CHAPTER V.3. THE (SYMMETRIC) MONOIDAL STRUCTURE ON THE CATEGORY OF CORRESPONDENCES

Contents

Introduction

0.1. **Why do we want it?** The goal of this chapter is to provide a general framework for theorems along the lines that the functor

$$\text{IndCoh}_{\text{Corr}(\text{Sch}_{\text{aft}})} : \text{Corr}(\text{Sch}_{\text{aft}}) \to \text{DGCat}_{\text{cont}} \tag{0.1}$$

is symmetric monoidal.

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Why do we care about this? There are at least two applications for having such a formalism.

0.1.1. The first application is the following. We show that a symmetric monoidal structure on (0.1) encodes the duality on IndCoh; in this case, Serre self-duality of $\mathrm{IndCoh}(X)$ for $X \in \mathrm{Sch}_{\mathrm{aft}}$.

Namely, let $\mathbf{C}$ be an arbitrary category closed under finite products, and let

$$\Phi : \mathrm{Corr}(\mathbf{C}) \to \mathbf{O}$$

be a functor, where $\mathbf{O}$ be a symmetric monoidal category.

In Sect. 2.1.4 we show that the symmetric monoidal structure on $\mathbf{C}$, given by Cartesian products, gives rise to a symmetric monoidal structure on $\mathrm{Corr}(\mathbf{C})$. Assume now that the functor Φ is endowed with a symmetric monoidal structure.

It then follows from Proposition 2.3.4 that for every $\mathbf{c} \in \mathbf{C}$, the object $\Phi(\mathbf{c}) \in \mathbf{O}$ is canonically self-dual, so that the duality datum is provided by applying Φ to the 1-morphisms

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & \mathbf{c} \times \mathbf{c} \\ \downarrow & & \\ * & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & * \\ \downarrow & & \\ \mathbf{c} \times \mathbf{c}. & & \end{array}$$

In particular, for map $\mathbf{c}_1 \xrightarrow{f} \mathbf{c}_2$ in $\mathbf{C}$, the maps

$$\Phi(\mathbf{c}_1) \rightleftarrows \Phi(\mathbf{c}_2)$$

in $\mathbf{O}$, given by the diagrams

$$\begin{array}{ccc} \mathbf{c}_1 & \xrightarrow{\mathrm{id}} & \mathbf{c}_1 \\ {\scriptstyle f}\downarrow & & \\ \mathbf{c}_2 & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{c}_1 & \xrightarrow{f} & \mathbf{c}_2 \\ {\scriptstyle \mathrm{id}}\downarrow & & \\ \mathbf{c}_1 & & \end{array}$$

are the duals of each other.

Going back to the example of

$$\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}, \ \mathbf{O} = \mathrm{DGCat}_{\mathrm{cont}} \text{ and } \Phi = \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})},$$

the resulting identification

$$\mathrm{IndCoh}(X)^{\vee} \simeq \mathrm{IndCoh}(X)$$

is the ind-extension of the Serre duality anti-equivalence $\mathrm{Coh}(X)^{\mathrm{op}} \to \mathrm{Coh}(X)$.

0.1.2. Another application is the following. We show that a monoidal structure on (0.1) encodes the formation of *convolution categories*.

Let $\mathbf{C}$ be a category that admits finite limits. Let $\mathbf{c} \in \mathbf{C}$ be an object, and let $\mathbf{c}^\bullet \in \mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ be a *Segal object* [1] acting on $\mathbf{c}$. I.e., we have an identification $\mathbf{c}^0 = \mathbf{c}$ and we require that for any $n \geq 2$, the map

$$\mathbf{c}^1 \underset{\mathbf{c}}{\times} ... \underset{\mathbf{c}}{\times} \mathbf{c}^1,$$

given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

be an isomorphism.

In Theorem 4.4.2 we show that $\mathbf{c}^1$, regarded as an object of $\mathrm{Corr}(\mathbf{C})$, has a natural structure of associative algebra (with respect to the (symmetric) monoidal structure on $\mathrm{Corr}(\mathbf{C})$), where the binary operation on $\mathbf{c}^1$ is given by the diagram

$$\begin{array}{ccc} \mathbf{c}^2 & \longrightarrow & \mathbf{c}^1 \times \mathbf{c}^1 \\ \Big\downarrow & & \\ \mathbf{c}^1, & & \end{array}$$

in which the vertical map is given by the active map $[1] \to [2]$, and the horizontal map is given by the product of the two inert maps $[1] \to [2]$.

In particular, taking $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ and applying the (symmetric) monoidal functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}}) \to \mathrm{DGCat}_{\mathrm{cont}},$$

we obtain that for a Segal object $X^\bullet$ in the category of schemes, the category $\mathrm{IndCoh}(X^1)$ is endowed with a monoidal structure, given by *convolution*. I.e., it is given by pull-push along the diagram

$$\begin{array}{ccc} X_1 \underset{X_0}{\times} X_1 & \longrightarrow & X^1 \times X^1 \\ \Big\downarrow & & \\ X^1. & & \end{array}$$

0.2. **What is done in this Chapter?**

0.2.1. In Sect. 1 we make a general review, following [Lu2], of 'what it means to be (symmetric) monoidal'.

First, we define the notion of (commutative) monoid in an ∞-category.

As a result we obtain the notions of (symmetric) monoidal $(\infty, 1)$-category and $(\infty, 2)$-category.

We also review the notions of right-lax and left-lax (symmetric) monoidal functors between (symmetric) monoidal $(\infty, 1)$-categories and $(\infty, 2)$-categories. The latter leads to the notion of (commutative) algebra object in a (symmetric) monoidal $(\infty, 1)$-category.

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[1] Alternative terminology: *category-object*.

0.2.2. In Sect. 2, we show that if an $(\infty,1)$-category $\mathbf{C}$ has a (symmetric) monoidal structure, and $vert, horiz, adm$ are three classes of objects, preserved by the monoidal operation, then the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ acquires a (symmetric) monoidal structure.

In the applications, we will take $\mathbf{C}$ endowed with the *Cartesian* symmetric monoidal structure.

In Sect. 2.2 we show that the $(\infty,1)$-category

$$\mathrm{Corr}(\mathbf{C}) := \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{\mathrm{all;all}}$$

is endowed with a canonical anti-involution, given by swapping the roles of the vertical and horizontal arrows.

We prove that this anti-involution is canonically isomorphic to the *dualization* functor, when $\mathrm{Corr}(\mathbf{C})$ is considered as a symmetric monoidal category.

0.2.3. In Sect. 3 we show that the extension results of [Chapter V.1, Sects. 4 and 5] and [Chapter V.2] carry through to the (symmetric) monoidal world.

0.2.4. In Sect. 4 we give the following two constructions, starting from a Segal object $\mathbf{c}^\bullet$ in an $(\infty,1)$-category $\mathbf{C}$, acting on $\mathbf{c} = \mathbf{c}^0 \in \mathbf{C}$.

In the first construction (which does *not* appeal to the symmetric monoidal structure on the category of correspondences), we show that a Segal object as above defines an algebra object in the monoidal category

$$\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c},\mathbf{c})$$

of endomorphisms of $\mathbf{c}$ in the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

In the second construction (which does talk about the symmetric monoidal structure on the category of correspondences), we show that the object $\mathbf{c}^1$ has a natural structure of associative algebra in $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$.

1. (Symmetric) monoidal structures: recollections

In this section we review, mostly following [Lu2], and partly repeating the material of [Chapter I.1, Sect. 3], the notions of (commutative) monoid (in a given $(\infty,1)$-category), (symmetric) monoidal $(\infty,1)$-category, and (symmetric) monoidal $(\infty,2)$-category.

For a usual category $\mathcal{C}$, a monoid in it is an object $c \in \mathcal{C}$ equipped with a product operation $c \times c \to c$ and a unit map $* \to c$ that satisfy the usual axioms (in fact, three altogether).

The main feature in the ∞-setting is that if we were to imitate this definition when $\mathcal{C}$ is an $(\infty,1)$-category, in addition to the above binary operation we will need to supply a whole tail of higher operations (e.g., a homotopy between the two tertiary operations $c^{\times 3} \rightrightarrows c$), and axioms on the compatibilities between them. The problem is that this becomes too unwieldy to work with.

The main idea is that when defining monoids, instead of specifying just one object c, we specify the entire datum of its products and maps between them. Such a data is encoded by just one functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, where the original c is the value of our functor on $[1] \in \mathbf{\Delta}^{\mathrm{op}}$. This functor must satisfy some obvious condition (that expresses the fact that its value on $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ is the the product of n copies of c). This description was first formulated in [Seg].

The above approach to the definition creates a very convenient framework for working with monoids (and the related notions of monoidal categories, algebras in them, etc.). It also allows for an immediate generalization in the world of $(\infty,2)$-categories.

1.1. **Monoids and commutative monoids.** In this subsection we recall the notions of monoid and commutative monoid in the setting of $(\infty, 1)$-categories.

1.1.1. Let $\mathcal{C}$ be an $(\infty, 1)$-category with finite products (including the empty finite product, i.e., a final object). One can then talk about *monoids* in $\mathcal{C}$. By definition, they form a full subcategory, denoted $\mathrm{Monoid}(\mathcal{C})$, in $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, consisting of objects $\mathcal{R}^\bullet$, such that $\mathcal{R}^0 = *_{\mathcal{C}}$, and such that for any $n \geq 2$, the map

$$\mathcal{R}^n \to \mathcal{R}^1 \times ... \times \mathcal{R}^1, \tag{1.1}$$

given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

is an isomorphism.

1.1.2. Similarly, one can talk about *commutative monoids*, denoted $\mathrm{ComMonoid}(\mathcal{C})$ in $\mathcal{C}$. Instead of $\mathbf{\Delta}^{\mathrm{op}}$ we use the category Fin_* of pointed finite sets. The condition now is that $\mathcal{R}^{\{*\}} = *_{\mathcal{C}}$ and for every $(* \in I) \in \mathrm{Fin}_*$ the map

$$\mathcal{R}^I \to \prod_{i \in I - \{*\}} \mathcal{R}^{(* \in \{* \sqcup i\})} \tag{1.2}$$

is an isomorphism.

1.1.3. Recall that we have a canonically defined functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fin}_*$, see [Chapter I.1, Sect. 3.3.2].

Pre-composing, we obtain the forgetful functor

$$\mathrm{ComMonoid}(\mathcal{C}) \to \mathrm{Monoid}(\mathcal{C}).$$

In what follows we will focus on the symmetric monoidal case, while the monoidal case can be treated similarly.

1.2. **Symmetric monoidal categories.** In this subsection we recall the notion of symmetric monoidal $(\infty, 1)$-category, see [Chapter I.1, Sect. 3.3], from a slightly different perspective.

1.2.1. Applying Sect. 1.1 to $\mathcal{C} = 1\text{-Cat}$, we obtain the notion of *symmetric monoidal category.* Unstraightening associates to a symmetric monoidal category a co-Caretsian fibration

$$\mathcal{C}^{\otimes, \mathrm{Fin}_*} \to \mathrm{Fin}_* \tag{1.3}$$

and also a Cartesian fibration

$$\mathcal{C}^{\otimes, \mathrm{Fin}_*^{\mathrm{op}}} \to \mathrm{Fin}_*^{\mathrm{op}}. \tag{1.4}$$

The data of a symmetric monoidal category is equivalent to either (1.3) and (1.4) satisfying the condition that $\mathcal{C}^{\{*\}} = *$ and for every $(* \in I) \in \mathrm{Fin}_*$ the corresponding functor

$$\mathcal{C}^I \to \prod_{i \in I - \{*\}} \mathcal{C}^{(* \in \{* \sqcup i\})} \tag{1.5}$$

is an equivalence.

1.2.2. By a symmetric monoidal functor we shall mean a 1-morphism in the category

$$\mathrm{ComMonoid}(1\text{-Cat}).$$

Equivalently, this is a functor over Fin_*

$$\mathcal{C}_1^{\otimes,\mathrm{Fin}_*} \to \mathcal{C}_2^{\otimes,\mathrm{Fin}_*} \tag{1.6}$$

that sends coCartesian arrows to coCartesian arrows, and still equivalently, a functor over $(\mathrm{Fin}_*)^{\mathrm{op}}$

$$\mathcal{C}_1^{\otimes,\mathrm{Fin}_*^{\mathrm{op}}} \to \mathcal{C}_2^{\otimes,\mathrm{Fin}_*^{\mathrm{op}}} \tag{1.7}$$

that sends Cartesian arrows to Cartesian arrows.

1.2.3. We shall say that a map $(* \in I) \to (* \in J)$ is *inert* (resp., *idle*) if any element in $J - \{*\}$ has exactly (resp., at most one) preimage.

By a *right-lax* symmetric monoidal functor between $\mathcal{C}_1$ and $\mathcal{C}_2$ we shall mean a functor as in (1.6) that is only required to send coCartesian arrows that lie over *idle* maps in Fin_* to coCartesian arrows.

By a *non-unital right-lax* symmetric monoidal functor between $\mathcal{C}_1$ and $\mathcal{C}_2$ we shall mean a functor as in (1.6) that is only required to send coCartesian arrows that lie over *inert* maps in Fin_* to coCartesian arrows.

Similarly, we obtain that notions of *left-lax* and *non-unital left-lax* symmetric monoidal functors: use (1.7) instead of (1.6) and 'Cartesian' instead of 'coCartesian'.

1.2.4. By a *commutative algebra* in a symmetric monoidal category $\mathcal{C}$ we shall mean a non-unital right-lax symmetric monoidal functor $* \to \mathcal{C}$.

We let $\mathrm{ComAlg}(\mathcal{C})$ denote the category of commutative algebras in $\mathcal{C}$. (Note that these are the *unital* commutative algebras!)

By construction, a right-lax monoidal functor $\mathcal{C}_1 \to \mathcal{C}_2$ induces a functor

$$\mathrm{ComAlg}(\mathcal{C}_1) \to \mathrm{ComAlg}(\mathcal{C}_2).$$

1.3. **The Cartesian symmetric monoidal structure.** Let $\mathcal{C}$ be again an $(\infty,1)$-category with finite products.

It is then intuitively clear that the operation of Cartesian product defines on $\mathcal{C}$ a symmetric monoidal structure, called the *Cartesian symmetric monoidal structure*. We will formalize this in the present subsection, following [Lu2, Sect. 2.4.1].

1.3.1. We start with the functor

$$\mathrm{Fin}_*^{\mathrm{op}} \to 1\text{-Cat}, \quad (* \in I) \mapsto \mathcal{C}^{I-\{*\}}, \tag{1.8}$$

see [Chapter I.1, Sect. 3.3.3].

Straightening defines a Cartesian fibration

$$\mathcal{C}^{\times,\mathrm{Fin}_*} \to \mathrm{Fin}_*\,. \tag{1.9}$$

However, the condition that $\mathcal{C}$ admits finite products implies that (1.9) is also a coCartesian fibration, thereby giving rise to the datum as in (1.3).

It is clear that the functors in (1.5) are equivalences. Hence, (1.9) corresponds to a canonically defined symmetric monoidal structure on $\mathcal{C}$. This is the Cartesian symmetric monoidal structure.

1.3.2. Note that any functor $\mathcal{C}_1 \to \mathcal{C}_2$ gives rise to a functor over Fin_*

$$\mathcal{C}_1^{\times,\mathrm{Fin}_*} \to \mathcal{C}_2^{\times,\mathrm{Fin}_*}$$

and thus defines a left-lax functor from $\mathcal{C}_1$ to $\mathcal{C}_2$, when both are considered as equipped with the Cartesian symmetric monoidal structure.

This functor is (strictly) symmetric monoidal if and only if the initial functor $\mathcal{C}_1 \to \mathcal{C}_2$ commutes with finite products.

1.3.3. Note now that on the one hand, we have the notion of commutative monoid in $\mathcal{C}$, and on the other hand, we have the notion of commutative algebra in $\mathcal{C}$, considered as a symmetric monoidal category.

However, that these two notions coincide, i.e., the categories $\mathrm{ComMonoid}(\mathcal{C})$ and $\mathrm{ComAlg}(\mathcal{C})$ are canonically equivalent, see [Lu2, Proposition 2.4.2.5].

Let us see explicitly the functor

$$\mathrm{ComAlg}(\mathcal{C}) \to \mathrm{ComMonoid}(\mathcal{C}).$$

Indeed, the operation of assigning to $(* \in I) \in \mathrm{Fin}_*$, the functor of Cartesian product along $I - \{*\}$

$$\mathcal{C}^{I-\{*\}} \to \mathcal{C}$$

defines a functor

$$\mathcal{C}^{\times,\mathrm{Fin}_*} \to \mathcal{C} \times \mathrm{Fin}_*$$

over Fin_*.

Given a section $\mathrm{Fin}_* \to \mathcal{C}^{\times,\mathrm{Fin}_*}$, we thus obtain a functor $\mathrm{Fin}_* \to \mathcal{C}$. It is easy to see that the requirement on the above section to be right-lax symmetric monoidal implies that the resulting object of $\mathcal{C}^{\mathrm{Fin}_*}$ is a commutative monoid.

1.4. **Symmetric monoidal $(\infty, 2)$-categories.** In this subsection we introduce symmetric monoidal $(\infty, 2)$-categories. We refer the reader to [Chapter A.1] for our conventions regarding $(\infty, 2)$-categories.

1.4.1. The $(\infty, 1)$-category 2-Cat has finite products. Hence, we can talk about commutative monoids in 2-Cat (or, equivalently, according to Sect. 1.3.3 above, about commutative algebras in 2-Cat with respect to the Cartesian symmetric monoidal structure).

Thus, we obtain the notion of *symmetric monoidal* $(\infty, 2)$*-category.*

1.4.2. Applying the 2-categorical unstraigtening (see [Chapter A.2, Theorem 2.1.8]), we can encode the datum of a symmetric monoidal $(\infty, 2)$-category $\mathbb{S}$ by a 2-coCartesian fibration

$$\mathbb{S}^{\otimes,\mathrm{Fin}_*} \to \mathrm{Fin}_*, \tag{1.10}$$

or equivalently a 2-Cartesian fibration

$$\mathbb{S}^{\otimes,\mathrm{Fin}_*^{\mathrm{op}}} \to \mathrm{Fin}_*^{\mathrm{op}}. \tag{1.11}$$

As in the case of $(\infty, 1)$-categories, the datum of a symmetric monoidal $(\infty, 2)$-category is equivalent to that of (1.10) (or (1.11)) such that the corresponding functors

$$\mathbb{S}^I \to \prod_{i \in I - \{*\}} \mathbb{S}^{(* \in \{* \sqcup i\})}$$

are equivalences.

1.4.3. As in the case of $(\infty,1)$-categories, this leads to the notion of (resp., non-unital) right-lax symmetric monoidal functor $\mathbb{S}_1 \to \mathbb{S}_2$ between symmetric monoidal $(\infty,2)$-categories $\mathbb{S}_1$ and $\mathbb{S}_2$: this is a functor

$$\mathbb{S}_1^{\otimes,\mathrm{Fin}_*} \to \mathbb{S}_2^{\otimes,\mathrm{Fin}_*}$$

satisfying the same condition for idle (resp., inert) arrows in Fin_*.

1.4.4. Applying the above for $\mathbb{S}_1 = *$ and $\mathbb{S}_2 = \mathbb{S}$, we obtain the notion of commutative algebra object in $\mathbb{S}$.

Note, however, that since $*$ is a 1-category, commutative algebras in $\mathbb{S}$ (and homomorphisms between them) are the same as the corresponding notions for $\mathbb{S}^{\text{1-Cat}}$.

1.4.5. The feature of the 2-categorical situation is that, given right-lax (resp., non-unital) symmetric monoidal functors

$$\Phi', \Phi'' : \mathbb{S}_1 \to \mathbb{S}_2,$$

in addition to usual natural transformations between them equipped with a symmetric monoidal structure, one can talk about natural transformations between them equipped with a *right-lax* (resp., *left-lax*) symmetric monoidal structure.

By definition those are right-lax (resp., left-lax) natural transformations (see [Chapter A.1, Sect. 3.2.7] for what this means) between the corresponding functors

$$\Phi'^{\otimes,\mathrm{Fin}_*}, \Phi''^{\otimes,\mathrm{Fin}_*} : \mathbb{S}_1^{\otimes,\mathrm{Fin}_*} \rightrightarrows \mathbb{S}_2^{\otimes,\mathrm{Fin}_*},$$

that are strict over idle arrows in Fin_*.

One can also talk about natural transformations between them equipped with a *non-unital* right-lax (resp., left-lax) symmetric monoidal structure: replace the word 'idle' by 'inert' in the above definition.

Thus, given a symmetric monoidal $(\infty,2)$-category $\mathbb{S}$, we obtain the notion of *right-lax* (resp., *left-lax*) homomorphism

$$s' \to s''$$

between two commutative algebra objects in $\mathbb{S}$.

Similarly, we obtain the notion of *non-unital* right-lax (resp., left-lax) homomorphism.

1.4.6. As in the case of 1-Cat, if an $(\infty,2)$-category $\mathbb{S}$ has finite products, it acquires the Cartesian symmetric monoidal structure.

By Sects. 1.4.4 and 1.3.3, commutative algebra objects in $\mathbb{S}$ in the Cartesian symmetric monoidal structure are the same as commutative monads in $\mathbb{S}^{\text{1-Cat}}$.

1.4.7. Let us take $\mathbb{S} = \mathbf{1\text{-}Cat}$. By Sect. 1.4.6, we obtain that $\mathbf{1\text{-}Cat}$ is a symmetric monoidal $(\infty,2)$-category. Commutative algebra objects in it are the same as symmetric monoidal categories.

By unwinding the definitions, we obtain that given two symmetric monoidal categories $\mathcal{C}_1, \mathcal{C}_2$, regarded as commutative algebra objects in $\mathbf{1\text{-}Cat}$, *right-lax* (resp., *left-lax*) homomorphisms between them are the same as *right-lax* (resp., *left-lax*) symmetric monoidal functors

$$\mathcal{C}_1 \to \mathcal{C}_2$$

as defined in Sect. 1.2.3. Indeed, both identify with right-lax (resp., left-lax) natural transformations between functors

$$\mathrm{Fin}_* \to \mathbf{1\text{-}Cat}.$$

2. (Symmetric) monoidal structures and correspondences

Recall that in [Chapter V.1, Sect. 1] we associated to an $(\infty,1)$-category equipped with three classes of morphisms $vert, horiz, adm$ (satisfying some natural conditions) an $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

The first observation in that a Cartesian symmetric monoidal structure on $\mathbf{C}$ induces a symmetric monoidal structure on $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

We will now show that for the $(\infty,1)$-category

$$\mathrm{Corr}(\mathbf{C}) = \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{\mathrm{all;all}},$$

the operation of *dualization* with respect to its symmetric monoidal structure (induced by the Cartesian symmetric monoidal structure on $\mathbf{C}$) can be interpreted as the anti-involution that swaps the roles of vertical and horizontal arrows.

2.1. The (symmetric) monoidal structure on the functor Corr.

Let $\mathbf{C}$ be an $(\infty,1)$-category, equipped with three classes of morphisms $vert, horiz, adm$ as in [Chapter V.1, Sect. 1.1], so that we can form the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$.

Assume that $\mathbf{C}$ has finite products, and each of the above classes of morphisms is preserved by finite products.

In this subsection we show (which is completely tautological) that the $(\infty,2)$-category $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ acquires a symmetric monoidal structure.

2.1.1. Let Trpl be the $(\infty,1)$-category whose objects are given by $(\infty,1)$-categories $\mathbf{C}$ together with three classes of 1-morphisms $vert, horiz$ and adm as in [Chapter V.1, Sect. 1.1].

A 1-morphism $(\mathbf{C}_1, vert_1, horiz_1, adm_1) \to (\mathbf{C}_2, vert_2, horiz_2, adm_2)$ is a functor from $\mathbf{C}_1$ to $\mathbf{C}_2$ that preserves each of the three classes of 1-morphisms as well as the Cartesian squares from [Chapter V.1, Sect. 1.1].

We endow Trpl with the Cartesian symmetric monoidal structure.

2.1.2. The assignment

$$(\mathbf{C}, vert, horiz, adm) \mapsto {}''\mathbf{Grid}^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz} \tag{2.1}$$

is clearly a functor

$$\mathrm{Trpl} \to 1\text{-}\mathrm{Cat}^{\mathbf{\Delta}^{\mathrm{op}}},$$

whose essential image belongs to the essnetial image of the functor $\mathrm{Seq}_\bullet$.

Hence, we obtain that the assignment

$$(\mathbf{C}, vert, horiz, adm) \mapsto \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

is a functor

$$\mathrm{Corr} : \mathrm{Trpl} \to 2\text{-}\mathrm{Cat}\,. \tag{2.2}$$

By Sect. 1.3.2, the above functor Corr carries a left-lax symmetric monoidal structure. However, it is easy to see that this left-lax symmetric structure is actually strict, e.g., because this is the case for the functors (2.1) and $\mathrm{Seq}_\bullet$.

Thus, we obtain that (2.2) has a natural symmetric monoidal structure.

2.1.3. Let $(\mathbf{C}, vert, horiz, adm)$ be an object of Trpl, and let $\mathbf{C}$ be endowed with a symmetric monoidal structure. Assume that each of the classes of the morphisms $vert$, $horiz$, and adm is preserved by the tensor product functor

$$\mathbf{C} \times \mathbf{C} \to \mathbf{C}.$$

Since the forgetful functor

$$\text{Trpl} \to \text{1-Cat}$$

that remembers the underlying category $\mathbf{C}$ is 1-fully faithful, by [Chapter II.1, Lemma 2.2.7], the symmetric monoidal structure on $\mathbf{C} \in \text{1-Cat}$ gives rise to a symmetric monoidal structure on $(\mathbf{C}, vert, horiz, adm) \in \text{Trpl}$.

Hence, we obtain that

$$\text{Corr}(\mathbf{C})^{adm}_{vert;horiz} \in \text{2-Cat}$$

acquires a structure of symmetric monoidal $(\infty, 2)$-category.

2.1.4. In our main application, we will work with the *Cartesian* symmetric monoidal structure on $\mathbf{C}$.

Thus, in this case we assume that the classes of 1-morphisms $(vert, horiz, adm)$ are preserved by finite products, and we obtain that

$$\text{Corr}(\mathbf{C})^{adm}_{vert;horiz} \in \text{2-Cat}$$

acquires a structure of symmetric monoidal $(\infty, 2)$-category.

In the sequel, unless explicitly stated otherwise, when discussing a symmetric monoidal structure on the $\text{Corr}(\mathbf{C})^{adm}_{vert;horiz}$, we shall mean the one, coming from the Cartesian symmetric monoidal structure on $\mathbf{C}$.

2.2. **The canonical anti-involution on the category of correspondences.** In this subsection we show that the $(\infty, 1)$-category $\text{Corr}(\mathbf{C})$ carries a canonical anti-involution, given swapping the roles of vertical and horizontal arrows.

We will also show that this anti-involution is canonically isomorphic to the dualization functor on $\text{Corr}(\mathbf{C})$, when the latter is regarded as a symmetric monoidal $(\infty, 1)$-category.

2.2.1. Let us take $adm = \text{isom}$, so that $\text{Corr}(\mathbf{C})^{adm}_{vert;horiz} = \text{Corr}(\mathbf{C})_{vert;horiz}$ is an $(\infty, 1)$-category.

Let us also take $vert = horiz = \text{all}$. Note that in this case,

$$\text{Corr}(\mathbf{C}) := \text{Corr}(\mathbf{C})_{\text{all},\text{all}}$$

carries a canonical anti-involution, denoted ϖ.

At the level of objects ϖ acts as identity. At the level of 1-morphisms it sends

$$\begin{array}{ccc} \mathbf{c}_{1,0} & \xrightarrow{\alpha_0} & \mathbf{c}_0 \\ {\scriptstyle \alpha_1}\downarrow & & \\ \mathbf{c}_1 & & \end{array}$$

to

$$\begin{array}{ccc} \mathbf{c}_{1,0} & \xrightarrow{\alpha_1} & \mathbf{c}_1 \\ {\scriptstyle \alpha_0}\downarrow & & \\ \mathbf{c}_0. & & \end{array}$$

2.2.2. The formal definition is as follows. To define ϖ, we need to construct an involutive identification

$$\mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{C})) \simeq \mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{C})) \circ (\mathrm{rev})^{\mathrm{op}}, \tag{2.3}$$

where

$$\mathrm{rev} : \mathbf{\Delta} \to \mathbf{\Delta}$$

is the reversal involution on $\mathbf{\Delta}$, see [Chapter I.1, Sect. 1.1.9].

By definition,

$$\mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{C})) = \mathrm{Grid}_\bullet^{\geq\,\mathrm{dgnl}}(\mathbf{C}),$$

and (2.3) comes from the involutive identification

$$([\bullet] \times [\bullet])^{\geq\,\mathrm{dgnl}} \simeq ([\bullet] \times [\bullet])^{\geq\,\mathrm{dgnl}} \circ \mathrm{rev}, \tag{2.4}$$

as functors $\mathbf{\Delta} \to 1\text{-}\mathrm{Cat}^{\mathrm{ordn}}$, given by reflecting half-grids over the NW-SE diagonal.

I.e., for every $[n] \in \mathbf{\Delta}$, the corresponding involution on

$$([n] \times [n])^{\geq\,\mathrm{dgnl}}$$

is $(i,j) \mapsto (n-j, n-i)$.

2.2.3. Suppose now that the $(\infty,1)$-category $\mathbf{C}$, in addition, admits finite products.

In this case, by Sect. 2.1.4, the Cartesian symmetric monoidal structure on $\mathbf{C}$ induces a symmetric monoidal structure on $\mathrm{Corr}(\mathbf{C})$. The unit for this symmetric monoidal structure is the final object $* \in \mathbf{C}$, viewed as an object of $\mathrm{Corr}(\mathbf{C})$. (Note, however, that $*$ is *not* the final object in $\mathrm{Corr}(\mathbf{C})$.)

Unwinding the definitions, we see that in this case, ϖ has a natural structure of symmetric monoidal functor.

2.3. **Relationship to the dualization functor.** In this subsection we will show that the anti-involution ϖ, defined above, is the dualization functor on $\mathrm{Corr}(\mathbf{C})$, when the latter is regarded as a symmetric monoidal category.

2.3.1. Recall that if a symmetric monoidal category $\mathbf{O}$ is such that every object $\mathbf{o} \in \mathbf{O}$ is dualizable, there is a canonical anti-involution

$$\mathbf{O} \overset{\sim}{\to} \mathbf{O}^{\mathrm{op}} \tag{2.5}$$

that takes $\mathbf{o} \in \mathbf{O}$ to its monoidal dual $\mathbf{o}^\vee$, see [Chapter I.1, Sect. 4.1.4].

This dualization functor is characterized by an isomorphism

$$\mathrm{Maps}(\mathbf{o}, \mathbf{o}') \overset{\sim}{\to} \mathrm{Maps}(\mathbf{1}_{\mathbf{O}}, \mathbf{o}^\vee \otimes \mathbf{o}') \tag{2.6}$$

as functors $\mathbf{O}^{\mathrm{op}} \times \mathbf{O} \to \mathrm{Spc}$, see [Chapter I.1, Sect. 4.1.2].

2.3.2. We take $\mathbf{O} := \mathrm{Corr}(\mathbf{C})$, where the latter is regarded as a symmetric monoidal category by the procedure of Sect. 2.1.4.

Let us observe that every object $\mathbf{c} \in \mathrm{Corr}(\mathbf{C})$ is dualizable and in fact self-dual. The duality data is supplied by the 1-morphisms in $\mathrm{Corr}(\mathbf{C})$:

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & * \\ \big\downarrow & & \\ \mathbf{c} \times \mathbf{c} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{c} & \longrightarrow & \mathbf{c} \times \mathbf{c} \\ \big\downarrow & & \\ * & & \end{array} \tag{2.7}$$

2.3.3. We will prove:

Proposition 2.3.4. *The anti-involution*

$$\varpi : \mathrm{Corr}(\mathbf{C}) \to \mathrm{Corr}(\mathbf{C})^{\mathrm{op}}$$

is canonically isomorphic to the dualization functor.

2.3.5. *Variant.* Let $(\mathbf{C}, vert, horiz, adm)$ be an object of Trpl as in Sect. 2.1.4, with $adm = \mathrm{isom}$ and $vert = horiz$. Then the involution ϖ restricts to an involution on $\mathrm{Corr}(\mathbf{C})_{vert,horiz}$.

Assume now that for every $\mathbf{c} \in \mathbf{C}$, the diagonal map $\mathbf{c} \to \mathbf{c} \times \mathbf{c}$ and the tautological maps $\mathbf{c} \to *$ belong to $vert = horiz$. In this case every object of $\mathrm{Corr}(\mathbf{C})_{vert,horiz}$ is dualizable, and the assertion of Proposition 2.3.4 holds verbatim (indeed, replace the original $\mathbf{C}$ by $\mathbf{C}_{vert} = \mathbf{C}_{horiz}$).

2.4. **A digression: the twisted arrows category.**

2.4.1. For an integer n let tw_n denote the (ordinary) category

$$-n \to ... \to -1 \to -0 \to 0 \to 1 \to ... \to n.$$

We have the natural functors

$$[n]^{\mathrm{op}} \to \mathrm{tw}_n \leftarrow [n].$$

The assignment $n \rightsquigarrow \mathrm{tw}_n$ is naturally a functor

$$\mathrm{tw}_\bullet : \mathbf{\Delta} \to \text{1-Cat}^{\mathrm{ordn}} \subset \text{1-Cat},$$

equipped with the natural transformations

$$[\bullet]^{\mathrm{op}} \to \mathrm{tw}_\bullet \leftarrow [\bullet].$$

2.4.2. For a $(\infty, 1)$-category $\mathbf{D}$, set

$$\mathrm{Tw}_n(\mathbf{D}) := \mathrm{Maps}_{\text{1-Cat}}(\mathrm{tw}_n, \mathbf{D}).$$

Thus, $\mathrm{Tw}_\bullet(\mathbf{D})$ is an object of

$$\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Spc}) = \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}},$$

which is easily seen to be a complete Segal space, equipped with the maps

$$\mathrm{Seq}_\bullet(\mathbf{D}^{\mathrm{op}}) \leftarrow \mathrm{Tw}_\bullet(\mathbf{D}) \to \mathrm{Seq}_\bullet(\mathbf{D}). \tag{2.8}$$

2.4.3. We define the *twisted arrow category* of $\mathbf{D}$, denoted $\mathrm{Tw}(\mathbf{D})$ so that

$$\mathrm{Seq}_\bullet(\mathrm{Tw}(\mathbf{D})) = \mathrm{Tw}_\bullet(\mathbf{D}).$$

The maps (2.8) give rise to a functor

$$\mathrm{Tw}(\mathbf{D}) \to \mathbf{D}^{\mathrm{op}} \times \mathbf{D}. \tag{2.9}$$

It is not difficult to see (see [Lu6, Proposition 4.2.5]) that the functor (2.9) is a co-Cartesian fibration, which is the unstraightening of the Yoneda functor

$$\mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathrm{Spc}.$$

2.5. **Proof of Proposition 2.3.4.**

2.5.1. We will prove Proposition 2.3.4 by exhibiting a canonical isomorphism between the functors

$$\mathrm{Corr}(\mathbf{C})^{\mathrm{op}} \times \mathrm{Corr}(\mathbf{C}) \to \mathrm{Spc}$$

given by

$$\mathrm{Maps}_{\mathrm{Corr}(\mathbf{C})}(-,-) \text{ and } \mathrm{Maps}_{\mathrm{Corr}(\mathbf{C})}(*, \varpi(-) \otimes -), \tag{2.10}$$

respectively.

Unstraightening, we need to construct an isomorphism between the coCartesian fibrations in spaces that correspond to the two functors in (2.10).

2.5.2. By Sect. 2.4, the functor $\mathrm{Maps}_{\mathrm{Corr}(\mathbf{C})}(-,-)$ corresponds to the coCartesian fibration

$$\mathrm{Tw}(\mathrm{Corr}(\mathbf{C})) \to (\mathrm{Corr}(\mathbf{C}))^{\mathrm{op}} \times \mathrm{Corr}(\mathbf{C}).$$

The functor

$$\mathrm{Maps}_{\mathrm{Corr}(\mathbf{C})}(*,-) : \mathrm{Corr}(\mathbf{C}) \to \mathrm{Spc}$$

is given by the coCartesian fibration $\mathrm{Corr}(\mathbf{C})_{*/}$.

Thus, in order to construct an isomorphism between the functors (2.10) we need to construct a functor

$$\mathrm{Tw}(\mathrm{Corr}(\mathbf{C})) \to \mathrm{Corr}(\mathbf{C})_{*/} \tag{2.11}$$

that fits into a pullback diagram

$$\begin{array}{ccc} \mathrm{Tw}(\mathrm{Corr}(\mathbf{C})) & \longrightarrow & \mathrm{Corr}(\mathbf{C})_{*/} \\ \downarrow & & \downarrow \\ (\mathrm{Corr}(\mathbf{C}))^{\mathrm{op}} \times \mathrm{Corr}(\mathbf{C}) & \xrightarrow{\mathrm{mult} \circ (\varpi \times \mathrm{Id})} & \mathrm{Corr}(\mathbf{C}) \end{array} \tag{2.12}$$

where $\mathrm{mult} : \mathrm{Corr}(\mathbf{C}) \times \mathrm{Corr}(\mathbf{C}) \to \mathrm{Corr}(\mathbf{C})$ is the functor of tensor product.

2.5.3. We have

$$\mathrm{Seq}_n(\mathrm{Corr}(\mathbf{C})_{*/}) \simeq \mathrm{Grid}_{n+1}^{\geq \mathrm{dgnl}}(\mathbf{C}) \underset{\mathbf{C}}{\times} *,$$

where $\mathrm{Grid}_{n+1}^{\geq \mathrm{dgnl}}(\mathbf{C}) \to \mathbf{C}$ is evaluation on the object

$$(0,0) \in ([n+1] \times [n+1])^{\geq \mathrm{dgnl}}.$$

The sought-for maps

$$\mathrm{Seq}_n(\mathrm{Tw}(\mathrm{Corr}(\mathbf{C}))) \to \mathrm{Seq}_n(\mathrm{Corr}(\mathbf{C})_{*/})$$

are defined as follows.

An object

$$\underline{\mathbf{c}} \in \mathrm{Maps}((\mathrm{tw}_n \times \mathrm{tw}_n)^{\geq \mathrm{dgnl}}, \mathbf{C})$$

gets sent to

$$\underline{\mathbf{c}}' \in \mathrm{Maps}(([n+1] \times [n+1])^{\geq \mathrm{dgnl}}, \mathbf{C}),$$

given by the formula:

$$\underline{\mathbf{c}}'_{i,j} = \begin{cases} * \text{ if } i = j = 0 \\ \underline{\mathbf{c}}_{-(j-1),j-1} \text{ if } i = 0, j \geq 1 \\ \underline{\mathbf{c}}_{i-1,j-1} \times \underline{\mathbf{c}}_{-(j-1),-(1-i)} \text{ if } i \geq 1. \end{cases}$$

In other words, we fold the half-grid over the NW-SE diagonal. For example, for $n = 0$, this map sends a diagram

$$\begin{array}{ccc} \mathbf{c}_{0,-0} & \longrightarrow & \mathbf{c}_{-0,-0} \\ \big\downarrow & & \\ \mathbf{c}_{0,0} & & \end{array}$$

to

$$\begin{array}{ccc} \mathbf{c}_{0,-0} & \longrightarrow & * \\ \big\downarrow & & \\ \mathbf{c}_{0,0} \times \mathbf{c}_{-0,-0} & & \end{array}$$

and for $n = 1$ a diagram

$$\begin{array}{ccccccc} \mathbf{c}_{-1,1} & \longrightarrow & \mathbf{c}_{-1,0} & \longrightarrow & \mathbf{c}_{-1,-0} & \longrightarrow & \mathbf{c}_{-1,-1} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ \mathbf{c}_{-0,1} & \longrightarrow & \mathbf{c}_{-0,0} & \longrightarrow & \mathbf{c}_{-0,-0} & & \\ \big\downarrow & & \big\downarrow & & & & \\ \mathbf{c}_{0,1} & \longrightarrow & \mathbf{c}_{0,0} & & & & \\ \big\downarrow & & & & & & \\ \mathbf{c}_{1,1} & & & & & & \end{array}$$

to

$$\begin{array}{ccccc} \mathbf{c}_{-1,1} & \longrightarrow & \mathbf{c}_{-0,0} & \longrightarrow & * \\ \big\downarrow & & \big\downarrow & & \\ \mathbf{c}_{0,1} \times \mathbf{c}_{-1,-0} & \longrightarrow & \mathbf{c}_{0,0} \times \mathbf{c}_{-0,-0} & & \\ \big\downarrow & & & & \\ \mathbf{c}_{1,1} \times \mathbf{c}_{-1,-1}. & & & & \end{array}$$

The fact that (2.12) is a pullback diagram is an easy verification.

3. Extension results in the symmetric monoidal context

In [Chapter V.1, Sects. 4 and 5] and in [Chapter V.2] we proved several theorems that say that a functor out of a certain $(\infty, 2)$-category of correspondences can be uniquely extended to a functor out of another $(\infty, 2)$-category of correspondences.

In this section we will show that these extension procedures are compatible with symmetric monoidal structures.

3.1. **'No cost' and factorization extensions.** In this subsection, we let $\mathbf{C}$ be a symmetric monoidal category, equipped with three classes of morphisms $vert, horiz, adm$ as in Sect. 2.1.3[2].

We will study how the extension paradigm in [Chapter V.1, Sects. 4 and 5] interacts with the symmetric monoidal structures.

[2] We *do not* need the symmetric monoidal structure on $\mathbf{C}$ to be Cartesian.

3.1.1. Recall the setting of [Chapter V.1, Sect. 4]. I.e., we start with a $\mathbf{C}$, equipped with four classes of morphisms $vert, horiz, adm, adm'$, satisfying the assumptions of [Chapter V.1, Sect. 4.1.1].

Let $\mathbb{S}$ be an $(\infty, 2)$-category and let us be given a functor

$$\Phi^{adm'}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm'}_{vert,horiz} \to \mathbb{S}.$$

Assume that all four classes of morphisms are preserved by the functor of tensor product $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, so that

$$\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \text{ and } \mathrm{Corr}(\mathbf{C})^{adm'}_{vert,horiz}$$

acquire symmetric monoidal structures by Sect. 2.1.3.

Assume also that $\mathbb{S}$ is equipped with a symmetric monoidal structure. Assume also that

$$\Phi^{adm}_{vert,horiz} := \Phi^{adm'}_{vert,horiz}|_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}}$$

is equipped with a symmetric monoidal structure.

We claim:

Proposition 3.1.2. *The symmetric monoidal structure on $\Phi^{adm}_{vert,horiz}$ extends uniquely to one on $\Phi^{adm'}_{vert,horiz}$.*

Proof. Indeed, [Chapter V.1, Theorem 4.1.3] implies that for any n, the functor

$$(\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz})^{\times n} \to (\mathrm{Corr}(\mathbf{C})^{adm'}_{vert,horiz})^{\times n}$$

is a categorical epimorphism.

Now, our assertion follows from the next observation:

Lemma 3.1.3. *Let $\mathbf{O}$ be a symmetric monoidal category, and let $\mathbf{o}_1 \to \mathbf{o}_2$ be a homomrphism between commutative algebras on $\mathbf{O}$, such that for any n, the map $(\mathbf{o}_1)^{\otimes n} \to (\mathbf{o}_2)^{\otimes n}$ is a categorical epimorphism. Then for another commutative algebra $\mathbf{o}'$ in $\mathbf{O}$, restriction defines an isomorphism from the space of homomorphisms $\mathbf{o}_2 \to \mathbf{o}'$ and the subspace of homomorphisms $\mathbf{o}_1 \to \mathbf{o}'$ that factor through $\mathbf{o}_2$ as maps of plain objects of $\mathbf{O}$.*

□

3.1.4. The above discussion applies verbatim to the setting of [Chapter V.1, Sect. 5]. I.e., we start with $\mathbf{C}$, equipped with four classes of morphisms $vert, horiz, adm$ and $co\text{-}adm$, satisfying the assumptions of [Chapter V.1, Sect. 5.1].

Let $\mathbb{S}$ be an $(\infty, 2)$-category and let us be given a functor

$$\Phi^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S}.$$

Assume now that $\mathbf{C}$ all four classes of morphisms are preserved by the functor of tensor product $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, so that

$$\mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert,co\text{-}adm} \text{ and } \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}$$

acquire symmetric monoidal structures by Sect. 2.1.3.

Assume also that

$$\Phi^{\mathrm{isom}}_{vert,co\text{-}adm} := \Phi^{adm}_{vert,horiz}|_{\mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{vert,co\text{-}adm}}$$

is equipped with a symmetric monoidal structure.

We have (with the same proof as above):

Proposition 3.1.5. *The symmetric monoidal structure on* $\Phi^{\mathrm{isom}}_{vert,co\text{-}adm}$ *extends uniquely to one on* $\Phi^{adm}_{vert,horiz}$.

3.2. **Right Kan extensions and symmetric monoidal structures.** In this subsection we will review the notion of right Kan extension of 1-morphisms in $(\infty,2)$-categories, and how it interacts with symmetric monoidal structures.

As an application we will show that the extension procedure in [Chapter V.2, Sect. 6] is (lax!) compatible with symmetric monoidal structures.

3.2.1. Let $\mathbb{S}$ be an $(\infty,2)$-category, and let $\alpha : s_1 \to s_2$ be a 1-morphism in $\mathbb{S}$. Given another object $s' \in \mathbb{S}$, restriction defines a functor

$$\mathbf{Maps}_{\mathbb{S}}(s_2,s') \to \mathbf{Maps}_{\mathbb{S}}(s_1,s').$$

The (partially defined) right adjoint functor to the above restriction functor is called the functor of *right Kan extension*, and is denoted by RKE_α.

3.2.2. Suppose now that $\mathbb{S}$ has a symmetric monoidal structure. Let s_1 and s_2 be commutative algebra objects in $\mathbb{S}$, and let

$$\alpha : s_1 \to s_2$$

be a homomorphism.

Let now s' be another commutative algebra object in $\mathbb{S}$, and let

$$\phi_1 : s_1 \to s'$$

be a *right-lax* homomorphism (see Sect. 1.4.5 for what this means).

Suppose that RKE_α is defined on ϕ_1 regarded as a plain object of $\mathbf{Maps}_{\mathbb{S}}(s_1,s')$. Suppose, moreover, that for every n, the canonical map from the composition

$$s_2^{\otimes n} \overset{\otimes}{\to} s_2 \overset{\mathrm{RKE}_\alpha(\phi_1)}{\longrightarrow} s'$$

to the right Kan extension along $\alpha^{\otimes n} : s_1^{\otimes n} \to s_2^{\otimes n}$ of the composition

$$s_1^{\otimes n} \overset{\otimes}{\to} s_1 \overset{\phi_1}{\longrightarrow} s'$$

is an isomorphism (in particular, the latter right Kan extension is also defined).

By unwinding the construction, we obtain that in this case

$$\phi_2 := \mathrm{RKE}_\alpha(\phi_1)$$

has a unique structure of *right-lax* homomorphism $s_2 \to s'$ so that the co-unit of the adjunction

$$\phi_2 \circ \alpha \to \phi_1$$

has the structure of a map between right-lax homomorphisms.

3.2.3. Consider now the situation of [Chapter V.2, Theorem 6.1.5], where $(\mathbf{C}, vert, horiz, adm)$ and $(\mathbf{D}, vert, horiz, adm)$ are as in Sect. 2.1.4. Assume also that the functor F takes products to products, so that it induces a symmetric monoidal functor

$$F^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz}.$$

Let $\mathbb{S}$ be a symmetric monoidal $(\infty,2)$-category, and assume that

$$\Phi^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S}$$

is endowed with a right-lax symmetric monoidal structure.

We claim:

Proposition 3.2.4. *Suppose that for any n and any map*

$$\mathbf{c} \to \mathbf{d}_1 \times ... \times \mathbf{d}_n$$

that is in horiz (here $\mathbf{c} \in \mathbf{C}$, $\mathbf{d}_i \in \mathbf{D}$), each of the projections $\mathbf{c} \to \mathbf{d}_i$ is also in horiz. Then

$$\Psi^{adm}_{vert,horiz} := \mathrm{RKE}_{F^{adm}_{vert,horiz}}(\Phi^{adm}_{vert,horiz}) : \mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz} \to \mathbb{S}$$

acquires a uniquely defined right-lax symmetric monoidal structure, for which the natural transformation

$$\Psi^{adm}_{vert,horiz} \circ F^{adm}_{vert,horiz} \to \Phi^{adm}_{vert,horiz}$$

has the structure of a map between right-lax symmetric monoidal functors.

Proof. We need to show that the isomorphism condition from Sect. 3.2.2 is satisfied.

By [Chapter V.2, Theorem 6.1.5], it is enough to check the corresponding 1-categorical statement. Thus, we need to show that for an integer n and an n-tuple of objects $\mathbf{d}_1, ..., \mathbf{d}_n$, the map

$$\lim_{\mathbf{c} \in \mathbf{C}, \alpha : \mathbf{c} \to \mathbf{d}_1 \times ... \times \mathbf{d}_n, \alpha \in horiz} \Phi(\mathbf{c}) \to \lim_{\mathbf{c}_i \in \mathbf{C}, \alpha_i : \mathbf{c}_i \to \mathbf{d}_i, \alpha_i \in horiz} \Phi(\mathbf{c}_1 \times ... \times \mathbf{c}_n)$$

is an isomorphism.

However, the condition of the proposition implies that the corresponding map of index categories is cofinal.

□

3.3. **Symmetric monoidal structure on the bivariant extension.** In this subsection we show that the extension procedure of [Chapter V.2, Sect. 1] is compatible with symmetric monoidal structures.

3.3.1. Let us now be in the setting of [Chapter V.2, Sect. 1], where both $(\mathbf{C}, vert, horiz, adm)$ and $(\mathbf{D}, vert, horiz, adm)$ are as in Sect. 2.1.4. We let the functor

$$F : \mathbf{C} \to \mathbf{D}$$

be endowed with a symmetric monoidal structure.

Let the target $(\infty, 2)$-category $\mathbb{S}$ be equipped with a symmetric monoidal structure. Assume also that the tensor product functor

$$\mathbb{S} \times \mathbb{S} \to \mathbb{S}$$

is such that the underlying functor $\mathbb{S}^{\text{1-Cat}} \times \mathbb{S}^{\text{1-Cat}} \to \mathbb{S}^{\text{1-Cat}}$ commutes with *colimits in each variable.*

3.3.2. Let us be given a functor

$$\Phi^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} \to \mathbb{S},$$

satisfying the assumptions of [Chapter V.2, Sect. 1.1.6], and let

$$\Psi^{adm}_{vert,horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz} \to \mathbb{S}$$

be its unique extension, satisfying the requirements of [Chapter V.2, Theorem 1.1.9].

Assume that $\Phi^{adm}_{vert,horiz}$ is equipped with a symmetric monoidal structure. We claim:

Proposition 3.3.3. *Suppose that for any n and any map*

$$\mathbf{c} \to \mathbf{d}_1 \times ... \times \mathbf{d}_n$$

that is in horiz (resp., vert, adm), each of the projections $\mathbf{c} \to \mathbf{d}_i$ is also in horiz (resp., vert, adm). Assume also that for every $\mathbf{d} \in \mathbf{D}$, the maps

$$* \to \mathbf{d} \text{ and } \mathbf{d} \to \mathbf{d} \times \mathbf{d}$$

are in adm, and that the functor

$$\mathrm{Maps}_{\mathbb{S}}(\mathbf{1}_{\mathbb{S}}, -) : \mathbb{S}^{\text{1-Cat}} \to \mathrm{Spc}$$

is conservative. Then then functor $\Psi^{adm}_{vert,horiz}$ carries a unique symmetric monoidal structure, which induces the given one on

$$\Phi^{adm}_{vert,horiz} \simeq \Psi^{adm}_{vert,horiz}|_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz}}.$$

Proof. It is enough to show that for any integer n, both circuits of the diagram

$$\begin{array}{ccc} (\mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz})^{\times n} & \xrightarrow{(\Psi^{adm}_{vert,horiz})^{\times n}} & \mathbb{S}^{\times n} \\ {\scriptstyle \text{product map}}\downarrow & & \downarrow{\scriptstyle \text{product map}} \\ \mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz} & \xrightarrow{\Psi^{adm}_{vert,horiz}} & \mathbb{S} \end{array} \tag{3.1}$$

identify with the *canonical* extension, given by [Chapter V.2, Theorem 1.1.9], of the functor given by the (canonically identified) two circuits of the diagram

$$\begin{array}{ccc} (\mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz})^{\times n} & \xrightarrow{(\Phi^{adm}_{vert,horiz})^{\times n}} & \mathbb{S}^{\times n} \\ {\scriptstyle \text{product map}}\downarrow & & \downarrow{\scriptstyle \text{product map}} \\ \mathrm{Corr}(\mathbf{C})^{adm}_{vert,horiz} & \xrightarrow{\Phi^{adm}_{vert,horiz}} & \mathbb{S}. \end{array}$$

We claim that the two circuits in (3.1) satisfy the assumptions of [Chapter V.2, Corollary 1.1.10].

We first check condition (i) for the anti-clockwise circuit. We need to show that for $\mathbf{d}_1, .., \mathbf{d}_n \in \mathbf{D}$, the map

$$\underset{\mathbf{c}_i \in \mathbf{C}, \beta_i : \mathbf{c}_i \to \mathbf{d}_i, \beta_i \in vert}{\mathrm{colim}} \Phi(\mathbf{c}_1 \times ... \times \mathbf{c}_n) \to \underset{\mathbf{c} \in \mathbf{C}, \beta : \mathbf{c} \to \mathbf{d}_1 \times ... \times \mathbf{d}_n, \beta \in vert}{\mathrm{colim}} \Phi(\mathbf{c}) =: \Psi(\mathbf{c}_1 \times ... \times \mathbf{c}_n)$$

is an isomorphism. However, this follows from the fact that the corresponding map of index categories is cofinal. Condition (ii) for the anti-clockwise circuit follows in the same way as condition (i).

Let us check (i) for the clockwise circuit. We need to show that for $\mathbf{d}_1, .., \mathbf{d}_n \in \mathbf{D}$, the map

$$\underset{\mathbf{c}_i \in \mathbf{C}, \beta_i : \mathbf{c}_i \to \mathbf{d}_i, \beta_i \in vert}{\mathrm{colim}} \Phi(\mathbf{c}_1) \otimes ... \otimes \Phi(\mathbf{c}_n) \to \bigotimes_i \left(\underset{\mathbf{c}_i \in \mathbf{C}, \beta_i : \mathbf{c}_i \to \mathbf{d}_i, \beta_i \in vert}{\mathrm{colim}} \Phi(\mathbf{c}_i) \right)$$

is an isomorphism. However, this follows from the commutation of the tensor product on $\mathbb{S}^{\text{1-Cat}}$ with colimits in each variable.

In particular, we obtain that the restrictions of the two circuits in (3.1) to

$$(\mathbf{D}_{vert})^{\times n} \subset (\mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz})^{\times n},$$

and further to

$$(\mathbf{D}_{adm})^{\times n} \subset (\mathrm{Corr}(\mathbf{D})^{adm}_{vert,horiz})^{\times n},$$

are canonically identified.

Conditions (iii) and (iv) for the clockwise circuit follows by the same argument as condition (i). Hence, we obtain that they also hold for the anti-clockwise circuit, since the two functors are identified on $(\mathbf{D}_{adm})^{\times n}$.

Note that this, in particular, establishes the assertion of the proposition in the case when $horiz = adm = vert$. Thus, the restriction of $\Psi^{adm}_{vert,horiz}$ to $\mathrm{Corr}(\mathbf{D})^{adm}_{adm,adm}$ has a symmetric monoidal structure. By further restricting to $\mathrm{Corr}(\mathbf{D})_{adm,adm}$, and applying Proposition 2.3.4, we obtain that for every $\mathbf{d} \in \mathbf{D}$, the object $\Psi(\mathbf{d}) \in \mathbb{S}$ is *dualizable*.

It remains to check condition (ii) for the clockwise circuit. I.e., we need to show that the map

$$\bigotimes_i \left(\lim_{\mathbf{c}_i \in \mathbf{C}, \alpha_i : \mathbf{c}_i \to \mathbf{d}_i, \alpha_i \in horiz} \Phi^!(\mathbf{c}_i) \right) \to \lim_{\mathbf{c}_i \in \mathbf{C}, \alpha_i : \mathbf{c}_i \to \mathbf{d}_i, \alpha_i \in horiz} \Phi^!(\mathbf{c}_1) \otimes ... \otimes \Phi^!(\mathbf{c}_n)$$

is an isomorphism. However, this follows as in [Chapter I.3, Proposition 3.1.7] from the fact that each $\Phi^!(\mathbf{c}_i)$ and each

$$\lim_{\mathbf{c}_i \in \mathbf{C}, \alpha_i : \mathbf{c}_i \to \mathbf{d}_i, \alpha_i \in horiz} \Phi^!(\mathbf{c}_i) \simeq \Psi^!(\mathbf{d}_i)$$

is dualizable, using [Chapter I.1, Lemma 4.1.6(a)].

□

4. Monads and associative algebras in the category of correspondences

It turns out that the category of correspondences is well adapted to the formalism of convolution algebras and convolution categories.

Let $\mathbf{c}^\bullet$ be a *Segal object* of $\mathbf{C}$ (such data are also called *category-objects*); see Sect. 4.1.3 for the definition. Let $p_s, p_t : \mathbf{c}^1 \to \mathbf{c}^0 = \mathbf{c}$ be the source and target maps, respectively, corresponding to the two maps $[0] \to [1]$.

Consider the 1-morphism

$$\begin{array}{ccc} \mathbf{c}^1 & \xrightarrow{p_s} & \mathbf{c} \\ {\scriptstyle p_t}\downarrow & & \\ \mathbf{c} & & \end{array}$$

as an object of the monoidal $(\infty,1)$-category $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c})$.

When $\mathbf{C}$ is an ordinary category, it is clear that the above 1-morphism is an associative algebra object in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c})$, and that all associative algebra objects are obtained in this way. The first result of this section, Proposition 4.1.5 shows that the same is true in the ∞-setting.

As a corollary, we obtain that if Φ is a functor from $\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$ with values in $\mathbf{1}\text{-}\mathbf{Cat}$, a Segal object $\mathbf{c}^\bullet$ (under appropriate conditions) defines a monad on the category $\Phi(\mathbf{c})$.

The second result of this section, Theorem 4.4.2 is (essentially) the following. It says that for a Segal object $\mathbf{c}^\bullet$, the object $\mathbf{c}^1 \in \mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$ has a natural structure of associative algebra in the (symmetric) monoidal category $\mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$.

As a consequence, we obtain that if Φ is a *monoidal functor* from $\mathrm{Corr}(\mathbf{C})_{\mathrm{all;all}}$ to 1-Cat, the category $\Phi(\mathbf{c}^1)$ acquires a canonical monoidal structure, given by convolution.

4.1. **Monads and Segal objects.** In this subsection we will articulate the following idea:

The category of algebras in the monoidal category of endomorphisms of an object $\mathbf{c} \in \mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}$ is canonically equivalent to the category if Segal objects acting on $\mathbf{c}$, i.e., simplicial objects $\mathbf{c}^\bullet$ with $\mathbf{c}^0 = \mathbf{c}$.

4.1.1. Note that if $\mathbb{S}$ is an $(\infty, 2)$-category and $s \in \mathbb{S}$ is an object, then the $(\infty, 1)$-category $\mathbf{Maps}_{\mathbb{S}}(s, s)$ acquires a natural monoidal structure. Indeed, the corresponding functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \text{1-Cat}$$

is given by

$$\mathrm{Seq}_n(\mathbb{S}) \underset{\mathrm{Seq}_0(\mathbb{S}) \times ... \times \mathrm{Seq}_0(\mathbb{S})}{\times} \{s \times ... \times s\},$$

where $\mathrm{Seq}_n(\mathbb{S}) \to \mathrm{Seq}_0(\mathbb{S}) \times ... \times \mathrm{Seq}_0(\mathbb{S})$ corresponds to the map

$$[0] \sqcup ... \sqcup [0] \simeq [n]^{\mathrm{Spc}} \to [n].$$

By definition, a *monad* acting on s is an associative algebra in the monoidal $(\infty, 1)$-category $\mathbf{Maps}_{\mathbb{S}}(s, s)$.

4.1.2. Let $\mathbf{C}$ be an $(\infty, 1)$-category with finite limits, and take $vert = horiz = adm = \mathrm{all}$. Take $\mathbb{S} = \mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}$.

We will be interested in the category of monads in $\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}$ acting on a given $\mathbf{c} \in \mathbf{C}$.

4.1.3. Recall now that if $\mathbf{C}$ is any $(\infty, 1)$-category with finite limits, one can talk about Segal objects[3] in $\mathbf{C}$, acting on a given $\mathbf{c} \in \mathbf{C}$. Denote this category by $\mathrm{Seg}(\mathbf{c})$.

By definition, this is a full subcategory $\mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects $\mathbf{c}^\bullet$ equipped with an identification $\mathbf{c}^0 = \mathbf{c}$, consisting of objects for which for every $n \geq 2$, the map

$$\mathbf{c}^n \to \mathbf{c}^1 \underset{\mathbf{c}}{\times} ... \underset{\mathbf{c}}{\times} \mathbf{c}^1,$$

given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1,$$

is an isomorphism.

This condition can be equivalently formulated as saying that for any $\mathbf{c}' \in \mathbf{C}$, the simplicial space $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}', \mathbf{c}^\bullet)$ is a Segal space (but not necessarily a complete Segal space).

4.1.4. We will prove:

Proposition-Construction 4.1.5. *There exists a canonical equivalence between* $\mathrm{Seg}(\mathbf{c})$ *and the category* $\mathrm{AssocAlg}(\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c}))$.

[3]An alternative terminology for this is 'category-objects' in $\mathbf{C}$, acting on $\mathbf{c} \in \mathbf{C}$.

4.1.6. *Variant.* Let $(\mathbf{C}, vert, horiz, adm)$ be an object of Trpl. Let $\mathbf{c}^\bullet$ be an object of $\mathrm{Seg}(\mathbf{c})$. Suppose that:

- The 'source' map $\mathbf{c}^1 \to \mathbf{c}^0$ (i.e., one corresponding to $(0 \in [0]) \mapsto (0 \in [1])$) belongs to $horiz$;
- The 'target' map $\mathbf{c}^1 \to \mathbf{c}^0$ (i.e., one corresponding to $(0 \in [0]) \mapsto (1 \in [1])$) belongs to $vert$;
- The multiplication map $\mathbf{c}^2 \to \mathbf{c}^1$ (i.e., one corresponding to the active map $[1] \to [2]$) belongs to *adm*.

In this case, we obtain that the algebra object in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c})$ corresponding to $\mathbf{c}^\bullet$ by Proposition 4.1.5, defines an algebra object in the 1-full monoidal subcategory

$$\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c}, \mathbf{c}) \subset \mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c}).$$

4.2. **Proof of Proposition 4.1.5.** The equivalence stated in the proposition is completely evident when $\mathbf{C}$ is an ordinary category, and the reader should check it before proceeding to the ∞-case.

The proof in the latter case will use (a little bit) of diagram manipulation.

4.2.1. For $(\infty, 2)$-category and $s \in \mathbb{S}$, the monoidal $(\infty, 1)$-category $\mathbf{Maps}_{\mathbb{S}}(s, s)$ is described as follows: the corresponding functor $\mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat}$ sends $[n]$ to

$$\mathrm{Seq}_n(\mathbb{S}) \underset{\mathrm{Seq}_0(\mathbb{S})^{\times n+1}}{\times} *,$$

where the maps $* \to \mathrm{Seq}_0(\mathbb{S})$ are given by $s \in \mathbb{S}$.

Applying this to $\mathbb{S} = \mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}$ and $s = \mathbf{C}$, we obtain that the monoidal category $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c})$ is given by the functor $\mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat}$ sends $[n]$ to

$$\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C}) \underset{\mathbf{C}^{n+1}}{\times} *.$$

I.e., this is the category whose objects are half-grids with all squares being Cartesian, and with the diagonal entries identified with $\mathbf{c}$.

4.2.2. Thus, associative algebras in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}, \mathbf{c})$ are right-lax natural transformations

$$[n] \mapsto \left(* \Rightarrow \mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C}) \underset{\mathbf{C}^{n+1}}{\times} \{\mathbf{c}, ..., \mathbf{c}\} \right), \tag{4.1}$$

with the corresponding natural transformations being isomorphisms over *inert* arrows in $\mathbf{\Delta}^{\mathrm{op}}$.

We will now describe the above category of natural transformations (4.1) slightly differently.

4.2.3. Consider the functor

$$\mathbf{\Delta} \to 1\text{-Cat}^{\mathrm{ordn}}, \quad [n] \mapsto ([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}.$$

Let $\mathbf{I}$ denote the corresponding *Cartesian* fibration over $\mathbf{\Delta}^{\mathrm{op}}$. Let $\mathbf{I}_n$ denote the fiber of $\mathbf{I}$ over $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, i.e., $\mathbf{I}_n = ([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$.

Let $\mathbf{I}' \subset \mathbf{I}$ be the full subcategory, such that for each n, the subcategory $\mathbf{I}'_n \subset \mathbf{I}_n$ corresponds to the element $(0, n) \in ([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$, i.e., the NW corner of the half-grid. It is easy to see that the projection

$$\mathbf{I}' \to \mathbf{\Delta}^{\mathrm{op}}$$

is an equivalence.

Let $\mathbf{I}'' \subset \mathbf{I}$ be the full subcategory, such that for every n, the subcategory $\mathbf{I}''_n \subset \mathbf{I}_n$ corresponds to the union of the elements (i,i), i.e., the diagonal entries.

4.2.4. Then the category of natural transformations (4.1) is a full subcategory in the category of functors

$$\mathbf{I} \to \mathbf{C},$$

such that:

- The restriction of the functor to $\mathbf{I}''$ is identified with the constant functor with value $\mathbf{c}$;
- For every n and every square in $\mathbf{I}_n$, the resulting square in $\mathbf{C}$ is Cartesian;
- Arrows in $\mathbf{I}$ that are Cartesian over inert arrows in $\mathbf{\Delta}^{\mathrm{op}}$ get sent to isomorphisms in $\mathbf{C}$.

4.2.5. We note now that restriction along $\mathbf{I}' \to \mathbf{I}$ associates to a functor $\mathbf{I} \to \mathbf{C}$ a simplicial object in $\mathbf{C}$. Functors satisfying the assumptions of Sect. 4.2.4 are easily seen to give rise to functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ that satisfy the conditions of being a Segal object.

Vice versa, starting from a functor

$$\mathbf{\Delta}^{\mathrm{op}} \simeq \mathbf{I}' \to \mathbf{C}, \tag{4.2}$$

we apply right Kan extension along the embedding $\mathbf{I}' \to \mathbf{I}$, and thus obtain a functor $\mathbf{I} \to \mathbf{C}$.

It is easy to see that if (4.2) is a Segal object, then the resulting functor $\mathbf{I} \to \mathbf{C}$ satisfies the assumptions of Sect. 4.2.4.

This defines the desired equivalence

$$\mathrm{AssocAlg}(\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})) \to \mathrm{Seg}(\mathbf{c}).$$

4.3. **Action on a module: first version.** In this subsection we will add a few remarks concerning the action of the monoidal category $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$ on the categories of the form $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}',\mathbf{c})$ for $\mathbf{c}' \in \mathbf{C}$.

4.3.1. Let s and s' be objects in an $(\infty,2)$-category $\mathbb{S}$. Then the category $\mathbf{Maps}_{\mathbb{S}}(s',s)$ is naturally a module for the monoidal category $\mathbf{Maps}_{\mathbb{S}}(s,s)$.

Applying this to $\mathbb{S} = \mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}$, we obtain that for $\mathbf{c}' \in \mathbf{C}$, the monoidal category $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$ acts naturally on the category

$$\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}',\mathbf{c}).$$

4.3.2. Let now $\mathbf{c}^\bullet$ be a monad acting on $\mathbf{c}$ *over* an object $\mathbf{c}'$, i.e., is a monad in the category $\mathbf{C}_{/\mathbf{c}'}$. Let γ denote the morphism $\mathbf{c} \to \mathbf{c}'$.

Then it follows from the construction that the object $(\mathbf{c}' \to \mathbf{c}) \in \mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}',\mathbf{c})$, corresponding to the diagram

$$\begin{array}{ccc} \mathbf{c} & \xrightarrow{\gamma} & \mathbf{c}' \\ {\scriptstyle \mathrm{id}}\downarrow & & \\ \mathbf{c} & & \end{array} \tag{4.3}$$

is a module over the algebra in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$, corresponding to $\mathbf{c}^\bullet$.

4.3.3. Note now that if $s' \xrightarrow{f} s$ is a 1-morphism in an $(\infty,2)$-category $\mathbb{S}$ such that f admits a left adjoint, then the composition $(f \circ f^L) \in \mathbf{Maps}_{\mathbb{S}}(s,s)$ has a natural structure of algebra (this follows, e.g., from the description of the procedure of passage to the adjoint morphism given in [Chapter A.3, Theorem 1.2.4]). In fact, this is a universal algebra object in $\mathbf{Maps}_{\mathbb{S}}(s,s)$ that acts on $f \in \mathbf{Maps}_{\mathbb{S}}(s',s)$.

Note that the 1-morphism (4.3) admits a left adjoint given by

$$\begin{array}{ccc} \mathbf{c} & \xrightarrow{\mathrm{id}} & \mathbf{c} \\ {\scriptstyle\gamma}\big\downarrow & & \\ \mathbf{c}'. & & \end{array} \tag{4.4}$$

Hence, we obtain that the algebra in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$, corresponding to $\mathbf{c}^\bullet$, admits a canonical homomorphism to the algebra corresponding to the composition of (4.3) and (4.4). This map corresponds to the morphism

$$\mathbf{c}^1 \to \mathbf{c} \underset{\mathbf{c}'}{\times} \mathbf{c},$$

expressing the fact that $\mathbf{c}^1$ acts on $\mathbf{c}$ *over* $\mathbf{c}'$.

4.3.4. Let $\mathbf{c}^\bullet$ now be the Čech nerve of the map $\beta : \mathbf{c} \to \mathbf{c}'$. In this case we claim that the above homomorphism of algebras is an isomorphism. Indeed, it suffices to check this fact at the level of the underlying objects of $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$, and the required isomorphism follows from the Cartesian diagram

$$\begin{array}{ccc} \mathbf{c}^1 & \xrightarrow{p_s} & \mathbf{c} \\ {\scriptstyle p_t}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ \mathbf{c} & \xrightarrow{\gamma} & \mathbf{c}'. \end{array}$$

4.3.5. *Variant.* Retaining the assumptions of Sect. 4.1.6, assume that the map $\gamma : \mathbf{c} \to \mathbf{c}'$ belongs to *horiz*, and that the target map $p_s : \mathbf{c}^1 \to \mathbf{c}$ belongs to *adm*.

Then we obtain that the object of $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c}',\mathbf{c})$, given by (4.4), is a module for the algebra in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c},\mathbf{c})$, corresponding to $\mathbf{c}^\bullet$.

Furthermore, if $\gamma \in adm$, in which case the corresponding morphism

$$(\mathbf{c}' \to \mathbf{c}) \in \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}$$

admits a left adjoint, the isomorphism from Sect. 4.3.4 holds at the level of algebras in the monoidal category $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c},\mathbf{c})$.

4.4. **From monads/Segal objects to algebras.** In this subsection we formulate the main result of this section, Theorem 4.4.2. It says that if $\mathbf{c}^\bullet$ is a Segal object of $\mathbf{C}$ acting on $\mathbf{c}$, then its first term $\mathbf{c}^1$ acquires a natural algebra structure in $\mathrm{Corr}(\mathbf{C})$.

4.4.1. Let $\mathbf{C}$ be an $(\infty,1)$-category with finite limits, and take $vert = horiz = adm =$ all. We let

$$\mathrm{Corr}(\mathbf{C}) := \mathrm{Corr}(\mathbf{C})^{\mathrm{isom}}_{\mathrm{all;all}}$$

be endowed with a symmetric monoidal structure as in Sect. 2.1.4.

We will prove the following:

Theorem-Construction 4.4.2. *There exists a canonical right-lax homomorphism of monoidal $(\infty,1)$-categories*

$$\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c}) \to \mathrm{Corr}(\mathbf{C}).$$

4.4.3. Let us explain the content of Theorem 4.4.2 when $\mathbf{C}$ is an ordinary category. In this case the sought-for functor

$$\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c}) \to \mathrm{Corr}(\mathbf{C})$$

is explicitly defined as follows:

It sends an object of $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$, given by

$$\begin{array}{ccc} \widetilde{\mathbf{c}} & \longrightarrow & \mathbf{c} \\ \big\downarrow & & \\ \mathbf{c} & & \end{array}$$

to $\widetilde{\mathbf{c}} \in \mathrm{Corr}(\mathbf{C})$. The monoidal structure is defined as follows: for a pair of objects

$$\begin{array}{ccc} \widetilde{\mathbf{c}}_1 & \longrightarrow & \mathbf{c} \\ \big\downarrow & & \\ \mathbf{c} & & \end{array}$$

and

$$\begin{array}{ccc} \widetilde{\mathbf{c}}_2 & \longrightarrow & \mathbf{c} \\ \big\downarrow & & \\ \mathbf{c} & & \end{array}$$

their tensor product in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$ is given by

$$\begin{array}{ccc} \widetilde{\mathbf{c}}_1 \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}_2 & \longrightarrow & \mathbf{c} \\ \big\downarrow & & \\ \mathbf{c} & & \end{array}$$

and the corresponding 1-morphism $\widetilde{\mathbf{c}}_1 \times \widetilde{\mathbf{c}}_2 \to \widetilde{\mathbf{c}}_1 \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}_2$ (note the direction of the arrow!) in $\mathrm{Corr}(\mathbf{C})$ is given by the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{c}}_1 \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}_2 & \longrightarrow & \widetilde{\mathbf{c}}_1 \times \widetilde{\mathbf{c}}_2 \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \\ \widetilde{\mathbf{c}}_1 \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}_2. & & \end{array}$$

4.4.4. As a formal consequence of Theorem 4.4.2, combined with Proposition 4.1.5, we obtain:

Corollary 4.4.5. *For any $\mathbf{c} \in \mathbf{C}$, there is a canonically defined functor*

$$\mathrm{Seg}(\mathbf{c}) \to \mathrm{AssocAlg}(\mathrm{Corr}(\mathbf{C})).$$

4.4.6. We note that if $\mathbf{c}^\bullet$ is an object of $\operatorname{Seg}(\mathbf{c})$, the object of $\operatorname{Corr}(\mathbf{C})$, underlying the corresponding algebra is $\mathbf{c}^1$, and the product map is given by the diagram

$$\begin{array}{ccc} \mathbf{c}^2 & \longrightarrow & \mathbf{c}^1 \times \mathbf{c}^1 \\ \downarrow & & \\ \mathbf{c}^1, & & \end{array}$$

where the horizontal arrow corresponds to the two inert maps $[1] \to [2]$.

4.4.7. *Variant.* Let $(\mathbf{C}, vert, horiz, \text{isom})$ be an object of Trpl. Assume that $vert \subset horiz$. Let $\mathbf{c}^\bullet$ be an object of $\operatorname{Seg}(\mathbf{c})$. Assume that:

- The 'source' map $\mathbf{c}^1 \to \mathbf{c}^0$ belongs to $horiz$;
- The 'target' map $\mathbf{c}^1 \to \mathbf{c}^0$ belongs to $vert$;
- The multiplication map $\mathbf{c}^2 \to \mathbf{c}^1$ belongs to $vert$.

In this case, Corollary 4.4.5 implies that to $\mathbf{c}^\bullet$ there corresponds a canonically defined algebra object in $\operatorname{Corr}(\mathbf{C})_{vert;horiz}$.

4.5. **Action on a module: second version.** In this subsection we will add some comments on how the construction in Theorem 4.4.2 interacts with an action on modules. The upshot is that the associative algebra in $\operatorname{Corr}(\mathbf{C})$, corresponding to a Segal object $\mathbf{c}^\bullet$, acts on $\mathbf{c}^0$.

4.5.1. Let us return to the setting of Sect. 4.3. For a pair of objects $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$ we consider the monoidal category $\mathbf{Maps}_{\operatorname{Corr}(\mathbf{C})^{\text{all}}_{\text{all;all}}}(\mathbf{c}, \mathbf{c})$ and its module category $\mathbf{Maps}_{\operatorname{Corr}(\mathbf{C})^{\text{all}}_{\text{all;all}}}(\mathbf{c}', \mathbf{c})$.

It will follow from the construction in Theorem 4.4.2 that the right-lax monoidal functor

$$\mathbf{Maps}_{\operatorname{Corr}(\mathbf{C})^{\text{all}}_{\text{all;all}}}(\mathbf{c}, \mathbf{c}) \to \operatorname{Corr}(\mathbf{C})$$

extends to a right-lax map between module categories

$$\mathbf{Maps}_{\operatorname{Corr}(\mathbf{C})^{\text{all}}_{\text{all;all}}}(\mathbf{c}', \mathbf{c}) \to \operatorname{Corr}(\mathbf{C}),$$

which at the level of objects sends

$$\begin{array}{ccc} \widetilde{\mathbf{c}}' & \longrightarrow & \mathbf{c}' \\ \downarrow & & \\ \mathbf{c} & & \end{array} \tag{4.5}$$

to $\widetilde{\mathbf{c}}'$.

At the level of objects, this right-lax monoidal structure is defined as follows. For an object of $\mathbf{Maps}_{\operatorname{Corr}(\mathbf{C})^{\text{all}}_{\text{all;all}}}(\mathbf{c}, \mathbf{c})$, given by the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{c}} & \longrightarrow & \mathbf{c} \\ \downarrow & & \\ \mathbf{c} & & \end{array}$$

and the object of $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c}',\mathbf{c})$ given by the diagram (4.5), the corresponding 1-morphism in $\mathrm{Corr}(\mathbf{C})$ is given by

$$\begin{array}{ccc} \widetilde{\mathbf{c}} \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}' & \longrightarrow & \widetilde{\mathbf{c}} \times \widetilde{\mathbf{c}}' \\ {\scriptstyle \mathrm{id}}\downarrow & & \\ \widetilde{\mathbf{c}} \underset{\mathbf{c}}{\times} \widetilde{\mathbf{c}}'. & & \end{array}$$

4.5.2. Let us take $\mathbf{c}' = *$, and consider the object of $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(*,\mathbf{c})$, given by

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & * \\ {\scriptstyle \mathrm{id}}\downarrow & & \\ \mathbf{c}. & & \end{array} \tag{4.6}$$

By Sect. 4.3.2, we obtain that for any Segal object $\mathbf{c}^\bullet$ acting on $\mathbf{c}$, the object (4.6) is naturally a module over the corresponding algebra in $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$.

Hence, applying Sect. 4.5.1, we obtain that the corresponding algebra $\mathbf{c}^1 \in \mathrm{Corr}(\mathbf{C})$ acts on the object $\mathbf{c} \in \mathrm{Corr}(\mathbf{C})$.

Explicitly, the corresponding action map is given by the diagram

$$\begin{array}{ccc} \mathbf{c}^1 & \xrightarrow{\mathrm{id}\times p_s} & \mathbf{c}^1 \times \mathbf{c} \\ {\scriptstyle p_t}\downarrow & & \\ \mathbf{c}. & & \end{array}$$

4.5.3. Note now that if an object $\mathbf{o}$ in the (symmetric) monoidal category $\mathbf{O}$ is dualizable, there exists a universal associative algebra, denoted $\underline{\mathrm{End}}_{\mathbf{O}}(\mathbf{o})$ in $\mathbf{O}$, acting on $\mathbf{o}$. The object of $\mathbf{O}$ underlying $\underline{\mathrm{End}}_{\mathbf{O}}(\mathbf{o})$ is $\mathbf{o}^\vee \otimes \mathbf{o}$.

Applying this to $\mathbf{O} = \mathrm{Corr}(\mathbf{C})$ and $\mathbf{o} = \mathbf{c}$, we obtain a canonically defined homomorphism of algebras

$$\mathbf{c}^1 \to \underline{\mathrm{End}}_{\mathrm{Corr}(\mathbf{C})}(\mathbf{c}). \tag{4.7}$$

Identifying $\mathbf{c}^\vee \simeq \mathbf{c}$ (see Proposition 2.3.4), the map in $\mathrm{Corr}(\mathbf{C})$, underlying the above homomorphism is

$$\mathbf{c}^1 \overset{p_s \times p_t}{\longrightarrow} \mathbf{c} \times \mathbf{c}. \tag{4.8}$$

4.5.4. Let us now take $\mathbf{c}^\bullet$ to be the Čech nerve of the map $\mathbf{c} \to *$, i.e., $\mathbf{c}^n = \mathbf{c}^{\times(n+1)}$. Note that in this case we obtain that the homomorphism (4.7) is an isomorphism. Indeed, this is so because the map of the underlying objects of $\mathrm{Corr}(\mathbf{C})$, i.e., (4.7) is an isomorphism.

4.5.5. *Variant.* Let us be in the situation of Sect. 4.4.7. Assume in addition that $\mathbf{c}$ is such that the diagonal map $\mathbf{c} \to \mathbf{c} \times \mathbf{c}$ and the tautological map $\mathbf{c} \to *$ belong to *horiz*.

In this case we still obtain that $\mathbf{c} \in \mathrm{Corr}(\mathbf{C})_{vert;horiz}$ is a module over $\mathbf{c}^1$, where the latter is viewed as an associative algebra.

If we assume that the diagonal map $\mathbf{c} \to \mathbf{c}\times\mathbf{c}$ and the tautological map $\mathbf{c} \to *$ belong to *vert* as well, then $\mathbf{c}$ is dualizable in $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$, and the isomorphism of Sect. 4.5.4 happens at the level of associative algebras in $\mathrm{Corr}(\mathbf{C})_{vert;horiz}$.

4.6. Proof of Theorem 4.4.2: introduction.

4.6.1. Let

$$\mathbf{E} := \mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})^{\otimes,\mathbf{\Delta}^{\mathrm{op}}} \text{ and } \mathbf{F} := \mathrm{Corr}(\mathbf{C})^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}$$

be the coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ corresponding to $\mathbf{Maps}_{\mathrm{Corr}(\mathbf{C})^{\mathrm{all}}_{\mathrm{all;all}}}(\mathbf{c},\mathbf{c})$ and $\mathrm{Corr}(\mathbf{C})$, respectively.

We need to construct a functor $\mathbf{E} \to \mathbf{F}$ over $\mathbf{\Delta}^{\mathrm{op}}$.

4.6.2. Recall (see Sect. 4.2.1) that $\mathbf{E}$ is the coCartesian fibration attached to the functor $\mathbf{\Delta}^{\mathrm{op}} \to$ 1-Cat, given by

$$[n] \mapsto \mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C}) \underset{\mathbf{C}^{n+1}}{\times} \{\mathbf{c}, ..., \mathbf{c}\}. \tag{4.9}$$

Consider another functor $\mathbf{\Delta}^{\mathrm{op}} \to$ 1-Cat,

$$[n] \mapsto \mathbf{Grid}^{> \mathrm{dgnl}}_n(\mathbf{C}), \tag{4.10}$$

where $\mathbf{Grid}^{> \mathrm{dgnl}}_n$ is defined in the same way as as $\mathbf{Grid}^{\geq \mathrm{dgnl}}$, with the difference that we use $([n] \times [n]^{\mathrm{op}})^{> \mathrm{dgnl}}$ instead of $([n] \times [n]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$.

Restriction defines a natural transformation (4.9) $\Rightarrow$ (4.10). Let

$$\mathbf{E}' \to \mathbf{\Delta}^{\mathrm{op}}$$

be the coCartesian fibration corresponding to the functor (4.10). We obtain a functor

$$\mathbf{E} \to \mathbf{E}'.$$

4.6.3. The sought-for functor $\mathbf{E} \to \mathbf{F}$ will be obtained as a composition of the above functor $\mathbf{E} \to \mathbf{E}'$ and a functor $\mathbf{E}' \to \mathbf{F}$ that we will now proceed to define.

Fix an integer k and an object $\alpha \in \mathrm{Seq}_k(\mathbf{\Delta}^{\mathrm{op}})$. We will construct a map

$$\mathrm{Seq}_k(\mathbf{E}') \underset{\mathrm{Seq}_k(\mathbf{\Delta}^{\mathrm{op}})}{\times} \{\alpha\} \to \mathrm{Seq}_k(\mathbf{F}) \underset{\mathrm{Seq}_k(\mathbf{\Delta}^{\mathrm{op}})}{\times} \{\alpha\}, \tag{4.11}$$

functorially in $[k] \in \mathbf{\Delta}^{\mathrm{op}}$ and α.

4.7. Proof of Theorem 4.4.2: Step 1.

We shall first give an explicit description of the space

$$\mathrm{Seq}_k(\mathbf{E}') \underset{\mathrm{Seq}_k(\mathbf{\Delta}^{\mathrm{op}})}{\times} \{\alpha\}.$$

4.7.1. We begin with the following general observation. Let $\mathbf{I}$ be an index $(\infty,1)$-category, and let $\mathbf{D} \to \mathbf{I}^{\mathrm{op}}$ be a Cartesian fibration, corresponding to a functor $\mathbf{I} \to$ 1-Cat. Let $\mathbf{D}' \to \mathbf{I}$ be the coCartesian fibration, corresponding to the same functor. I.e., $\mathbf{D}'$ and $\mathbf{D}$ have the same fibers over objects of $\mathbf{I}$.

Note that the space $\mathrm{Seq}_k(\mathbf{D}')$ can be described as follows in terms of $\mathbf{D}$. Namely, $\mathrm{Seq}_k(\mathbf{D}')$ consists of functors

$$([k] \times [k]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to \mathbf{D}$$

with the property that all the vertical arrows in $([k] \times [k]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$ map to isomorphisms in $\mathbf{I}^{\mathrm{op}}$, and all the horizontal arrows map to arrows in $\mathbf{D}'$ that are Cartesian over $\mathbf{I}^{\mathrm{op}}$.

4.7.2. Note that assignment

$$(i,j) \mapsto ([\alpha(j)] \times [\alpha(j)]^{\mathrm{op}})^{>\,\mathrm{dgnl}}$$

gives a functor

$$([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}} \to 1\text{-}\mathrm{Cat}^{\mathrm{ordn}}.$$

Let $\mathbf{I}_\alpha$ denote the corresponding coCartesian fibration over $([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}}$.

Consider the space

$$\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}).$$

4.7.3. It follows from Sect. 4.7.1 that the space $\mathrm{Seq}_k(\mathbf{E}') \underset{\mathrm{Seq}_k(\boldsymbol{\Delta}^{\mathrm{op}})}{\times} \{\alpha\}$ is a full subspace

$$'\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \subset \mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}),$$

consisting of functors that satisfy the following conditions:

- For every $(i,j) \in ([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}}$, the resulting functor
$$([\alpha(j)] \times [\alpha(j)]^{\mathrm{op}})^{>\,\mathrm{dgnl}} \to \mathbf{C}$$
sends squares to Cartesian squares in $\mathbf{C}$.
- Every arrow in $\mathbf{I}_\alpha$, which is coCartesian over a horizontal arrow in $([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}}$, gets sent to an isomorphism in $\mathbf{C}$.

4.8. **Proof of Theorem 4.4.2: Step 2.** We shall now describe the space

$$\mathrm{Seq}_k(\mathbf{F}) \underset{\mathrm{Seq}_k(\boldsymbol{\Delta}^{\mathrm{op}})}{\times} \{\alpha\}.$$

4.8.1. Let $\mathbf{C}'$ be a monoidal $(\infty,1)$-category with finite limits, such that the monoidal operation commutes with finite limits. Then $(\mathbf{C}', \mathrm{all}, \mathrm{all}, \mathrm{isom})$ is naturally an associative algebra object in Trpl.

Hence, by Sect. 2.1.2, the $(\infty,1)$-category $\mathrm{Corr}(\mathbf{C}')$ acquires a monoidal structure. Consider the corresponding coCartesian fibration $\mathrm{Corr}(\mathbf{C}')^{\otimes,\boldsymbol{\Delta}^{\mathrm{op}}}$ over $\boldsymbol{\Delta}^{\mathrm{op}}$.

Let $\mathbf{C}'^{\otimes,\boldsymbol{\Delta}} \to \boldsymbol{\Delta}$ be the *Cartesian* fibration corresponding to the monoidal structure on $\mathbf{C}'$.

Then the space $\mathrm{Seq}_k(\mathrm{Corr}(\mathbf{C}')^{\otimes,\boldsymbol{\Delta}^{\mathrm{op}}})$ admits the following description. It consists of functors

$$([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}} \to \mathbf{C}'^{\otimes,\boldsymbol{\Delta}},$$

such that vertical arrows in $([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}}$ get sent to arrows in $\mathbf{C}'^{\otimes,\boldsymbol{\Delta}}$ that project to isomorphisms in $\boldsymbol{\Delta}$;

4.8.2. We apply this to $\mathbf{C}' = \mathbf{C}$ equipped with the Cartesian monoidal structure. We obtain that the space

$$\mathrm{Seq}_k(\mathbf{F}) \underset{\mathrm{Seq}_k(\boldsymbol{\Delta}^{\mathrm{op}})}{\times} \{\alpha\} \tag{4.12}$$

can be described as follows.

Recall the coCartesian fibration $\mathbf{I}_\alpha \to ([k] \times [k]^{\mathrm{op}})^{\geq\,\mathrm{dgnl}}$, see Sect. 4.7.2. Then (4.12) identifies with the subspace

$$''\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \subset \mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}),$$

consisting of functors that satisfy the following condition:

- For every $(i,j) \in ([k] \times [k]^{\mathrm{op}})^{\geq \mathrm{dgnl}}$, the tautological extension of the corresponding functor
$$([\alpha(j)] \times [\alpha(j)]^{\mathrm{op}})^{> \mathrm{dgnl}} \to \mathbf{C}$$
to a functor
$$([\alpha(j)] \times [\alpha(j)]^{\mathrm{op}})^{\geq \mathrm{dgnl}} \to \mathbf{C},$$
where we send the diagonal entries to $* \in \mathbf{C}$ has the property that it sends squares to Cartesian squares in $\mathbf{C}$.

4.8.3. Let us decipher the above condition. Set $n = \alpha(j)$. Let the functor

$$([n] \times [n]^{\mathrm{op}})^{> \mathrm{dgnl}} \to \mathbf{C} \tag{4.13}$$

be given by $\underline{\mathbf{c}} \in \mathbf{Grid}_n^{> \mathrm{dgnl}}(\mathbf{C})$.

Our condition says that all the squares in $\underline{\mathbf{c}}$ must be Cartesian (so it is the same as the corresponding condition for $'\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C})$, see Sect. 4.7.3). In addition, we require that for every $m = 0, ..., n-2$, the map

$$\mathbf{c}_{m,m+2} \to \mathbf{c}_{m+1,m+2} \times \mathbf{c}_{m,m+1}$$

be an isomorphism.

Note that, the datum of a functor (4.13) as above is equivalent to that of an object of $\mathbf{C}^{\times n}$ (as it should be).

4.9. **Proof of Theorem 4.4.2: Step 3.** We shall now complete the construction of a map (4.11) by constructing a map

$$'\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \to {}''\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}).$$

4.9.1. Consider the *category*

$$\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}),$$

and the corresponding full subcategories

$$'\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \hookrightarrow \mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \hookleftarrow {}''\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}).$$

It is easy to see, however, that the embedding

$$''\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \hookrightarrow \mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C})$$

admits a left adjoint.

4.9.2. Composing with this left adjoint, we obtain a functor

$$'\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \hookrightarrow \mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \to {}''\mathbf{Maps}(\mathbf{I}_\alpha, \mathbf{C}).$$

Passing to the underlying spaces, we obtain the desired map

$$'\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}) \to {}''\mathrm{Maps}(\mathbf{I}_\alpha, \mathbf{C}).$$