & & \\ X_2 & & \end{array}$$

to

$$\begin{array}{ccc} X_{12} & \xrightarrow{g} & X_2 \\ {\scriptstyle f}\big\downarrow & & \\ X_1. & & \end{array}$$

Moreover, by [Chapter V.3, Proposition 2.3.4], we have:

Theorem 4.2.3. *The inclusion*

$$(\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}})^{\mathrm{dualizable}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}$$

is an isomorphism. The anti-involution ϖ identifies canonically with the dualization functor

$$\left((\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}})^{\mathrm{dualizable}}\right)^{\mathrm{op}} \to (\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}})^{\mathrm{dualizable}}.$$

4.2.4. Combining Theorem 4.2.3 with Theorem 4.1.2 we obtain:

Theorem 4.2.5. *We have the following commutative diagram of functors*

$$\begin{array}{ccc} (\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}})^{\mathrm{op}} & \xrightarrow{\left(\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}}\right)^{\mathrm{op}}} & \left(\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\right)^{\mathrm{op}} \\ {\scriptstyle \varpi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{dualization}} \\ \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} & \xrightarrow{\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}}} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}. \end{array}$$

4.2.6. Let us explain the concrete meaning of this theorem. It says that for $X \in \mathrm{Sch}_{\mathrm{aft}}$ there is a canonical equivalence

$$\mathbf{D}^{\mathrm{Serre}}_X : \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X),$$

and for a map $f : X \to Y$ an isomorphism

$$(f^!)^\vee \simeq f^{\mathrm{IndCoh}}_*,$$

where $(f^!)^\vee$ is viewed as a functor

$$\mathrm{IndCoh}(X) \overset{(\mathbf{D}^{\mathrm{Serre}}_X)^{-1}}{\longrightarrow} \mathrm{IndCoh}(X)^\vee \overset{(f^!)^\vee}{\longrightarrow} \mathrm{IndCoh}(Y)^\vee \overset{\mathbf{D}^{\mathrm{Serre}}_Y}{\longrightarrow} \mathrm{IndCoh}(Y).$$

4.2.7. Let us write down explicitly the unit and co-unit for the identification $\mathbf{D}^{\mathrm{Serre}}_X$:

The co-unit, denoted ϵ_X is given by

$$\mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X) \overset{\boxtimes}{\simeq} \mathrm{IndCoh}(X \times X) \overset{\Delta^!_X}{\longrightarrow} \mathrm{IndCoh}(X) \overset{(p_X)^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}) = \mathrm{Vect},$$

where $p_X : X \to \mathrm{pt}$.

The unit, denoted μ_X is given by

$$\mathrm{Vect} = \mathrm{IndCoh}(\mathrm{pt}) \overset{p^!_X}{\longrightarrow} \mathrm{IndCoh}(X) \overset{(\Delta_X)^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(X \times X) \overset{\boxtimes}{\simeq} \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X).$$

Remark 4.2.8. One does not need to rely on Theorems 4.2.3 and 4.1.2 in order to show that the maps μ_X and ϵ_X, defined above, give rise to an identification

$$\mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X).$$

Indeed, the fact that the composition

$$\mathrm{IndCoh}(X) \overset{\mathrm{Id}_{\mathrm{IndCoh}(X)}\otimes\mu_X}{\longrightarrow} \mathrm{IndCoh}(X)\otimes\mathrm{IndCoh}(X)\otimes\mathrm{IndCoh}(X) \overset{\epsilon_X\otimes\mathrm{Id}_{\mathrm{IndCoh}(X)}}{\longrightarrow} \mathrm{IndCoh}(X)$$

is isomorphic to the identity functor follows by base change from the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X\times X & \xrightarrow{\mathrm{id}_X\times p_X} & X \\ {\scriptstyle\Delta_X}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}_X\times\Delta_X} & & \\ X\times X & \xrightarrow{\Delta_X\times\mathrm{id}_X} & X\times X\times X & & \\ {\scriptstyle p_X\times\mathrm{id}_X}\big\downarrow & & & & \\ X, & & & & \end{array}$$

and similarly for the other composition. A similar diagram chase implies the isomorphism

$$(f^!)^\vee \simeq f_*^{\mathrm{IndCoh}}.$$

Remark 4.2.9. Let us also note that one does not need the (difficult) Theorem 2.1.4 either in order to construct the pairing ϵ_X:

Indeed, both functors involved in ϵ_X, namely, $\Delta_X^!$ and $(p_X)_*^{\mathrm{IndCoh}}$ are ‘elementary’.

If one believes that the functor ϵ_X defined in the above way is the co-unit of a duality (which is a property, and not an extra structure), then one can recover the object $\omega_X \in \mathrm{IndCoh}(X)$. Namely,

$$\omega_X := (p_X\times\mathrm{id}_X)_*^{\mathrm{IndCoh}}(\mu_X(k)).$$

4.2.10. *Relation to the usual Serre duality.* By passage to compact objects, the equivalence

$$\mathbf{D}_X^{\mathrm{Serre}} : \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X)$$

gives rise to an equivalence

$$\mathbb{D}_X^{\mathrm{Serre}} : (\mathrm{Coh}(X))^{\mathrm{op}} \simeq \mathrm{Coh}(X).$$

It is shown in [Ga1, Proposition 8.3.5] that $\mathbb{D}_X^{\mathrm{Serre}}$ is the usual Serre duality anti-equivalence of $\mathrm{Coh}(X)$, given by internal Hom into ω_X.

4.3. **An alternative construction of the !-pullback.** In this subsection we show how one can avoid using the formalism of correspondences if one only wants to construct the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch_{aft}}} : (\mathrm{Sch_{aft}})^{\mathrm{op}} \to \mathrm{DGCat_{cont}}.$$

4.3.1. Note that even without having the formalism of the !-pullback, we know that for $X \in \mathrm{Sch_{aft}}$, the functor

$$\epsilon_X : \mathrm{IndCoh}(X)\otimes\mathrm{IndCoh}(X) \to \mathrm{Vect},$$

defined as

$$\mathrm{IndCoh}(X)\otimes\mathrm{IndCoh}(X) \overset{\sim}{\to} \mathrm{IndCoh}(X\boxtimes X) \overset{\Delta_X^!}{\longrightarrow} \mathrm{IndCoh}(X) \overset{(p_X)_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{Vect}$$

gives[2] rise to the co-unit of an adjunction.

[2]Here we use the assumption that our schemes are assumed separated, so the morphism Δ_X is a closed embedding, and thus $\Delta_X^!$ is a priori defined as the right adjoint of $(\Delta_X)_*^{\mathrm{IndCoh}}$.

Indeed, the corresponding unit of the adjunction μ_X can be defined as follows. Choose an open embedding $X \overset{j}{\hookrightarrow} \overline{X}$, where $\overline{X}$ is proper and set

$$\widetilde{\omega}_X := j^{\mathrm{IndCoh},*} \circ (p_{\overline{X}})^!(k).$$

Then one readily checks that the object

$$(\Delta_X)_*^{\mathrm{IndCoh}}(\widetilde{\omega}_X) \simeq \mathrm{IndCoh}(X \boxtimes X) \simeq \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X),$$

viewed as a functor $\mathrm{Vect} \to \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X)$ provides the unit of the adjunction.

4.3.2. Thus, if we start with the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad X \mapsto \mathrm{IndCoh}(X), \quad (X \overset{f}{\to} Y) \mapsto f_*^{\mathrm{IndCoh}}, \tag{4.4}$$

we obtain that it takes values in the full subcategory

$$(\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{dualizable}} \subset \mathrm{DGCat}_{\mathrm{cont}}.$$

Applying the dualization functor

$$((\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{dualizable}})^{\mathrm{op}} \to (\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{dualizable}},$$

from (4.4), we obtain the desired functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}} : (\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad X \mapsto \mathrm{IndCoh}(X), \quad (X \overset{f}{\to} Y) \mapsto f^!. \tag{4.5}$$

In other words, for a morphism $f : X \to Y$, the functor

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

is *defined* as the dual of f_*^{IndCoh} under the self-dualities given by ϵ_X and ϵ_Y, respectively.

4.3.3. Let $\omega_X \in \mathrm{IndCoh}(X)$ denote the object $(p_X)^!(k)$. Unwinding the definitions we obtain that ω_X identifies with

$$(p_X \times \mathrm{id})_*^{\mathrm{IndCoh}} \circ \mu_X(k).$$

One can also give an explicit construction of the functor

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

for a morphism $f : X \to Y$. Namely,

$$f^!(\mathcal{F}) \simeq (\mathrm{Graph}_f)^!(\omega_X \boxtimes \mathcal{F}),$$

where $\mathrm{Graph}_f : X \to X \times Y$ is a closed embedding because Y is separated, and so $(\mathrm{Graph}_f)^!$ is defined as the right adjoint of $(\mathrm{Graph}_f)_*^{\mathrm{IndCoh}}$.

4.3.4. Since (4.4) is symmetric monoidal, the functor (4.5) also acquires a natural symmetric monoidal structure.

As in Sect. 4.1.3, the symmetric monoidal structure on $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$ makes $\mathrm{IndCoh}(X)$ into a symmetric monoidal category under the operation of $\overset{!}{\otimes}$-tensor product. By construction, $\omega_X \in \mathrm{IndCoh}(X)$ is the unit of this symmetric monoidal structure.

Note, however, that the construction of the *non-unital* symmetric monoidal structure on $\mathrm{IndCoh}(X)$ only uses the !-pullback functor for diagonal morphisms, which are closed embeddings.

Thus, the object $\omega_X \in \mathrm{IndCoh}(X)$ can be uniquely characterized as being *the* unit in the above non-unital symmetric monoidal category[3].

[3]The essential uniqueness of a unit is established in [Lu2, ???].

Remark 4.3.5. The idea that the isomorphism

$$\omega_X \simeq \omega_X \overset{!}{\otimes} \omega_X$$

characterizes ω_X uniquely is borrowed from [YZ, Theorem 5.11 and Proposition 6.1].

4.3.6. Finally, let us see that for $f : X \to Y$, the functor $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y)$ constructed above identifies with the functor that we had initially denoted by $f^!$, i.e., the right adjoint of f_*^{IndCoh}. To distinguish the two, let us keep the notation $f^!$ for the latter functor.

We need to construct an identification between $(f^!)^\vee$ and f_*^{IndCoh}. Unwinding the definitions, the functor

$$(f^!)^\vee : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$$

is given by

$$\mathcal{F} \mapsto (\mathrm{id}_Y \times p_X)_*^{\mathrm{IndCoh}} \circ (\mathrm{id}_Y \times \Delta_X)^! \circ (\mathrm{id}_Y \times f \times \mathrm{id}_X)(\mu_Y(k) \boxtimes \mathcal{F}).$$

I.e., this is pull-push of $\mu_Y(k) \boxtimes \mathcal{F} \in \mathrm{IndCoh}(Y \times Y \times X)$ along the clockwise circuit of the following diagram

$$\begin{array}{ccc} Y \times Y \times X & \xleftarrow{\mathrm{id}_Y \times \mathrm{Graph}_f} & Y \times X \\ {\scriptstyle \mathrm{id}_Y \times p_Y \times f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y \times p_X} \\ Y \times Y & \xleftarrow{\Delta_Y} & Y, \end{array}$$

in which the horizontal arrows are closed embeddings.

Applying base change, we replace the above functor by push-pull along the counterclockwise circuit, and we obtain

$$\begin{aligned} &(\Delta_Y)^! \circ (\mathrm{id}_Y \times p_Y \times f)_*^{\mathrm{IndCoh}}(\mu_Y(k) \boxtimes \mathcal{F}) \simeq \\ &\quad \simeq (\Delta_Y)^! \left((\mathrm{id}_Y \times p_Y)_*^{\mathrm{IndCoh}}(\mu_Y(k)) \boxtimes f_*^{\mathrm{IndCoh}}(\mathcal{F})\right) \simeq (\Delta_Y)^!(\omega_Y \boxtimes f^!(\mathcal{F})) \simeq f_*^{\mathrm{IndCoh}}(\mathcal{F}), \end{aligned}$$

as required.

5. Convolution monoidal categories and algebras

In this section[4] we will apply the the formalism of IndCoh as a functor out of the category of correspondences to carry out the following construction and its generalizations:

Let $\mathcal{R} \rightrightarrows X$ be a Segal object in the category of schemes (see below for what this means). Then the category $\mathrm{IndCoh}(\mathcal{R})$ has a natural monoidal structure, and $\omega_{\mathcal{R}} \in \mathrm{IndCoh}(\mathcal{R})$ defines a monad acting on $\mathrm{IndCoh}(X)$.

5.1. **Convolution algebras.**

In this subsection we will show that monoid-objects give rise to convolution algebras.

[4] The contents of this section were suggested to us by S. Raskin.

5.1.1. Let $\mathcal{R}^\bullet$ be a *Segal object* in $\mathrm{PreStk}_{\mathrm{laft}}$ acting on a given $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

I.e., $\mathcal{R}^\bullet$ is a simplicial object in $\mathrm{PreStk}_{\mathrm{laft}}$, equipped with an identification $\mathcal{R}^0 = \mathcal{X}$, and such that for any $n \geq 2$, the map

$$\mathcal{R}^n \to \mathcal{R}^1 \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} \mathcal{R}^1,$$

given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

is an isomorphism.

Remark 5.1.2. An alternative terminology for such $\mathcal{R}^\bullet$ is *category-object*. Indeed, the above condition is equivalent to requiring that for any $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, the simplicial space

$$\mathrm{Maps}(\mathcal{Y}, \mathcal{R}^\bullet)$$

be a Segal space. Note we *do not* require it to be a *complete* Segal space.

5.1.3. In what follows we will denote $\mathcal{R} = \mathcal{R}^1$. We will informally think of a Segal object $\mathcal{R}^\bullet$ as the prestack $\mathcal{R}$, equipped with the source and target maps

$$p_s, p_t : \mathcal{R} \rightrightarrows \mathcal{X},$$

and the multiplication map

$$\mathrm{mult} : \mathcal{R} \underset{t,\mathcal{X},s}{\times} \mathcal{R} \to \mathcal{R}.$$

For the duration of this subsection we will assume:

- The target map $p_t : \mathcal{R} \rightrightarrows \mathcal{X}$ is schematic;
- The multiplication map $\mathrm{mult} : \mathcal{R} \underset{t,\mathcal{X},s}{\times} \mathcal{R} \to \mathcal{R}$ is proper.

5.1.4. Applying [Chapter V.3, Proposition 4.1.4 and Variant 4.1.6], we obtain that $\mathcal{R}^\bullet$ defines a monad $\mathsf{M}_{\mathcal{R}}$, acting on $\mathcal{X}$, i.e., an associative algebra object in the monoidal $(\infty, 1)$-category

$$\mathbf{Maps}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch;all}}}(\mathcal{X}, \mathcal{X}).$$

Concretely, the 1-morphism $\mathcal{X} \to \mathcal{X}$, corresponding to $\mathsf{M}_{\mathcal{R}}$ is given by the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{p_s} & \mathcal{X} \\ {\scriptstyle p_t}\downarrow & & \\ \mathcal{X}, & & \end{array}$$

and the multiplication on $\mathsf{M}_{\mathcal{R}}$ is given by the diagram

$$\begin{array}{ccccc} \mathcal{R} \underset{\mathcal{X}}{\times} \mathcal{R} & & & & \\ & \searrow & & \searrow & \\ & & \mathcal{R} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \\ & & \mathcal{X}. & & \end{array} \tag{5.1}$$

5.1.5. Applying the functor

$$\text{IndCoh}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{sch-qc;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{sch-qc;all}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}},$$

we obtain that to $\mathsf{M}_{\mathcal{R}}$ there corresponds a monad $\text{IndCoh}(\mathsf{M}_{\mathcal{R}})$ acting on $\text{IndCoh}(\mathcal{X})$.

It follows from the definitions that as an endo-functor of $\text{IndCoh}(\mathcal{X})$, the monad $\text{IndCoh}(\mathsf{M}_{\mathcal{R}})$ is given by

$$(p_t)^{\text{IndCoh}}_* \circ (p_s)^!.$$

I.e., the above construction formalizes the idea of a pull-push monad, corresponding to a Segal object $\mathcal{R}^\bullet$.

5.1.6. Assume now that $\mathcal{R}^\bullet$ is a groupoid object of $\text{PreStk}_{\text{laft}}$, equal to the Čech nerve of a proper schematic map $g : \mathcal{X} \to \mathcal{Y}$.

In this case, it follows from [Chapter V.3, Sect. 4.3.4 and Variant 4.3.5] that the monad $\mathsf{M}_{\mathcal{R}}$ is canonically isomorphic to one corresponding to the composite of g (viewed as a 1-morphism in the $(\infty, 2)$-category $\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{sch-qc;all}}$) with its right adjoint.

5.1.7. Assume now that $\mathcal{R}^\bullet$ is a groupoid object of $\text{PreStk}_{\text{laft}}$, with the maps $p_s, p_t : \mathcal{R} \to \mathcal{X}$ being proper.

In then according to Sect. 3.4.6, the endo-functor $(p_t)^{\text{IndCoh}}_* \circ (p_s)^!$ acquires an (a priori different) structure of monad.

We claim, however that the above two ways of introducing a structure of monad on the endo-functor $(p_t)^{\text{IndCoh}}_* \circ (p_s)^!$ coincide. Indeed, this follows from Sect. 5.1.6, applied to

$$\mathcal{Y} := |\mathcal{R}^\bullet|.$$

5.2. **Convolution monoidal categories.** In this subsection we will show that IndCoh of a Segal object in $\text{PreStk}_{\text{laft}}$ carries a natural monoidal structure.

5.2.1. Let $\mathcal{R}^\bullet$ be a Segal object in $\text{PreStk}_{\text{laft}}$ acting on $\mathcal{X}$. We impose the following conditions:

The maps $p_t : \mathcal{R} \to \mathcal{X}$ *and* $\text{mult} : \mathcal{R} \underset{t,\mathcal{X},s}{\times} \mathcal{R} \to \mathcal{R}$ *are both schematic.*

5.2.2. By [Chapter V.3, Theorem 4.4.2 and Variant 4.4.7], the object $\mathcal{R}$ acquires a natural structure of algebra in the (symmetric) monoidal category

$$\text{Corr}(\text{PreStk}_{\text{laft}})_{\text{sch;all}}.$$

Moreover, according to [Chapter V.3, Sect. 4.5.2 and Variant 4.5.5], the object

$$\mathcal{X} \in \text{Corr}(\text{PreStk}_{\text{laft}})_{\text{sch;all}}$$

is naturally a module for $\mathcal{R}$.

Applying the right-lax (symmetric) monoidal functor

$$\text{IndCoh}_{\text{Corr}(\text{PreStk}_{\text{laft}})_{\text{sch;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})_{\text{sch;all}} \to \text{DGCat}_{\text{cont}},$$

we obtain that the DG category $\text{IndCoh}(\mathcal{R})$ acquires a structure of monoidal DG category (i.e., a structure of associative algebra in $\text{DGCat}_{\text{cont}}$), and $\text{IndCoh}(\mathcal{X})$ acquires a structure of $\text{IndCoh}(\mathcal{R})$-module.

Unwinding the definitions, we obtain that the binary operation on $\mathrm{IndCoh}(\mathcal{R})$ is given by the *convolution* product, i.e., pull-push along the diagram

$$\begin{array}{ccc} \mathcal{R} \underset{\mathcal{X}}{\times} \mathcal{R} & \longrightarrow & \mathcal{R} \times \mathcal{R} \\ \big\downarrow & & \\ \mathcal{R}, & & \end{array}$$

and the action of $\mathrm{IndCoh}(\mathcal{R})$ on $\mathrm{IndCoh}(\mathcal{X})$ is given by pull-push along the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{p_s \times \mathrm{id}} & \mathcal{X} \times \mathcal{R} \\ {\scriptstyle p_t}\big\downarrow & & \\ \mathcal{X}. & & \end{array}$$

5.2.3. Consider a particular case when $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{aft}}$, and $\mathcal{X}^n = X^{\times(n+1)}$. So

$$\mathcal{R} = X \times X.$$

We obtain that $\mathrm{IndCoh}(X \times X)$ acquires a structure of monoidal category, equipped with an action on $\mathrm{IndCoh}(X)$.

I.e., we obtain a monoidal functor

$$\mathrm{IndCoh}(X \times X) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{IndCoh}(X)). \tag{5.2}$$

By construction, as a functor of plain categories, (5.2) identifies with

$$\mathrm{IndCoh}(X \times X) \simeq \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X) \overset{(\mathbf{D}_X^{\mathrm{Serre}})^{-1} \otimes \mathrm{Id}}{\longrightarrow} \mathrm{IndCoh}(X)^\vee \otimes \mathrm{IndCoh}(X) \simeq$$
$$\simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{IndCoh}(X)).$$

In particular, the functor (5.2) is an equivalence of monoidal categories.

5.3. **The case of** QCoh**.** In this subsection we will explain the variant of the constructions in this subsection for QCoh instead of IndCoh.

5.3.1. First, starting from the functor

$$\mathrm{Sch} \overset{\mathrm{QCoh}_{\mathrm{Sch}}}{\longrightarrow} \mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

and using the usual base change property for QCoh, we apply [Chapter V.1, Theorem 3.2.2(a)] and we obtain a functor

$$\mathrm{QCoh}_{\mathrm{Corr(Sch)}^{\mathrm{all}}_{\mathrm{all;all}}} : \mathrm{Corr(Sch)}^{\mathrm{all}}_{\mathrm{all;all}} \to (\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}})^{2\text{-}\mathrm{op}}.$$

Moreover, by [Chapter V.3, Proposition 3.1.5], the above functor carries a natural (symmetric) monoidal structure.

5.3.2. Further, applying [Chapter V.2, Theorem 6.1.5], from the functor $\mathrm{QCoh}_{\mathrm{Corr(Sch)}^{\mathrm{all}}_{\mathrm{all;all}}}$ constructed above, we obtain the functor

$$\mathrm{QCoh}_{\mathrm{Corr(PreStk)}^{\mathrm{sch}}_{\mathrm{sch;all}}} : \mathrm{Corr(PreStk)}^{\mathrm{sch}}_{\mathrm{sch;all}} \to (\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}})^{2\text{-}\mathrm{op}}.$$

Applying [Chapter V.3, Proposition 3.2.4], we obtain that the latter functor carries a right-lax (symmetric) monoidal structure.

5.3.3. Hence, the above discussion of convolution categories and algebras applies almost verbatim, when we replace IndCoh by QCoh, with the only difference that in whatever applies to 2-categorical phenomena, the direction of 2-morphisms gets reversed.

In particular, the geometric constructions that gave rise to *algebras* in the monoidal categories $\mathrm{IndCoh}(\mathcal{R})$ will produce *co-algebras* in the monoidal categories $\mathrm{QCoh}(\mathcal{R})$.