CHAPTER V.2. EXTENSION THEOREMS FOR THE CATEGORY OF CORRESPONDENCES

Contents

Introduction

This Chapter should be regarded as a complement to [Chapter V.1]. Here we prove two more extension theorems that allow, starting from a functor

$$\Phi^{adm}_{vert;horiz} : \text{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S},$$

and a functor $F : \mathbf{C} \to \mathbf{D}$, to canonically produce a functor

$$\Psi^{adm}_{vert;horiz} : \text{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S}.$$

Date: November 21, 2021.

The nature of the extensions in this chapter will be very different from that in [Chapter V.1, Theorems 4.1.3 and 5.2.4]: in *loc. cit.* we enlarged our 2-category of correspondences by allowing more 1-morphisms and 2-morphisms. In the present section, we will enlarge the class of objects.

0.1. **The bivariant extension procedure.** The first of our two extension results, Theorem 1.1.9, is a general framework designed to treat the following situation. We start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and we want to extend it to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

0.1.1. Here is how this extension is supposed to behave.

Let us first restrict our attention to the 1-full subcategories

$$\mathrm{Sch}_{\mathrm{aft}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \text{ and } \mathrm{indinfSch}_{\mathrm{aft}} \subset \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}},$$

and the fully faithful embedding

$$\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{indinfSch}_{\mathrm{aft}}\,. \tag{0.1}$$

We want the restriction $\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{aft}}}$ of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}}$ to $\mathrm{indinfSch}_{\mathrm{aft}}$ to be given by *left* Kan extension of

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}}|_{\mathrm{Sch}_{\mathrm{aft}}} := \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$$

along (0.1).

0.1.2. Consider now the 1-full subcategories

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \subset \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \text{ and } (\mathrm{indinfSch}_{\mathrm{aft}})^{\mathrm{op}} \subset \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}},$$

and the fully faithful embedding

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to (\mathrm{indinfSch}_{\mathrm{aft}})^{\mathrm{op}}. \tag{0.2}$$

We want the restriction $\mathrm{IndCoh}^!_{\mathrm{indinfSch}_{\mathrm{aft}}}$ of $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}}$ to $(\mathrm{indinfSch}_{\mathrm{aft}})^{\mathrm{op}}$ to be given by *right* Kan extension of

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}}|_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}} =: \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}$$

along (0.2).

0.1.3. So, we see that our extension behaves as a left Kan extension along vertical directions and as a right Kan extension along the horizontal directions. But these two patterns of behavior are closely linked via the 2-categorical structure.

Namely, the left Kan extension behavior along the vertical direction and the right Kan extension behavior along the vertical direction, once restricted to 1-full subcategories corresponding to nil-closed maps, are formal consequences of each other (this is obtained by combining [Chapter III.3, Corollay 4.2.3 and Theorem 4.3.2], [Chapter V.1, Theorem 3.2.2] and Proposition 2.2.7 below).

The idea is that there are 'enough' of nil-closed maps to fix the behavior of our extension on all objects and morphisms.

0.1.4. The 2-categorical features are essential in the proof of Theorem 1.1.9. Namely, we perceive the datum of a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

as that of simplicial functor between $(\infty,1)$-categories

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{\bullet}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{Seq}^{\mathrm{ext}}_{\bullet}(\mathbb{S}).$$

The corresponding functors

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbb{S})$$

are obtained as *left Kan extensions* of the corresponding functors

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbb{S}).$$

That is to say, here we use in an essential way the fact that $'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{all;all}$ is a category, and not just a space: we approximate diagrams $\underline{\mathbf{d}} \in \mathrm{Grid}_n(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}$ by diagrams $\underline{\mathbf{c}} \in \mathrm{Grid}_n(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}$, where the maps $\underline{\mathbf{c}} \to \underline{\mathbf{d}}$ are required to be term-wise nil-closed.

So, it would be impossible to prove an analog of Theorem 1.1.9, if instead of the $(\infty,2)$-categories

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \text{ and } \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}$$

we used the underlying $(\infty,1)$-categories

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \text{ and } \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})_{\mathrm{all;all}}.$$

0.1.5. The bulk of the proof of Theorem 1.1.9 is concentrated in Proposition 1.2.5 that guarantees that the left Kan extension extensions of

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbb{S})$$

along

$$'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_{n}(\mathrm{indinfSch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}$$

'does the right thing', i.e., produces the expected strings of 1-morphisms.

In the process of proving Proposition 1.2.5 we will need to make a digression and study the behavior of colimits in the categories $\mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbb{S})$, and how these colimits behave with respect to restriction functors

$$\mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbb{S}) \to \mathrm{Seq}^{\mathrm{ext}}_{m}(\mathbb{S}),$$

corresponding to maps $[m] \to [n]$ in the category $\mathbf{\Delta}$.

0.2. **The horizontal extension procedure.** Our second extension result, Theorem 6.1.5 is a general framework designed to treat the following situation. We start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and we want to extend it to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\ \&\ \mathrm{proper}}_{\mathrm{sch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{proper}}_{\mathrm{sch;all}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

0.2.1. The above extension procedure is an instance of the 2-categorical right Kan extension. We do not develop the general theory of right Kan extensions for $(\infty,2)$-categories in this book.

What saves the day here is the fact that the present situation has a feature that makes it particularly simple:

Whatever that right Kan

$$\mathrm{RKE}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\ \&\ \mathrm{proper}}_{\mathrm{sch;all}}}(\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}})$$

is, its restriction to the underlying $(\infty,1)$-categories is given by the usual (i.e., 1-categorical) right Kan extension, i.e., the canonical map

$$\mathrm{RKE}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\ \&\ \mathrm{proper}}_{\mathrm{sch;all}}}(\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}})|_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{all;all}}} \to$$
$$\to \mathrm{RKE}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{sch;all}}}(\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}})$$

is an isomorphism.

0.2.2. A general statement of when a functor $F : \mathbb{T}_1 \to \mathbb{T}_2$ between $(\infty,2)$-categories has the property that for any $\Phi : \mathbb{T}_1 \to \mathbb{S}$, the map

$$\mathrm{RKE}_F(\Phi)|_{\mathbb{T}_2^{1\text{-Cat}}} \to \mathrm{RKE}_{F|_{\mathbb{T}_1^{1\text{-Cat}}}}(\Phi|_{\mathbb{T}_1^{1\text{-Cat}}})$$

is an isomorphism, is given in Lemma 6.3.3.

The idea of the condition of this lemma says that for any $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$, the morphisms in the category

$$\mathbf{Maps}_{\mathbb{T}_2}(t_2, F(t_1))$$

'come' from 2-morphisms in $\mathbb{T}_1$.

This guarantees that the 1-categorical right Kan extension

$$\mathrm{RKE}_{F|_{\mathbb{T}_1^{1\text{-Cat}}}}(\Phi|_{\mathbb{T}_1^{1\text{-Cat}}}) : \mathbb{T}_2^{1\text{-Cat}} \to \mathbb{S}$$

can be canonically extended to 2-morphisms using the data of Φ itself.

1. Functors obtained by bivariant extension

In this section we will describe one of the two extension results of this chapter that allows to extend a functor from a given $(\infty,2)$-category of correspondences to a larger one.

A typical situation in which the procedure described in this section will be applied is when we want to extend IndCoh as a functor out of the category of correspondences of schemes to that of ind-inf-schemes. I.e., we start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and we want to extend it to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{aft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

1.1. **Set-up for the bivariant extension.** In this subsection we will describe the context of our extension procedure and state the main result of this section, Theorem 1.1.9.

1.1.1. Let $(\mathbf{C}, vert, horiz, adm)$ be as in [Chapter V.1, Sect. 1.1.1]. We will also assume that all three classes $vert$, $horiz$ and adm satisfy the '2 out of 3' property.

Suppose we have a functor

$$\Phi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S},$$

where $\mathbb{S} \in 2\text{-Cat}$.

Denote

$$\Phi := \Phi^{adm}_{vert;horiz}|_{\mathbf{C}_{vert}} \text{ and } \Phi^! := \Phi^{adm}_{vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}};$$

$$\Phi_{adm} := \Phi|_{\mathbf{C}_{adm}} \simeq \Phi^{adm}_{vert;horiz}|_{\mathbf{C}_{adm}}$$

and

$$\Phi^!_{adm} := \Phi^!|_{(\mathbf{C}_{adm})^{\mathrm{op}}} \simeq \Phi^{adm}_{vert;horiz}|_{(\mathbf{C}_{adm})^{\mathrm{op}}}.$$

For an object $\mathbf{c} \in \mathbf{C}$ we will simply write $\Phi(\mathbf{c})$ for $\Phi^{adm}_{vert;horiz}(\mathbf{c})$.

In our main application we will take $\mathbb{S}$ to be $\mathrm{DGCat}_{\mathrm{cat}}$ and $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$ with $vert = horiz = \mathrm{all}$ and $adm = \text{nil-closed}$. We take $\Phi^{adm}_{vert;horiz}$ to be the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\text{nil-closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

1.1.2. Let $(\mathbf{D}, vert, horiz, adm)$ be another datum as above, and assume that $\mathbf{D}$ admits all fiber products. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor that preserves the corresponding classes of 1-morphisms, i.e., that it gives rise to well-defined functors

$$F_{vert} : \mathbf{C}_{vert} \to \mathbf{D}_{vert},\ F_{horiz} : \mathbf{C}_{horiz} \to \mathbf{D}_{horiz} \text{ and } F_{adm} : \mathbf{C}_{adm} \to \mathbf{D}_{adm},$$

and that each of the above functors (including F itself) is fully faithful.

We will assume that F takes Cartesian squares as in [Chapter V.1, Diagram (1.1)] to Cartesian squares. Hence, F induces a functor

$$F^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}.$$

1.1.3. We will also assume that for every $\mathbf{d} \in \mathbf{D}$ there exists $\mathbf{c} \in \mathbf{C}$ equipped with a map $\mathbf{c} \to \mathbf{d}$ that belongs to adm.

1.1.4. The goal of this section is to extend the functor $\Phi^{adm}_{vert;horiz}$ to a functor

$$\Psi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S},$$

under certain conditions on the $(\infty, 2)$-category $\mathbb{S}$ (see Sect. 1.1.5) and on the functor $\Phi^{adm}_{vert;horiz}$ (see Sect. 1.1.6).

1.1.5. *Conditions on the target* 2*-category.* Let $\mathbb{S}$ be a target $(\infty, 2)$-category. We will impose the following conditions on $\mathbb{S}$:

(1) For any $s', s'' \in \mathbb{S}$, the category $\mathbf{Maps}_{\mathbb{S}}(s', s'')$ is presentable;
(2) For any $s', s'' \in \mathbb{S}$, the category $\mathbf{Maps}_{\mathbb{S}}(s', s'')$ is pointed, i.e., the map from the initial object to the final object is an isomorphism;
(3) For any $s', s'' \in \mathbb{S}$, and a fixed $\widetilde{s}' \overset{\alpha}{\to} s'$ (resp., $s'' \overset{\beta}{\to} \widetilde{s}''$), the functors

$$\mathbf{Maps}_{\mathbb{S}}(s', s'') \xrightarrow{-\circ\alpha} \mathbf{Maps}_{\mathbb{S}}(\widetilde{s}', s'') \text{ and } \mathbf{Maps}_{\mathbb{S}}(s', s'') \xrightarrow{\beta\circ -} \mathbf{Maps}_{\mathbb{S}}(s', \widetilde{s}'')$$

preserve colimits;
(4) $\mathbb{S}^{1\text{-Cat}}$ is presentable;
(5) $\mathbb{S}^{1\text{-Cat}}$ is pointed;

(6) For $s \in \mathbb{S}$ there exist objects $[1] \otimes s$ and $s^{[1]}$ equipped with functorial identifications

$$\operatorname{Maps}([1] \otimes s, s') \simeq \operatorname{Maps}([1], \mathbf{Maps}_{\mathbb{S}}(s, s')) \text{ and } \operatorname{Maps}(s', s^{[1]}) \simeq \operatorname{Maps}([1], \mathbf{Maps}_{\mathbb{S}}(s', s)),$$

respectively.

These conditions are satisfied, e.g., if $\mathbb{S}$ is $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ ot $\mathrm{DGCat}_{\mathrm{cont}}$.

1.1.6. *Conditions on the functor $\Phi^{adm}_{vert;horiz}$.* Denote

$$\Psi_{adm} := \mathrm{LKE}_{F_{adm}}(\Phi_{adm}), \quad \Psi^!_{adm} := \mathrm{RKE}_{F^{\mathrm{op}}_{adm}}(\Phi^!_{adm}),$$
$$\Psi_{vert} := \mathrm{LKE}_{F_{vert}}(\Phi_{vert}), \quad \Psi^!_{horiz} := \mathrm{RKE}_{F^{\mathrm{op}}_{horiz}}(\Phi^!_{horiz}).$$

We impose the following conditions on the interaction of $\Phi^{adm}_{vert;horiz}$ and F:

(1) The functor Ψ_{vert} satisfies the left Beck-Chevalley condition with respect to $adm \subset vert$.

(2) The canonical map $\Psi_{adm} \to \Psi_{vert}|_{\mathbf{D}_{adm}}$ is an isomorphism.

(3) The functor $\Psi^!$ satisfies the right Beck-Chevalley condition with respect to $adm \subset horiz$.

(4) The canonical map $\Psi^!_{horiz}|_{(\mathbf{D}_{adm})^{\mathrm{op}}} \to \Psi^!_{adm}$ is an isomorphism.

1.1.7. Finally, we impose one more technical condition:

$(*)$ For every morphism $\mathbf{d}' \to \mathbf{d}$ in $\mathbf{D}$, the map

$$\underset{\mathbf{c} \in (\mathbf{D}_{adm})_{/\mathbf{d}} \underset{\mathbf{D}_{adm}}{\times} \mathbf{C}_{adm}}{\operatorname{colim}} \Psi_{adm}(\mathbf{d}' \underset{\mathbf{d}}{\times} \mathbf{c}) \to \Psi_{adm}(\mathbf{d}')$$

is an isomorphism.

1.1.8. The main result of the present section is the following:

Theorem 1.1.9. *Under the above circumstances there exists a uniquely defined functor*

$$\Psi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S},$$

equipped with an identification,

$$\Phi^{adm}_{vert;horiz} \simeq \Psi^{adm}_{vert;horiz} \circ F^{adm}_{vert;horiz},$$

such that the induced natural transformation

$$\mathrm{LKE}_{F_{adm}}(\Phi_{adm}) \to \Psi^{adm}_{vert;horiz}|_{\mathbf{D}_{adm}}$$

is an isomorphism.

In addition, the functor $\Psi^{adm}_{vert;horiz}$ has the following properties:

- *The induced natural transformation*

$$\mathrm{LKE}_{F_{vert}}(\Phi) \to \Psi^{adm}_{vert;horiz}|_{\mathbf{D}_{vert}}$$

is an isomorphism;

- *The induced natural transformation*

$$\Psi^{adm}_{vert;horiz}|_{(\mathbf{D}_{horiz})^{\mathrm{op}}} \to \mathrm{RKE}_{(F_{horiz})^{\mathrm{op}}}(\Phi^!)$$

is an isomorphism.

Note that we can reformulate the uniqueness part of Theorem 1.1.9 as follows:

Corollary 1.1.10. *Let*

$$\widetilde{\Psi}^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S}$$

be a functor such that the following maps are isomorphisms:

(i) $\mathrm{LKE}_{F_{vert}}(\widetilde{\Psi}^{adm}_{vert;horiz} \circ F_{vert}|_{\mathbf{C}_{vert}}) \to \widetilde{\Psi}^{adm}_{vert;horiz}|_{\mathbf{D}_{vert}}$*;*

(ii) $\Psi^{adm}_{vert;horiz}|_{\mathbf{D}^{\mathrm{op}}_{horiz}} \to \mathrm{LKE}_{F^{\mathrm{op}}_{horiz}}(\widetilde{\Psi}^{adm}_{vert;horiz} \circ F^{\mathrm{op}}_{horiz}|_{\mathbf{C}^{\mathrm{op}}_{horiz}})$*;*

(iii) $\mathrm{LKE}_{F_{adm}}(\widetilde{\Psi}^{adm}_{vert;horiz} \circ F_{adm}|_{\mathbf{C}_{adm}}) \to \widetilde{\Psi}^{adm}_{vert;horiz}|_{\mathbf{D}_{adm}}$

(iv) *The map*

$$\underset{\mathbf{c} \in (\mathbf{D}_{adm})_{/\mathbf{d}} \underset{\mathbf{D}_{adm}}{\times} \mathbf{C}_{adm}}{\mathrm{colim}} \widetilde{\Psi}_{adm}(\mathbf{d}' \underset{\mathbf{d}}{\times} \mathbf{c}) \to \widetilde{\Psi}_{adm}(\mathbf{d}')$$

for every morphism $\mathbf{d}' \to \mathbf{d}$ *in* $\mathbf{D}$*.*

Then the functor $\Phi^{adm}_{vert;horiz} := \widetilde{\Psi}^{adm}_{vert;horiz} \circ F^{adm}_{vert;horiz}$ *satisfies the assumptions of Theorem 1.1.9, and* $\widetilde{\Psi}^{adm}_{vert;horiz}$ *identifies canonically with the extension given by that theorem.*

1.2. **Construction of the functor.** The rest of this section and the following four sections are devoted to the proof of Theorem 1.1.9.

We will construct $\Psi^{adm}_{vert;horiz}$ as a simplicial functor

$$\Psi_\bullet : {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{D})^{adm}_{vert;horiz} \to \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}).$$

1.2.1. The functor $F : \mathbf{C} \to \mathbf{D}$ gives rise to a simplicial functor

$$F_n : {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz} \to {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{D})^{adm}_{vert;horiz},$$

which is fully faithful for each n.

We define the functor

$$\Psi_n : {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{D})^{adm}_{vert;horiz} \to \mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{S}),$$

by

$$\Psi_n := \mathrm{LKE}_{F_n}(\Phi_n),$$

where

$$\Phi_\bullet : {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_\bullet(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Seq}^{\mathrm{ext}}_\bullet(\mathbb{S}),$$

is the simplicial functor corresponding to $\Phi^{adm}_{vert;horiz}$. (It will follow from Corollary 2.3.5 that this left Kan extension exists.)

1.2.2. Consider the following set up. Let $\mathbf{I}$ be an index category, and let $F_{\mathbf{I}} : \mathbf{C}_{\mathbf{I}} \to \mathbf{D}_{\mathbf{I}}$ be a map between co-Cartesian fibrations over $\mathbf{I}$ that takes co-Cartesian arrows to co-Cartesian ones. Let $\mathbf{S}_{\mathbf{I}}$ be another co-Cartesian fibration over $\mathbf{I}$, and let $\Phi_{\mathbf{I}} : \mathbf{C}_{\mathbf{I}} \to \mathbf{S}_{\mathbf{I}}$ be a functor over $\mathbf{I}$ that also takes co-Cartesian arrows to co-Cartesian arrows.

For each arrow $r : i \to j$ in $\mathbf{I}$ there are canonical natural transformations

$$\mathrm{LKE}_{F_i}(\Phi_j \circ r_{\mathbf{C}}) \to \mathrm{LKE}_{F_j}(\Phi_j) \circ r_{\mathbf{D}} \tag{1.1}$$

and

$$\mathrm{LKE}_{F_i}(r_{\mathbf{S}} \circ \Phi_i) \to r_{\mathbf{S}} \circ \mathrm{LKE}_{F_i}(\Phi_i) \tag{1.2}$$

as functors $\mathbf{D}_i \to \mathbf{S}_j$, while

$$F_j \circ r_{\mathbf{C}} \simeq r_{\mathbf{D}} \circ F_i \text{ and } r_{\mathbf{S}} \circ \Phi_i \simeq \Phi_j \circ r_{\mathbf{C}},$$

where

$$r_{\mathbf{C}} : \mathbf{C}_i \to \mathbf{C}_j,\ r_{\mathbf{D}} : \mathbf{D}_i \to \mathbf{D}_j,\ r_{\mathbf{S}} : \mathbf{S}_i \to \mathbf{S}_j$$

denote the corresponding functors.

We have:

Lemma 1.2.3. *Assume that the maps* (1.1) *and* (1.2) *are isomorphisms. Then relative left Kan extension defines a functor* $\Psi_{\mathbf{I}} : \mathbf{D}_{\mathbf{I}} \to \mathbf{S}_{\mathbf{I}}$*, which has the property that it sends co-Cartesian arrows to co-Cartesian ones. Furthermore, for every* $i \in \mathbf{I}$*, the natural map*

$$\mathrm{LKE}_{F_i}(\Phi_i) \to \Psi_{\mathbf{I}}|_{\mathbf{C}_i}$$

is an isomorphism.

1.2.4. We will apply Lemma 1.2.3 to

$$\mathbf{I} := \mathbf{\Delta}^{\mathrm{op}}, \quad F_{\mathbf{I}} := F_\bullet, \quad \mathbf{S}_{\mathbf{I}} := \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}), \quad \Phi_{\mathbf{I}} := \Phi_\bullet.$$

For a map $r : [m] \to [n]$ in $\mathbf{\Delta}$, let $r_{\mathbb{S}}^*$ (resp., $r_{\mathbf{C}}^*$, $r_{\mathbf{D}}^*$) denotes the functor $\mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S}) \to \mathrm{Seq}_m^{\mathrm{ext}}(\mathbb{S})$ (resp., $'\mathbf{Grid}_n^{\geq\,\mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm} \to {'\mathbf{Grid}}_m^{\geq\,\mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$ and similarly for $\mathbf{D}$).

Consider the resulting functors

$$\mathrm{LKE}_{F_n}(\Phi_m \circ r_{\mathbf{C}}^*) \to \mathrm{LKE}_{F_m}(\Phi_m) \circ r_{\mathbf{D}}^*. \tag{1.3}$$

and

$$\mathrm{LKE}_{F_n}(r_{\mathbb{S}}^* \circ \Phi_n) \to r_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_n). \tag{1.4}$$

The bulk of the proof of Theorem 1.1.9 will amount to the proof of the next proposition:

Proposition 1.2.5.

(a) *The maps* (1.3) *are isomorphisms.*

(b) *The maps* (1.4) *are isomorphisms.*

1.3. **Proof of Theorem 1.1.9, existence.** In this subsection we will assume Proposition 1.2.5 and will deduce the existence part of Theorem 1.1.9. The uniqueness assertion will be proved in Sect. 3.2.

1.3.1. First, assuming Proposition 1.2.5, and using Lemma 1.2.3, we obtain that that the functors

$$\Psi_n := \mathrm{LKE}_{F_n}(\Phi_n)$$

give rise to a simplicial functor

$$\Psi_\bullet : {'\mathbf{Grid}}_\bullet^{\geq\,\mathrm{dgnl}}(\mathbf{D})_{vert;horiz}^{adm} \to \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}).$$

By the $(\mathfrak{L}^{\mathrm{ext}}, \mathrm{Seq}^{\mathrm{ext}})$-adjunction, from $\Psi_\bullet$ we obtain a functor

$$\mathfrak{L}^{\mathrm{ext}}({'\mathbf{Grid}}_\bullet^{\geq\,\mathrm{dgnl}}(\mathbf{D})_{vert;horiz}^{adm}) \to \mathbb{S},$$

and finally, using [Chapter V.1, Corollary 1.4.6], the sought-for functor

$$\Psi_{vert;horiz}^{adm} : \mathrm{Corr}(\mathbf{D})_{vert;horiz}^{adm} \to \mathbb{S}.$$

1.3.2. The composite

$$'\mathbf{Grid}_\bullet^{\geq \text{dgnl}}(\mathbf{C})^{adm}_{vert;horiz} \to {}'\mathbf{Grid}_\bullet^{\geq \text{dgnl}}(\mathbf{D})^{adm}_{vert;horiz} \xrightarrow{\Psi_\bullet} \text{Seq}^{\text{ext}}_\bullet(\mathbb{S})$$

identifies with $\Phi_\bullet$. This gives an identification

$$\Phi^{adm}_{vert;horiz} \simeq \Psi^{adm}_{vert;horiz} \circ F^{adm}_{vert;horiz}.$$

Denote

$$\Psi := \Psi^{adm}_{vert;horiz}|_{\mathbf{D}_{vert}} \text{ and } \Psi^! := \Psi^{adm}_{vert;horiz}|_{(\mathbf{D}_{horiz})^{\text{op}}};$$

$$\Psi_{adm} := \Psi|_{\mathbf{D}_{adm}} \simeq \Psi^{adm}_{vert;horiz}|_{\mathbf{D}_{adm}}$$

and

$$\Psi^!_{adm} := \Psi^!|_{(\mathbf{D}_{adm})^{\text{op}}} \simeq \Psi^{adm}_{vert;horiz}|_{(\mathbf{D}_{adm})^{\text{op}}}.$$

For an object $\mathbf{d} \in \mathbf{D}$ we shall simply write $\Psi(\mathbf{d})$ for $\Psi^{adm}_{vert;horiz}(\mathbf{d})$.

1.3.3. Let us now show that that the induced natural transformation

$$\text{LKE}_{F_{adm}}(\Phi_{adm}) \to \Psi_{adm} \tag{1.5}$$

is an isomorphism.

It suffices to show that this natural transformation is an isomorphism on objects. I.e., we have to show that for $\mathbf{d} \in \mathbf{D}$, the map

$$\underset{\gamma:\mathbf{c}\to\mathbf{d},\gamma\in adm}{\text{colim}} \Phi(\mathbf{c}) \to \Psi(\mathbf{d})$$

is an isomorphism, where the above map is given by a compatible family of maps

$$\Phi(\mathbf{c}) \overset{\sim}{\to} \Psi(\mathbf{c}) \overset{\Psi(\gamma)}{\to} \Psi(\mathbf{d}). \tag{1.6}$$

By the definition of the functor Ψ_n for $n = 0$, we have a tautological isomorphism

$$\underset{\gamma:\mathbf{c}\to\mathbf{d},\gamma\in adm}{\text{colim}} \Phi(\mathbf{c}) = \Psi_0(\mathbf{d}) = \Psi(\mathbf{d}). \tag{1.7}$$

Now, by the definition of Ψ_n for $n = 1$, the compatible family of maps

$$\Phi(\mathbf{c}) \to \Psi(\mathbf{d})$$

that comprise (1.6) identifies with that in (1.7).

1.3.4. Let us show that the natural transformation

$$\text{LKE}_{F_{vert}}(\Phi) \to \Psi$$

is an isomorphism.

Again, it is enough to do so at the level of objects. Consider the commutative diagram

$$\begin{array}{ccc}
\text{LKE}_{F_{adm}}(\Phi_{adm}) & \longrightarrow & \Psi_{adm} \\
\downarrow & & \downarrow = \\
\text{LKE}_{F_{vert}}(\Phi)|_{\mathbf{D}_{adm}} & \longrightarrow & \Psi|_{\mathbf{D}_{adm}}.
\end{array}$$

The top horizontal arrow in this diagram is an isomorphism by Sect. 1.3.3 above. The left vertical arrow is an isomorphism by the second condition in Sect. 1.1.6. This implies that the bottom horizontal arrow is an isomorphism, as desired.

1.3.5. Let us now show that the natural transformation

$$\Psi^! \to \mathrm{RKE}_{(F_{horiz})^{\mathrm{op}}}(\Phi^!)$$

is an isomorphism.

As in Sect. 1.3.4, it suffices to show that the natural transformation

$$\Psi^!_{adm} \to \mathrm{RKE}_{(F_{adm})^{\mathrm{op}}}(\Phi^!_{adm})$$

is an isomorphism.

Since the functor $\Psi^!_{adm}$ is obtained from Ψ_{adm} by passing to right adjoints (by [Chapter V.1, Theorem 3.2.2]), we need to show that for $\mathbf{d} \in \mathbf{D}$ the map

$$\Psi(\mathbf{d}) \to \lim_{\gamma:\mathbf{c}\to\mathbf{d},\gamma\in adm} \Phi(\mathbf{c}),$$

comprised of functors

$$\Psi(\mathbf{d}) \to \Phi(\mathbf{c}),$$

right adjoint to those in (1.7), is an isomorphism. However, this follows from Proposition 2.2.5 below.

2. Limits and colimits of sequences

In this section we prepare for the proof of Proposition 1.2.5 by making a digression on the behavior of limits and colimits in categories of the form $\mathrm{Seq}^{\mathrm{ext}}_n(\mathbb{S})$, where $\mathbb{S}$ is an $(\infty,2)$-category as in Sect. 1.1.5.

2.1. **Limits and colimits of presentable categories.** In this subsection we recall the behavior of limits and colimits in the $(\infty,1)$-category, whose objects are presentable categories, and whose 1-morphisms are colimit-preserving functors.

2.1.1. Recall from [Chapter I.1, Sect. 2.5] that $1\text{-}\mathrm{Cat}_{\mathrm{Prs}}$ denotes the 1-full subcategory of 1-Cat, whose objects are presentable $(\infty,1)$-categories, and whose morphisms are continuous (i.e., *colimit-preserving*) functors. Recall that the embedding

$$1\text{-}\mathrm{Cat}_{\mathrm{Prs}} \hookrightarrow 1\text{-}\mathrm{Cat}$$

commutes with *limits*, see [Chapter I.1, Lemma 2.5.2(b)].

Let us recall the setting of [Chapter I.1, Proposition 2.5.7], which we will extensively use. Let

$$\mathbf{I} \to 1\text{-}\mathrm{Cat}_{\mathrm{Prs}}, \quad i \mapsto \mathbf{C}_i \tag{2.1}$$

be a functor, and consider the object

$$\mathbf{C}' := \underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{C}_i \in 1\text{-}\mathrm{Cat}_{\mathrm{Prs}}.$$

For an index i, let ins_i denote the tautological functor

$$\mathbf{C}_i \to \mathbf{C}'.$$

Assume now that for every arrow $i \to j$ in $\mathbf{I}$, the corresponding functor $\mathbf{C}_i \to \mathbf{C}_j$ admits a *continuous* right adjoint. Consider the resulting functor

$$\mathbf{I}^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}_{\mathrm{Prs}}, \quad i \mapsto \mathbf{C}_i,$$

obtained from (2.1) by passing to right adjoints. Consider the object

$$\mathbf{C}'' := \lim_{i\in\mathbf{I}^{\mathrm{op}}} \mathbf{C}_i \in 1\text{-}\mathrm{Cat}_{\mathrm{Prs}}.$$

For an index i, let ev_i denote the tautological functor

$$\mathbf{C}'' \to \mathbf{C}_i.$$

Then [Chapter I.1, Proposition 2.5.7] says that each of the functors ev_i admits a left adjoint, and that the resulting functor

$$\mathbf{C}' \to \mathbf{C}'', \quad (\mathrm{ev}_i)^L : \mathbf{C}_i \to \mathbf{C}'' \tag{2.2}$$

is an equivalence. In other words, we have an equivalence $\mathbf{C}' \simeq \mathbf{C}''$, under which, the adjoint pair

$$\mathrm{ins}_i : \mathbf{C}_i \rightleftarrows \mathbf{C}' : (\mathrm{ins}_i)^R$$

identifies with

$$(\mathrm{ev}_i)^L : \mathbf{C}_i \rightleftarrows \mathbf{C}'' : \mathrm{ev}_i\,.$$

As a formal consequence, we obtain:

Corollary 2.1.2.
(a) *The natural transformation*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{ins}_i \circ (\mathrm{ins}_i)^R \to \mathrm{Id}_{\mathbf{C}'}$$

is an isomorphism, where the colimit is taken in either the category $\mathrm{Maps}_{1\text{-}\mathrm{Cat}_{\mathrm{Prs}}}(\mathbf{C}', \mathbf{C}')$ *or* $\mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{C}', \mathbf{C}')$.
(b) *The functor* $\mathbf{C}'' \to \mathbf{C}'$*, given by*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{ins}_i \circ \mathrm{ev}_i$$

provides an inverse to the functor $\mathbf{C}' \to \mathbf{C}''$ *from* (2.2)*, where the colimit is taken in either the category* $\mathrm{Maps}_{1\text{-}\mathrm{Cat}_{\mathrm{Prs}}}(\mathbf{C}'', \mathbf{C}')$ *or* $\mathrm{Maps}_{1\text{-}\mathrm{Cat}}(\mathbf{C}'', \mathbf{C}')$.
(c) *For* $\mathbf{D} \in 1\text{-}\mathrm{Cat}_{\mathrm{Prs}}$*, the natural map*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i) \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}')$$

is an isomorphism, where the colimit is taken in the category $1\text{-}\mathrm{Cat}_{\mathrm{Prs}}$.

Proof. Let us prove point (a). It suffices to show that the natural transformation

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ (\mathrm{ev}_i)^L \circ \mathrm{ev}_i \to \mathrm{Id}_{\mathbf{C}''}$$

is an equivalence. I.e., we have to show that for $\underline{\mathbf{c}}, \underline{\widetilde{\mathbf{c}}} \in \mathbf{C}''$, given by compatible systems of objects $\mathbf{c}_i, \widetilde{\mathbf{c}}_i \in \mathbf{C}_i$, respectively, the map

$$\mathrm{Maps}_{\mathbf{C}''}(\underline{\mathbf{c}}, \underline{\widetilde{\mathbf{c}}}) \to \lim_{i\in\mathbf{I}} \mathrm{Maps}_{\mathbf{C}''}((\mathrm{ev}_i)^L \circ \mathrm{ev}_i(\underline{\mathbf{c}}), \underline{\widetilde{\mathbf{c}}})$$

is an isomorphism. We rewrite the right-hand side as

$$\lim_{i\in\mathbf{I}} \mathrm{Maps}_{\mathbf{C}_i}((\mathrm{ev}_i(\underline{\mathbf{c}}), \mathrm{ev}_i(\underline{\widetilde{\mathbf{c}}})),$$

and now the assertion becomes manifest.

Point (b) follows formally from (a). Point (c) follows formally from (b) and the commutative diagram

$$\begin{array}{ccc}
\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i) & \longrightarrow & \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}') \\
{\scriptstyle\sim}\big\uparrow & & \big\uparrow{\scriptstyle\sim} \\
\lim_{i\in\mathbf{I}^{\mathrm{op}}} \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i) & \overset{\sim}{\longrightarrow} & \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}'').
\end{array}$$

□

2.2. **Limits and colimits in $\mathbb{S}^{\text{1-Cat}}$.** Let

$$\mathbf{1\text{-}Cat}_{\mathrm{Prs}} \subset \mathbf{1\text{-}Cat}$$

be the 1-full subcategory, corresponding to $\text{1-Cat}_{\mathrm{Prs}} \subset \text{1-Cat}$. We can view the equivalence (2.2) as a result about functors $\mathbf{I} \to \mathbf{1\text{-}Cat}_{\mathrm{Prs}}$.

In this subsection we will generalize the equivalence (2.2) by replacing $\mathbf{1\text{-}Cat}_{\mathrm{Prs}}$ by a more general $(\infty, 2)$-category $\mathbb{S}$.

2.2.1. *Limits of mapping categories.* Let $\mathbb{S}$ be an $(\infty, 2)$-category satisfying assumptions (1), (3), (4) and (6) of Sect. 1.1.5.

Lemma 2.2.2.
(a) *For* $\mathbf{I} \in \text{1-Cat}$ *and a functor* $\mathbf{I} \to \mathbb{S},\ i \mapsto s_i$ *with* $s := \underset{i \in \mathbf{I}}{\mathrm{colim}}\, s_i$*, for any* $s' \in \mathbb{S}$*, the resulting map*

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \lim_{i \in \mathbf{I}^{\mathrm{op}}} \mathbf{Maps}_{\mathbb{S}}(s_i, s')$$

is an isomorphism in $\text{1-Cat}_{\mathrm{Prs}}$.
(b) *For* $\mathbf{I} \in \infty\text{-Cat}$ *and a functor* $I \to \mathbb{S},\ i \mapsto s_i$ *with* $s := \lim_{i \in \mathbf{I}} s_i$*, for any* $s' \in \mathbb{S}$*, the resulting map*

$$\mathbf{Maps}_{\mathbb{S}}(s', s) \to \lim_{i \in \mathbf{I}} \mathbf{Maps}_{\mathbb{S}}(s', s_i)$$

is an isomorphism in $\text{1-Cat}_{\mathrm{Prs}}$.

Remark 2.2.3. By definition, the maps in the lemma a priori induce isomorphisms of the underlying spaces.

Proof. Follows from the above remark by replacing s' by the objects $s'^{[1]}$ and $[1] \otimes s'$, respectively.

□

2.2.4. Let

$$\mathbf{I} \to \mathbb{S},\ i \mapsto s_i \tag{2.3}$$

be a functor, and assume that for every arrow $i \to j$, the corresponding 1-morphism $s_i \to s_j$ admits a right adjoint. Consider the functor

$$\mathbf{I}^{\mathrm{op}} \to \mathbb{S},\ i \mapsto s_i,$$

obtained from (2.3) by passing to right adjoints.

Denote

$$s' := \underset{i \in \mathbf{I}}{\mathrm{colim}}\, s_i \text{ and } s'' := \lim_{i \in \mathbf{I}^{\mathrm{op}}} s_i$$

and let

$$\mathrm{ins}_i : s_i \to s' \text{ and } \mathrm{ev}_i : s'' \to s_i$$

denote the corresponding 1-morphisms.

We are going to prove:

Proposition 2.2.5.
(a) *Each of the 1-morphisms* ev_i *admits a left adjoint. The 1-morphism* $s' \to s''$*, given by the compatible family*

$$(\mathrm{ev}_i)^L : s_i \to s'',$$

is an isomorphism. Under this identification, the functors $(\mathrm{ev}_i)^L$ *correspond to* ins_i*, and the functors* ev_i *correspond to the the right adjoints* $(\mathrm{ins}_i)^R$ *of* ins_i.

(b) *The map*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{ins}_i \circ (\mathrm{ins}_i)^R \to \mathrm{id}_{s'}$$

is an isomorphism, where the colimit is taken in $\mathbf{Maps}_{\mathbb{S}}(s', s')$.

(c) *The inverse 1-morphism to one in point (a) is given by*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\ \mathrm{ins}_i \circ \mathrm{ev}_i,$$

where the colimit is taken in $\mathbf{Maps}_{\mathbb{S}}(s'', s')$.

(d) *For any* $t \in \mathbb{S}$*, the natural functor*

$$\underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{Maps}_{\mathbb{S}}(t, s_i) \to \mathbf{Maps}_{\mathbb{S}}(t, s')$$

is an equivalence, where the colimit is taken in $1\text{-}\mathrm{Cat}_{\mathrm{Prs}}$.

Proof. To show that the 1-morphism ev_j admits a left adjoint, it is enough to show that for any $t \in \mathbb{S}$, the induced morphism

$$\mathbf{Maps}_{\mathbb{S}}(t, s'') \to \mathbf{Maps}_{\mathbb{S}}(t, s_j)$$

admits a left adjoint and that for a 1-morphism $t_0 \to t_1$, the diagram

$$\begin{array}{ccc}
\mathbf{Maps}_{\mathbb{S}}(t_1, s'') & \longleftarrow & \mathbf{Maps}_{\mathbb{S}}(t_1, s_j) \\
\downarrow & & \downarrow \\
\mathbf{Maps}_{\mathbb{S}}(t_0, s'') & \longleftarrow & \mathbf{Maps}_{\mathbb{S}}(t_0, s_j)
\end{array}$$

obtained by passing to left adjoints along the horizontal arrows in the commutative diagram

$$\begin{array}{ccc}
\mathbf{Maps}_{\mathbb{S}}(t_1, s'') & \longrightarrow & \mathbf{Maps}_{\mathbb{S}}(t_1, s_j) \\
\downarrow & & \downarrow \\
\mathbf{Maps}_{\mathbb{S}}(t_0, s'') & \longrightarrow & \mathbf{Maps}_{\mathbb{S}}(t_0, s_j),
\end{array}$$

that a priori *commutes up to a natural transformation*, actually commutes.

By Lemma 2.2.2, we have

$$\mathbf{Maps}_{\mathbb{S}}(t, s'') \simeq \lim_{i\in\mathbf{I}^{\mathrm{op}}} \mathbf{Maps}_{\mathbb{S}}(t, s_i),$$

and now the assertion follows from the equivalence (2.2).

We will now show that the composite

$$s' \to s'' \to s',$$

where the first arrow is the map point (a) and the second arrow is the map in point (c), is isomorphic to $\mathrm{id}_{s'}$. We need to show that for every $j \in \mathbf{I}$, the composition

$$s_j \xrightarrow{\mathrm{ins}_j} s' \to s'' \to s' \tag{2.4}$$

is canonically isomorphic to ins_j, in a way functorial in j.

Note now that for any $t \in \mathbb{S}$ we have a commutative diagram

$$\begin{array}{ccccc}
\mathbf{Maps}_{\mathbb{S}}(t, s') & \longrightarrow & \mathbf{Maps}_{\mathbb{S}}(t, s'') & \longrightarrow & \mathbf{Maps}_{\mathbb{S}}(t, s') \\
\uparrow & & \uparrow & & \uparrow \\
\underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{Maps}_{\mathbb{S}}(t, s_i) & \longrightarrow & \lim_{i\in\mathbf{I}^{\mathrm{op}}} \mathbf{Maps}_{\mathbb{S}}(t, s_i) & \longrightarrow & \underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{Maps}_{\mathbb{S}}(t, s_i),
\end{array} \tag{2.5}$$

where the colimits in the bottom row are taken in $1\text{-Cat}_{\text{Prs}}$. Now, Corollary 2.1.2(b) asserts that the bottom composition is canonically isomorphic to the identity on $\underset{i\in\mathbf{I}}{\text{colim}}\,\mathbf{Maps}_{\mathbb{S}}(t,s_i)$.

Applying this to $t=s_j$ and the tautological map

$$s_j\to\underset{i\in\mathbf{I}}{\text{colim}}\,\mathbf{Maps}_{\mathbb{S}}(s_j,s_i),$$

we obtain that the composition (2.4) is indeed isomorphic to ins_j.

Thus, we obtain that the composite $s'\to s''\to s'$ is indeed isomorphic to the identity map. In particular, the composite in the top row of (2.5) is also isomorphic to the identity map. Now, since the middle vertical map in (2.5) is an isomorphism (by Lemma 2.2.2(b)), we obtain that the left vertical map is an isomorphism as well.

This proves point (d) of the proposition. Now, since all maps in the bottom row of (2.5) are isomorphisms, we obtain that the same is true for the top row. This proves point (a) of the proposition.

Point (c) has been established already. Point (b) follows formally from point (c).

□

2.2.6. As an application of Proposition 2.2.5, we shall now prove the following. Let

$$F:\mathbf{C}\to\mathbf{D}$$

be a functor between $(\infty,1)$-categories. Let

$$\Phi:\mathbf{C}\to\mathbb{S}$$

be a functor, and set $\Psi:=\text{LKE}_F(\Phi)$.

Assume that for every arrow $\mathbf{c}_1\to\mathbf{c}_2$ in $\mathbf{C}$, the resulting 1-morphism $\Phi(\mathbf{c}_1)\to\Phi(\mathbf{c}_2)$ admits a right adjoint. Let $\Phi^!:\mathbf{C}^{\text{op}}\to\mathbb{S}$ be the functor, obtained from Φ by passing to right adjoints.

We claim:

Proposition 2.2.7. *Under the above circumstances, the functor* $\Psi^!:=\text{RKE}_{F^{\text{op}}}(\Phi^!)$ *is obtained from* Ψ *by passing to right adjoints.*

Proof. Let us first show that for any arrow $\mathbf{d}_1\to\mathbf{d}_2$ in $\mathbf{D}$, the 1-morphism

$$\Psi(\mathbf{d}_1)\to\Psi(\mathbf{d}_2)$$

admits a right adjoint.

We have:

$$\Psi(\mathbf{d}_1)=\underset{\mathbf{c}_1,F(\mathbf{c}_1)\to\mathbf{d}_1}{\text{colim}}\Phi(\mathbf{c}_1)\text{ and }\Psi(\mathbf{d}_2)=\underset{\mathbf{c}_2,F(\mathbf{c}_2)\to\mathbf{d}_2}{\text{colim}}\Phi(\mathbf{c}_2),$$

and the 1-morphism $\Psi(\mathbf{d}_1)\to\Psi(\mathbf{d}_2)$ is obtained from the map of index categories

$$(\mathbf{c}_1,F(\mathbf{c}_1)\to\mathbf{d}_1)\mapsto(\mathbf{c}_1,F(\mathbf{c}_1)\to\mathbf{d}_1\to\mathbf{d}_2).$$

The right adjoint in question is given, in terms of the isomorphism of Proposition 2.2.5(a) by

$$\underset{\mathbf{c}_2,F(\mathbf{c}_2)\to\mathbf{d}_2}{\lim}\Phi(\mathbf{c}_2)\to\underset{\mathbf{c}_1,F(\mathbf{c}_1)\to\mathbf{d}_1}{\lim}\Phi(\mathbf{c}_1).$$

Let $\Psi^!$ denote the functor obtained from Ψ by passing to right adjoints. Let us construct a natural transformation

$$\Psi^!\to\text{RKE}_{F^{\text{op}}}(\Phi^!),\tag{2.6}$$

which by definition amounts to a natural transformation

$$\Psi^! \circ F^{\mathrm{op}} \to \Phi^!.$$

This natural transformation is obtained by passing to right adjoints in the canonical natural transformation

$$\Phi \to \Psi \circ F.$$

Let us now show that the natural transformation (2.6) is an isomorphism. It is enough to check this at the level of objects. We have

$$\Psi^!(\mathbf{d}) = \underset{\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}}{\mathrm{colim}} \Psi(\mathbf{c}) \text{ and } \mathrm{RKE}_{F^{\mathrm{op}}}(\Phi^!)(\mathbf{d}) = \underset{\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}}{\lim} \Phi(\mathbf{c}).$$

By unwinding the definitions, we obtain that the resulting map

$$\underset{\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}}{\mathrm{colim}} \Psi(\mathbf{c}) \to \underset{\mathbf{c}, F(\mathbf{c}) \to \mathbf{d}}{\lim} \Phi(\mathbf{c})$$

is the map of Proposition 2.2.5(a).

□

2.3. **Colimits in** $\mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S})$**.** In this subsection we will record results pertaining to the behavior of colimits in the category $\mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S})$: these are Propositions 2.3.2, 2.3.4 and Corollary 2.3.7.

2.3.1. *Colimits of strings.* We begin with the following observation:

Proposition 2.3.2. *For a morphism* $r : [0, m] \to [0, n]$ *in* $\mathbf{\Delta}$*, the functor*

$$r_{\mathbb{S}}^* : \mathrm{Seq}_n^{\mathrm{ext}}(\mathbb{S}) \to \mathrm{Seq}_m^{\mathrm{ext}}(\mathbb{S})$$

commutes with colimits in the following cases:

(a) $m = 0$.

(b) $m = 1$ *and* r *sends* $0 \mapsto i$ *and* $1 \mapsto i + 1$ *for* $0 \leq i < n$.

(c) *For* r *inert; i.e. of the form* $i \mapsto k + i$ *for some* $0 \leq k \leq n - m$.

(d) r *is a surjection.*

Proof. We claim that in each of the above cases, the functor $r_{\mathbb{S}}^*$ admits a right adjoint. Clearly, (a) and (b) are particular cases of (c).

The right adjoint in question sends a string

$$s^0 \to s^1 \to \ldots \to s^{m-1} \to s^m$$

to

$$* \to \ldots \to * \to s^0 \to s^1 \to \ldots \to s^{m-1} \to s^m \to * \to \ldots \to *,$$

where $*$ denotes the initial/final object of $\mathbb{S}$. Here we are using the fact that for $s', s'' \in \mathbb{S}$, the map

$$\mathbf{Maps}(s', *) \times \mathbf{Maps}(*, s'') \to \mathbf{Maps}(s', s'')$$

sends the unique object in $\mathbf{Maps}(s', *) \times \mathbf{Maps}(*, s'')$ to the initial/final object in $\mathbf{Maps}(s', s'')$.

For r surjective, by transitivity, it suffices to consider the case when r collapses the interval $\{i-1\} \to \{i\}$ in $[n]$ to $\{i-1\} \in [n-1]$. In this case, the right adjoint in question sends

$$s_0 \to \ldots \to s^{i-2} \to s^{i-1} \to s^i \to s^{i+1} \to \ldots \to s^n$$

to

$$s^0 \to \ldots \to s^{i-2} \to s'^{i-1} \to s^{i+1} \to \ldots \to s^n,$$

where

$$s'^{i-1} = s^{i-1} \underset{s^i}{\times} (s^i)^{[1]},$$

where $(s^i)^{[1]} \to s^i$ corresponds to the map $[0] \to [1]$ given by $0 \mapsto 0$.

□

2.3.3. *Colimits of 1-morphisms.* We shall now describe explicitly how to compute colimits in $\mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$. Let $\mathbf{I}$ be an $(\infty,1)$-category, and let

$$i \mapsto \underline{s}_i := (s_i^0 \xrightarrow{\beta_i} s_i^1)$$

be a functor $\mathbf{I} \to \mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$.

Let us denote

$$s^0 \simeq \operatorname*{colim}_{i\in\mathbf{I}} s_i^0 \text{ and } s^1 \simeq \operatorname*{colim}_{i\in\mathbf{I}} s_i^1. \tag{2.7}$$

Let ins_i^0 and ins_i^1 denote the canonical 1-morphisms

$$s_i^0 \to s^0 \text{ and } s_i^1 \to s^1,$$

respectively.

Note that by Proposition 2.3.2(b), if the colimit

$$\operatorname*{colim}_{i\in\mathbf{I}} \underline{s}_i \in \mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$$

exists, the 0-th (resp., 1-st) component of the corresponding object identifies with s_0 (resp., s_1).

Assume now that for every 1-morphism $i \to j$ in $\mathbf{I}$, the 1-morphism $s_i^0 \to s_j^0$ admits a right adjoint. In this case, the 1-morphism ins_i^0 also admits a right adjoint, which we denote by $(\mathrm{ins}_i^0)^R$.

Proposition 2.3.4. *Under the above circumstances, the colimit* $\underline{s} := \operatorname*{colim}_{i\in\mathbf{I}} \underline{s}_i \in \mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$ *exists. The resulting* 1*-morphism*

$$s^0 \to s^1$$

identifies canonically with

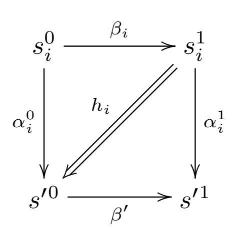
$$\beta := \operatorname*{colim}_{i\in\mathbf{I}} \mathrm{ins}_i^1 \circ \beta^i \circ (\mathrm{ins}_i^0)^R \in \mathbf{Maps}_{\mathbb{S}}(s_0, s_1).$$

Proof. Let

$$s'^0 \xrightarrow{\beta'} s'^1$$

be an object in $\mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$. We need to show that a compatible system of data

(2.8)



is equivalent to that of

$$
\begin{array}{ccc}
s^0 & \xrightarrow{\beta} & s^1 \\
\downarrow{\scriptstyle \alpha^0} & \overset{h}{\swarrow} & \downarrow{\scriptstyle \alpha^1} \\
s'^0 & \xrightarrow[\beta']{} & s'^1
\end{array} \tag{2.9}
$$

The 1-morphisms α^0 and α^1 are uniquely recovered from the compatible families α^0_i and α^1_i respectively, by (2.7).

Since compositions of 1-morphisms commute with colimits, by the construction of β, the data of h is equivalent to that of a compatible system of 2-morphisms

$$\alpha^1_i \circ \beta_i \circ (\text{ins}^0_i)^R \simeq \alpha^1 \circ \text{ins}^1_i \circ \beta_i \circ (\text{ins}^0_i)^R \to \beta' \circ \alpha^0,$$

which by adjunction is equivalent to

$$\alpha^1_i \circ \beta_i \to \beta' \circ \alpha^0 \circ \text{ins}^0_i \simeq \beta' \circ \alpha^0_i,$$

as desired.

□

Combing with Proposition 2.3.2(b), we obtain:

Corollary 2.3.5. *Let $i \mapsto \underline{s}_i$ be a functor $\mathbf{I} \to \text{Seq}^{\text{ext}}_n(\mathbb{S})$, such that for every $k \in \{0, ..., n-1\}$, and every 1-morphism $i \to j$ in $\mathbf{I}$, the corresponding 1-morphism $s^k_i \to s^k_j$ admits a right adjoint. Then the colimit*

$$\underline{s} := \underset{i \in \mathbf{I}}{\text{colim}}\, \underline{s}_i \in \text{Seq}^{\text{ext}}_n(\mathbb{S})$$

exists.

2.3.6. *The product situation.* Let $\underline{s}$

$$s^0 \to s^1 \to ... \to s^n \tag{2.10}$$

be an object of $\text{Seq}^{\text{ext}}_n(\mathbb{S})$.

Let us be given, for each $k = 0, ..., n$ an index category $\mathbf{I}_k$ and a colimit diagram

$$\underset{i_k \in \mathbf{I}_k}{\text{colim}}\, s_{i_k} \to s^k,$$

such that each of the 1-morphisms $\text{ins}_{i_k} : s_{i_k} \to s^k$ and $\alpha_{i_k, i'_k} : s_{i_k} \to s_{i'_k}$ (for $(i_k \to i'_k) \in \mathbf{I}_k$) admits a right adjoint.

Set $\mathbf{I} := \mathbf{I}_0 \times ... \times \mathbf{I}_n$. We define an $\mathbf{I}$-diagram $\underline{i} \mapsto \underline{s}_{\underline{i}}$ in $\text{Seq}^{\text{ext}}_n(\mathbb{S})$ by setting for $\underline{i} = (i_0, ..., i_k)$

$$s^k_{\underline{i}} = s_{i_k}, \quad s_{i_{k-1}} \xrightarrow{\text{ins}_{i_{k-1}}} s^{k-1} \to s^k \xrightarrow{(\text{ins}_{i_k})^R} s_{i_k}.$$

By construction, the above $\mathbf{I}$-family $\underline{s}_{\underline{i}}$ is equipped with a map to the object $\underline{s}$ of (2.10).

From Propositions 2.3.4 and 2.3.2(b), we obtain:

Corollary 2.3.7. *Assume that each of the index categories* $\mathbf{I}_k$ *is contractible. Then:*
(a) *The map*

$$\underset{\underline{i}\in\mathbf{I}}{\operatorname{colim}}\, \underline{s}_{\underline{i}} \to \underline{s}$$

is an isomorphism.
(b) *For any* $r:[m]\to[n]$, *the map*

$$r^*_{\mathbb{S}}(\underset{\underline{i}\in\mathbf{I}}{\operatorname{colim}}\, \underline{s}_{\underline{i}}) \to \underset{\underline{i}\in\mathbf{I}}{\operatorname{colim}}\,(r^*_{\mathbb{S}}(\underline{s}_{\underline{i}})).$$

is an isomorphism.

3. The core of the proof

3.1. **Calculation of** $\mathrm{LKE}_{F_1}(\Phi_1)$**.** Recall the notation

$$\Psi := \mathrm{LKE}_{F_{vert}}(\Phi) : \mathbf{D}_{vert} \to \mathbb{S}$$

and

$$\Psi^! := \mathrm{RKE}_{(F_{horiz})^{\mathrm{op}}}(\Phi^!) : (\mathbf{D}_{horiz})^{\mathrm{op}} \to \mathbb{S}.$$

Note that we can identify the values of Ψ and $\Psi^!$ on objects of $\mathbf{D}$ by Proposition 2.2.7 and the conditions (2) and (4) in Sect. 1.1.6, i.e., that

$$\mathrm{LKE}_{F_{adm}}(\Phi|_{\mathbf{C}_{adm}}) \simeq \Psi|_{\mathbf{D}_{adm}}$$

and

$$\Psi^!|_{(\mathbf{D}_{adm})^{\mathrm{op}}} \simeq \mathrm{RKE}_{(F_{adm})^{\mathrm{op}}}(\Phi^!|_{(\mathbf{C}_{adm})^{\mathrm{op}}}).$$

Let an object $\underline{\mathbf{d}} \in \mathrm{Seq}_1^{\mathrm{ext}}(\mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz})$ be given by a diagram

(3.1)
$$\begin{array}{ccc} \mathbf{d}_{0,1} & \xrightarrow{\alpha_{\mathbf{d}}} & \mathbf{d}_{0,0} \\ {\scriptstyle\beta_{\mathbf{d}}}\big\downarrow & & \\ \mathbf{d}_{1,1}. & & \end{array}$$

The goal of this subsection is to construct a canonical identification of $\mathrm{LKE}_{F_1}(\Psi_1)(\underline{\mathbf{d}}_1)$, which is a 1-morphism

$$\Psi(\mathbf{d}_{0,0}) \to \Psi(\mathbf{d}_{1,1}),$$

with $\Psi(\beta_{\mathbf{d}})\circ\Psi^!(\alpha_{\mathbf{d}})$.

3.1.1. The 1-morphism

$$\mathrm{LKE}_{F_n}(r^*_{\mathbb{S}}\circ\Phi_n)(\underline{\mathbf{d}})$$

is the colimit in $\mathrm{Seq}_1^{\mathrm{ext}}(\mathbb{S})$ over

$$(\underline{\gamma} : \underline{\mathbf{c}} \to \underline{\mathbf{d}}) \in ({}'\mathbf{Grid}_1^{\geq\,\mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}}$$

of

$$\Phi(\beta_{\mathbf{c}})\circ\Phi^!(\alpha_{\mathbf{c}}),$$

for the morphisms

$$\begin{array}{ccc} \mathbf{c}_{0,1} & \xrightarrow{\alpha_{\mathbf{c}}} & \mathbf{c}_{0,0} \\ {\scriptstyle\beta_{\mathbf{c}}}\big\downarrow & & \\ \mathbf{c}_{1,1}. & & \end{array}$$

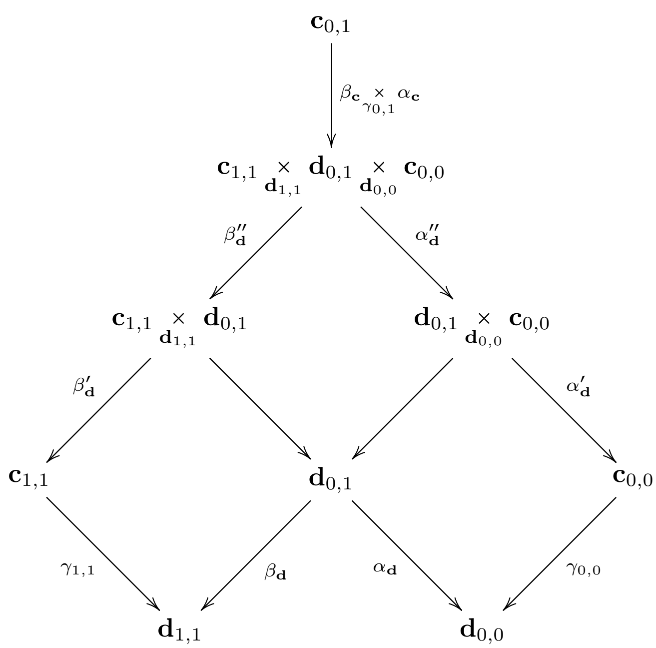
We calculate this colimit using Lemma 2.3.4, and we obtain

$$\underset{\underline{\gamma}:\underline{\mathbf{c}}\to\underline{\mathbf{d}}}{\operatorname{colim}} \Psi_{adm}(\gamma_{1,1}) \circ (\Phi(\beta_{\mathbf{c}}) \circ \Phi^!(\alpha_{\mathbf{c}})) \circ (\Psi_{adm}(\gamma_{0,0}))^R,$$

which can be rewritten as

$$\underset{\underline{\gamma}:\underline{\mathbf{c}}\to\underline{\mathbf{d}}}{\operatorname{colim}} \Psi(\gamma_{1,1}) \circ \Psi(\beta_{\mathbf{c}}) \circ \Psi^!(\alpha_{\mathbf{c}}) \circ \Psi^!(\gamma_{0,0}). \tag{3.2}$$

3.1.2. Consider the following diagram:



We rewrite the expression in (3.2) as the colimit over

$$\{\underline{\gamma} : \underline{\mathbf{c}} \to \underline{\mathbf{d}}\} \in (\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}}$$

of

$$\Psi(\gamma_{1,1}) \circ \Psi(\beta'_{\mathbf{d}}) \circ \Psi(\beta''_{\mathbf{d}}) \circ \Psi(\beta_{\mathbf{c}} \underset{\gamma_{0,1}}{\times} \alpha_{\mathbf{c}}) \circ \Psi^!(\beta_{\mathbf{c}} \underset{\gamma_{0,1}}{\times} \alpha_{\mathbf{c}}) \circ \Psi^!(\alpha''_{\mathbf{d}}) \circ \Psi^!(\alpha'_{\mathbf{d}}) \circ \Psi^!(\gamma_{0,0}). \tag{3.3}$$

3.1.3. The forgetful functor

$$(\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}} \to (\mathbf{C}_{adm})_{/\mathbf{d}_{0,0}} \times (\mathbf{C}_{adm})_{/\mathbf{d}_{1,1}}$$

is a co-Cartesian fibration. For a fixed object

$$\gamma_{0,0} : \mathbf{c}_{0,0} \to \mathbf{d}_{0,0} \text{ and } \gamma_{1,1} : \mathbf{c}_{1,1} \to \mathbf{d}_{1,1},$$

the fiber of $(\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}}$ over it is canonically equivalent to

$$(\mathbf{C}_{adm})_{/\mathbf{c}_{1,1} \underset{\mathbf{d}_{1,1}}{\times} \mathbf{d}_{0,1} \underset{\mathbf{d}_{0,0}}{\times} \mathbf{c}_{0,0}}.$$

Hence, by Proposition 2.2.5(b), the colimit of the expressions (3.3) over the above fiber is canonically isomorphic to

$$\Psi(\gamma_{1,1}) \circ \Psi(\beta'_{\mathbf{d}}) \circ \Psi(\beta''_{\mathbf{d}}) \circ \Psi^!(\alpha''_{\mathbf{d}}) \circ \Psi^!(\alpha'_{\mathbf{d}}) \circ \Psi^!(\gamma_{0,0}).$$

3.1.4. Applying the Beck-Chevalley isomorphisms (i.e., Conditions (1) and (3) from Sect. 1.1.6), we rewrite the latter expression as

$$\Psi(\gamma_{1,1}) \circ \Psi^!(\gamma_{1,1}) \circ \Psi(\beta_{\mathbf{d}}) \circ \Psi^!(\alpha_{\mathbf{d}}) \circ \Psi(\gamma_{0,0}) \circ \Psi^!(\gamma_{0,0}). \tag{3.4}$$

We obtain that the colimit over $\underline{\mathbf{c}} \in (\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}}$ is isomorphic to the colimit over

$$(\mathbf{c}_{0,0} \times \mathbf{c}_{1,1}) \in (\mathbf{C}_{adm})_{/\mathbf{d}_{0,0}} \times (\mathbf{C}_{adm})_{/\mathbf{d}_{1,1}}$$

of the expressions (3.4). Applying Proposition 2.2.5(b) again, we obtain that the latter is canonically isomorphic to

$$\Psi(\beta_{\mathbf{d}}) \circ \Psi^!(\alpha_{\mathbf{d}}),$$

as asserted.

3.2. **Proof of uniqueness in Theorem 1.1.9.** In this subsection we will continue to assume Proposition 1.2.5 and will deduce the uniqueness assertion in Theorem 1.1.9.

3.2.1. Let

$$\widetilde{\Psi}^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S}$$

be a functor.

Denote

$$\widetilde{\Psi} := \widetilde{\Psi}^{adm}_{vert;horiz}|_{\mathbf{D}_{vert}}, \; \widetilde{\Psi}^! := \widetilde{\Psi}^{adm}_{vert;horiz}|_{(\mathbf{D}_{horiz})^{\mathrm{op}}}.$$

Let us be given a natural transformation

$$\Phi^{adm}_{vert;horiz} \to \widetilde{\Psi}^{adm}_{vert;horiz} \circ F^{adm}_{vert;horiz}. \tag{3.5}$$

Let $\widetilde{\Psi}_\bullet$ denote the functor of simplicial categories

$$'\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz} \xrightarrow{\Psi_\bullet} \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}),$$

corresponding to $\widetilde{\Psi}$.

By the construction of $\Psi^{adm}_{vert;horiz}$, we obtain a natural transformation between simplicial functors

$$\Psi_\bullet \to \widetilde{\Psi}_\bullet,$$

as functors from the simplicial category $'\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz}$ to the simplicial category $\mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S})$.

3.2.2. Restricting along

$$''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz} \hookrightarrow {}'\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz},$$

we obtain a natural transformation between simplicial functors

$$\Psi_\bullet|_{''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz}} \to \widetilde{\Psi}_\bullet|_{''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz}}, \tag{3.6}$$

as functors from the simplicial category

$$''\mathbf{Grid}_\bullet^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz} \simeq \mathrm{Seq}_\bullet(\mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz})$$

to the simplicial category $\mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S})$, while both functors take values in

$$\mathrm{Seq}_\bullet(\mathbb{S}) \subset \mathrm{Seq}_\bullet^{\mathrm{ext}}(\mathbb{S}).$$

3.2.3. Suppose now that (3.5) is an isomorphism, and that the natural transformation

$$\mathrm{LKE}_{F_{adm}}(\Phi_{adm}) \to \widetilde{\Psi}^{adm}_{vert;horiz}|_{\mathbf{D}_{adm}}, \tag{3.7}$$

induced by (3.5), is also an isomorphism.

We will show that in this case, the natural transformation (3.6) is an isomorphism. By the Segal condition, it suffices to do so on 0-simplices and 1-simplices.

For 0-simplices, this is just the fact that the map (3.7) is an isomorphism. For 1-simplices, we need to show that for a 1-morphism $f : \mathbf{d} \to \mathbf{d}'$ in $\mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}$ the 2-morphism in

$$\begin{array}{ccc} \Psi(\mathbf{d}) & \longrightarrow & \Psi(\mathbf{d}') \\ {\scriptstyle\sim}\downarrow & \overset{h}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle\sim} \\ \widetilde{\Psi}(\mathbf{d}) & \longrightarrow & \widetilde{\Psi}(\mathbf{d}') \end{array} \tag{3.8}$$

is an isomorphism. It is enough to consider separately the cases when $f : \mathbf{d} \to \mathbf{d}'$ is vertical or horizontal.

3.2.4. Note that the natural transformation (3.5) gives rise to a natural transformation

$$\Phi \to \widetilde{\Psi} \circ F_{vert},$$

and hence to

$$\Psi := \mathrm{LKE}_{F_{vert}}(\Phi) \to \widetilde{\Psi}. \tag{3.9}$$

Restricting to $\mathbf{D}_{adm}$ we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{LKE}_{F_{adm}}(\Phi_{adm}) & \xrightarrow{\mathrm{id}} & \mathrm{LKE}_{F_{adm}}(\Phi_{adm}) \\ \downarrow & & \downarrow \\ \Psi|_{\mathbf{D}_{adm}} & \longrightarrow & \widetilde{\Psi}|_{\mathbf{D}_{adm}}. \end{array} \tag{3.10}$$

Parenthetically, note that the vertical arrows in (3.10) are isomorphisms. Hence, $\Psi|_{\mathbf{D}_{adm}} \to \widetilde{\Psi}|_{\mathbf{D}_{adm}}$ is an isomorphism. Therefore, (3.9) is also an isomorphism, because it is such on objects.

3.2.5. We claim that for $f : \mathbf{d} \to \mathbf{d}'$ being a vertical morphism $\mathbf{d} \overset{\beta}{\to} \mathbf{d}'$, the diagram (3.8) represents the natural transformation (3.9) evaluated on β.

Indeed, it follows from the calculation of Ψ_1 in Sect. 3.1 that $\Psi_1(\beta)$ can be written as a colimit of the category

$$\begin{array}{ccc} \mathbf{c} & \xrightarrow{\beta_{\mathbf{c}}} & \mathbf{c}' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ \mathbf{d} & \xrightarrow{\beta} & \mathbf{d}' \end{array}$$

of the objects $(\Phi(\mathbf{c}) \overset{\Phi(\beta_{\mathbf{c}})}{\longrightarrow} \Phi(\mathbf{c}')) \in \mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$. Thus, to show that (3.8) is represented by

$$\begin{array}{ccc} \Psi(\mathbf{d}) & \xrightarrow{\Psi(\beta)} & \Psi(\mathbf{d}') \\ \downarrow & & \downarrow \\ \widetilde{\Psi}(\mathbf{d}) & \xrightarrow{\widetilde{\Psi}(\beta)} & \widetilde{\Psi}(\mathbf{d}'), \end{array}$$

we need to construct a compatible of diagrams

$$\begin{array}{ccc} \Phi(\mathbf{c}) & \xrightarrow{\Phi(\beta_{\mathbf{c}})} & \Phi(\mathbf{c}') \\ {\scriptstyle\Psi(\gamma)}\downarrow & & \downarrow{\scriptstyle\Psi(\gamma')} \\ \Psi(\mathbf{d}) & \xrightarrow{\Psi(\beta)} & \Psi(\mathbf{d}') \\ \downarrow & & \downarrow \\ \widetilde{\Psi}(\mathbf{d}) & \xrightarrow{\widetilde{\Psi}(\beta)} & \widetilde{\Psi}(\mathbf{d}'), \end{array}$$

in which the outer squares are identified with

$$\begin{array}{ccc} \Phi(\mathbf{c}) & \longrightarrow & \Phi(\mathbf{c}') \\ {\scriptstyle\widetilde{\Psi}(\gamma)}\downarrow & & \downarrow{\scriptstyle\widetilde{\Psi}(\gamma')} \\ \widetilde{\Psi}(\mathbf{d}) & \xrightarrow{\widetilde{\Psi}(\beta)} & \widetilde{\Psi}(\mathbf{d}'). \end{array}$$

However, this is given by the diagram (3.10).

3.2.6. As in Sect. 3.2.4, from the *isomorphism* (3.5), we obtain a natural transformation

$$\widetilde{\Psi}^! \circ (F_{horiz})^{\mathrm{op}} \to \Phi^!,$$

and hence a natural transformation

$$\widetilde{\Psi}^! \to \Psi^!, \tag{3.11}$$

which is also an isomorphism by the same logic.

We claim that for a 1-morphism $f : \mathbf{d} \to \mathbf{d}'$ in $\mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}$, given by a horizontal morphism $\alpha : \mathbf{d}' \to \mathbf{d}$, the diagram (3.8) represents the *inverse* of the natural transformation (3.11) evaluated on α.

Indeed, using the calculation of Ψ_1 in Sect. 3.1, we obtain $\Psi_1(f)$ can be written as a colimit of the category

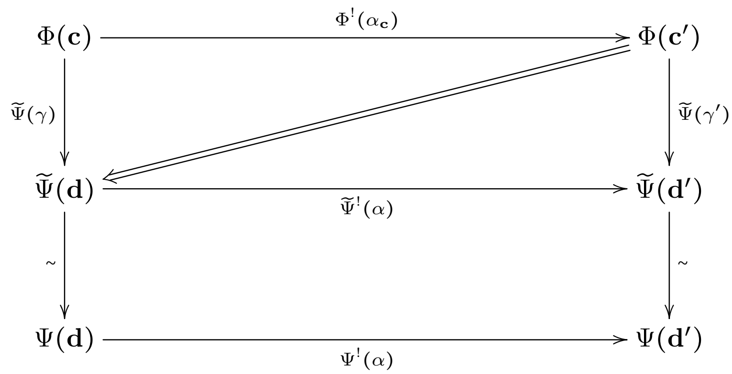
$$\begin{array}{ccc} \mathbf{c} & \xleftarrow{\alpha_{\mathbf{c}}} & \mathbf{c}' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ \mathbf{d} & \xleftarrow{\alpha} & \mathbf{d}' \end{array}$$

of the objects $(\Phi(\mathbf{c}) \xrightarrow{\Phi^!(\alpha_{\mathbf{c}})} \Phi(\mathbf{c}')) \in \mathrm{Seq}^{\mathrm{ext}}_1(\mathbb{S})$.

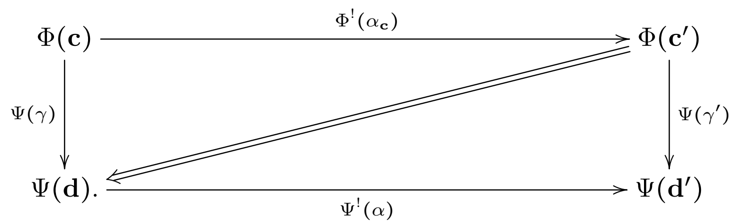
Thus, to show that (3.8) is represented by

$$\begin{array}{ccc} \Psi(\mathbf{d}) & \xrightarrow{\Psi^!(\alpha)} & \Psi(\mathbf{d}') \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ \widetilde{\Psi}(\mathbf{d}) & \xrightarrow{\widetilde{\Psi}^!(\alpha)} & \widetilde{\Psi}(\mathbf{d}'), \end{array}$$

we need to construct a compatible family of diagrams



where the outer square is



Now, the required family of diagrams is obtained by passing to left adjoints along the γ-arrows in the family of *commutative* diagrams

$$\begin{array}{ccc}
\widetilde{\Psi}(\mathbf{d}) & \xrightarrow{\widetilde{\Psi}^!(\alpha)} & \widetilde{\Psi}(\mathbf{d}') \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\Psi(\mathbf{d}) & \xrightarrow{\Psi^!(\alpha)} & \Psi(\mathbf{d}') \\
{\scriptstyle \Psi^!(\gamma)}\downarrow & & \downarrow{\scriptstyle \Psi^!(\gamma')} \\
\Phi(\mathbf{c}) & \xrightarrow[\Phi^!(\alpha_{\mathbf{c}})]{} & \Phi(\mathbf{c}')
\end{array}$$

4. Proof of Proposition 1.2.5: easy reduction steps

4.1. **Reductions for Proposition 1.2.5(a).**

4.1.1. We claim that the map (1.3) is an isomorphism if r is of the form $i \mapsto i+k$ for $0 \le k \le n-m$ (in particular, it is an isomorphism for $m = 0$).

In fact, we claim that in this case, the map

$$\mathrm{LKE}_{F_n}(G \circ r^*_{\mathbf{C}}) \to \mathrm{LKE}_{F_m}(G) \circ r^*_{\mathbf{D}}$$

is an isomorphism for any functor G out of $'\mathbf{Grid}_m^{\geq \mathrm{dgnl}}(\mathbf{C})^{adm}_{vert;horiz}$.

Note that if the latter statement holds for $r : [m] \to [n]$ and $q : [l] \to [m]$, then it holds for the composition $r \circ q : [l] \to [n]$. This reduces the assertion to the case of r being a map $[n-1] \to [n]$, which is either $i \mapsto i$ or $i \mapsto i+1$.

Fix an object $\underline{\mathbf{d}}_n \in {}'\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{D})^{adm}_{vert;horiz}$. We need to show that the map

$$\underset{\underline{\mathbf{c}}_n, F_n(\underline{\mathbf{c}}_n) \to \underline{\mathbf{d}}_n}{\mathrm{colim}} G(r^*_{\mathbf{C}}(\underline{\mathbf{c}}_n)) \to \underset{\underline{\mathbf{c}}_m, F_m(\underline{\mathbf{c}}_m) \to r^*_{\mathbf{D}}(\underline{\mathbf{d}}_n)}{\mathrm{colim}} G(\underline{\mathbf{c}}_m)$$

is an isomorphism.

We will show that in each of the above cases, the functor of index categories, i.e.,

$$\{\underline{\mathbf{c}}_n, F_n(\underline{\mathbf{c}}_n) \to \underline{\mathbf{d}}_n\} \to \{\underline{\mathbf{c}}_m, F_m(\underline{\mathbf{c}}_m) \to r^*_{\mathbf{D}}(\underline{\mathbf{d}}_n)\}, \quad \underline{\mathbf{c}}_n \mapsto r^*_{\mathbf{C}}(\underline{\mathbf{c}}_n) \tag{4.1}$$

is cofinal. By symmetry, it is sufficient to consider the case of r being the map $i \mapsto i$.

The functor (4.1) is (obviously) a co-Cartesian fibration. Hence, it is enough to show that it has contractible fibers.

For a given

$$\underline{\mathbf{c}}_{n-1} \to r^*_{\mathbf{D}}(\underline{\mathbf{d}}),$$

the fiber over it is the category of diagrams

$$\{\mathbf{c}_{n,n} \to \mathbf{d}_{n,n}, \mathbf{c}_{n-1,n} \to \mathbf{c}_{n-1,n-1} \underset{\mathbf{d}_{n-1,n-1}}{\times} \mathbf{d}_{n-1,n} \underset{\mathbf{d}_{n,n}}{\times} \mathbf{c}_{n,n}\},$$

where both maps are in *adm*.

This category is a co-Cartesian fibration over the category of

$$\{\mathbf{c}_{n,n} \to \mathbf{d}_{n,n}\}.$$

We claim that this category is contractible. Indeed, this follows from Lemma 4.1.2 below.

So, it is enough to show that each fiber of this co-Cartesian fibration , i.e.,

$$\{\mathbf{c}_{n-1,n} \to \mathbf{c}_{n-1,n-1} \underset{\mathbf{d}_{n-1,n-1}}{\times} \mathbf{d}_{n-1,n} \underset{\mathbf{d}_{n,n}}{\times} \mathbf{c}_{n,n}\}$$

is contractible. However, this also follows from Lemma 4.1.2.

Lemma 4.1.2. *For a given* $\mathbf{d} \in \mathbf{D}$*, the category* $\mathbf{C}_{adm} \underset{\mathbf{D}_{adm}}{\times} (\mathbf{D}_{adm})_{/\mathbf{d}}$ *is contractible.*

Proof. We claim that $\mathbf{C}_{adm} \underset{\mathbf{D}_{adm}}{\times} (\mathbf{D}_{adm})_{/\mathbf{d}}$ is in fact cofiltered (i.e., its opposite category is filtered).

Indeed, the category $(\mathbf{D}_{adm})_{/\mathbf{d}}$ has products, and therefore is cofiltered. Note now that every object of $(\mathbf{D}_{adm})_{/\mathbf{d}}$ admits a map from an object in $\mathbf{C}_{adm} \underset{\mathbf{D}_{adm}}{\times} (\mathbf{D}_{adm})_{/\mathbf{d}}$, by Sect. 1.1.3.

Now, we have the following general assertion: let $\mathbf{E}' \to \mathbf{E}$ be a fully faithful embedding with $\mathbf{E}$ filtered. Assume that every object of $\mathbf{E}$ admits a morphism to an object of $\mathbf{E}'$. Then $\mathbf{E}'$ is also filtered (and its embedding into $\mathbf{E}$ is cofinal).

□

4.1.3. We will now show that it is enough to prove that (1.3) is an isomorphism for $m = 1$. Indeed, in order to show that

$$\mathrm{LKE}_{F_n}(\Phi_m \circ r^*_{\mathbf{C}}) \to \mathrm{LKE}_{F_m}(\Phi_m) \circ r^*_{\mathbf{D}}$$

is an isomorphism, it is enough to show that the induced natural transformation

$$q^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_m \circ r^*_{\mathbf{C}}) \to q^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_m}(\Phi_m) \circ r^*_{\mathbf{D}}$$

is an isomorphism for every $m \geq 1$ and $q : [1] \to [m]$ of the form

$$0 \mapsto i,\, 1 \mapsto i+1.$$

We have a commutative diagram

$$\begin{array}{ccc}
q^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_m \circ r^*_{\mathbf{C}}) & \longrightarrow & q^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_m}(\Phi_m) \circ r^*_{\mathbf{D}} \\
\uparrow & & \uparrow \\
\mathrm{LKE}_{F_n}(q^*_{\mathbb{S}} \circ \Phi_m \circ r^*_{\mathbf{C}}) & \longrightarrow & \mathrm{LKE}_{F_m}(q^*_{\mathbb{S}} \circ \Phi_m) \circ r^*_{\mathbf{D}} \\
\simeq \downarrow & & \downarrow \simeq \\
\mathrm{LKE}_{F_n}(\Phi_1 \circ q^*_{\mathbf{C}} \circ r^*_{\mathbf{C}}) & & \mathrm{LKE}_{F_m}(\Phi_1 \circ q^*_{\mathbf{C}}) \circ r^*_{\mathbf{D}} \\
= \downarrow & & \downarrow \\
\mathrm{LKE}_{F_n}(\Phi_1 \circ (r \circ q)^*_{\mathbf{C}}) & & \mathrm{LKE}_{F_m}(\Phi_1) \circ q^*_{\mathbf{D}} \circ r^*_{\mathbf{D}} \\
\mathrm{id} \downarrow & & \downarrow = \\
\mathrm{LKE}_{F_n}(\Phi_1 \circ (r \circ q)^*_{\mathbf{C}}) & \longrightarrow & \mathrm{LKE}_{F_1}(\Phi_1) \circ (r \circ q)^*_{\mathbf{D}}.
\end{array}$$

By assumption, the bottom horizontal arrow is an isomorphism. The second-from-the-bottom right vertical arrow is an isomorphism by Sect. 4.1.1. The upper left and upper right vertical arrows are isomorphisms by Proposition 2.3.2(b).

Hence, the top horizontal arrow is also an isomorphism, as required.

4.1.4. We will now show that it is enough to show that the map (1.3) is an isomorphism for $r : [1] \to [n]$ of the form $0 \mapsto 0$ and $1 \mapsto n$.

Indeed, given a map $r : [1] \to [n]$ decompose it as $p \circ q$, where $q : [1] \to [m]$ is of the form $0 \mapsto 0$ and $1 \mapsto m$ and p is of the form $i \mapsto i + k$ for $0 \leq k \leq n - m$.

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{LKE}_{F_n}(\Phi_1 \circ (p \circ q)^*_{\mathbf{C}}) & \longrightarrow & \mathrm{LKE}_{F_1}(\Phi_1) \circ (p \circ q)^*_{\mathbf{D}} \\
= \downarrow & & \downarrow = \\
\mathrm{LKE}_{F_n}(\Phi_1 \circ q^*_{\mathbf{C}} \circ p^*_{\mathbf{C}}) & & \mathrm{LKE}_{F_1}(\Phi_1) \circ q^*_{\mathbf{D}} \circ p^*_{\mathbf{D}} \\
\downarrow & & \downarrow \mathrm{id} \\
\mathrm{LKE}_{F_m}(\Phi_1 \circ q^*_{\mathbf{C}}) \circ p^*_{\mathbf{D}} & \longrightarrow & \mathrm{LKE}_{F_1}(\Phi_1) \circ q^*_{\mathbf{D}} \circ p^*_{\mathbf{D}}.
\end{array}$$

By assumption, the bottom horizontal map is an isomorphism. The lower left vertical is an isomorphism by Sect. 4.1.1. This implies that the top horizontal map is an isomorphism, as required.

4.1.5. To summarize, in order to prove Proposition 1.2.5(a) it remains to consider the following two cases:

(I) r is the map $[1] \to [n]$ given by $0 \mapsto 0$ and $1 \mapsto n$ with $n > 1$.

(II) r is the degeneracy map $[1] \to [0]$.

4.2. **Reductions for Proposition 1.2.5(b).**

4.2.1. First, we note that the map (1.4) is an isomorphism for $m = 0$, by Proposition 2.3.2(a).

4.2.2. We will show that it is sufficient it is sufficient to prove that the map (1.4) is an isomorphism for $m = 1$.

Indeed, in order to show that (1.4) is an isomorphism, it is sufficient to show that the induced natural transformation

$$q_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(r_{\mathbb{S}}^* \circ \Phi_n) \to q_{\mathbb{S}}^* \circ r_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_n)$$

is an isomorphism, for every $m \geq 1$ and $q : [1] \to [m]$ of the form

$$0 \mapsto i,\ 1 \mapsto i+1.$$

We have a commutative diagram

$$\begin{array}{ccc}
q_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(r_{\mathbb{S}}^* \circ \Phi_n) & \longrightarrow & q_{\mathbb{S}}^* \circ r_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_n) \\
\uparrow & & \\
\mathrm{LKE}_{F_n}(q_{\mathbb{S}}^* \circ r_{\mathbb{S}}^* \circ \Phi_n) & & \Big\downarrow = \\
= \Big\downarrow & & \\
\mathrm{LKE}_{F_n}((r \circ q)_{\mathbb{S}}^* \circ \Phi_n) & \longrightarrow & (r \circ q)_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_n),
\end{array}$$

where the upper left vertical arrow is an isomorphism by Proposition 2.3.2(b).

This establishes the announced reduction step.

4.2.3. We will now further reduce the verification of the fact that (1.4) is an isomorphism to the case when $r : [1] \to [n]$ sends $0 \mapsto 0$ and $1 \mapsto n$.

Given a map $r : [1] \to [n]$ we can factor it as

$$[1] \xrightarrow{p} [m] \xrightarrow{q} [n],$$

where q is of the form $i \mapsto i + k$ for $0 \leq k \leq n - m$, and p sends $0 \mapsto 0$ and $1 \mapsto m$.

We claim that if the map

$$\mathrm{LKE}_{F_n}(p_{\mathbb{S}}^* \circ \Phi_m) \to p_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_m)$$

is an isomorphism, then so is the map

$$\mathrm{LKE}_{F_n}(r_{\mathbb{S}}^* \circ \Phi_m) \to r_{\mathbb{S}}^* \circ \mathrm{LKE}_{F_n}(\Phi_m).$$

Indeed, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{LKE}_{F_n}(r^*_{\mathbb{S}} \circ \Phi_n) & \longrightarrow & r^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_n) \\
{\scriptstyle =}\uparrow & & \uparrow{\scriptstyle =} \\
\mathrm{LKE}_{F_n}(p^*_{\mathbb{S}} \circ q^*_{\mathbb{S}} \circ \Phi_n) & & p^*_{\mathbb{S}} \circ q^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_n) \\
{\scriptstyle \sim}\uparrow & & \uparrow \\
\mathrm{LKE}_{F_n}(p^*_{\mathbb{S}} \circ \Phi_m \circ q^*_{\mathbf{C}}) & \longrightarrow & p^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(q^*_{\mathbb{S}} \circ \Phi_n) \\
\downarrow & & \downarrow{\scriptstyle \sim} \\
\mathrm{LKE}_{F_m}(p^*_{\mathbb{S}} \circ \Phi_m) \circ q^*_{\mathbf{D}} & & p^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_m \circ q^*_{\mathbf{C}}) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\
\mathrm{LKE}_{F_m}(p^*_{\mathbb{S}} \circ \Phi_m) \circ q^*_{\mathbf{D}} & \longrightarrow & p^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_m}(\Phi_m) \circ q^*_{\mathbf{D}}
\end{array}$$

By assumption, the bottom horizontal map is an isomorphism, and we wish to deduce that so is the top horizontal map. We claim that all the vertical maps are isomorphisms.

The second-from-the-bottom left vertical map and the lower right vertical map are isomorphisms by Sect. 4.1.1. The second-from-the-top right vertical map is an isomorphism by Proposition 2.3.2(c).

5. End of the proof of Proposition 1.2.5

5.1. **Proof of Proposition 1.2.5(b); the main case.** We need to show that the map

$$\mathrm{LKE}_{F_n}(r^*_{\mathbb{S}} \circ \Phi_n) \to r^*_{\mathbb{S}} \circ \mathrm{LKE}_{F_n}(\Phi_n) \tag{5.1}$$

is an isomorphism for r being the map $[1] \to [n]$, $0 \mapsto 0$ and $1 \mapsto n$.

Fix an object $\underline{\mathbf{d}}_n \in {}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{D})^{adm}_{vert;horiz}$. We need to show that the map

$$\underset{\underline{\mathbf{c}}_n}{\mathrm{colim}}\, r^*_{\mathbb{S}} \circ \Phi_n(\underline{\mathbf{c}}_n) \to r^*_{\mathbb{S}}(\underset{\underline{\mathbf{c}}_n}{\mathrm{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \tag{5.2}$$

is an isomorphism, where $\underline{\mathbf{c}}_n$ runs over the category

$$({}'\mathbf{Grid}^{\geq \mathrm{dgnl}}_n(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}_n}.$$

5.1.1. We first calculate the colimit

$$\underset{\underline{\mathbf{c}}_n}{\mathrm{colim}}\, \Phi_n(\underline{\mathbf{c}}_n) \in \mathrm{Seq}_n(\mathbb{S}), \tag{5.3}$$

which is a string

$$\Psi(\mathbf{d}_{0,0}) \to \Psi(\mathbf{d}_{1,1}) \to ... \to \Psi(\mathbf{d}_{n,n}).$$

We claim that (5.3) is given by

$$\Psi(\mathbf{d}_{i-1,i-1}) \overset{\Psi(\beta_{i-1,i}) \circ \Psi^!(\alpha_{i,i-1})}{\longrightarrow} \Psi(\mathbf{d}_{i,i}) \tag{5.4}$$

$i = 1, ..., n$.

For $i = 1, ..., n$, let q denote the map $[1] \to [n]$, given by $0 \mapsto i-1$, $1 \mapsto i$. We will identify

$$q^*_{\mathbb{S}}(\underset{\underline{\mathbf{c}}_n}{\mathrm{colim}}\, \Phi_n(\underline{\mathbf{c}}_n))$$

with (5.4).

Note that by Proposition 2.3.2(b), we have

$$q_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \simeq \underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, q_{\mathbb{S}}^* \circ \Phi_n(\underline{\mathbf{c}}_n). \tag{5.5}$$

For $i = 1, ..., n$, let $\underline{\mathbf{d}}_n^i$ be the object of $'\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{D})_{vert;horiz}^{adm}$ given by

$$\begin{array}{ccc} \mathbf{d}_{i-1,i} & \longrightarrow & \mathbf{d}_{i-1,i-1} \\ \downarrow & & \\ \mathbf{d}_{i,i}. & & \end{array}$$

We have a natural restriction functor

$$r_{\mathbf{C}}^* : ('\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n} \to ('\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n^i}. \tag{5.6}$$

The functor

$$r_{\mathbb{S}}^* \circ \Phi_n : ('\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n} \to \mathrm{Seq}_1(\mathbb{S})$$

identifies with $\Phi_1 \circ r_{\mathbf{C}}^*$.

Now, it is easy to see that the functor (5.6) is a co-Cartesian fibration. Moreover, it follows from Lemma 4.1.2 that its fibers are contractible. Hence, the functor (5.6) is cofinal.

Therefore, the colimit in (5.5) identifies with

$$\underset{\underline{\mathbf{c}}_1}{\operatorname{colim}}\, \Phi_1(\underline{\mathbf{c}}_1),$$

where $\underline{\mathbf{c}}_1$ runs over the category $('\mathbf{Grid}_1^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n^i}$.

Now, the fact that the latter colimit identifies with (5.4) is the calculation carried out in Sect. 3.1.

5.1.2. Consider the index category in (5.2), i.e., $('\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n}$. It is a co-Cartesian fibration over

$$\underline{\gamma} := (\mathbf{c}_{0,0} \xrightarrow{\gamma_{0,0}} \mathbf{d}_{0,0}), ..., (\mathbf{c}_{n,n} \xrightarrow{\gamma_{n,n}} \mathbf{d}_{n,n}).$$

Hence, the map in (5.2) can be written as a composite of

$$\underset{\underline{\gamma}}{\operatorname{colim}}\, \underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, r_{\mathbb{S}}^* \circ \Phi_n(\underline{\mathbf{c}}_n) \to \underset{\underline{\gamma}}{\operatorname{colim}}\, r_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \tag{5.7}$$

and

$$\underset{\underline{\gamma}}{\operatorname{colim}}\, r_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \to r_{\mathbb{S}}^*(\underset{\underline{\gamma}}{\operatorname{colim}}\, \underset{\underline{\mathbf{c}}_n}{\operatorname{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)), \tag{5.8}$$

where now $\underline{\mathbf{c}}_n$ runs over the fiber category

$$(('\mathbf{Grid}_n^{\geq \mathrm{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n})_{\underline{\gamma}}.$$

We will show that each of the maps (5.7) and (5.8) is an isomorphism.

5.1.3. We start with the following observation:

For a given $\underline{\gamma}$, let $\underline{\mathbf{d}}'_n$ be the object of $'\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{D})^{adm}_{vert;horiz}$ with

$$\mathbf{d}'_{i,i} = \mathbf{c}_{i,i} \text{ and } \mathbf{d}'_{i,i+1} = \mathbf{c}_{i+1,i+1} \underset{\mathbf{d}_{i+1,i+1}}{\times} \mathbf{d}_{i,i+1} \underset{\mathbf{d}_{i,i}}{\times} \mathbf{c}_{i,i};$$

the other coordinates of $\underline{\mathbf{d}}'$ are uniquely determined by the condition that the inner squares should be Cartesian.

Consider the category $('\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}'_n}$. Note, however, that cofinal in this category is the full subcategory consisting of $\underline{\mathbf{c}}'_n \to \underline{\mathbf{d}}'_n$ with $\mathbf{c}'_{i,i} = \mathbf{c}_{i,i}$ for $i = 0, ..., n$. And note that this subcategory identifies tautologically with the fiber category $(('\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}_n})_{\underline{\gamma}}$ appearing in the colimits (5.7) and (5.8).

Hence, we need to show that the maps

$$\underset{\underline{\gamma}}{\text{colim}}\, \underset{\underline{\mathbf{c}}_n}{\text{colim}}\, r_{\mathbb{S}}^* \circ \Phi_n(\underline{\mathbf{c}}_n) \to \underset{\underline{\gamma}}{\text{colim}}\, r_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\text{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \tag{5.9}$$

and

$$\underset{\underline{\gamma}}{\text{colim}}\, r_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\text{colim}}\, \Phi_n(\underline{\mathbf{c}}'_n)) \to r_{\mathbb{S}}^*(\underset{\underline{\gamma}}{\text{colim}}\, \underset{\underline{\mathbf{c}}_n}{\text{colim}}\, \Phi_n(\underline{\mathbf{c}}'_n)), \tag{5.10}$$

are isomorphisms, where $\underline{\mathbf{c}}'_n$ runs over the category $('\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}'_n}$.

5.1.4. We begin by showing that the map (5.10) is an isomorphism.

According to Sect. 5.1.1, the colimit $\underset{\underline{\mathbf{c}}_n}{\text{colim}}\, \Phi_n(\underline{\mathbf{c}}_n))$, which is a string

$$\Psi(\mathbf{c}_{0,0}) \to \Psi(\mathbf{c}_{1,1}) \to ... \to \Psi(\mathbf{c}_{n,n}),$$

is given by

$$\Psi(\mathbf{d}'_{i-1,i-1}) \overset{\Psi(\beta'_{i-1,i}) \circ \Psi^!(\alpha'_{i,i-1})}{\longrightarrow} \Psi(\mathbf{d}'_{i,i}). \tag{5.11}$$

Now, as in Sect. 3.1.4, by the Beck-Chevalley conditions, for each $i = 1, ..., n$, the composition $\Psi(\beta'_{i-1,i}) \circ \Psi^!(\alpha'_{i,i-1})$ identifies with

$$\Psi^!(\gamma_i) \circ \Psi(\beta_{i-1,i}) \circ \Psi^!(\alpha_{i,i-1}) \circ \Psi(\gamma_{i-1}).$$

Hence, the fact that (5.10) is an isomorphism follows from Corollary 2.3.7(b). Here, the corresponding categories

$$(\mathbf{c}_{i,i} \overset{\gamma_{i,i}}{\to} \mathbf{d}_{i,i}) = (\mathbf{C}_{adm})_{/\mathbf{d}_{i,i}}$$

are contractible by Lemma 4.1.2.

5.1.5. We will now show that the map (5.9) is an isomorphism. In fact, we claim that the map

$$\underset{\underline{\mathbf{c}}_n}{\text{colim}}\, r_{\mathbb{S}}^* \circ \Phi_n(\underline{\mathbf{c}}_n) \to r_{\mathbb{S}}^*(\underset{\underline{\mathbf{c}}_n}{\text{colim}}\, \Phi_n(\underline{\mathbf{c}}_n)) \tag{5.12}$$

is already an isomorphism.

Note, however, that the map (5.12) is the map (5.2) for $\underline{\mathbf{d}}_n$ replaced by $\underline{\mathbf{d}}'_n$. I.e., we have reduced the original problem to the case when $\mathbf{d}_{i,i} \in \mathbf{C}$.

We note that in this case the category $('\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{C})^{adm}_{vert;horiz})_{/\underline{\mathbf{d}}_n}$ identifies with the product

$$(\mathbf{C}_{adm})_{/\mathbf{d}_{0,1}} \times ... \times (\mathbf{C}_{adm})_{/\mathbf{d}_{i-1,i}} \times .... \times (\mathbf{C}_{adm})_{/\mathbf{d}_{n-1,n}}.$$

For every $\underline{\mathbf{c}} \in ('\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{C})_{vert;horiz}^{adm})_{/\underline{\mathbf{d}}_n}$, the object $r_{\mathbb{S}}^* \circ \Phi_n(\underline{\mathbf{c}}_n)$, which is a map

$$\Psi(\mathbf{d}_{0,0}) \to \Psi(\mathbf{d}_{n,n})$$

equals the composite

$$(\Psi(\beta_{n-1,n}) \circ \Psi(\gamma_{n-1,n}) \circ \Psi^!(\gamma_{n-1,n}) \circ \Psi^!(\alpha_{n,n-1})) \circ ... \circ (\Psi(\beta_{0,1}) \circ \Psi(\gamma_{0,1}) \circ \Psi^!(\gamma_{0,1}) \circ \Psi^!(\alpha_{1,0})).$$

Hence, as in Sect. 3.1.3, the left-hand side in (5.12) identifies with

$$(\Psi(\beta_{n-1,n}) \circ \Psi^!(\alpha_{n,n-1})) \circ ... \circ (\Psi(\beta_{0,1}) \circ \Psi^!(\alpha_{1,0})). \tag{5.13}$$

Now, according to Sect. 5.1.1, the right-hand side in (5.12) also identifies with (5.13). By unwinding the constructions, it is easy to see that the map in (5.12) corresponds to the identity endomorphism on (5.13) in terms of the above identifications.

5.2. **Proof of Proposition 1.2.5(a): the degeneracy map.** To finish the proof of Proposition 1.2.5(a), we need to treat the cases specified in Sect. 4.1.5. In this subsection we will consider the degeneracy map $[1] \to [0]$.

5.2.1. Fix an object $\mathbf{d} \in \mathbf{D} = '\mathbf{Grid}_0^{\geq \text{dgnl}}(\mathbf{C})_{vert;horiz}^{adm}$. We need to show that the map

$$\underset{\mathbf{c},\gamma:\mathbf{c}\to\mathbf{d}}{\text{colim}} \Phi_1(r_{\mathbf{C}}^*(\mathbf{c})) \to \underset{\underline{\mathbf{c}}_1,\underline{\mathbf{c}}_1\to r_{\mathbf{D}}^*(\mathbf{d})}{\text{colim}} \Phi_1(\underline{\mathbf{c}}_1)$$

is an isomorphism.

5.2.2. We compose the above map with the isomorphism

$$\underset{\underline{\mathbf{c}}_1,\underline{\mathbf{c}}_1\to r_{\mathbf{D}}^*(\mathbf{d})}{\text{colim}} \Phi_1(\underline{\mathbf{c}}_1) \simeq \text{id}_{\Psi(\mathbf{d})}$$

of Sect. 3.1.

So, we need to show that the map

$$\underset{\mathbf{c},\gamma:\mathbf{c}\to\mathbf{d}}{\text{colim}} \Phi_1(r_{\mathbf{C}}^*(\mathbf{c})) \simeq \underset{\mathbf{c},\gamma:\mathbf{c}\to\mathbf{d}}{\text{colim}} \text{id}_{\Phi(\mathbf{c})} \to \text{id}_{\Psi(\mathbf{d})}$$

is an isomorphism.

However, this follows from Proposition 2.3.4.

5.3. **Proof of Proposition 1.2.5(a): the main case.** The case we now need to consider is that of r being the map $[1] \to [n]$, $n > 1$, with $0 \mapsto 0$ and $1 \mapsto n$.

5.3.1. Fix an object $\underline{\mathbf{d}}_n \in '\mathbf{Grid}_n^{\geq \text{dgnl}}(\mathbf{D})_{vert;horiz}^{adm}$. We need to show that the map

$$\underset{\underline{\mathbf{c}}_n,F_n(\underline{\mathbf{c}}_n)\to\underline{\mathbf{d}}_n}{\text{colim}} \Phi_1(r_{\mathbf{C}}^*(\underline{\mathbf{c}}_n)) \to \underset{\underline{\mathbf{c}}_1,F_1(\underline{\mathbf{c}}_1)\to r_{\mathbf{D}}^*(\underline{\mathbf{d}}_n)}{\text{colim}} \Phi_1(\underline{\mathbf{c}}_1) \tag{5.14}$$

is an isomorphism.

Consider the maps

$$\alpha_{n,0} : \mathbf{d}_{n,0} \to \mathbf{d}_{0,0} \text{ and } \beta_{0,n} : \mathbf{d}_{n,0} \to \mathbf{d}_{n,n}.$$

By Sect. 3.1, the right-hand side in (5.14), which is a 1-morphism $\Psi(\mathbf{d}_{0,0}) \to \Psi(\mathbf{d}_{n,n})$, identifies with

$$\Psi(\beta_{0,n}) \circ \Psi^!(\alpha_{n,0}).$$

By Proposition 2.3.4, the left-hand side in (5.14) identifies with

$$\underset{\underline{\mathbf{c}}_n,F_n(\underline{\mathbf{c}}_n)\to\underline{\mathbf{d}}_n}{\text{colim}} \Psi(\beta_{0,n}) \circ \Psi(\gamma_{n,0}) \circ \Psi^!(\gamma_{n,0}) \circ \Psi^!(\alpha_{n,0}).$$

5.3.2. Thus, it suffices to show that the map

$$\underset{\underline{\mathbf{c}}_n, F_n(\underline{\mathbf{c}}_n)\to \underline{\mathbf{d}}_n}{\operatorname{colim}} \Psi(\gamma_{n,0}) \circ \Psi^!(\gamma_{n,0}) \to \mathrm{id}_{\Psi(\mathbf{d}_{n,0})}$$

is an isomorphism.

However, this follows by induction from Condition $(*)$ in Sect. 1.1.6.

6. Functors obtained by horizontal extension

In this section we prove the second extension result in this chapter. A typical situation that it applies to is when we start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and we want to extend it to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch\ \&\ proper}}_{\mathrm{sch;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

6.1. **Set up for the horizontal extension.** In this subsection we formulate the main result of this section, Theorem 6.1.5.

6.1.1. Let $(\mathbf{C}, vert, horiz, adm)$ and $(\mathbf{D}, vert, horiz, adm)$ both be as in [Chapter V.1, Sect. 1.1.1], and let $F : \mathbf{C} \to \mathbf{D}$ be a functor that preserves the corresponding classes of 1-morphisms, i.e., that it gives rise to well-defined functors

$$F_{vert} : \mathbf{C}_{vert} \to \mathbf{D}_{vert},\ F_{horiz} : \mathbf{C}_{horiz} \to \mathbf{D}_{horiz} \text{ and } F_{adm} : \mathbf{C}_{adm} \to \mathbf{D}_{adm}.$$

Furthermore, suppose that F takes Cartesian squares in [Chapter V.1, Diagram (1.1)] to Cartesian squares. Hence, F induces a functor

$$F^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}.$$

6.1.2. Now, suppose we have a functor

$$\Phi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathbb{S},$$

where $\mathbb{S}$ is a $(\infty,2)$-category, and let

$$\Phi^! := \Phi^{adm}_{vert;horiz}|_{(\mathbf{C}_{horiz})^{\mathrm{op}}} : (\mathbf{C}_{horiz})^{\mathrm{op}} \to \mathbb{S}^{1\text{-Cat}}.$$

Our interest in this section the *right Kan extension* of $\Phi^{adm}_{vert;horiz}$ under $F^{adm}_{vert;horiz}$.

By definition, such a right Kan extension is a functor (if it exists)

$$\Psi^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S},$$

universal with respect to the property of being endowed with a natural transformation

$$\Psi^{adm}_{vert;horiz} \circ F^{adm}_{vert;horiz} \to \Phi^{adm}_{vert;horiz}.$$

6.1.3. We make the following assumptions on $\mathbb{S}$:

- The $(\infty,1)$-category $\mathbb{S}^{1\text{-Cat}}$ admits limits;
- For every $s \in \mathbb{S}$ there exists and object $[1]\otimes s \in \mathbb{S}$ equipped with a functorial identification
$$\mathrm{Maps}([1]\otimes s, s') \simeq \mathrm{Maps}([1], \mathbf{Maps}_{\mathbb{S}}(s,s')).$$
respectively.

Note that in this case the conclusion of Lemma 2.2.2(b) is applicable to $\mathbb{S}$.

6.1.4. We do not intend to develop the general theory of right Kan extensions in the 2-categorical context. However, we will prove the following result:

Theorem 6.1.5. *Assume that for any* $\mathbf{c} \in \mathbf{C}$ *the functor* F *induces an equivalencea*

$$(\mathbf{C}_{vert})_{/\mathbf{c}} \to (\mathbf{D}_{vert})_{/F(\mathbf{c})} \text{ and } (\mathbf{C}_{adm})_{/\mathbf{c}} \to (\mathbf{D}_{adm})_{/F(\mathbf{c})}.$$

Then the right Kan extension

$$\mathrm{RKE}_{F^{adm}_{vert;horiz}}(\Phi^{adm}_{vert;horiz}) : \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz} \to \mathbb{S}$$

exists and the natural maps

$$\mathrm{RKE}_{F^{adm}_{vert;horiz}}(\Phi^{adm}_{vert;horiz})|_{\mathrm{Corr}(\mathbf{D})_{vert;horiz}} \to \mathrm{RKE}_{F_{vert;horiz}}(\Phi_{vert;horiz}) \tag{6.1}$$

and

$$\mathrm{RKE}_{F_{vert;horiz}}(\Phi_{vert;horiz})|_{(\mathbf{D}_{horiz})^{\mathrm{op}}} \to \mathrm{RKE}_{(F_{horiz})^{\mathrm{op}}}(\Phi^!) \tag{6.2}$$

are isomorphisms.

The rest of this section is devoted to the proof of Theorem 6.1.5.

6.2. **Proof of Theorem 6.1.5: the easy case.** As a warm-up, we shall first prove the easy case of Theorem 6.1.5, namely, when adm = isom. In this case, the assertion of the theorem amounts to the isomorphism (6.2).

6.2.1. Fix an object $\mathbf{d} \in \mathbf{D}$. We need to show that the map

$$\lim_{\mathbf{c} \in (\mathrm{Corr}(\mathbf{C})_{vert;horiz})_{\mathbf{d}/}} \Phi(\mathbf{c}) \to \lim_{\mathbf{c} \in ((\mathbf{C}_{horiz})_{/\mathbf{d}})^{\mathrm{op}}} \Phi(\mathbf{c})$$

is an isomorphism.

We claim that the functor of index categories, i.e.,

$$(\mathbf{C}_{horiz})_{/\mathbf{d}} \to ((\mathrm{Corr}(\mathbf{C})_{vert;horiz})_{\mathbf{d}/})^{\mathrm{op}}$$

is cofinal.

6.2.2. We claim that the above functor admits a left adjoint. Note that the objects of the category $(\mathrm{Corr}(\mathbf{C})_{vert;horiz})_{\mathbf{d}/}$ are diagrams

$$\begin{array}{ccc} \widetilde{\mathbf{d}} & \xrightarrow{\alpha} & \mathbf{d} \\ \beta \big\downarrow & & \\ F(\mathbf{c}) & & \end{array}$$

with $\mathbf{c} \in \mathbf{C}$, $\alpha \in horiz$ and $\beta \in vert$.

However, the condition of the theorem implies that the vertical arrow

$$\widetilde{\mathbf{d}} \xrightarrow{\beta} F(\mathbf{c})$$

is of the form

$$F(\widetilde{\mathbf{c}}) \to F(\mathbf{c})$$

for a canonically defined $\widetilde{\mathbf{c}} \xrightarrow{\beta'} \mathbf{c}$ in $\mathbf{C}$. So, the above diagram has the form

$$\begin{array}{ccc} F(\widetilde{\mathbf{c}}) & \xrightarrow{\alpha} & \mathbf{d} \\ F(\beta') \big\downarrow & & \\ F(\mathbf{c}). & & \end{array}$$

The left adjoint in question sends such a diagram to

$$F(\widetilde{\mathbf{c}}) \to \mathbf{d}.$$

6.3. **Proof of Theorem 6.1.5: the principle.** We do not intend to tackle the general theory of right Kan extensions in $(\infty,2)$-categories. However, it turns out that under the condition of the theorem, the 2-categorical right Kan extension amounts to the 1-categorical one.

6.3.1. We consider the following general paradigm. Let

$$F : \mathbb{T}_1 \to \mathbb{T}_2$$

be a functor between $(\infty,2)$-categories, and let

$$\Phi : \mathbb{T}_1 \to \mathbb{S}$$

be another functor, where $\mathbb{S}$ satisfies the assumptions of Sect. 6.1.3.

6.3.2. We now make the following assumption on the functor F. For $t_2 \in \mathbb{T}_2$ let $\mathbf{I}$ denote the index category, whose objects are pairs

$$(t_1' \in \mathbb{T}_1, g : t_2 \to F(t_1')),$$

and whose morphisms are *commutative diagrams* (i.e., we only allow invertible 2-morphisms).

For an object $t_1 \in \mathbb{T}_1$, we have an $\mathbf{I}^{\mathrm{op}}$-diagram of categories

$$(t_1' \in \mathbb{T}_1, g : t_2 \to F(t_1')) \mapsto \mathrm{Maps}_{\mathbb{T}_1}(t_1', t_1).$$

We have a naturally defined functor

$$\underset{(t_1',g)\in\mathbf{I}^{\mathrm{op}}}{\mathrm{colim}}\ \mathrm{Maps}_{\mathbb{T}_1}(t_1', t_1) \to \mathrm{Maps}_{\mathbb{T}_2}(t_2, F(t_1)), \quad (t_1' \overset{f}{\to} t_1) \mapsto F(f) \circ g, \tag{6.3}$$

where the colmit is taken in 1-Cat.

We claim:

Lemma 6.3.3. *Suppose that the functor* (6.3) *is an equivalence. Then*

$$\mathrm{RKE}_F(\Phi) : \mathbb{T}_2 \to \mathbb{S}$$

exists and the canonical map

$$\mathrm{RKE}_F(\Phi)|_{\mathbb{T}_2^{1\text{-Cat}}} \to \mathrm{RKE}_{F|_{\mathbb{T}_1^{1\text{-Cat}}}}(\Phi|_{\mathbb{T}_1^{1\text{-Cat}}})$$

is an isomorphism.

Remark 6.3.4. Intuitively, the lemma says that 2-functoriality of

$$\mathrm{RKE}_{F|_{\mathbb{T}_1^{1\text{-Cat}}}}(\Phi|_{\mathbb{T}_1^{1\text{-Cat}}}) : \mathbb{T}_2^{1\text{-Cat}} \to \mathbb{S}^{1\text{-Cat}}$$

is already built in, because 2-morphisms between arrows

$$t_2 \to F(t_1)$$

all come from 2-morphisms in $\mathbb{T}_1$, and thus are encoded by the 2-functoriality of Φ.

6.4. **Proof of Theorem 6.1.5: the general case.** We will prove that (6.1) by applying Lemma 6.3.3 to our functor

$$F^{adm}_{vert;horiz} : \mathrm{Corr}(\mathbf{C})^{adm}_{vert;horiz} \to \mathrm{Corr}(\mathbf{D})^{adm}_{vert;horiz}.$$

6.4.1. Fix an object $\mathbf{d} \in \mathbf{D}$ and $\mathbf{c} \in \mathbf{C}$. Consider the corresponding category $\mathbf{I}$. First we note that as in Sect. 6.2, cofinal in $\mathbf{I}^{\mathrm{op}}$ is its full subcategory $\mathbf{I}'^{\mathrm{op}}$, consisting of horizontal morphisms $F(\mathbf{c}') \to \mathbf{D}$.

6.4.2. Thus, we need to show that the functor

$$\underset{\mathbf{c}', F(\mathbf{c}')\to\mathbf{d}}{\text{colim}} \text{Maps}_{\text{Corr}(\mathbf{C})^{adm}_{vert;horiz}}(\mathbf{c}',\mathbf{c}) \to \text{Maps}_{\text{Corr}(\mathbf{D})^{adm}_{vert;horiz}}(\mathbf{d}, F(\mathbf{c}))$$

is an equivalence.

However, we claim that the above functor admits an explicit inverse: it sends

$$\begin{array}{ccc} \mathbf{d}' & \longrightarrow & \mathbf{d} \\ {\scriptstyle\beta_{\mathbf{d}}}\downarrow & & \\ F(\mathbf{c}) & & \end{array}$$

to the object $(F(\mathbf{c}') \to \mathbf{d}) \in \mathbf{I}'$ and

$$\begin{array}{ccc} \mathbf{c}' & \longrightarrow & \mathbf{c}' \\ {\scriptstyle\beta_{\mathbf{c}}}\downarrow & & \\ \mathbf{c}, & & \end{array}$$

where $\mathbf{d}' = F(\mathbf{c}')$, $\beta_{\mathbf{D}} = F(\beta_{\mathbf{c}})$.