CHAPTER A.2. STRAIGHTENING AND YONEDA FOR $(\infty, 2)$-CATEGORIES

Contents

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Introduction

0.1. **What is done in this Chapter?** The goal of this Chapter is to construct the 2-categorical Yoneda embedding

$$\mathrm{Yon}_{\mathbb{S}}:\mathbb{S}\hookrightarrow \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat}),\quad \mathbb{S}\in 2\text{-Cat}, \tag{0.1}$$

which will, in turn, be needed for the proof of the Adjunction Theorem in [Chapter A.3].

As in the case of $(\infty,1)$-categories, in the present 2-categorical context, a natural approach to the construction of the functor $\mathrm{Yon}_{\mathbb{S}}$ is via the straightening/unstraightening procedure.

The latter is an equivalence between the $(\infty,2)$-category of functors $\mathbb{S}^{1\text{-op}}\to\mathbf{1\text{-}Cat}$ and the $(\infty,2)$-category of *1-Cartesian* fibrations over $\mathbb{S}$.

0.1.1. Let us comment on the notion of 1-Cartesian fibration over a given $\mathbb{S}\in 2$-Cat.

The space of such will be a full subspace in $(\mathbf{2\text{-}Cat}_{/\mathbb{S}})^{\mathrm{Spc}}$, and it is singled out by certain explicit conditions; the actual definition is given in Sect. 1.2.1. The definition is rigged so that the datum of a 1-Cartesian fibration over $\mathbb{S}$ is equivalent to that of a functor $\mathbb{S}^{1\text{-op}}\to\mathbf{1\text{-}Cat}$.

As to the 2-categorical structure, there are actually two natural $(\infty,2)$-categories

$$(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\mathrm{strict}}\subset(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{2\text{-strict}},$$

one being a 1-full subcategory in the other.

In Sect. 1 we state the sub-main result of this Chapter, Corollary 1.2.6, that says that there is a canonical 'straightening/unstraightening' equivalence

$$(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{2\text{-strict}}\simeq \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat})_{\text{right-lax}}, \tag{0.2}$$

which induces an equivalence

$$(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\mathrm{strict}}\simeq \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat}).$$

0.1.2. Here is, however, a catch: the above straightening/unstraightening assertion (i.e., the equivalence (0.2)) is *too weak* to be amenable to a natural proof.

Namely, the equivalence (0.2) does not contain enough functoriality (the mechanics of how this happens can be seen by tracing through the proof of the main theorem of this Chapter, Theorem 1.1.8; see also Sect. 0.1.5 below).

0.1.3. To remedy this, we engage a more ambitious straightening/unstraightening procedure.

Namely, in Sect. 1 we introduce the notion of 2-Cartesian fibration (over a given $(\infty,2)$-category $\mathbb{S}$). Again, the space of such is a full subspace of $(\mathbf{2\text{-}Cat}_{/\mathbb{S}})^{\mathrm{Spc}}$, and it is singled out by certain explicit conditions specified in Sect. 1.1.1.

As in the case of 1-Cartesian fibrations, there two natural $(\infty,2)$-categories

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\mathrm{strict}}\subset(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{2\text{-strict}},$$

one being a 1-full subcategory in the other.

The 2-categorical straightening/unstraightening assertion, Theorem 1.1.8, which is the main result of this Chapter, says that there exists a canonical equivalence

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{2\text{-strict}}\simeq \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat})_{\text{right-lax}}, \tag{0.3}$$

which induces an equivalence

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\mathrm{strict}}\simeq \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat}).$$

0.1.4. The proof of Theorem 1.1.8 is spread over Sects. 2-4. Let us indicate its main steps.

In Sect. 2 we establish the particular case of the isomorphism (0.3), when $\mathbb{S}$ is the interval $[n]$. This is done by a combinatorial procedure, which essentially amounts to unwinding the definitions.

In Sect. 3 we realize 2-Cartesian fibrations over the Gray product $\mathbb{S}_1 \circledast \mathbb{S}_2$ as an explicit full subspace in $\text{2-Cat}_{/\mathbb{S}_1\times\mathbb{S}_2}$.

In Sect. 4 we use the results of the previous two sections to establish the isomorphism (0.3) *at the level of spaces underlying the* $(\infty,2)$*-categories on both sides*, in the case when $\mathbb{S} = [m]\circledast[n]$.

Using [Chapter A.1, Theorems 4.1.3 and 5.2.3] we deduce from this that the isomorphism (0.3) holds *at the level of spaces* for any $\mathbb{S} \in \text{2-Cat}$.

0.1.5. So, far, the same strategy would have worked if we worked with 1-Cartesian fibrations and **1-Cat** instead of **2-Cat** as a target.

However, now, in the 2-Cartesian context, we observe that the statement that we are trying to prove has enough functoriality, that it allows to formally deduce the equivalence (0.3) from just knowing it at the level of the underlying spaces.

0.2. What else is done in this Chapter?

0.2.1. As was mentioned before, our actual goal is to construct the Yoneda embedding (0.1) (and prove its fully faithfulness).

Having proved the 2-categorical straightening theorem in the earlier sections, the construction of the Yoneda embedding and the proof of its properties is carried out in Sect. 5.

0.2.2. In addition, this Chapter contains two sections in the Appendix.

In Sect. A, given $\mathbb{S} \in \text{2-Cat}$, we give an explicit description of the universal non-unital right-lax functor out of $\mathbb{S}$:

$$\mathbb{S} \overset{\iota_{\mathbb{S}}}{\dashrightarrow} \text{RLax}_{\text{non-untl}}(\mathbb{S}),$$

so that any non-unital right-lax functor $F : \mathbb{S} \dashrightarrow \mathbb{T}$ is obtained as

$$\widetilde{F} \circ \iota_{\mathbb{S}},$$

for a canonically defined *strict* functor $\widetilde{F} : \text{RLax}_{\text{non-untl}}(\mathbb{S}) \to \mathbb{T}$.

The explicit description of $\text{RLax}_{\text{non-untl}}(\mathbb{S})$ is used in Sect. 3.

0.2.3. In Sect. B we discuss the condition on a functor $\mathbb{S} \to \mathbb{T}$ between $(\infty,2)$-categories to be a *localization on 1-morphisms*. Informally, this means that $\mathbb{T}$ is obtained from $\mathbb{S}$ by inverting certain 2-morphisms.

This notion is used in the description of 2-Cartesian fibrations over Gray products, also in Sect. 3.

1. Straightening for $(\infty,2)$-categories

In this section we define the notion of a 2-Cartesian fibration of $(\infty,2)$-categories and formulate the main result in this Chapter: this is the *straightening theorem* that says that 2-Cartesian fibrations over a given $(\infty,2)$-category $\mathbb{S}$ are equivalent to functors $\mathbb{S}^{1\text{-op}} \to \mathbf{2\text{-}Cat}$.

1.1. **The notion of 2-Cartesian fibration.** In this subsection we will introduce the notion of 2-Cartesian fibration between $(\infty,2)$-categories.

When defining it, one should basically 'follow one's nose', keeping in mind that a 2-Cartesian fibration over $\mathbb{S}$ should be the same as a functor $\mathbb{S}^{1\text{-op}} \to \mathbf{2\text{-}Cat}$, while adapting the definition of Cartesian fibration in the context of $(\infty,1)$-categories.

1.1.1. Let $F : \mathbb{T} \to \mathbb{S}$ be a functor between $(\infty,2)$-categories. We shall say that a 1-morphism $t_0 \overset{\alpha}{\to} t_1$ is 2-Cartesian over $\mathbb{S}$ if for every $t \in \mathbb{T}$, the functor

$$\mathbf{Maps}_{\mathbb{T}}(t,t_0) \to \mathbf{Maps}_{\mathbb{T}}(t,t_1) \underset{\mathbf{Maps}_{\mathbb{S}}(F(t),F(t_1))}{\times} \mathbf{Maps}_{\mathbb{S}}(F(t),F(t_0)),$$

given by composition with α, is an equivalence of $(\infty,1)$-categories.

Definition 1.1.2. *We shall say that F is a 2-Cartesian fibration if the following conditions hold:*

(1) *For every $t \in \mathbb{T}$ and a 1-morphism $s' \overset{\beta}{\to} F(t)$ there exists a 2-Cartesian 1-morphism $t' \overset{\alpha}{\to} t$ with $F(\alpha) \simeq \beta$.*

(2) *For every $t', t \in \mathbb{T}$, the functor*

$$\mathbf{Maps}(t',t) \to \mathbf{Maps}(F(t'),F(t))$$

is a coCartesian fibration (of $(\infty,1)$-categories), and for any $\widetilde{t'} \to t'$ and $t \to \widetilde{t}$, the corresponding functors

$$\mathbf{Maps}(t',t) \to \mathbf{Maps}(\widetilde{t'},t) \text{ and } \mathbf{Maps}(t',t) \to \mathbf{Maps}(t',\widetilde{t}),$$

given by composition, send arrows that are coCartesian over $\mathbf{Maps}(F(t'),F(t))$ to arrows that are coCartesian over $\mathbf{Maps}(F(\widetilde{t'}),F(t))$ and $\mathbf{Maps}(F(t'),F(\widetilde{t}))$, respectivelly.

1.1.3. Let us assume that condition (1) above holds, and let us write down the second condition in more explicit terms.

Let $\alpha_{\mathbb{S}} : s' \to s$ be a 1-morphism in $\mathbb{S}$, and let t be an object of $\mathbb{T}$ that lies over s. Then condition (1) implies that there exists a canonically defined object $t' \in \mathbb{T}$ that lies over s' and a 1 -morphism

$$\alpha_{\mathbb{T}} : t' \to t$$

that covers $\alpha_{\mathbb{S}}$.

Suppose now that we are given a pair of 1-morphisms

$$\alpha^1_{\mathbb{S}}, \alpha^2_{\mathbb{S}} : s' \rightrightarrows s$$

and a 2-morphism $\alpha^1_{\mathbb{S}} \xrightarrow{\phi_{\mathbb{S}}} \alpha^2_{\mathbb{S}}$. Then the second condition says that there exists a 1-morphism

$$\beta : t^{1\prime} \to t^{2\prime}$$

and a 2-morphism

$$\alpha^1_{\mathbb{T}} \xrightarrow{\phi_{\mathbb{T}}} \alpha^2_{\mathbb{T}} \circ \beta,$$

with the following property: for any $\widetilde{t'}$ in the fiber of $\mathbb{T}$ over s', and a pair of morphisms

$$\gamma_1 : \widetilde{t'} \to t^{1\prime} \text{ and } \gamma_2 : \widetilde{t'} \to t^{2\prime},$$

composition with $\phi_{\mathbb{T}}$ defines an isomorphism from the space of 2-morphisms

$$\beta \circ \gamma_1 \to \gamma_2$$

to the space of 2-morphisms

$$\alpha^1_{\mathbb{T}} \circ \gamma_1 \to \alpha^2_{\mathbb{T}} \circ \gamma_2$$

covering $\phi_{\mathbb{S}}$.

Furthermore, the formation of β is compatible in the natural sense with compositions

$$(\alpha^1_{\mathbb{S}}, \alpha^2_{\mathbb{S}}) \mapsto (\widetilde{\alpha}_{\mathbb{S}} \circ \alpha^1_{\mathbb{S}}, \widetilde{\alpha}_{\mathbb{S}} \circ \alpha^2_{\mathbb{S}}), \quad \widetilde{\alpha}_{\mathbb{S}} : s \to \widetilde{s}$$

and

$$(\alpha^1_{\mathbb{S}}, \alpha^2_{\mathbb{S}}) \mapsto (\alpha^1_{\mathbb{S}} \circ \widetilde{\alpha}'_{\mathbb{S}}, \alpha^2_{\mathbb{S}} \circ \widetilde{\alpha}'_{\mathbb{S}}), \quad \widetilde{\alpha}'_{\mathbb{S}} : \widetilde{s}' \to s'.$$

1.1.4. Let $\mathbf{2\text{-}Cart}_{/\mathbb{S}} \subset \mathbf{2\text{-}Cat}_{/\mathbb{S}}$ denote the full subcategory spanned by 2-Cartesian fibrations.

Let $(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{1-strict}} \subset \mathbf{2\text{-}Cart}_{/\mathbb{S}}$ be the 1-full subcategory, where we allow as 1-morphisms those functors $\mathbb{T}_1 \to \mathbb{T}_2$ over $\mathbb{S}$ that send 1-morphisms in $\mathbb{T}_1$ that are 2-Cartesian over $\mathbb{S}$ to 1-morphisms in $\mathbb{T}_2$ that are 2-Cartesian over $\mathbb{S}$.

1.1.5. Let $(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{2-strict}} \subset \mathbf{2\text{-}Cart}_{/\mathbb{S}}$ be the 1-full subcategory, where we impose the following condition on 1-morphisms:

Given $F_1 : \mathbb{T}_1 \to \mathbb{S}$ and $F_2 : \mathbb{T}_2 \to \mathbb{S}$, we consider those functors $G : \mathbb{T}_1 \to \mathbb{T}_2$ over $\mathbb{S}$ such that the corresponding functors

$$\mathbf{Maps}_{\mathbb{T}_1}(t'_1, t) \to \mathbf{Maps}_{\mathbb{T}_2}(G(t'_1), G(t))$$

send arrows that are coCartesian over $\mathbf{Maps}_{\mathbb{S}}(F_1(t'_1), F_1(t))$ to arrows that are coCartesian over

$$\mathbf{Maps}_{\mathbb{S}}(F_2 \circ G(t'_1), F_2 \circ G(t)) \simeq \mathbf{Maps}_{\mathbb{S}}(F_1(t'_1), F_1(t)).$$

1.1.6. Let $(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{strict}} \subset \mathbf{2\text{-}Cart}_{/\mathbb{S}}$ be the 1-full subcategory equal to

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{1-strict}} \cap (\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{2-strict}}.$$

Denote also

$$\text{2-Cart}_{/\mathbb{S}} := (\mathbf{2\text{-}Cart}_{/\mathbb{S}})^{\text{1-Cat}}, \ (\text{2-Cart}_{/\mathbb{S}})_{\text{2-strict}} := ((\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{2-strict}})^{\text{1-Cat}}$$

and

$$(\text{2-Cart}_{/\mathbb{S}})_{\text{strict}} := ((\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{strict}})^{\text{1-Cat}}.$$

1.1.7. Our goal in the next few sections will be to prove:

Theorem-Construction 1.1.8.
(a) *There exists a canonical equivalence*

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{2-strict}} \simeq \text{Funct}(\mathbb{S}^{\text{1-op}}, \mathbf{2\text{-}Cat})_{\text{right-lax}},$$

functorial in $\mathbb{S}$.
(b) *Under the equivalence of point (a), the 1-full subcategories*

$$(\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{strict}} \subset (\mathbf{2\text{-}Cart}_{/\mathbb{S}})_{\text{2-strict}} \ \textit{and} \ \text{Funct}(\mathbb{S}^{\text{1-op}}, \mathbf{2\text{-}Cat}) \subset \text{Funct}(\mathbb{S}^{\text{1-op}}, \mathbf{2\text{-}Cat})_{\text{right-lax}}$$

correspond to one another.

1.2. **The notion of 1-Cartesian fibration.** According to Theorem 1.1.8, 2-Cartesian fibrations over $\mathbb{S}$ correspond to functors $\mathbb{S}^{\text{1-op}} \to \mathbf{2\text{-}Cat}$.

In this subsection we will define the notion of *1-Cartesian fibration*. Those will form a full subcategory among 2-Cartesian fibrations, and they will correspond to functors $\mathbb{S}^{\text{1-op}} \to \mathbf{1\text{-}Cat}$.

1.2.1. Let $F : \mathbb{T} \to \mathbb{S}$ be a functor between $(\infty,2)$-categories.

Definition 1.2.2. *We shall say that F is a 1-Cartesian fibration if the following conditions hold:*

(1) *The induced functor*

$$\mathbb{T}^{\text{1-Cat}} \to \mathbb{S}^{\text{1-Cat}}$$

is a Cartesian fibration;

(2) *For every $t', t \in \mathbb{T}$, the functor*

$$\mathbf{Maps}_{\mathbb{T}}(t',t) \to \mathbf{Maps}_{\mathbb{S}}(F(t'),F(t))$$

is a coCartesian fibration in spaces.

If $F : \mathbb{T} \to \mathbb{S}$ is a 1-Cartesian fibration, we will say that a 1-morphism in $\mathbb{T}$ is *Cartesian* if the corresponding morphism in $\mathbb{T}^{\text{1-Cat}}$ is Cartesian over $\mathbb{S}^{\text{1-Cat}}$.

1.2.3. Let $\mathbf{1\text{-}Cart}_{/\mathbb{S}}$ denote the full subcategory of $\mathbf{2\text{-}Cat}_{/\mathbb{S}}$ formed by 1-Cartesian fibrations.

We let $(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}$ be the 1-full subcategory of $\mathbf{1\text{-}Cart}_{/\mathbb{S}}$, where we restrict morphisms to those functors $\mathbb{T}_1 \to \mathbb{T}_2$ over $\mathbb{S}$, such that send arrows in $(\mathbb{T}_1)^{\text{1-Cat}}$ Cartesian over $\mathbb{S}^{\text{1-Cat}}$ to arrows in $(\mathbb{T}_2)^{\text{1-Cat}}$ with the same property.

Denote also

$$\text{1-Cart}_{/\mathbb{S}} := (\mathbf{1\text{-}Cart}_{/\mathbb{S}})^{\text{1-Cat}} \text{ and } (\text{1-Cart}_{/\mathbb{S}})_{\text{strict}} := ((\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}})^{\text{1-Cat}}.$$

1.2.4. We claim:

Lemma 1.2.5.

(a) *For a functor $F : \mathbb{T} \to \mathbb{S}$ the following conditions are equivalent:*

(i) *F is a 1-Cartesian fibration;*

(ii) *F is a 2-Cartesian fibration and the fiber of F over every $s \in \mathbb{S}$ is an $(\infty,1)$-category.*

(b) *If $\mathbb{T} \to \mathbb{S}$ is a 1-Cartesian fibration, then a 1-morphism in $\mathbb{T}$ is 2-Cartesian over $\mathbb{S}$ if and only if it is Cartesian.*

Hence, combining this lemma with Theorem 1.1.8 and [Chapter A.1, Proposition 6.3.2], we obtain:

Corollary 1.2.6.

(a) *There exists a canonical equivalence*

$$\mathbf{1\text{-}Cart}_{/\mathbb{S}} \simeq \text{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat})_{\text{right-lax}},$$

functorial in $\mathbb{S} \in$ 2-Cat.

(b) *Under the equivalence of point (a), the 1-full subcategories*

$$(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}} \subset \mathbf{1\text{-}Cart}_{/\mathbb{S}} \text{ and } \text{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat}) \subset \text{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat})_{\text{right-lax}}$$

correspond to one another.

1.2.7. Let $\mathbb{S}=\mathbf{S}$ be an $(\infty,1)$-category. We note:

Lemma 1.2.8. *A functor $\mathbb{T}\to\mathbf{S}$ is a 1-Cartesian fibration if and only if the following conditions hold:*

- $\mathbb{T}=\mathbf{T}\in 1\text{-Cat}$*;*
- *The resulting functor $\mathbf{T}\to\mathbf{S}$ is a Cartesian fibration.*

I.e., we obtain that in the above case, the notion of 1-Cartesian fibration reduces to the usual notion of 1-Cartesian fibration on $(\infty,1)$-categries.

It will follow from the construction that the equivalence of Corollary 1.2.6(b) in this case, i.e.,

$$(\mathbf{1\text{-}Cart}_{/\mathbf{S}})_{\text{strict}}\simeq \text{Funct}(\mathbf{S}^{1\text{-op}},\mathbf{1\text{-}Cat}),$$

induces at the level of the underlying $(\infty,1)$-categories, i.e.,

$$(\text{Cart}_{/\mathbf{S}})_{\text{strict}} \text{ and } \text{Maps}(\mathbf{S}^{1\text{-op}},\mathbf{1\text{-}Cat}),$$

the equivalence of [Chapter I.1, Sect. 1.4.5].

Remark 1.2.9. Let us take $\mathbb{S}=\mathbf{S}=[n]^{\text{op}}$. We obtain that in this case the equivalence of Corollary 1.2.6(a) at the level of the underlying $(\infty,1)$-categories amounts to the *definition* of the $(\infty,1)$-category $\text{Seq}_n^{\text{ext}}(\mathbf{1\text{-}Cat})$, see [Chapter A.1, Sect. 5.3].

The idea of the proof of Theorem 1.1.8 is to give a similar interpretation of $\text{Seq}_n^{\text{ext}}(\mathbf{2\text{-}Cat})$, namely, as 2-Cartesian fibrations over $[n]^{\text{op}}$. This will be furnished by Theorem 2.0.1.

The rest of the proof of Theorem 1.1.8 will amount to bootstrapping the statement for any $\mathbb{S}\in 2\text{-Cat}$ from the case $\mathbb{S}=[n]^{\text{op}}$, and lifting the 1-categorical equivalence to a 2-categorical one.

1.3. **Variants.** In this subsection we will introduce the companion notions of *2-coCartesian* and *1-coCartesian* fibrations over an $(\infty,2)$-category.

1.3.1. We shall say that a functor between $(\infty,2)$-categories $\mathbb{T}\to\mathbb{S}$ is *2-coCartesian* (resp., *1-coCartesian*) fibration if the corresponding functor $\mathbb{T}^{1\&2\text{-op}}\to\mathbb{S}^{1\&2\text{-op}}$ is a 2-Cartesian (resp., 1-Cartesian) fibration.

Similarly, we introduce the 1-full subcategories

$$(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\text{strict}}\subset(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\text{2-strict}}\subset\mathbf{2\text{-}coCart}_{/\mathbb{S}}\subset\mathbf{2\text{-}Cat}_{/\mathbb{S}}$$

and

$$(\mathbf{1\text{-}coCart}_{/\mathbb{S}})_{\text{strict}}\subset(\mathbf{1\text{-}coCart}_{/\mathbb{S}})_{\text{2-strict}}\subset\mathbf{1\text{-}coCart}_{/\mathbb{S}}\subset\mathbf{2\text{-}Cat}_{/\mathbb{S}}.$$

1.3.2. From Theorem 1.1.8 we obtain:

Corollary 1.3.3.

(a) *There exists a canonical equivalence*

$$(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}}\simeq\text{Funct}(\mathbb{S},\mathbf{2\text{-}Cat})_{\text{left-lax}},$$

functorial in $\mathbb{S}$.

(b) *Under the equivalence of point (a) the 1-full subcategories*

$$(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\text{strict}}\subset(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}} \text{ and } \text{Funct}(\mathbb{S},\mathbf{2\text{-}Cat})\subset\text{Funct}(\mathbb{S},\mathbf{2\text{-}Cat})_{\text{left-lax}}$$

correspond to one another.

Similarly, from Corollary 1.2.6 we obtain:

Corollary 1.3.4.

(a) *There exists a canonical equivalence*

$$\mathbf{1\text{-}coCart}_{/\mathbb{S}} \simeq \mathrm{Funct}(\mathbb{S}, \mathbf{1\text{-}Cat})_{\text{left-lax}},$$

functorial in $\mathbb{S} \in 2\text{-Cat}$.

(b) *Under the equivalence of point (a), the 1-full subcategories*

$$(\mathbf{1\text{-}coCart}_{/\mathbb{S}})_{\text{strict}} \subset \mathbf{1\text{-}coCart}_{/\mathbb{S}} \text{ and } \mathrm{Funct}(\mathbb{S}, \mathbf{1\text{-}Cat}) \subset \mathrm{Funct}(\mathbb{S}, \mathbf{1\text{-}Cat})_{\text{left-lax}}$$

correspond to one another.

1.3.5. We note that in addition to the notions of 2-Cartesian and 2-coCartesian (resp., 1-Cartesian and 1-coCartesian) fibration, there exist two more notions, induced by the involution $\mathbb{S} \mapsto \mathbb{S}^{2\text{-op}}$ on 2-Cat.

These notions correspond to functors from $\mathbb{S}^{1\text{-op}}$ and $\mathbb{S}^{2\text{-op}}$ with values in $\mathbf{2\text{-}Cat}$ and $\mathbf{1\text{-}Cat}$, respectively.

2. Straightening over intervals

In this section we will establish the following particular case of Theorem 1.1.8:

We will take the base $\mathbb{S}$ to be the interval $[n]$, and we will identify the $(\infty,1)$-categories underlying the $(\infty,2)$-categories appearing on the two sides in Theorem 1.1.8.

More precisely, our goal is to prove the following:

Theorem-Construction 2.0.1.

(a) *There exists a canonical equivalence of simplicial categories*

$$\mathrm{Seq}^{\mathrm{ext}}_{\bullet}(\mathbf{2\text{-}Cat}) \simeq (2\text{-}\mathrm{Cart}_{/[\bullet]^{\mathrm{op}}})_{2\text{-strict}}.$$

(b) *For an individual* n*, under the equivalence*

$$\mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbf{2\text{-}Cat}) \simeq (2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{2\text{-strict}},$$

the 1-full subcategories

$$\mathrm{Seq}^{\mathrm{ext}}_{n}(2\text{-}\mathrm{Cat}) \subset \mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbf{2\text{-}Cat}) \text{ and } (2\text{-}\mathrm{coCart}_{/[n]^{\mathrm{op}}})_{\text{strict}} \subset (2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{2\text{-strict}}$$

correspond to one another.

Remark 2.0.2. Note that since $[n]^{\mathrm{op}}$ is a 1-category, the inclusion

$$(2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{2\text{-strict}} \subset 2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}}$$

is an equivalence.

2.1. **The main construction.** We now proceed to defining the functor in one direction

$$2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}} \to \mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbf{2\text{-}Cat}).$$

The idea of the construction is pretty straightforward: we think of an object of $\mathrm{Seq}^{\mathrm{ext}}_{n}(\mathbf{2\text{-}Cat})$ as a string

$$\mathbb{T}^0 \to \mathbb{T}^1 \to \ldots \to \mathbb{T}^n$$

of $(\infty,2)$-categories, which we encode by means of a functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cart}_{/[n]^{\mathrm{op}}},$$

see [Chapter A.1, Sect. 6.1.3].

The value of this functor on $[m] \in \mathbf{\Delta}^{\mathrm{op}}$ is the category of strings $t_0^i \to t_1^i \to \ldots \to t_m^i$ in $\mathbb{T}^i$, where i varies along $[n]$. We interpret such strings as strings in the 'total' $(\infty,2)$-category over $[n]$ that project to a single object in $[n]$.

The total category in question is precisely the object $\mathbb{T} \in 2\text{-Cart}_{/[n]^{\mathrm{op}}}$ that we start from. We will now make turn this idea into an actual construction.

2.1.1. Given $(\mathbb{T} \to [n]^{\mathrm{op}}) \in 2\text{-Cart}_{/[n]^{\mathrm{op}}}$ we define an object

$$\mathbf{E}_{\bullet,n} \in \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cart}_{/[n]^{\mathrm{op}}})$$

as follows:

We let

$$\mathbf{E}_{m,n} \coloneqq \mathrm{Seq}_m^{\mathrm{ext}}(\mathbb{T}) \underset{\mathrm{Seq}_m^{\mathrm{ext}}([n]^{\mathrm{op}})}{\times} [n]^{\mathrm{op}},$$

where $[n]^{\mathrm{op}} \to \mathrm{Seq}_m^{\mathrm{ext}}([n]^{\mathrm{op}})$ is the functor

$$[n]^{\mathrm{op}} = \mathrm{Funct}(\{*\}, [n]^{\mathrm{op}}) \to \mathrm{Funct}([m], [n]^{\mathrm{op}}) = \mathrm{Seq}_m^{\mathrm{ext}}([n]^{\mathrm{op}}).$$

It is straightforward to check that $\mathbf{E}_{m,n}$, viewed as a category over $[n]^{\mathrm{op}}$, is a Cartesian fibration, and that the object $\mathbf{E}_{\bullet,n}$ thus constructed defines an object of $\mathrm{Seq}_n^{\mathrm{ext}}(\mathbf{2\text{-}Cat})$.

Furthermore, this construction is clearly functorial in $\mathbb{T}$, thereby giving rise to a functor

$$2\text{-Cart}_{/[n]^{\mathrm{op}}} \to \mathrm{Seq}_n^{\mathrm{ext}}(\mathbf{2\text{-}Cat}). \tag{2.1}$$

Furthermore, it is clear that the above functor sends the 1-full subcategory

$$(2\text{-Cart}_{/[n]^{\mathrm{op}}})_{\mathrm{strict}} \subset 2\text{-Cart}_{/[n]^{\mathrm{op}}}$$

to the 1-full subcategory

$$\mathrm{Seq}_n^{\mathrm{ext}}(2\text{-Cat}) \subset \mathrm{Seq}_n^{\mathrm{ext}}(\mathbf{2\text{-}Cat}).$$

Remark 2.1.2. Note that the construction presented above is a generalization of the construction in [Chapter A.1, Proposition 6.3.2]. The reason that we cannot finish the proof of Theorem 2.0.1 as easily as in the case of [Chapter A.1, Proposition 6.3.2] is that we do not yet know that for given $\mathbb{S}_0, \mathbb{S}_1 \in 2\text{-Cat}$, the category

$$2\text{-Cart}_{/[1]^{\mathrm{op}}} \underset{2\text{-Cat} \times 2\text{-Cat}}{\times} \{\mathbb{S}_0 \times \mathbb{S}_1\}$$

identifies with

$$\mathbf{Maps}_{\mathbf{2\text{-}Cat}}(\mathbb{S}_0, \mathbb{S}_1) \simeq \mathrm{Funct}(\mathbb{S}_0, \mathbb{S}_1)^{1\text{-Cat}}.$$

2.2. **Proof of Theorem 2.0.1: the inverse map.** We will define a functor

$$\mathrm{Seq}_n^{\mathrm{ext}}(\mathbf{2\text{-}Cat}) \to 2\text{-Cart}_{/[n]^{\mathrm{op}}} \tag{2.2}$$

inverse to (2.1).

We now want to recover the 'total' $(\infty,2)$-category $\mathbb{T}$ over $[n]$, i.e., for each m, we want to recover the corresponding category of strings

$$t_0 \to t_1 \to \ldots \to t_n,$$

while we know the category of strings that project to a single element in $[n]$.

We will recover all strings by a variant of the construction used in [Chapter A.1, Sect. 1.6] to define the unstraightening procedure for $(\infty,1)$-categories.

2.2.1. In order to define the functor (2.2), we will need the following combinatorial construction. Let $\mathrm{Tot}(\mathbf{\Delta})$ be the coCartesian fibration over $\mathbf{\Delta}$ corresponding to the tautological functor

$$\mathbf{\Delta} \to 1\text{-Cat}\,.$$

Note that $\mathrm{Tot}(\mathbf{\Delta})$ is an ordinary category, whose objects are pairs $([n]\in\mathbf{\Delta}, i\in[n])$, and such that the set of morphisms $([n_0],i_0)\to([n_1],i_1)$ is the set of morphisms $\phi:[n_0]\to[n_1]$ such that $\phi(i_0)=i_1$.

We let $p:\mathrm{Tot}(\mathbf{\Delta})\to\mathbf{\Delta}$ the tautological projection $([n],i)\mapsto[n]$. We let $\mathrm{Tot}(\mathbf{\Delta})_{[m]}$ the fiber of $\mathrm{Tot}(\mathbf{\Delta})$ over $[m]\in\mathbf{\Delta}$; tautologically $\mathrm{Tot}(\mathbf{\Delta})_{[m]}\simeq[m]$.

We note now that in addition to p, there is another canonically defined functor

$$q:\mathrm{Tot}(\mathbf{\Delta})\to\mathbf{\Delta}.$$

Namely, we set

$$q([n],i):=[i],\quad q(([n_0],i_0)\overset{\phi}{\to}([n_1],i_1))=([i_0]\overset{\phi|_{[i_0]}}{\longrightarrow}[i_1]).$$

In particular, restricting to $\mathrm{Tot}(\mathbf{\Delta})_{[m]}$, we obtain the functor

$$q_{[m]}:[m]\to\mathbf{\Delta},\quad i\mapsto[i].$$

2.2.2. Going back to the desired functor (2.2), let $\mathbf{E}_{\bullet,n}$ be an object of $\mathrm{Seq}_n^{\mathrm{ext}}(\mathbf{2\text{-}Cat})$, thought of as a functor

$$\mathbf{\Delta}^{\mathrm{op}}\to\mathrm{Cart}_{/[n]^{\mathrm{op}}}\,.$$

We can view[1] the data of $\mathbf{E}_{\bullet,n}$ as an $(\infty,1)$-category $\mathbf{E}^{\oint}$ over $\mathbf{\Delta}^{\mathrm{op}}\times[n]^{\mathrm{op}}$, such that:

- The composition $\mathbf{E}^{\oint}\to\mathbf{\Delta}^{\mathrm{op}}\times[n]^{\mathrm{op}}\to\mathbf{\Delta}^{\mathrm{op}}$ is a coCartesian fibration;
- The composition $\mathbf{E}^{\oint}\to\mathbf{\Delta}^{\mathrm{op}}\times[n]^{\mathrm{op}}\to[n]^{\mathrm{op}}$ is a Cartesian fibration;
- The functor $\mathbf{E}^{\oint}\to\mathbf{\Delta}^{\mathrm{op}}\times[n]^{\mathrm{op}}$, viewed as a functor between coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, belongs to $(\mathrm{coCart}_{/\mathbf{\Delta}^{\mathrm{op}}})_{\mathrm{strict}}$;
- The functor $\mathbf{E}^{\oint}\to\mathbf{\Delta}^{\mathrm{op}}\times[n]^{\mathrm{op}}$, viewed as a functor between Cartesian fibrations over $[n]^{\mathrm{op}}$, belongs to $(\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{\mathrm{strict}}$.

2.2.3. We construct the object $\mathbb{T}\in 2\text{-}\mathrm{Cart}_{/[n]^{\mathrm{op}}}$ corresponding to $\mathbf{E}_{\bullet,n}$ as follows. We define the category

$$\mathrm{Funct}([m]^{\mathrm{op}},\mathbb{T})_{\text{right-lax}}$$

(which will be the same as $\mathrm{Seq}_m^{\mathrm{ext}}(\mathbb{T})$, up to the involution rev) to be a certain full subcategory in the $(\infty,1)$-category of pairs (ϕ,Φ), where ϕ is a functor $[m]\to[n]$, and Φ is a lift of the functor

$$(\phi^{\mathrm{op}},(\mathrm{rev}\circ q_{[m]})^{\mathrm{op}}):[m]^{\mathrm{op}}\to[n]^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}$$

to a functor

$$[m]^{\mathrm{op}}\to\mathbf{E}^{\oint}\,.$$

[1]See the elementary [Chapter A.3, Proposition 2.1.3] for a general assertion to this effect.

2.2.4. We single out $\text{Funct}([m]^{\text{op}},\mathbb{T})_{\text{right-lax}}$ by imposing the following condition on objects.

Fix $i=1,...,m$. Consider a coCartesian lift in $\mathbf{E}^{\oint}$

$$\Phi(i)\to e'$$

of the 1-morphism

$$(\text{rev}\circ q_{[m]}(i)\to \text{rev}\circ q_{[m]}(i-1))\in \mathbf{\Delta}^{\text{op}}.$$

Consider a Cartesian lift in $\mathbf{E}^{\oint}$

$$e''\to \Phi(i-1)$$

of the 1-morphism

$$(\phi(i)\to\phi(i-1))\in [n]^{\text{op}}.$$

Note that by the last two properties of $\mathbf{E}^{\oint}$ listed in Sect. 2.2.2, we have a canonical map

$$e'\to e''.$$

We require that this map be an isomorphism.

2.2.5. Clearly, the assignment

$$m\mapsto \text{Funct}([m]^{\text{op}},\mathbb{T})_{\text{right-lax}}$$

extends to a functor $\mathbf{\Delta}^{\text{op}}\to\text{1-Cat}$.

We set

$$\text{Seq}^{\text{ext}}_{\bullet}(\mathbb{T}):=\text{Funct}([\bullet]^{\text{op}},\mathbb{T})_{\text{right-lax}}\circ\text{rev},$$

where rev is the reversal involution on $\mathbf{\Delta}^{\text{op}}$.

Using [Chapter A.1, Theorem 5.2.3(a)], we show:

Lemma 2.2.6. *The simplicial category* $\text{Seq}^{\text{ext}}_{\bullet}(\mathbb{T})$ *belongs to the essential image of the functor*

$$\text{Seq}^{\text{ext}}_{\bullet}:\text{2-Cat}\to\text{Funct}(\mathbf{\Delta}^{\text{op}},\text{1-Cat}).$$

Let $\mathbb{T}$ denote the resulting object of 2-Cat.

2.2.7. By construction, the simplicial category $\text{Seq}^{\text{ext}}_{\bullet}(\mathbb{T})$ maps to the simplicial category

$$m\mapsto\text{Funct}([m],[n]^{\text{op}}).$$

Hence, the $(\infty,2)$-category $\mathbb{T}$, constructed above, comes equipped with a functor

$$\mathbb{T}\to[n]^{\text{op}}.$$

It is a straightforward verification that the above functor $\mathbb{T}\to[n]^{\text{op}}$ is a 2-Cartesian fibration.

2.2.8. Thus, we have constructed a functor

$$\text{Seq}^{\text{ext}}_{n}(\mathbf{2\text{-}Cat})\to\text{2-Cart}_{/[n]^{\text{op}}}.$$

It is again a straightforward verification that this functor is inverse to (2.1).

3. Locally 2-Cartesian and 2-Cartesian fibrations over Gray products

As was mentioned before, the assertion of Theorem 1.1.8 will be deduced from that of Theorem 2.0.1 by a certain bootstrapping procedure.

However, in order to do so, we will need to enlarge the entities that appear in both the left-hand and the right-hand side. For the left-hand side, the corresponding notion is that of *locally 2-Cartesian fibration.*

3.1. **The notion of locally 2-Cartesian fibration.** The idea of the notion of locally 2-Cartesian fibration is the following: whereas 2-Cartesian fibrations over $\mathbb{S}$ correspond to functors

$$\mathbb{S}^{1\text{-op}} \to \mathbf{2\text{-}Cat},$$

locally 2-Cartesian fibrations correspond to *right-lax* functors

$$\mathbb{S}^{1\text{-op}} \dashrightarrow \mathbf{2\text{-}Cat}.$$

3.1.1. Let $F : \mathbb{T} \to \mathbb{S}$ be a functor between $(\infty,2)$-categories. We shall say that a 1-morphism α in $\mathbb{T}$ is locally 2-Cartesian over $\mathbb{S}$, if the resulting 1-morphism in $[1] \underset{F(\alpha),\mathbb{S}}{\times} \mathbb{T}$ is 2-Cartesian with respect to the projection

$$[1] \underset{F(\alpha),\mathbb{S}}{\times} \mathbb{T} \to [1].$$

Definition 3.1.2. *We shall say that F is a locally 2-Cartesian fibration if the following conditions hold:*

(1) *For every $t \in \mathbb{T}$ and a 1-morphism $s' \overset{\beta}{\to} F(t)$ there exists a locally 2-Cartesian 1-morphism $t' \overset{\alpha}{\to} t$ with $F(\alpha) \simeq \beta$.*

(2) *Condition (2) in Definition 1.1.2 holds.*

3.1.3. Note that if $F : \mathbb{T} \to \mathbb{S}$ is a a locally 2-Cartesian fibration, then for every 1-morphism $s_0 \to s_1$, the functor

$$[1] \underset{\mathbb{S}}{\times} \mathbb{T} \to [1]$$

is a 2-Cartesian fibration. In particular, by Theorem 2.0.1 and [Chapter A.1, Proposition 6.2.2], it gives rise to a well-defined functor

$$\mathbb{T}_{s_1} \to \mathbb{T}_{s_0}.$$

We will refer to it as the *pullback functor* along the given 1-morphism.

3.1.4. The next assertion follows from the definitions:

Lemma 3.1.5.

(a) *A functor $F : \mathbb{T} \to \mathbb{S}$ is a 2-Cartesian fibration if and only it is a locally 2-Cartesian fibration and the induced functor $\mathbb{T}^{1\text{-Cat}} \to \mathbb{S}^{1\text{-Cat}}$ is a Cartesian fibration of $(\infty,1)$-categories.*

(b) *If $F : \mathbb{T} \to \mathbb{S}$ is a 2-Cartesian fibration, then any 1-morphism in $\mathbb{T}$ that is locally 2-Cartesian over $\mathbb{S}$ is automatically 2-Cartesian.*

(c) *If $F : \mathbb{T} \to \mathbb{S}$ is a locally 2-Cartesian fibration, then a 1-morphism in $\mathbb{T}$ is locally 2-Cartesian over $\mathbb{S}$ if and only if the corresponding 1-morphism in $\mathbb{T}^{2\text{-ordn}}$ is is locally 2-Cartesian over $\mathbb{S}^{2\text{-ordn}}$.*

(d) *If $F : \mathbb{T} \to \mathbb{S}$ is a locally 2-Cartesian fibration, then it is 2-Cartesian if and only if the corresponding functor $\mathbb{T}^{2\text{-ordn}} \to \mathbb{S}^{2\text{-ordn}}$ is.*

3.1.6. Let $\text{2-Cart}^{\text{loc}}_{/\mathbb{S}}$ denote the the full subcategory of $\text{2-Cat}_{/\mathbb{S}}$ formed by locally 2-Cartesian fibrations in $(\infty,1)$-categories. Let

$$(\text{2-Cart}^{\text{loc}}_{/\mathbb{S}})_{1\text{-strict}} \supset (\text{2-Cart}^{\text{loc}}_{/\mathbb{S}})_{\text{strict}} \subset (\text{2-Cart}^{\text{loc}}_{/\mathbb{S}})_{2\text{-strict}}$$

be the 1-full subcategories, defined by the same conditions as in Sects. 1.1.4-1.1.6.

3.2. **Locally 2-Cartesian fibrations vs 2-Cartesian fibrations over** $\text{RLax}_{\text{non-untl}}(\mathbb{S})$**.** In this subsection we will formulate, and begin the proof of, the main assertion of this section, Theorem 3.2.2.

3.2.1. Let
$$\mathbb{S} \dashrightarrow \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}),$$
be the universal non-unital right-lax functor, see Sect. A.

We are going to prove:

Theorem-Construction 3.2.2. *There exists a canonical fully faithful embedding*
$$(2\text{-Cart}^{\text{loc}}_{/\mathbb{S}})^{\text{Spc}} \overset{\Phi}{\hookrightarrow} (2\text{-Cart}_{/\,\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})})^{\text{Spc}},$$
functorial in $\mathbb{S}$*, whose essential image consists of those 2-Cartesian fibrations for which the pullback functors along quasi-invertible arrows (see Sect. A.3) are equivalences.*

Remark 3.2.3. The above proposition is stated as an isomorphism of spaces. However, it will follow from the construction that this equivalence extends to one between the corresponding $(\infty,2)$-categories (both 2-strict and strict versions).

Remark 3.2.4. If we assume Theorem 1.1.8, then Theorem 3.2.2 implies that the space
$$(2\text{-Cart}^{\text{loc}}_{/\mathbb{S}})^{\text{Spc}}$$
is isomorphic to space of right-lax functors
$$\mathbb{S}^{1\text{-op}} \dashrightarrow \mathbf{2\text{-}Cat}.$$

3.2.5. In the rest of this subsection we will construct the map in the easy direction, i.e.,
$$(2\text{-Cart}^{\text{loc}}_{/\mathbb{S}})^{\text{Spc}} \overset{\Psi}{\leftarrow} (2\text{-Cart}_{/\,\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})})^{\text{Spc}}.$$

Consider the coCartesian fibrations
$$\mathbb{S}^{\oint} \to \mathbf{\Delta}^{\text{op}} \text{ and } \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\text{op}},$$
and the adjoint functors
$$\iota^{\oint}_{\mathbb{S}} : \mathbb{S}^{\oint} \rightleftarrows \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} : \rho^{\oint}_{\mathbb{S}},$$
see Sect. A.

3.2.6. Starting from a 2-Cartesian fibration $\widetilde{\mathbb{T}} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$, define
$$\Psi(\widetilde{\mathbb{T}})^{\oint} := \widetilde{\mathbb{T}}^{\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}}{\times} \mathbb{S}^{\oint},$$
where the functor $\mathbb{S}^{\oint} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$ is $\iota^{\oint}_{\mathbb{S}}$.

We have:

Lemma 3.2.7.

(a) *The composite functor*
$$\Psi(\widetilde{\mathbb{T}})^{\oint} \to \mathbb{S}^{\oint} \to \mathbf{\Delta}^{\text{op}}$$
is a coCartesian fibration.

(b) *The functor* $\mathbf{\Delta}^{\text{op}} \to 1\text{-Cat}$*, corresponding to the coCartesian fibration of point (a) lies in the essential image of the functor* $\mathrm{Seq}_\bullet : 2\text{-Cat} \to \mathrm{Funct}(\mathbf{\Delta}^{\text{op}}, 1\text{-Cat})$. *Denote the resulting* $(\infty,2)$*-category by* $\Psi(\widetilde{\mathbb{T}})$.

(c) *The functor*
$$\Psi(\widetilde{\mathbb{T}})^{\oint} \to \mathbb{S}^{\oint}$$
maps arrows that are coCartesian over $\mathbf{\Delta}^{\text{op}}$ *to arrows that are coCartesian over* $\mathbf{\Delta}^{\text{op}}$.

(d) *The functor* $\Psi(\widetilde{\mathbb{T}}) \to \mathbb{S}$ *arising from point (c) is a locally 2-Cartesian fibration.*

3.3. **Proof of Theorem 3.2.2, the inverse map.** In this subsection we define the sought-for map

$$(\text{2-Cart}^{\text{loc}}_{/\mathbb{S}})^{\text{Spc}} \overset{\Phi}{\to} (\text{2-Cart}_{/\,\text{RLax}_{\text{non-untl}}(\mathbb{S})})^{\text{Spc}}.$$

3.3.1. Given $\mathbb{T} \in \text{2-Cart}^{\text{loc}}_{/\mathbb{S}}$, consider the corresponding functor

$$\mathbb{T}^{\oint} \to \mathbb{S}^{\oint}.$$

We define

$$'\Phi(\mathbb{T})^{\oint} := \mathbb{T}^{\oint} \underset{\mathbb{S}^{\oint}}{\times} \text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint},$$

where the functor $\text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbb{S}^{\oint}$ is $\rho^{\oint}_{\mathbb{S}}$.

We will define the sought-for $(\infty,1)$-category $\Phi(\mathbb{T})^{\oint}$ is a certain full subcategory of $'\Phi(\mathbb{T})^{\oint}$.

3.3.2. Fix an object of

$$\{\gamma\} \underset{((\mathbf{\Delta}_{\text{actv}})_{[m]/})^{\text{op}}}{\times} \left(\text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[m]\}\right) \simeq \text{Seq}_n(\mathbb{S}), \quad \gamma : [m] \to [n],$$

given by

$$\underline{s} = s_0 \to s_1 \to \ldots \to s_n,$$

see Sect. A.1.3 for the notation.

The fiber of $'\Phi(\mathbb{T})^{\oint}$ over the above object of $\text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$ is by definition

$$\text{Seq}_n(\mathbb{T}) \underset{\text{Seq}_n(\mathbb{S})}{\times} \{s_0 \to s_1 \to \ldots \to s_n\}, \tag{3.1}$$

i.e., this is the category of strings

$$\underline{t} = t_0 \to t_1 \to \ldots \to t_n$$

in $\mathbb{T}$ that project to $\underline{s}$.

3.3.3. The full subcategory of (3.1), corresponding to $\Phi(\mathbb{T})^{\oint} \subset {}'\Phi(\mathbb{T})^{\oint}$ consists of those $\underline{t}$, for which for every $i \in 1,...,n$ for which there exists a $j \in 1,...,m$ with

$$\gamma(j-1) \le i-1 < i \le \gamma(j),$$

the corresponding 1-morphism $t_{i-1} \to t_i$ in $\mathbb{T}$ is locally 2-Cartesian over $s_{i-1} \to s_i$.

3.3.4. We claim:

Lemma 3.3.5.

(a) *The composite functor*

$$\Phi(\mathbb{T})^{\oint} \to \text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\text{op}}$$

is a coCartesian fibration.

(b) *The functor* $\mathbf{\Delta}^{\text{op}} \to \text{1-Cat}$*, corresponding to the coCartesian fibration of point (a) lies in the essential image of the functor* $\text{Seq}_\bullet : \text{2-Cat} \to \text{Funct}(\mathbf{\Delta}^{\text{op}}, \text{1-Cat})$*. Denote the resulting* $(\infty,2)$*-category by* $\Phi(\mathbb{T})$.

(c) *The functor*

$$\Phi(\mathbb{T})^{\oint} \to \text{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$$

maps arrows that are coCartesian over $\mathbf{\Delta}^{\text{op}}$ *to arrows that are coCartesian over* $\mathbf{\Delta}^{\text{op}}$.

(d) *The functor* $\Phi(\mathbb{T}) \to \text{RLax}_{\text{non-untl}}(\mathbb{S})$ *arising from point (c) is a 2-Cartesian fibration, for which the pullback functors along quasi-invertible arrows are equivalences.*

3.4. **Proof of Theorem 3.2.2, computation of the compositions.** In this subsection we will conclude the proof of Theorem 3.2.2 by showing that the maps Φ and Ψ constructed above are mutually inverse.

3.4.1. Since the composition $\rho^{\oint}_{\mathbb{S}} \circ \iota^{\oint}_{\mathbb{S}}$ is isomorphic to $\mathrm{Id}_{\mathbb{S}^{\oint}}$, we obtain immediately that the composition

$$(2\text{-Cart}^{\mathrm{loc}}_{/\mathbb{S}})^{\mathrm{Spc}} \xrightarrow{\Psi} (2\text{-Cart}_{/\,\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})})^{\mathrm{Spc}} \xrightarrow{\Phi} (2\text{-Cart}^{\mathrm{loc}}_{/\mathbb{S}})^{\mathrm{Spc}}$$

is canonically isomorphic to the identity map.

3.4.2. Let now $\widetilde{\mathbb{T}} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$ be a 2-Cartesian fibration. We will now construct a functor

$$\widetilde{\mathbb{T}} \to {}'\Phi(\Psi(\widetilde{\mathbb{T}})).$$

The datum of such a functor is equivalent to that of a functor

$$\widetilde{\mathbb{T}}^{\oint} \to \widetilde{\mathbb{T}}^{\oint} \tag{3.2}$$

that fits into the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{T}}^{\oint} & \xrightarrow{(3.2)} & \widetilde{\mathbb{T}}^{\oint} \\ \downarrow & & \downarrow \\ \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} & \xrightarrow{\iota^{\oint}_{\mathbb{S}} \circ \rho^{\oint}_{\mathbb{S}}} & \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}. \end{array} \tag{3.3}$$

3.4.3. The construction of the functor (3.2) is based on the following lemma:

Lemma 3.4.4. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between $(\infty,1)$-categories, and let $\Gamma_{\mathbf{D}} : [1] \times \mathbf{D} \to \mathbf{D}$ be a functor such that $\Gamma_{\mathbf{D}}|_{\{1\}\times\mathbf{D}} = \mathrm{Id}_{\mathbf{D}}$. Suppose that for every $\mathbf{c} \in \mathbf{C}$ there* exists *a Cartesian arrow $\mathbf{c}' \to \mathbf{c}$ that covers the 1-morphism $\Gamma_{\mathbf{D}}|_{[1]\times\{F(\mathbf{c})\}}$ in $\mathbf{D}$. Then there exists a uniquely defined functor $\Gamma_{\mathbf{C}} : [1] \times \mathbf{C} \to \mathbf{C}$, such that:*

- *$\Gamma_{\mathbf{C}}$ is equipped with an identification $\Gamma_{\mathbf{C}}|_{\{1\}\times\mathbf{C}} = \mathrm{Id}_{\mathbf{C}}$;*
- *The diagram*

$$\begin{array}{ccc} [1] \times \mathbf{C} & \xrightarrow{\Gamma_{\mathbf{C}}} & \mathbf{C} \\ {\scriptstyle \mathrm{Id}_{[1]}\times F}\downarrow & & \downarrow{\scriptstyle F} \\ [1] \times \mathbf{D} & \xrightarrow{\Gamma_{\mathbf{D}}} & \mathbf{D} \end{array} \tag{3.4}$$

  commutes
- *For any $\mathbf{c} \in \mathbf{C}$, the 1-morphism given by $\Gamma_{\mathbf{C}}|_{[1]\times\{\mathbf{c}\}}$ is Cartesian over $\mathbf{D}$.*

3.4.5. We apply the above lemma to $\mathbf{D} := \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$,

$$\mathbf{C} := \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint},$$

with F induced by the projection $\widetilde{\mathbb{T}} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$. We let $\Gamma_{\mathbf{D}}$ be given by the natural transformation

$$\iota^{\oint}_{\mathbb{S}} \circ \rho^{\oint}_{\mathbb{S}} \to \mathrm{Id}_{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}},$$

corresponding to the $(\iota^{\oint}_{\mathbb{S}}, \rho^{\oint}_{\mathbb{S}})$-adjunction.

Applying Lemma 3.4.4 we obtain a functor

$$[1]\times \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint} \to$$
$$\to \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}.$$

Restricting to $\{0\}\in[1]$, and composing with

$$\widetilde{\mathbb{T}}^{\oint} \hookrightarrow \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint},$$

we obtain a functor

$$\widetilde{\mathbb{T}}^{\oint} \to \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}. \tag{3.5}$$

Now, by unwinding the definitions, we obtain that the above functor (3.5) factors through the 1-full subcategory

$$\widetilde{\mathbb{T}}^{\oint} \subset \widetilde{\mathbb{T}}^{\mathrm{ext},\oint} \underset{\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}}{\times} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\mathrm{ext},\oint}.$$

The resulting functor

$$\widetilde{\mathbb{T}}^{\oint} \to \widetilde{\mathbb{T}}^{\oint}$$

is the desired functor (3.2).

3.4.6. By further unwinding the definitions, we obtain that the essential image of the functor

$$\widetilde{\mathbb{T}} \to {}'\Phi(\Psi(\widetilde{\mathbb{T}}))$$

constructed above, belongs to $\Phi(\Psi(\widetilde{\mathbb{T}})) \subset {}'\Phi(\Psi(\widetilde{\mathbb{T}}))$.

Finally, if $\widetilde{\mathbb{T}} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$ is such that the pullback functors along quasi-invertible arrows (see Sect. A.3) are equivalences, then the resulting functor

$$\widetilde{\mathbb{T}} \to \Phi(\Psi(\widetilde{\mathbb{T}}))$$

is an equivalence.

□(Theorem 3.2.2)

3.5. **Gray products and 2-Cartesian fibrations.** In this subsection we will use Theorem 3.2.2 to give an explicit description of 2-Cartesian fibrations over Gray products.

3.5.1. Recall the condition on a functor between $(\infty,2)$-categories to be a *localization on 1-morphisms*, see Sect. B.1. The following is straightforward:

Lemma 3.5.2. *Let $\widetilde{\mathbb{S}} \to \mathbb{S}$ be a localization on 1-morphisms. Then the map*

$$\mathbf{2\text{-}Cart}_{/\mathbb{S}} \to \mathbf{2\text{-}Cart}_{/\widetilde{\mathbb{S}}},$$

defined by pullback, is fully faithful. Its essential image consists of those $F:\widetilde{\mathbb{T}}\to\widetilde{\mathbb{S}}$ that satisfy the following condition:

For every $t\in\widetilde{\mathbb{T}}$, a pair of 1-morphisms $\beta_0,\beta_1 : s'\to F(t)$ and a 2-morphism

$$\phi \in \mathrm{Maps}_{\mathbf{Maps}_{\widetilde{\mathbb{S}}}(s',F(t))}(\beta_0,\beta_1),$$

if we denote by $\alpha_0 : t'\to t$ the 2-Cartesian lift of β_0 and by $\psi\in\mathrm{Maps}_{\mathbf{Maps}_{\widetilde{\mathbb{T}}}(t',t)}(\alpha_0,\alpha_1)$ the coCartesian lift of ϕ, if the image of ϕ in $\mathbb{S}$ is invertible*, then the 1-morphism $\alpha_1 : t'\to t$ is 2-Cartesian over β_1.*

3.5.3. We fix $\mathbb{S}_1, \mathbb{S}_2 \in$ 2-Cat. We shall now describe the space

$$(\text{2-Cart}_{/\mathbb{S}_1 \circledast \mathbb{S}_2})^{\text{Spc}}$$

in a way functorial in $\mathbb{S}_1$ and $\mathbb{S}_2$. Indeed, combining Lemma 3.5.2 applied to

$$\text{RLax}_{\text{non-untl}}(\mathbb{S}_1 \times \mathbb{S}_2) \to \mathbb{S}_1 \circledast \mathbb{S}_2,$$

with Proposition 3.2.2, we obtain:

Corollary 3.5.4. *There exists a canonically defined fully faithful embedding*

$$(\text{2-Cart}_{/\mathbb{S}_1 \circledast \mathbb{S}_2})^{\text{Spc}} \hookrightarrow (\text{2-Cart}^{\text{loc}}_{/\mathbb{S}_1 \times \mathbb{S}_2})^{\text{Spc}}.$$

Its essential image consists of those $\mathbb{T} \to \mathbb{S}_1 \times \mathbb{S}_2$ that satisfy:

- *For every pair of composable 1-morphisms in $\mathbb{T}$, locally Cartesian over $\mathbb{S}_1 \times \mathbb{S}_2$, that cover two morphisms in $\mathbb{S}_1 \times \mathbb{S}_2$ both of which project to isomorphisms under $\mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_1$, their composition is locally Cartesian over $\mathbb{S}_1 \times \mathbb{S}_2$.*
- *For every pair of composable 1-morphisms in $\mathbb{T}$, locally Cartesian over $\mathbb{S}_1 \times \mathbb{S}_2$, that cover two morphisms in $\mathbb{S}_1 \times \mathbb{S}_2$ both of which project to isomorphisms under $\mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_2$, their composition is locally Cartesian over $\mathbb{S}_1 \times \mathbb{S}_2$.*
- *For every pair of 1-morphisms $(s_1' \xrightarrow{\alpha_1} s_1) \in \mathbb{S}_1$ and $(s_2' \xrightarrow{\alpha_2} s_2) \in \mathbb{S}_2$ and locally Cartesian 1-morphisms $t'' \xrightarrow{\beta} t'$ and $t' \xrightarrow{\gamma} t$ covering $(\alpha_1, \text{id}_{s_2'})$ and $(\text{id}_{s_1}, \alpha_2)$, respectively, the 1-morphism $\gamma \circ \beta$ is locally Cartesian over (α_1, α_2).*

Corollary 3.5.5.

(a) *The essential image of the (fully faithful) map*

$$(\text{2-Cart}_{/\mathbb{S}_1 \circledast \mathbb{S}_2})^{\text{Spc}} \to (\text{2-Cart}^{\text{loc}}_{/\mathbb{S}_1 \times \mathbb{S}_2})^{\text{Spc}} \subset (\text{2-Cat}_{/\mathbb{S}_1 \times \mathbb{S}_2})^{\text{Spc}}$$

consists of those

$$\mathbb{T} \to \mathbb{S}_1 \times \mathbb{S}_2$$

such that:

1. *The composition $\mathbb{T} \to \mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{S}_1$ is a 2-Cartesian fibration;*
2. *The functor $\mathbb{T} \to \mathbb{S}_1 \times \mathbb{S}_2$, viewed as a map in* $\text{2-Cart}_{/\mathbb{S}_1}$, *belongs to* $(\text{2-Cart}_{/\mathbb{S}_1})_{\text{strict}}$*;*
3. *For every $s_1 \in \mathbb{S}_1$, the resulting functor $\mathbb{T}_{s_1} \to \mathbb{S}_2$ is a 2-Cartesian fibration.*
4. *For every 1-morphism $s_1 \to s_1'$ in $\mathbb{S}_1$, the pullback functor $\mathbb{T}_{s_1'} \to \mathbb{T}_{s_1}$, which by the previous point is a 1-morphism in* $\text{2-Cart}_{/\mathbb{S}_2}$, *belongs* $(\text{2-Cart}_{/\mathbb{S}_2})_{2\text{-strict}}$.

(b) *The subspace*

$$(\text{2-Cart}_{/\mathbb{S}_1 \times \mathbb{S}_2})^{\text{Spc}} \subset (\text{2-Cart}_{/\mathbb{S}_1 \circledast \mathbb{S}_2})^{\text{Spc}}$$

corresponds to replacing in condition (4) the category $(\text{2-Cart}_{/\mathbb{S}_2})_{2\text{-strict}}$ *by its 1-full subcategory*

$$(\text{2-Cart}_{/\mathbb{S}_2})_{\text{strict}} \subset (\text{2-Cart}_{/\mathbb{S}_2})_{2\text{-strict}}.$$

4. Proof of Theorem 1.1.8

4.1. **Proof of Theorem 1.1.8, Step 1: identifying the underlying spaces.** In this subsection we will establish the assertion of Theorem 1.1.8 at the level of the underlying spaces.

4.1.1. First, we notice that Theorem 2.0.1 can be reformulated as follows:

Corollary 4.1.2. *There exists a canonical equivalence of bi-simplicial spaces that send m, n to*

$$\text{Sq}_{m,n}(\mathbf{2\text{-}Cat}) \text{ and } \text{Maps}([m], \text{2-Cart}_{/[n]^{\text{op}}}),$$

respectively.

4.1.3. Applying the 1-fully faithful embedding

$$\text{Funct}([m],\text{2-Cat}) \simeq \text{Seq}_m^{\text{ext}}(\text{2-Cat}) \simeq (\text{2-Cart}_{/[m]^{\text{op}}})_{\text{strict}} \hookrightarrow \text{2-Cart}_{/[m]^{\text{op}}} \hookrightarrow \text{2-Cat}_{/[m]^{\text{op}}} \tag{4.1}$$

(where the second isomorphism is Theorem 2.0.1(b)), we obtain a fully faithful map

$$\text{Maps}([m],\text{2-Cat}_{/[n]^{\text{op}}}) \to ((\text{2-Cat}_{/[m]^{\text{op}}})_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}} \simeq (\text{2-Cat}_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}}.$$

Restricting to $\text{2-Cart}_{/[n]^{\text{op}}} \hookrightarrow \text{2-Cat}_{/[n]^{\text{op}}}$, we obtain a fully faithful map

$$\text{Maps}([m],\text{2-Cart}_{/[n]^{\text{op}}}) \to (\text{2-Cat}_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}}. \tag{4.2}$$

Lemma 4.1.4. *The essential image of the map* (4.2) *lies in*

$$(\text{2-Cart}^{\text{loc}}_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}} \subset (\text{2-Cat}_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}}$$

and coincides with the essential image of fully faithful embedding

$$(\text{2-Cart}_{/[n]^{\text{op}}\circledast[m]^{\text{op}}})^{\text{Spc}} \hookrightarrow (\text{2-Cart}^{\text{loc}}_{/[n]^{\text{op}}\times[m]^{\text{op}}})^{\text{Spc}}$$

of Corollary 3.5.4.

Proof. Follows from Corollary 3.5.5(a). □

4.1.5. Thus, combining Lemma 4.1.4 and Corollary 4.1.2 we obtain a canonical identification of bi-simplicial spaces

$$\text{Sq}_{m,n}(\mathbf{2\text{-}Cat}) \simeq (\text{2-Cart}_{/[n]^{\text{op}}\circledast[m]^{\text{op}}})^{\text{Spc}}. \tag{4.3}$$

We can now establish the assertion of Theorem 1.1.8 at the level of the underlying spaces:

Corollary 4.1.6. *For* $\mathbb{S} \in \text{2-Cat}$*, there exists a canonical equivalence*

$$(\text{2-Cart}_{/\mathbb{S}})^{\text{Spc}} \simeq \text{Maps}_{\text{2-Cat}}(\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat}),$$

functorial in $\mathbb{S}$.

Proof. It follows from [Chapter A.1, Theorems 4.1.3 and 5.2.3(a)] that for $\mathbb{S} \in \text{2-Cat}$, the restriction map

$$(\text{2-Cat}_{/\mathbb{S}})^{\text{Spc}} \to \text{Maps}_{\text{Funct}(\boldsymbol{\Delta}^{\text{op}}\times\boldsymbol{\Delta}^{\text{op}},\text{Spc})}(\text{Sq}_{\bullet,\bullet}(\mathbb{S}),(\text{2-Cat}_{/[\bullet]\circledast[\bullet]})^{\text{Spc}})$$

is an isomorphism, and under this isomorphism the subspaces

$$(\text{2-Cart}_{/\mathbb{S}})^{\text{Spc}} \subset (\text{2-Cat}_{/\mathbb{S}})^{\text{Spc}}$$

and

$$\text{Maps}_{\text{Funct}(\boldsymbol{\Delta}^{\text{op}}\times\boldsymbol{\Delta}^{\text{op}},\text{Spc})}(\text{Sq}_{\bullet,\bullet}(\mathbb{S}),(\text{2-Cart}_{/[\bullet]\circledast[\bullet]})^{\text{Spc}}) \subset$$
$$\subset \text{Maps}_{\text{Funct}(\boldsymbol{\Delta}^{\text{op}}\times\boldsymbol{\Delta}^{\text{op}},\text{Spc})}(\text{Sq}_{\bullet,\bullet}(\mathbb{S}),(\text{2-Cat}_{/[\bullet]\circledast[\bullet]})^{\text{Spc}})$$

correspond to one another.

Hence, the assertion of the corollary follows from the isomorphism (4.3) using the canonical identification of bi-cosimplicial objects of 2-Cat:

$$([m]\circledast[n])^{1\text{-op}} \simeq [n]^{\text{op}}\circledast[m]^{\text{op}}.$$

□

4.2. **Proof of Theorem 1.1.8, Step 2: identifying the underlying $(\infty,1)$-categories.** In this subsection we will construct the identification of the $(\infty,1)$-categories underlying the two sides in Theorem 1.1.8(b).

4.2.1. We need to construct an isomorphism of simplicial spaces

$$\mathrm{Maps}_{1\text{-Cat}}([m],(2\text{-Cart}_{/\mathbb{S}})_{\mathrm{strict}}) \simeq \mathrm{Maps}_{2\text{-Cat}}(\mathbb{S}^{1\text{-op}}\times[m],\mathbf{2\text{-}Cat}), \quad [m]\in\mathbf{\Delta}.$$

Taking into account Corollary 4.1.6, we need to construct an isomorphism of simplicial spaces

$$\mathrm{Maps}([m],(2\text{-Cart}_{/\mathbb{S}})_{\mathrm{strict}}) \simeq (2\text{-Cart}_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}, \quad [m]\in\mathbf{\Delta}. \tag{4.4}$$

4.2.2. Given $[m]\in\mathbf{\Delta}$, using the 1-fully faithful embedding

$$\mathrm{Funct}([m],2\text{-Cat})\simeq \mathrm{Seq}^{\mathrm{ext}}_m(2\text{-Cat})\simeq(2\text{-Cart}_{/[m]^{\mathrm{op}}})_{\mathrm{strict}}\hookrightarrow 2\text{-Cart}_{/[m]^{\mathrm{op}}}\hookrightarrow 2\text{-Cat}_{/[m]^{\mathrm{op}}}$$

of (4.1) (which, we note, uses the statement of Theorem 2.0.1), we obtain a fully faithful map

$$\mathrm{Maps}([m],2\text{-Cat}_{/\mathbb{S}})\hookrightarrow((2\text{-Cat}_{/[m]^{\mathrm{op}}})_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}\simeq(2\text{-Cat}_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}.$$

Composing with the embedding

$$\mathrm{Maps}([m],(2\text{-Cart}_{/\mathbb{S}})_{\mathrm{strict}})\hookrightarrow\mathrm{Maps}([m],2\text{-Cart}_{/\mathbb{S}})\hookrightarrow\mathrm{Maps}([m],2\text{-Cat}_{/\mathbb{S}}),$$

we obtain a fully faithful map

$$\mathrm{Maps}([m],(2\text{-Cart}_{/\mathbb{S}})_{\mathrm{strict}})\to(2\text{-Cat}_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}. \tag{4.5}$$

Lemma 4.2.3. *The essential image of the map* (4.5) *equals*

$$(2\text{-Cart}_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}\subset(2\text{-Cat}_{/\mathbb{S}\times[m]^{\mathrm{op}}})^{\mathrm{Spc}}.$$

Proof. Follows from Corollary 3.5.5(b). □

Thus, we obtain the required identification (4.4).

4.3. Proof of Theorem 1.1.8, Step 3: end of the argument.

4.3.1. Given $\mathbb{T}\in 2\text{-Cat}$, we need to construct an isomorphism of spaces

$$\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}})\simeq\mathrm{Maps}_{2\text{-Cat}}(\mathbb{T}\circledast\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat}),$$

functorial in $\mathbb{T}$ and $\mathbb{S}$, so that the subspaces

$$\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\mathrm{strict}})\subset\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}})$$

and

$$\mathrm{Maps}_{2\text{-Cat}}(\mathbb{T}\times\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat})\subset\mathrm{Maps}_{2\text{-Cat}}(\mathbb{T}\circledast\mathbb{S}^{1\text{-op}},\mathbf{2\text{-}Cat})$$

correspond to one another.

Taking into account Corollary 4.1.6, we need to construct an isomorphism of spaces

$$\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}})\simeq(2\text{-coCart}_{/\mathbb{S}\circledast\mathbb{T}^{1\text{-op}}})^{\mathrm{Spc}}, \tag{4.6}$$

so that

$$\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\mathrm{strict}})\subset\mathrm{Maps}(\mathbb{T},(\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}})$$

maps to

$$(2\text{-coCart}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\mathrm{Spc}}\subset(2\text{-coCart}_{/\mathbb{S}\circledast\mathbb{T}^{1\text{-op}}})^{\mathrm{Spc}}.$$

4.3.2. The equivalence of $(\infty,1)$-categories

$$\mathbf{Maps}_{2\text{-Cat}}(\mathbb{T}, \mathbf{2\text{-}Cat}) \simeq (2\text{-coCart}_{/\mathbb{T}^{1\text{-op}}})_{\text{strict}}$$

established in Step 2, gives rise to a 1-fully faithful embedding

$$\mathbf{Maps}_{2\text{-Cat}}(\mathbb{T}, \mathbf{2\text{-}Cat}) \hookrightarrow 2\text{-Cat}_{/\mathbb{T}^{1\text{-op}}}.$$

From here, we obtain a fully faithful embedding

$$\text{Maps}(\mathbb{T}, \mathbf{2\text{-}Cat}_{/\mathbb{S}}) \hookrightarrow ((2\text{-Cat}_{/\mathbb{T}^{1\text{-op}}})_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}} = (2\text{-Cat}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}}.$$

Composing with

$$\text{Maps}(\mathbb{T}, (\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}}) \subset \text{Maps}(\mathbb{T}, \mathbf{2\text{-}Cat}_{/\mathbb{S}}),$$

we obtain a fully faithful map

$$\text{Maps}(\mathbb{T}, (\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}}) \hookrightarrow (2\text{-Cat}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}}. \tag{4.7}$$

We claim:

Lemma 4.3.3.
(a) *The essential image of the map* (4.7) *is contained in* $(2\text{-Cat}^{\text{loc}}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}}$ *and equals the essential image of the fully faithful embedding*

$$(2\text{-coCart}_{/\mathbb{S}\circledast\mathbb{T}^{1\text{-op}}})^{\text{Spc}} \hookrightarrow (2\text{-Cat}^{\text{loc}}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}}$$

of Corollary 3.5.4.
(b) *Under the resulting isomorphism*

$$\text{Maps}(\mathbb{T}, (\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}}) \simeq (2\text{-coCart}_{/\mathbb{S}\circledast\mathbb{T}^{1\text{-op}}})^{\text{Spc}}$$

the subspace

$$\text{Maps}(\mathbb{T}, (\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{\text{strict}}) \subset \text{Maps}(\mathbb{T}, (\mathbf{2\text{-}coCart}_{/\mathbb{S}})_{2\text{-strict}})$$

maps to

$$(2\text{-coCart}_{/\mathbb{S}\times\mathbb{T}^{1\text{-op}}})^{\text{Spc}} \subset (2\text{-coCart}_{/\mathbb{S}\circledast\mathbb{T}^{1\text{-op}}})^{\text{Spc}}.$$

Proof. Follows from Corollary 3.5.5. □

The last lemma establishes the desired isomorphism (4.6).

□(Theorem 1.1.8)

5. The Yoneda embedding

The goal of this section is to discuss the several incarnations of what can be called the Yoneda lemma in the context of $(\infty,2)$-categories.

For example, we will show that to $s \in \mathbb{S}$ there corresponds a Yoneda functor

$$h_s : \mathbb{S} \to \mathbf{1\text{-}Cat}, \quad h_s(s') = \mathbf{Maps}_{\mathbb{S}}(s,s'), \tag{5.1}$$

and for any $\mathbb{S} \overset{F}{\to} \mathbf{1\text{-}Cat}$ we have an equivalence

$$\mathbf{Maps}_{\text{Funct}(\mathbb{S},\mathbf{1\text{-}Cat})}(h_s, F) \simeq F(s). \tag{5.2}$$

By letting s vary, we will construct the Yoneda embedding

$$\text{Yon} : \mathbb{S} \hookrightarrow \text{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat}).$$

5.1. **The right-lax slice construction.** In order to construct the Yoneda functors, we will use Corollary 1.3.4 in order to interpret the datum of a functor $\mathbb{S}\to\mathbf{1\text{-}Cat}$ as a 1-coCartesian fibration.

In this subsection we will construct the corresponding 1-coCartesian fibrations (up to reversing the arrows).

5.1.1. Let $\mathbb{S}$ be an $(\infty,2)$-category, and $s\in\mathbb{S}$ an object. We define the $(\infty,2)$-category $\mathbb{S}_{/\!/s}$ to be

$$\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\underset{\mathbb{S}}{\times}\{s\},$$

where the fiber product is formed using functor

$$\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\to\mathbb{S}.$$

given by evaluation at $1\in[1]$.

5.1.2. Let $p_s:\mathbb{S}_{/\!/s}\to\mathbb{S}$ be the functor given by evaluation at $0\in[1]$. By definition, the fiber of p over $s'\in\mathbb{S}$ is an $(\infty,2)$-category

$$\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\underset{\mathbb{S}\times\mathbb{S}}{\times}\{(s',s)\},$$

which by [Chapter A.1, Corollary 3.4.8] is an $(\infty,1)$-category, equipped with a canonical identification with $\mathbf{Maps}_{\mathbb{S}}(s',s)$.

5.1.3. We claim:

Lemma 5.1.4. *The functor $p_s:\mathbb{S}_{/\!/s}\to\mathbb{S}$ is a 1-Cartesian fibration.*

Proof. Let $\alpha:s_0'\to s$ be an object of $\mathbb{S}_{/\!/s}$, and and let $\beta:s_1'\to s_0'$ is a 1-morphism in $\mathbb{S}$. Then it is easy to see that the commutative diagram

$$\begin{array}{ccc} s_0' & \xrightarrow{\alpha} & s \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\mathrm{id}_s} \\ s_1' & \xrightarrow{\alpha\circ\beta} & s \end{array}$$

represents a Cartesian arrow in $(\mathbb{S}_{/\!/s})^{\text{1-Cat}}$ over β: indeed this is an assertion at the level of the underlying $(\infty,1)$-categories.

To finish the proof of the lemma, given a pair of objects

$$\underline{s}_0=(\alpha_0:s_0'\to s)\text{ and }\underline{s}_1=(\alpha_1:s_1'\to s)$$

of $\mathbb{S}_{/s}$, we need to show that the functor

$$\mathbf{Maps}_{\mathbb{S}_{/\!/s}}(\underline{s}_0,\underline{s}_1)\to\mathbf{Maps}_{\mathbb{S}}(s_0,s_1)$$

is a coCartesian fibration in spaces.

The category $\mathbf{Maps}_{\mathbb{S}_{/\!/s}}(\underline{s}_0,\underline{s}_1)$ has as objects pairs (β,ϕ), where $\beta:s_0'\to s_1'$ and ϕ is a 2-morphism $\alpha_0\to\alpha_1\circ\beta$. Morphisms from (β,ϕ) to $(\widetilde{\beta},\widetilde{\phi})$ is the space of 2-morphisms $\psi:\beta\to\widetilde{\beta}$, equipped with an identification $\widetilde{\phi}\simeq\alpha_1(\psi)\circ\phi$. This makes it clear that the assignment

$$(\beta,\phi)\mapsto\beta$$

is coCartesian fibration in spaces.

□

5.1.5. Applying Corollary 1.2.6, from the 1-Cartesian fibration $\mathbb{S}_{//s} \to \mathbb{S}$ we obtain a functor

$$\widetilde{h}_s : \mathbb{S}^{1\text{-op}} \to \mathbf{1\text{-}Cat}.$$

The value of this functor on a given $s' \in \mathbb{S}$ is

$$(\mathbb{S}_{//s})_{s'} \simeq \mathbf{Maps}_{\mathbb{S}}(s', s).$$

5.2. **The 2-categorical Yoneda lemma.** In this subsection we will establish the isomorphism (5.2).

5.2.1. For a pair of 1-Cartesian fibrations in $(\infty,1)$-categories $\mathbb{T}_0, \mathbb{T}_1$ over $\mathbb{S}$, let us denote by

$$\mathbf{Maps}^{\text{strict}}_{\mathbb{S}}(\mathbb{T}_0, \mathbb{T}_1) := \mathbf{Maps}_{(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}}(\mathbb{T}_0, \mathbb{T}_1),$$

where the notation $(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}$ is as in Sect. 1.2.3.

I.e., $\mathbf{Maps}^{\text{strict}}_{\mathbb{S}}(\mathbb{T}_0, \mathbb{T}_1)$ is the full subcategory of $\mathbf{Maps}_{\mathbb{S}}(\mathbb{T}_0, \mathbb{T}_1)$ that consists of those functors that map 1-morphisms in $\mathbb{T}_0$ that are Cartesian over $\mathbb{S}$ to 1-morphisms in $\mathbb{T}_1$ with the same property.

5.2.2. We claim:

Proposition 5.2.3. *For a 1-Cartesian fibration $F : \mathbb{T} \to \mathbb{S}$, evaluation at $(s \overset{\text{id}_s}{\to} s) \in \mathbb{S}_{//s}$ defines an equivalence*

$$\mathbf{Maps}^{\text{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{T}) \to \mathbb{T}_s.$$

Proof. Let

$$(\text{Funct}([1], \mathbb{T})_{\text{right-lax}})^{\text{Cart}_{/\mathbb{S}}} \subset \text{Funct}([1], \mathbb{T})_{\text{right-lax}}$$

denote the full subcategory whose objects are 1-morphisms Cartesian over $\mathbb{S}$.

Evaluation defines functors

$$\text{ev}_0, \text{ev}_1 : (\text{Funct}([1], \mathbb{T})_{\text{right-lax}})^{\text{Cart}_{/\mathbb{S}}} \to \mathbb{T}.$$

Consider the fiber product

$$(\text{Funct}([1], \mathbb{T})_{\text{right-lax}})^{\text{Cart}_{/\mathbb{S}}} \underset{\text{ev}_1, \mathbb{T}}{\times} \mathbb{T}_s \simeq (\text{Funct}([1], \mathbb{T})_{\text{right-lax}})^{\text{Cart}_{/\mathbb{S}}} \underset{F \circ \text{ev}_1, \mathbb{S}}{\times} \{s\}.$$

It is easy to see that the functor (between $(\infty,2)$-categories over $\mathbb{S}$)

$$(\text{Funct}([1], \mathbb{T})_{\text{right-lax}})^{\text{Cart}_{/\mathbb{S}}} \underset{\text{ev}_1, \mathbb{T}}{\times} \mathbb{T}_s \to \mathbb{S}_{//s} \times \mathbb{T}_s$$

is an equivalence.

Hence, we obtain a functor (between $(\infty,2)$-categories over $\mathbb{S}$)

$$\mathbb{S}_{//s} \times \mathbb{T}_s \to (\text{Funct}([1], \mathbb{T})_{\text{lax}})^{\text{Cart}_{/\mathbb{S}}} \underset{\text{ev}_1, \mathbb{T}}{\times} \mathbb{T}_s \to (\text{Funct}([1], \mathbb{T})_{\text{lax}})^{\text{Cart}_{/\mathbb{S}}} \xrightarrow{\text{ev}_0} \mathbb{T}.$$

The latter gives rise to a functor

$$\mathbb{T}_s \to \mathbf{Maps}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{T}).$$

It is easy to see that the latter functor takes values in $\mathbf{Maps}^{\text{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{T})$ and provides an inverse to one in the statement of the proposition.

□

5.2.4. Applying Corollary 1.2.6, from Proposition 5.2.3 we obtain:

Corollary 5.2.5. *For $F:\mathbb{S}^{1\text{-op}}\to \mathbf{1\text{-}Cat}$, evaluation at $s\in\mathbb{S}$ defines an equivalence*

$$\mathbf{Maps}_{\mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat})}(\widetilde{h}_s,F)\simeq F(s).$$

5.3. **The 2-categorical Yoneda embedding.** We will now show how to turn $s\in\mathbb{S}$ into a parameter and thus obtain the Yoneda functor

$$\mathrm{Yon}_{\mathbb{S}}:\mathbb{S}\to \mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat}).$$

We will then show that $\mathrm{Yon}_{\mathbb{S}}$ is fully faithful.

5.3.1. For $\mathbb{S}\in$ 2-Cat, consider the $(\infty,2)$-category

$$\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}.$$

Evaluation on $0,1\in[1]$ defines two functors

$$\mathrm{ev}_0,\mathrm{ev}_1:\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\rightrightarrows\mathbb{S}.$$

As in Lemma 5.1.4 one shows:

Lemma 5.3.2.
(a) *The functor* $\mathrm{ev}_1:\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\to\mathbb{S}$ *is a 2-coCartesian fibration of $(\infty,2)$-categories.*
(b) *The functor*

$$(\mathrm{ev}_0\times\mathrm{ev}_1):\mathrm{Funct}([1],\mathbb{S})_{\text{right-lax}}\to\mathbb{S}\times\mathbb{S}$$

is a strict functor between 2-coCartesian fibrations over $\mathbb{S}$.

5.3.3. Applying Corollary 1.3.3, from the functor $\mathrm{ev}_0\times\mathrm{ev}_1$ we obtain a functor

$$\mathbb{S}\to\mathbf{2\text{-}Cat},$$

equipped with a natural transformation to the constant functor with value $\mathbb{S}$.

I.e., we obtain a functor

$$\mathbb{S}\to\mathbf{2\text{-}Cat}_{/\mathbb{S}} \tag{5.3}$$

5.3.4. Note, however, that by Lemma 5.1.4, the functor (5.3) takes values in the full subcategory

$$\mathbf{1\text{-}Cart}_{/\mathbb{S}}\subset\mathbf{2\text{-}Cat}_{/\mathbb{S}}.$$

Moreover, the functor (5.3) factors through the 1-full subcategory

$$(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}\subset\mathbf{1\text{-}Cart}_{/\mathbb{S}}.$$

I.e., we have a functor

$$\mathbb{S}\to(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}. \tag{5.4}$$

5.3.5. Applying the equivalence $(\mathbf{1\text{-}Cart}_{/\mathbb{S}})_{\text{strict}}\simeq\mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat})$, from (5.4), we obtain a functor

$$\mathrm{Yon}_{\mathbb{S}}:\mathbb{S}\to\mathrm{Funct}(\mathbb{S}^{1\text{-op}},\mathbf{1\text{-}Cat}), \tag{5.5}$$

or, equivalently, a functor

$$\mathbb{S}^{1\text{-op}}\times\mathbb{S}\to\mathbf{1\text{-}Cat}. \tag{5.6}$$

We will refer to the functor $\mathrm{Yon}_{\mathbb{S}}$ of (5.5) as the *2-categorical Yoneda functor*.

5.3.6. We claim:

Proposition 5.3.7. *The functor* (5.5) *is fully faithful.*

Proof. We need to show that for $s, s' \in \mathbb{S}$, the functor

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \mathbf{Maps}_{\mathrm{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat})}(\mathrm{Yon}_{\mathbb{S}}(s), \mathrm{Yon}_{\mathbb{S}}(s'))$$

is an equivalence.

Equivalently (by virtue of Corollary 1.2.6), we need to show that the composite functor

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \mathbf{Maps}_{\mathrm{Funct}(\mathbb{S}^{1\text{-op}}, \mathbf{1\text{-}Cat})}(\mathrm{Yon}(s), \mathrm{Yon}(s')) \to \mathbf{Maps}^{\mathrm{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{S}_{//s'}) \tag{5.7}$$

is an equivalence.

By construction, the above map (5.7) has the property that for any $t \in \mathbb{S}$ the induced map

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \mathbf{Maps}^{\mathrm{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{S}_{//s'}) \to$$
$$\to \mathbf{Maps}((\mathbb{S}_{//s})_t, (\mathbb{S}_{//s'})_t) \simeq \mathbf{Maps}(\mathbf{Maps}_{\mathbb{S}}(t, s), \mathbf{Maps}_{\mathbb{S}}(t, s'))$$

is the map

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \mathbf{Maps}(\mathbf{Maps}_{\mathbb{S}}(t, s), \mathbf{Maps}_{\mathbb{S}}(t, s')),$$

given by composition of 1-morphisms.

Taking $t = s$ and evaluating at id_s, we obtain that the composition

$$\mathbf{Maps}_{\mathbb{S}}(s, s') \to \mathbf{Maps}^{\mathrm{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{S}_{//s'}) \to$$
$$\to \mathbf{Maps}((\mathbb{S}_{//s})_s, (\mathbb{S}_{//s'})_s) \to \mathbf{Maps}(\mathbf{Maps}_{\mathbb{S}}(s, s), \mathbf{Maps}_{\mathbb{S}}(s, s')) \to \mathbf{Maps}_{\mathbb{S}}(s, s')$$

is the identity map.

Now, according to Proposition 5.2.3, the composition

$$\mathbf{Maps}^{\mathrm{strict}}_{\mathbb{S}}(\mathbb{S}_{//s}, \mathbb{S}_{//s'}) \to \mathbf{Maps}((\mathbb{S}_{//s})_s, (\mathbb{S}_{//s'})_s) \to$$
$$\to \mathbf{Maps}(\mathbf{Maps}_{\mathbb{S}}(s, s), \mathbf{Maps}_{\mathbb{S}}(s, s')) \to \mathbf{Maps}_{\mathbb{S}}(s, s')$$

is an isomorphism, implying that (5.7) is an equivalence as well.

□

Appendix A. The universal right-lax functor

A.1. **The construction.**

A.1.1. Consider the 1-fully faithful functor

$$\text{2-Cat} \to \text{2-Cat}_{\text{right-lax}_{\text{non-untl}}},$$

see [Chapter A.1, Sect. 3.1.5].

This functor is easily seen to commute with limits. Hence, it admits a left adjoint, to be denoted

$$\mathbb{S} \mapsto \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}).$$

It turns out that this functor can be described rather explicitly, and this description is useful.

A.1.2. Recall the notation $\mathbb{S}^{\oint}$, see [Chapter A.1, Sect. 3.1.1].

Starting from $\mathbb{S} \in$ 2-Cat, consider the following $(\infty,1)$-category:

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} := \mathbb{S}^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \mathrm{Actv},$$

where Actv is the full subcategory of $\mathrm{Funct}([1], \mathbf{\Delta}^{\mathrm{op}})$, spanned by active morphisms, and $\mathrm{Actv} \to \mathbf{\Delta}^{\mathrm{op}}$ is the functor of evaluation at $0 \in [1]$.

Evaluation on $1 \in [1]$ defines a functor

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\mathrm{op}}. \tag{A.1}$$

A.1.3. For example,

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[0]\} \simeq \mathrm{Seq}_0(\mathbb{S}).$$

The category $\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[1]\}$ is described as follows. It is a coCartesian fibration over $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{actv}}$ (where $\mathbf{\Delta}_{\mathrm{actv}}$ is the 1-full subcategory of $\mathbf{\Delta}$ where we restrict the arrows to active morphisms). We have

$$\{[n]\} \underset{\mathbf{\Delta}_{\mathrm{actv}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[1]\}\right) \simeq \mathrm{Seq}_n(\mathbb{S}).$$

For an active map $\alpha : [m] \to [n]$ the corresponding functor between the fibers identifies with the functor

$$\mathrm{Seq}_n(\mathbb{S}) \to \mathrm{Seq}_m(\mathbb{S}),$$

induced by α.

A.1.4. The projection $[1] \to [0]$ defines a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Actv}$, which in turn gives rise to a functor

$$\iota^{\oint}_{\mathbb{S}} : \mathbb{S}^{\oint} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} := \mathbb{S}^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \mathrm{Actv},$$

compatible with projections to $\mathbf{\Delta}^{\mathrm{op}}$.

We will prove:

Theorem A.1.5.

(i) *The functor* $\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\mathrm{op}}$ *of* (A.1) *is a coCartesian fibration, and the resulting functor* $\mathbf{\Delta}^{\mathrm{op}} \to$ 1-Cat *lies in the essential image of the functor* $\mathrm{Seq}_\bullet$*; denote the resulting* $(\infty,2)$*-category by* $\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$.

(ii) *The functor* $\iota^{\oint}_{\mathbb{S}}$ *sends coCartesian arrows over inert morphisms in* $\mathbf{\Delta}^{\mathrm{op}}$ *to coCartesian arrows. Denote the resulting lax functor* $\mathbb{S} \dashrightarrow \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$ *by* $\iota_{\mathbb{S}}$.

(iii) *For any* $\mathbb{T} \in$ 2-Cat*, the composite map*

$$\mathrm{Maps}_{\text{2-Cat}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}), \mathbb{T}) \to \mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}), \mathbb{T}) \to$$
$$\to \mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathbb{S}, \mathbb{T}),$$

where the second arrow is given by precomposition with $\iota_{\mathbb{S}}$*, is an isomorphism.*

A.1.6. Note that the functor

$$\iota_{\mathbb{S}}^{\oint} : \mathbb{S}^{\oint} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$$

admits a left adjoint; to be denoted $\lambda_{\mathbb{S}}^{\oint}$. This is a functor between categories over $\mathbf{\Delta}^{\mathrm{op}}$ that sends coCartesian edges to coCartesian edges.

For example, the corresponding functor

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} [1] \to \mathrm{Seq}_1(\mathbb{S})$$

is given, in terms of the description in Sect. A.1.3 by the compatible family of functors

$$\mathrm{Seq}_n(\mathbb{S}) \to \mathrm{Seq}_1(\mathbb{S}),$$

each corresponding to the unique active map $[1] \to [n]$.

Hence, we obtain that the functor $\lambda_{\mathbb{S}}^{\oint}$ corresponds to a functor

$$\lambda_{\mathbb{S}} : \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}) \to \mathbb{S}.$$

We claim:

Proposition A.1.7. *The functor $\lambda_{\mathbb{S}} : \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}) \to \mathbb{S}$ is the counit of the adjunction, i.e., corresponds to the identity functor on $\mathbb{S}$, considered as a non-unital right-lax functor.*

Proof. We need to show that the composite lax functor

$$\mathbb{S} \overset{\iota_{\mathbb{S}}}{\dashrightarrow} \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}) \overset{\lambda_{\mathbb{S}}}{\to} \mathbb{S}$$

identifies with the identity functor on $\mathbb{S}$.

For that we need to show that the composite functor

$$\lambda_{\mathbb{S}}^{\oint} \circ \iota_{\mathbb{S}}^{\oint} : \mathbb{S}^{\oint} \to \mathbb{S}^{\oint}$$

is the identity functor. But this follows from the fact that the functor $\iota_{\mathbb{S}}^{\oint}$ is fully faithful. □

A.2. **Proof of Theorem A.1.5.**

A.2.1. To prove point (i) of the theorem, let us explicitly describe the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat}$$

corresponding to the projection

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\mathrm{op}}.$$

Namely, this functor sends $[m]$ to a coCartesian fibration over $((\mathbf{\Delta}_{\mathrm{actv}})_{[m]/})^{\mathrm{op}}$, whose fiber over an active map $\gamma : [m] \to [n]$ is

$$\{\gamma\} \underset{((\mathbf{\Delta}_{\mathrm{actv}})_{[m]/})^{\mathrm{op}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[m]\}\right) = \mathrm{Seq}_n(\mathbb{S}),$$

and where for active map $\alpha : [n_1] \to [n_2]$ the corresponding functor

$$\mathrm{Seq}_{n_2}(\mathbb{S}) \to \mathrm{Seq}_{n_1}(\mathbb{S})$$

is induced by α.

For a map $\beta : [m_1] \to [m_2]$, the corresponding functor

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[m_2]\} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[m_1]\} \tag{A.2}$$

is described as follows.

For an active map $\gamma_2 : [m_2] \to [n_2]$, the category of factorizations of $\gamma_2 \circ \beta$ as

$$[m_1] \overset{\gamma_1'}{\to} [n_1'] \overset{\alpha'}{\to} [n_2]$$

has a final object

$$[m_1] \overset{\gamma_1}{\to} [n_1] \overset{\alpha}{\to} [n_2]$$

In fact, α is the injection of the sub-segment with the smallest element $\gamma_2 \circ \beta(0)$ and the largest element $\gamma_2 \circ \beta(m_1)$.

The corresponding functor in (A.2) sends

$$\{\gamma_2\} \underset{((\mathbf{\Delta}_{\mathrm{actv}})_{[m_2]/})^{\mathrm{op}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[m_2]\}\right) \to$$
$$\to \{\gamma_1\} \underset{((\mathbf{\Delta}_{\mathrm{actv}})_{[m_1]/})^{\mathrm{op}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\mathrm{op}}}{\times} \{[m_1]\}\right)$$

and equals the functor

$$\mathrm{Seq}_{n_2}(\mathbb{S}) \to \mathrm{Seq}_{n_1}(\mathbb{S})$$

is induced by α.

The verification of Conditions (0)-(2) for being an $(\infty,2)$-category is now straightforward. It is equally easy to see that the functor $\iota_{\mathbb{S}}^{\oint}$ sends coCartesian arrows over inert arrows in $\mathbf{\Delta}$ to coCartesian arrows.

A.2.2. Let us now show that the map

$$\mathrm{Maps}_{2\text{-Cat}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}), \mathbb{T}) \to \mathrm{Maps}_{2\text{-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathbb{S}, \mathbb{T})$$

is an isomorphism.

Given $\mathbb{T} \in 2\text{-Cat}$, the operation of relative left Kan extension along $\iota_{\mathbb{S}}^{\oint}$ gives rise to a fully faithful embedding of spaces

$$\mathrm{Maps}_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathbb{S}^{\oint}, \mathbb{T}^{\oint}) \to \mathrm{Maps}_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}, \mathbb{T}^{\oint}). \tag{A.3}$$

Let

$$\mathrm{Maps}'_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathbb{S}^{\oint}, \mathbb{T}^{\oint}) \subset \mathrm{Maps}_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathbb{S}^{\oint}, \mathbb{T}^{\oint})$$

be the subspace consisting of functors that send coCartesian arrows over inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to coCartesian morphisms. Let

$$\mathrm{Maps}'_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}, \mathbb{T}^{\oint}) \subset \mathrm{Maps}_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}, \mathbb{T}^{\oint})$$

be the subspace consisting of functors that send all coCartesian arrows to coCartesian morphisms. We will show that the map (A.3) defines an isomorphism

$$\mathrm{Maps}'_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathbb{S}^{\oint}, \mathbb{T}^{\oint}) \to \mathrm{Maps}'_{1\text{-Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}}(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}, \mathbb{T}^{\oint}). \tag{A.4}$$

A.2.3. Note that the functor

$$\iota_{\mathbb{S}}^{\oint} : \mathbb{S}^{\oint} \to \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint}$$

admits a right adjoint, to be denoted $\rho_{\mathbb{S}}^{\oint}$. Explicitly, for every m and $\gamma : [m] \to [n]$, the functor $\rho_{\mathbb{S}}^{\oint}$ makes the following diagram commutative

$$\begin{array}{ccc}
\{\gamma\} \underset{((\mathbf{\Delta}_{\text{actv}})_{[m]/})^{\text{op}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[m]\}\right) & \longrightarrow & \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \\
\downarrow{\sim} & & \downarrow{\rho_{\mathbb{S}}^{\oint}} \\
\mathrm{Seq}_n(\mathbb{S}) & \longrightarrow & \mathbb{S}^{\oint}.
\end{array}$$

In particular, we note that $\rho_{\mathbb{S}}^{\oint}$ does *not* respect the projections

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbf{\Delta}^{\text{op}} \text{ and } \mathbb{S}^{\oint} \to \mathbf{\Delta}^{\text{op}}.$$

We have the following general assertion:

Lemma A.2.4. *Suppose we have a diagram of $(\infty,1)$-categories*

$$\begin{array}{ccc}
\mathbf{C}' & \xrightarrow{\iota} & \mathbf{C} \\
& \searrow \quad \swarrow & \\
& \mathbf{I} &
\end{array}$$

such that ι is fully faithful and admits a right adjoint ρ. Then for any coCartesian fibration $\mathbf{D} \to \mathbf{I}$, relative left Kan extension gives a fully faithful embedding

$$\mathrm{Maps}_{1\text{-Cat}_{/\mathbf{I}}}(\mathbf{C}', \mathbf{D}) \hookrightarrow \mathrm{Maps}_{1\text{-Cat}_{/\mathbf{I}}}(\mathbf{C}, \mathbf{D})$$

with the image consisting of functors $F : \mathbf{C} \to \mathbf{D}$ over $\mathbf{I}$ such that for every $c \in \mathbf{C}$, the counit of the adjunction $\iota \circ \rho(c) \to c$ induces the arrow

$$F(\iota \circ \rho(c)) \to F(c)$$

in $\mathbf{D}$ that is coCartesian over $\mathbf{I}$.

A.2.5. Applying this lemma, we need to show that for a functor

$$F : \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \to \mathbb{T}^{\oint}$$

the following conditions are equivalent:

(1) F takes coCartesian arrows to coCartesian arrows;
(2) F takes the arrows coming from the counit of the adjunction $\iota_{\mathbb{S}}^{\oint} \circ \rho_{\mathbb{S}}^{\oint} \to \mathrm{id}$ and also arrows of the form $\iota_{\mathbb{S}}^{\oint}(f)$, where f is a coCartesian arrow in $\mathbb{S}^{\oint}$ lying over an inert map in $\mathbf{\Delta}^{\text{op}}$, to coCartesian arrows.

We have the following general observation:

Lemma A.2.6. *Let $\mathbf{D} \to \mathbf{I}$ be a coCartesian fibration of $(\infty,1)$-categories. Then an arrow in $\mathbf{D}$ is coCartesian over $\mathbf{I}$ if and only if its image in $\mathbf{D}^{1\text{-ordn}}$ is coCartesian over $\mathbf{I}^{1\text{-ordn}}$.*

This lemma allows to replace the verification of the equivalence of conditions (1) and (2) above to the case when $\mathbb{T}$ (and hence also $\mathbb{S}$) is an ordinary 2-category. In this case the assertion is straightforward.

A.3. **Quasi-invertible 1-morphisms.**

A.3.1. Since
$$\mathrm{Seq}_0(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})) \simeq \mathrm{Seq}_0(\mathbb{S}),$$
the categories $\mathbb{S}_0$ and $\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})$ have the same spaces of objects.

Note that the subcategory
$$(\mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})))^{\text{invert}} \subset \mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}))$$
identifies with
$$\mathrm{Seq}_0(\mathbb{S}) \simeq \{[0]\} \underset{\mathbf{\Delta}_{\text{actv}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[1]\}\right) \subset$$
$$\subset \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[1]\} = \mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})).$$

A.3.2. We shall say that a 1-morphism is *quasi-invertible* if it belongs to the full subcategory, to be denoted $(\mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})))^{\text{q-invert}}$, and equal to
$$\mathrm{Seq}_0(\mathbb{S}) \simeq (\mathrm{Seq}_1(\mathbb{S}))^{\text{invert}} \subset \mathrm{Seq}_1(\mathbb{S}) \simeq \{[1]\} \underset{\mathbf{\Delta}_{\text{actv}}}{\times} \left(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[1]\}\right) \subset$$
$$\subset \mathrm{RLax}_{\text{non-untl}}(\mathbb{S})^{\oint} \underset{\mathbf{\Delta}^{\text{op}}}{\times} \{[1]\} = \mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})).$$

Remark A.3.3. Note that we thus obtain two *different* fully faithful functors
$$\mathrm{Seq}_0(\mathbb{S}) \simeq (\mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})))^{\text{invert}} \hookrightarrow \mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}))$$
and
$$\mathrm{Seq}_0(\mathbb{S}) \simeq (\mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})))^{\text{q-invert}} \hookrightarrow \mathrm{Seq}_1(\mathrm{RLax}_{\text{non-untl}}(\mathbb{S})).$$
By construction, these functors are connected by a natural transformation (from the former to the latter).

A.3.4. We observe:

Lemma A.3.5. *A non-unital right-lax functor* $\mathbb{S} \dashrightarrow \mathbb{T}$ *is unital if and only if the corresponding functor*
$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S}) \to \mathbb{T}$$
sends quasi-invertible 1-morphisms to isomorphisms.

Appendix B. Localizations on 1-morphisms

B.1. **The notion of localization on 1-morphisms.**

B.1.1. Let $\mathbf{C}$ be an $(\infty,1)$-category, and let $\mathbf{C}' \subset \mathbf{C}$ be a 1-full subcategory with the same class of objects. (I.e., the datum of $\mathbf{C}$ amounts to specifying a class of 1-morphisms containing all isomorphisms and closed under compositions).

Recall that the localization of $\mathbf{C}$ with respect to $\mathbf{C}'$ is a pair
$$(\mathbf{C}, F_{\text{can}} : \mathbf{C} \to \widetilde{\mathbf{C}}_{\text{can}}),$$
universal with respect to functors $F : \mathbf{C} \to \widetilde{\mathbf{C}}$ that map 1-morphisms from $\mathbf{C}'$ to isomorphisms.

B.1.2. Let $F : \mathbb{S} \to \mathbb{T}$ be a functor between $(\infty,2)$-categories.

Definition B.1.3. *We shall say that* F *is a* localization on 1-morphisms *if:*

1. *The functor* $\mathrm{Seq}_0(\mathbb{S}) \to \mathrm{Seq}(\mathbb{T})_0$ *is an isomorphism (in* Spc*);*
2. *The functor* $\mathrm{Seq}_1(\mathbb{S}) \to \mathrm{Seq}(\mathbb{T})_1$ *is a localization.*

B.1.4. We claim:

Proposition B.1.5. *For a functor $F:\mathbb{S}\to\mathbb{T}$, the following are equivalent:*

(1) *F is a localization on 1-morphisms;*

(2) *The corresponding functor $\mathbb{S}^{\oint}\to\mathbb{T}^{\oint}$ is a localization.*

Proof. Follows from the next general lemma:

Lemma B.1.6. *Let $\mathbf{C}\to\mathbf{I}$ and $\mathbf{D}\to\mathbf{I}$ be coCartesian fibrations, and let $F:\mathbf{C}\to\mathbf{D}$ be a functor compatible with the projections to $\mathbf{I}$ such that F sends coCartesian arrows to coCartesian arrows. Then F is a localization if and only if for every $\mathbf{i}\in\mathbf{I}$ the corresponding functor $\mathbf{C}\underset{\mathbf{I}}{\times}\{\mathbf{i}\}\to\mathbf{D}\underset{\mathbf{I}}{\times}\{\mathbf{i}\}$ is a localization.*

□

As a corollary, we obtain:

Corollary B.1.7. *Let $\mathbb{S}\to\mathbb{T}$ be a localization on 1-morphisms. Then for any $\mathbb{X}\in$ 2-Cat, the maps*

$$\mathrm{Maps}_{\text{2-Cat}}(\mathbb{T},\mathbb{X})\to\mathrm{Maps}_{\text{2-Cat}}(\mathbb{S},\mathbb{X}),\ \mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}}}(\mathbb{T},\mathbb{X})\to\mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}}}(\mathbb{S},\mathbb{X})$$

and

$$\mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathbb{T},\mathbb{X})\to\mathrm{Maps}_{\text{2-Cat}_{\text{right-lax}_{\text{non-untl}}}}(\mathbb{S},\mathbb{X})$$

are fully faithful.

B.1.8. It is easy to see that if $\mathbb{S}\to\mathbb{T}$ is a localization on 1-morphisms, then for any $\mathbb{X}\in$ 2-Cat, so is the functor

$$\mathbb{S}\times\mathbb{X}\to\mathbb{S}\times\mathbb{T}.$$

From here we obtain:

Corollary B.1.9. *Let $\mathbb{S}\to\mathbb{T}$ be a localization on 1-morphisms. Then for any $\mathbb{X}\in$ 2-Cat, the functor*

$$\mathrm{Funct}(\mathbb{T},\mathbb{X})\to\mathrm{Funct}(\mathbb{S},\mathbb{X})$$

is fully faithful.

B.2. **Description of localizations.**

B.2.1. We have:

Proposition B.2.2. *Let $\mathbb{S}$ be an $(\infty,2)$-category. The following pieces of data are equivalent:*

(i) *The datum of a functor $\mathbb{S}\to\mathbb{T}$, which is a localization on 1-morphisms.*

(ii) *The datum of a functor $\mathbb{S}^{\text{2-ordn}}\to\mathbb{T}^{\text{2-ordn}}$, which is a localization on 1-morphisms.*

(iii) *The datum of a subset of isomorphism classes of morphisms in $\mathrm{Seq}_1(\mathbb{S})$ that contains all isomorphisms and is closed under the composition operation*

$$\pi_0(\mathrm{Seq}_1(\mathbb{S}))\underset{\pi_0(\mathrm{Sq}(\mathbb{S})_0)}{\times}\pi_0(\mathrm{Seq}_1(\mathbb{S}))\to\pi_0(\mathrm{Seq}_1(\mathbb{S})).$$

Proof. Follows from the next general lemma:

Lemma B.2.3. *Let* $\mathbf{C} \to \mathbf{I}$ *be a coCartesian fibration. Then the datum of a localization* $F : \mathbf{C} \to \mathbf{D}$*, such that* $\mathbf{D}$ *is also a coCartesian fibration over* $\mathbf{I}$ *and* F *sends coCartesian arrows to coCartesian arrows is equivalent to the datum of a localization* $\mathbf{C} \underset{\mathbf{I}}{\times} \{\mathbf{i}\} \to \mathbf{D}_{\mathbf{i}}$ *for each* $\mathbf{i} \in \mathbf{I}$*, such that for every 1-morphism* $\mathbf{i}_1 \to \mathbf{i}_2$ *in* $\mathbf{I}$ *the corresponding functor*

$$\mathbf{C} \underset{\mathbf{I}}{\times} \{\mathbf{i}_1\} \to \mathbf{C} \underset{\mathbf{I}}{\times} \{\mathbf{i}_2\}$$

sends the 1-morphisms that become isomorphisms on $\mathbf{D}_{\mathbf{i}_1}$ *to 1-morphisms that become isomorphisms on* $\mathbf{D}_{\mathbf{i}_2}$*.*

□

B.2.4. As a corollary we obtain:

Corollary B.2.5. *For* $\mathbb{S} \in 2\text{-Cat}$*, the canonical functors*

$$\lambda_{\mathbb{S}} : \mathrm{RLax}_{\text{non-untl}}(\mathbb{S}) \to \mathbb{S},$$

$$\mathrm{RLax}_{\text{non-untl}}(\mathbb{S} \times \mathbb{T}) \to \mathbb{S} \circledast \mathbb{T} \text{ and } \mathbb{S} \circledast \mathbb{T} \to \mathbb{S} \times \mathbb{T}$$

are localizations on 1-morphisms.