

**STABILITY OF THE ULA PROPERTY UNDER TAKING PERVERSE
COHOMOLOGIES**

1. We fix a smooth morphism $\pi : X \rightarrow S$, where S is also smooth of dimension n . We shall say that an object $\mathcal{M} \in D(X)$ is LA with respect to π if for every $\mathcal{N} \in D(S)$, the natural map

$$(1) \quad \mathcal{M} \otimes^* \pi^*(\mathcal{N})[-n] \rightarrow \mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})[n]$$

is an isomorphism.

2. Let us be given a function f on S , which is smooth as a morphism $S \rightarrow \mathbb{A}^1$. Consider the Cartesian squares

$$\begin{array}{ccccc} X' & \xrightarrow{i_X} & X & \xleftarrow{j_X} & \overset{\circ}{X} \\ \pi' \downarrow & & \downarrow \pi & & \downarrow \overset{\circ}{\pi} \\ S' & \xrightarrow{i_S} & S & \xleftarrow{j_S} & \overset{\circ}{S} \\ f' \downarrow & & \downarrow f & & \downarrow \overset{\circ}{f} \\ 0 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{A}^1 - 0 \end{array}$$

It is clear that if $\mathcal{M} \in D(X)$ is LA with respect to π , then the natural map

$$(2) \quad i_X^*(\mathcal{M})[-1] \rightarrow i_X^!(\mathcal{M})[1]$$

is an isomorphism and the resulting object in $D(X')$ is LA with respect to π' .

3. Consider the functors of unipotent vanishing and nearby cycles

$$\Phi_S : D(S) \rightarrow D(S'), \quad \Psi_S : D(\overset{\circ}{S}) \rightarrow D(S'), \quad \Phi_X : D(X) \rightarrow D(X'), \quad \Psi_X : D(\overset{\circ}{X}) \rightarrow D(X').$$

We normalize them so that they are t-exact for the perverse t-structure.

By a slight abuse of notation we will denote by the same symbol $\Psi_S : D(S) \rightarrow D(S')$ (resp., $\Psi_X : D(X) \rightarrow D(X')$) the composite $\Psi_S \circ j_S^*$ (resp., $\Psi_X \circ j_X^*$).

Lemma A. *Let M be LA with respect to π . Then for any $\mathcal{N} \in D(S)$, the natural map*

$$(3) \quad \Psi_X(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \rightarrow i_X^!(\mathcal{M}) \overset{\circ}{\otimes} (\pi')^!(\Psi_S(\mathcal{N}))$$

is an isomorphism.

Proof. We will use Beilinson's definition of unipotent nearby cycles:

$$\Psi_S(\mathcal{N}) \simeq \operatorname{colim}_n i_S^! \circ (j_S)_!(j_S^!(\mathcal{N}) \overset{\circ}{\otimes} f^!(\mathcal{L}_n))[1],$$

$$\Psi_X(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \simeq \operatorname{colim}_n i_X^! \circ (j_X)_!(j_X^!(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \overset{\circ}{\otimes} \overset{\circ}{\pi}^! \circ f^!(\mathcal{L}_n))[1],$$

where \mathcal{L}_n is the standard length n Jordan block local system on $\mathbb{A}^1 - 0$, normalized so that \mathcal{L}_1 is the dualizing complex.

The map (3) comes from the canonical map

$$(j_X)_!(j_X^!(\mathcal{M} \otimes \pi^!(\mathcal{N})) \overset{\circ}{\otimes} \overset{\circ}{\pi}^! \circ \overset{\circ}{f}^!(\mathcal{L}_n)) \rightarrow \mathcal{M} \overset{\circ}{\otimes} (j_X)_!(j_X^!(\pi^!(\mathcal{N})) \overset{\circ}{\otimes} \overset{\circ}{\pi}^! \circ \overset{\circ}{f}^!(\mathcal{L}_n)) \simeq \\ \simeq \mathcal{M} \overset{\circ}{\otimes} \pi^! \left((j_S)_!(j_S^!(\mathcal{N}) \overset{\circ}{\otimes} \overset{\circ}{f}^!(\mathcal{L}_n)) \right),$$

where the second arrow is an isomorphism due to the fact that π is smooth. We claim that the first arrow is also an isomorphism. I.e., we need to show that

$$i_X^* \left(\mathcal{M} \overset{\circ}{\otimes} (j_X)_!(j_X^!(\pi^!(\mathcal{N})) \overset{\circ}{\otimes} \overset{\circ}{\pi}^! \circ \overset{\circ}{f}^!(\mathcal{L}_n)) \right) \simeq i_X^* \left(\mathcal{M} \overset{\circ}{\otimes} (j_X)_! \circ \overset{\circ}{\pi}^!(j_S^!(\mathcal{N}) \overset{\circ}{\otimes} \overset{\circ}{f}^!(\mathcal{L}_n)) \right) = 0.$$

We claim that for any $\overset{\circ}{\mathcal{N}} \in D(\overset{\circ}{S})$, we have

$$(4) \quad i_X^* \left(\mathcal{M} \overset{\circ}{\otimes} (j_X)_! \circ \overset{\circ}{\pi}^!(\overset{\circ}{\mathcal{N}}) \right) = 0.$$

Indeed, using the fact that π is smooth, we have

$$(j_X)_! \circ \overset{\circ}{\pi}^!(\overset{\circ}{\mathcal{N}}) \simeq \pi^! \circ (j_S)_!(\overset{\circ}{\mathcal{N}}),$$

and using the fact that \mathcal{M} is LA with respect to π , we therefore have

$$\mathcal{M} \overset{\circ}{\otimes} (j_X)_! \circ \overset{\circ}{\pi}^!(\overset{\circ}{\mathcal{N}})[2n] \simeq \mathcal{M} \overset{\circ}{\otimes} \pi^* \circ (j_S)_!(\overset{\circ}{\mathcal{N}}) \simeq \mathcal{M} \overset{\circ}{\otimes} (j_X)_! \circ \overset{\circ}{\pi}^*(\overset{\circ}{\mathcal{N}}) \simeq (j_X)_!(\mathcal{M} \overset{\circ}{\otimes} \overset{\circ}{\pi}^*(\overset{\circ}{\mathcal{N}})),$$

which implies (4). □

Taking \mathcal{N} to be the dualizing sheaf we obtain:

Corollary B. *Let \mathcal{M} be LA with respect to π . Then the canonical map*

$$(5) \quad \Psi_X(\mathcal{M}) \rightarrow i_X^!(\mathcal{M})[1]$$

is an isomorphism.

Using the distinguished triangles

$$\Phi_X(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \rightarrow \Psi_X(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \rightarrow i_X^!(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N}))[1]$$

and

$$i_X^!(\mathcal{M}) \overset{\circ}{\otimes} (\pi')^!(\Phi_S(\mathcal{N})) \rightarrow i_X^!(\mathcal{M}) \overset{\circ}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) \rightarrow i_X^!(\mathcal{M}) \overset{\circ}{\otimes} (\pi')^!(i_S^!(\mathcal{N}))[1],$$

we obtain:

Corollary C. *Let \mathcal{M} be LA with respect to π . Then for any $\mathcal{N} \in D(S)$ the canonical map*

$$(6) \quad \Phi_X(\mathcal{M} \overset{\circ}{\otimes} \pi^!(\mathcal{N})) \rightarrow i_X^!(\mathcal{M}) \overset{\circ}{\otimes} (\pi')^!(\Phi_S(\mathcal{N}))$$

is an isomorphism.

4. Conversely, let $\mathcal{M} \in D(X)$ be such that:

- (i) $\mathcal{M}|_{\overset{\circ}{X}} \in D(\overset{\circ}{X})$ is LA with respect to $\overset{\circ}{\pi}$;
- (ii) $\Psi_X(\mathcal{M}) \in D(X')$ is LA with respect to π' ;
- (iii) For any $\mathcal{N} \in D(S)$, the canonical map (3) is an isomorphism.

Proposition D. *Under the above circumstances \mathcal{M} is LA with respect to π .*

Proof. First, applying assumption (iii) and taking \mathcal{N} to be the dualizing, we obtain that the map (5) is an isomorphism. Hence, by Verdier duality, so is the map

$$(7) \quad i_X^*(\mathcal{M})[-1] \rightarrow \Psi_X(\mathcal{M}).$$

Note that the composite

$$i_X^*(\mathcal{M})[-1] \rightarrow \Psi_X(\mathcal{M}) \rightarrow i_X^!(\mathcal{M})[1]$$

is the map (2). Hence, the latter is also an isomorphism.

We need to show that for any $\mathcal{N} \in D(S)$, the map (1) map is an isomorphism. By assumption (i), it is such over $\overset{\circ}{X}$. So, it suffices to show that the map

$$\Phi_X(\mathcal{M} \otimes^* \pi^*(\mathcal{N}))[-n] \rightarrow \Phi_X(\mathcal{M} \otimes^! \pi^!(\mathcal{N}))[n]$$

is an isomorphism.

We have a commutative diagram¹

$$\begin{array}{ccc} \Phi_X(\mathcal{M} \otimes^* \pi^*(\mathcal{N}))[-n] & \longrightarrow & \Phi_X(\mathcal{M} \otimes^! \pi^!(\mathcal{N}))[n] \\ \uparrow & & \downarrow \\ i_X^*(\mathcal{M})[-1] \otimes^* (\pi')^*(\Phi_S(\mathcal{N}))[-(n-1)] & \longrightarrow & i_X^!(\mathcal{M})[1] \otimes^! (\pi')^!(\Phi_S(\mathcal{N}))[n-1], \end{array}$$

where the left vertical map is obtained by Verdier duality from the map (6), and the lower horizontal arrow is obtained by tensoring the canonical map (2) by the map (1) for π' . I.e., it identifies with the map

$$\Psi_X(\mathcal{M}) \otimes^* (\pi')^*(\Phi_S(\mathcal{N}))[-(n-1)] \rightarrow \Psi_X(\mathcal{M}) \otimes^! (\pi')^!(\Phi_S(\mathcal{N}))[n-1].$$

Now, assumption (iii) implies that the right vertical map is an isomorphism. The left vertical map is an isomorphism by duality. The bottom horizontal map is an isomorphism by assumption (ii). Hence, the top horizontal map is an isomorphism, as required. \square

5. Assume for a moment that $\mathcal{M} \in D(X)$ is perverse.

Lemma E. *If \mathcal{M} is LA with respect to π , then each of the functors*

$$D(S) \rightarrow D(X)$$

appearing in (1) is t-exact.

Proof. Indeed, the left-hand side is left t-exact (for any \mathcal{M}), and the right-hand side is right t-exact (for any \mathcal{M}). \square

Theorem F. *Let $\mathcal{M} \in D(X)$ be LA with respect to π . Then so are all of its perverse cohomologies.*

¹NB: I have not yet checked the commutativity!

Proof. We need to show that for a given integer k , the k -th perverse cohomology $h^k(\mathcal{M})$ is LA over X . By the generic ULA-ness, $h^k(\mathcal{M})$ is LA over a dense open subset of S . Therefore, we can assume that the locus where it is *potentially not* LA is cut out by a function f that is a smooth morphism $S \rightarrow \mathbb{A}^1$.

We will show that $h^k(\mathcal{M})$ satisfies the assumptions of Proposition D.

Assumption (i) is our assumption assumption that $h^k(\mathcal{M})$ is LA when restricted to $\overset{\circ}{X}$. Assumption (ii) follows from the t-exactness of the functor Ψ_X and induction on the dimension on S .

Hence, it remains to show that for any $\mathcal{N} \in D(S)$, the map

$$(8) \quad \Psi_X(h^k(\mathcal{M}) \overset{\circlearrowleft}{\otimes} \pi^!(\mathcal{N})) \rightarrow i_X^!(h^k(\mathcal{M})) \overset{\circlearrowleft}{\otimes} (\pi')^!(\Psi_S(\mathcal{N}))$$

is an isomorphism. In doing so we can assume that \mathcal{N} is *perverse*.

Let us apply the shift by $[n]$ to the map in (8). We obtain a map

$$(9) \quad \Psi_X(h^k(\mathcal{M}) \overset{\circlearrowleft}{\otimes} \pi^!(\mathcal{N})[n]) \rightarrow i_X^!(h^k(\mathcal{M}))[1] \overset{\circlearrowleft}{\otimes} (\pi')^!(\Psi_S(\mathcal{N}))[n-1].$$

We claim that the two sides in (9) are obtained by taking the k -th perverse cohomology of the two sides in the map

$$(10) \quad \Psi_X(\mathcal{M} \overset{\circlearrowleft}{\otimes} \pi^!(\mathcal{N}))[n] \rightarrow i_X^!(\mathcal{M}) \overset{\circlearrowleft}{\otimes} (\pi')^!(\Psi_S(\mathcal{N}))[n]$$

obtained by applying the shift $[n]$ to the map (3), which is an isomorphism by Lemma A.

Indeed, for the left-hand side, this follows from the fact that the functor Ψ_X is t-exact and Lemma E.

For the right-hand side, again using Lemma E, it suffices to show that $i_X^!(h^k(\mathcal{M}))[1]$ is a perverse sheaf that is LA with respect to π' .

Since the functor Φ_X is t-exact, we obtain that $\Phi_X(h^k(\mathcal{M})) = 0$. Hence, the map

$$\Psi_X(h^k(\mathcal{M})) \rightarrow i_X^!(h^k(\mathcal{M}))[1]$$

is an isomorphism, and the required assertion follows from the LA-ness of $h^k(\Psi_X(\mathcal{M}))$. \square

6. Finally, we will prove:

Theorem F. *Let $\mathcal{M} \in D(X)$ be a perverse sheaf that LA with respect to π . Then so are all of its subquotients.*

Proof. It is enough to show that if $\mathcal{M}' \subset \mathcal{M}$ is a sub-object, then \mathcal{M}' is LA with respect to π . By the generic ULA-ness, \mathcal{M}' is LA over a dense open subset of S . Therefore, we can assume that the locus where it is *potentially not* LA is cut out by a function f that is a smooth morphism $S \rightarrow \mathbb{A}^1$.

We will show that $h^k(\mathcal{M})$ satisfies the assumptions of Proposition D.

Assumption (i) is our assumption assumption that \mathcal{M}' is LA when restricted to $\overset{\circ}{X}$. Assumption (ii) follows from the t-exactness of the functor Ψ_X and induction on the dimension on S .

Hence, it remains to show that for any $\mathcal{N} \in D(S)$, the map

$$(11) \quad \Psi_X(\mathcal{M}' \overset{!}{\otimes} \pi^!(\mathcal{N})) \rightarrow i_X^!(\mathcal{M}') \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N}))$$

is an isomorphism. In doing so we can assume that \mathcal{N} is *perverse*.

Consider the commutative diagram

$$(12) \quad \begin{array}{ccc} \Psi_X(\mathcal{M}' \overset{!}{\otimes} \pi^!(\mathcal{N})[n]) & \longrightarrow & i_X^!(\mathcal{M}') [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) [n-1] \\ \downarrow & & \downarrow \\ \Psi_X(\mathcal{M} \overset{!}{\otimes} \pi^!(\mathcal{N})[n]) & \longrightarrow & i_X^!(\mathcal{M}) [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) [n-1]. \end{array}$$

We claim that the upper row in (12) consists of perverse sheaves, and the vertical arrows are *injections*. For the left column this follows from Lemma E and the t-exactness of the functor Ψ_X .

Denote $\mathcal{M}'' := \mathcal{M}/\mathcal{M}'$. To prove the assertion about the right column in (12), using Lemma E, it is enough to show that both $i_X^!(\mathcal{M}') [1]$ and $i_X^!(\mathcal{M}'') [1]$ are perverse sheaves that are LA with respect to π' . However,

$$\Phi(\mathcal{M}) = 0 \Rightarrow \Phi(\mathcal{M}') = 0 \text{ and } \Phi(\mathcal{M}'') = 0,$$

hence the maps

$$\Psi_X(\mathcal{M}') \rightarrow i_X^!(\mathcal{M}') [1] \text{ and } \Psi_X(\mathcal{M}'') \rightarrow i_X^!(\mathcal{M}'') [1]$$

are isomorphisms. Hence, the assertion follows by induction on the dimension of S .

The bottom horizontal arrow in (12) is an isomorphism by Lemma A. Hence, we obtain that the top horizontal arrow is an *injection* of perverse sheaves. In particular,

$$\text{length}(\Psi_X(\mathcal{M}' \overset{!}{\otimes} \pi^!(\mathcal{N})[n])) \leq \text{length}(i_X^!(\mathcal{M}') [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) [n-1]).$$

Hence, in order to prove that it is an isomorphism, it remains to establish an inequality in the opposite direction.

We note that the same argument shows that

$$(13) \quad \text{length}(\Psi_X(\mathcal{M}'' \overset{!}{\otimes} \pi^!(\mathcal{N})[n])) \geq \text{length}(i_X^!(\mathcal{M}'') [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) [n-1]).$$

We apply (13) to the short exact sequence

$$0 \rightarrow \mathbb{D}(\mathcal{M}'') \rightarrow \mathbb{D}(\mathcal{M}) \rightarrow \mathbb{D}(\mathcal{M}') \rightarrow 0$$

and $\mathbb{D}(\mathcal{N})$. We obtain the inequality

$$(14) \quad \text{length}(\Psi_X(\mathbb{D}(\mathcal{M}') \overset{!}{\otimes} \pi^!(\mathbb{D}(\mathcal{N})) [n])) \geq \text{length}(i_X^!(\mathbb{D}(\mathcal{M}')) [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathbb{D}(\mathcal{N}))) [n-1]).$$

Taking the Verdier duals in (14) we obtain:

$$(15) \quad \text{length}(\Psi_X(\mathcal{M}' \overset{*}{\otimes} \pi^*\mathcal{N})[-n]) \geq \text{length}(i_X^*(\mathcal{M}') [-1] \overset{*}{\otimes} (\pi')^*(\Psi_S(\mathcal{N})) [-(n-1)]).$$

However, we have

$$\mathcal{M}' \overset{!}{\otimes} \pi^!(\mathcal{N})[n] \simeq \mathcal{M}' \overset{*}{\otimes} \pi^*\mathcal{N}[-n]$$

(since the restriction \mathcal{M}' to $\overset{\circ}{X}$ is LA with respect to $\overset{\circ}{\pi}$), and

$$i_X^!(\mathcal{M}') [1] \overset{!}{\otimes} (\pi')^!(\Psi_S(\mathcal{N})) [n-1] \simeq i_X^*(\mathcal{M}') [-1] \overset{*}{\otimes} (\pi')^*(\Psi_S(\mathcal{N})) [-(n-1)]$$

since $i_X^!(\mathcal{M}') [1] \simeq \Psi(\mathcal{M}') \simeq i_X^*(\mathcal{M}') [-1]$ and the latter is LA with respect to π' .

Thus, each of the sides in (15) equals the corresponding side of (13).

□