NOTES ON GEOMETRIC LANGLANDS: STACKS

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This paper isn't even a paper. I will try to collect some basic definitions and facts about stacks in the DG setting that will be used in other installments of the Notes. Nearly all of the material presented here I have learned from Jacob Lurie, but it is nearly all contained in [TV1] and [TV2]. Most of the statements left unproved in the text can be found in [Lu1].

Conventions. We'll be working with schemes and stacks over a fixed ground field k. Throughout the notes, we'll be assuming that $char(k) = 0$.

Note on the choice of topology. When defining stacks, we will impose the descent condition with respect to the fppf topology. Another natural choice would be the smooth ($\sim$ étale) topology. However, as our main object of interest is Artin stacks, a fundamental result of [To], generalizing a theorem of Artin's, says that the two notions of Artin stack are actually equivalent.

1. Prestacks

1.1. Definition.

1.1.1. Let $\mathrm{DGSch}^{\mathrm{aff}}$ be the category of affine DG schemes over k, i.e., the category opposite to that of connective DG algebras over k (i.e., DG algebras over k concentrated in non-positive cohomological degrees).

By a prestack we shall mean a functor $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd}$. We shall denote by PreStk the $(\infty, 1)$-category of prestacks.

1.1.2. We shall say that $S \in \mathrm{DGSch}^{\mathrm{aff}}$ is n-coconnective if $S = \mathrm{Spec}(A)$ and $\pi_i(A) = 0$ for $i > n$. We shall denote the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}$ by n-coconnective objects by ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$. For $n = 0$ we recover $\mathrm{Sch}^{\mathrm{aff}}$, the category of classical affine schemes.

The embedding ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ admits a right adjoint, denoted $S \mapsto \tau^{\leq n}(S)$ and given by the usual procedure of cohomological truncation above degree $-n-1$ at the level of rings, i.e.,

$$\tau^{\leq n}(\mathrm{Spec}(A)) := \mathrm{Spec}(\tau^{\geq -n}(A))$$

(the truncation functor on the category of connective k-algebras is compatible with the corresponding truncation functor on Vect_k).

We shall say that $S \in \mathrm{DGSch}^{\mathrm{aff}}$ is *eventually coconnective* if it belongs to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ for some n. We shall denote the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}$ spanned by eventually coconnective objects by ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$.

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1.1.3. Let ${}^{\leq n}\mathrm{PreStk}$ denote the category of functors $({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd}$. Restriction defines a functor $\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$, which we will denote by $\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y}$. It admits a fully faithful left adjoint, given by the left Kan extension

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}\,.$$

Thus, ${}^{\leq n}\mathrm{PreStk}$ is a co-localization of PreStk. We denote the resulting co-localization functor

$$\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

by $\mathcal{Y} \mapsto \tau^{\leq n}(\mathcal{Y})$.

Remark 1.1.4. The usage of the symbol $\tau^{\leq n}$ diverges from the one accepted by Lurie: he employs $\tau^{\leq n}$ to denote the corresponding truncation of the Postnikov tower, whereas we denote the latter by the symbol $\mathsf{P}^{\leq n}$, see Sect. 1.1.7 below.

Tautologically, if $\mathcal{Y}$ is representable by an affine DG scheme $S = \mathrm{Spec}(A)$, then the above two meanings of $\tau^{\leq n}$ coincide: the prestack $\tau^{\leq n}(\mathcal{Y})$ is representable by the affine DG scheme $\tau^{\leq n}(S)$.

1.1.5. We shall say that $\mathcal{Y} \in \mathrm{PreStk}$ is n-coconnective if it belongs to the essential image of the functor $\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$.

For example, an affine DG scheme is n-coconnective as an affine DG scheme in the sense of Sect. 1.1.2 if and only if its image under the Yoneda functor is n-coconnective as a prestack.

We shall often identify ${}^{\leq n}\mathrm{PreStk}$ with its essential image under the above functor, and thus think of ${}^{\leq n}\mathrm{PreStk}$ as a full subcategory of PreStk.

We shall say that $\mathcal{Y} \in \mathrm{PreStk}$ is eventually coconnective if it is n-coconnective for some n. We shall denote the full subcategory of eventually coconnective objects of PreStk by ${}^{<\infty}\mathrm{PreStk}$.

1.1.6. *Classical prestacks.* Let $n = 0$. We shall call objects of ${}^{\leq 0}\mathrm{PreStk}$ "classical" prestacks, and use for it also the alternative notation ${}^{cl}\mathrm{PreStk}$.

We shall also denote the corresponding restriction functor $\mathcal{Y} \mapsto {}^{cl}\mathcal{Y}$, and the corresponding localization functor

$$\mathrm{PreStk} \to {}^{cl}\mathrm{PreStk} \to \mathrm{PreStk}$$

by $\mathcal{Y} \mapsto \tau^{cl}(\mathcal{Y})$.

1.1.7. *Truncatedness.* For a fixed n, we shall say that $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ is k-truncated if, as a functor

$$({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd},$$

it takes values in the full subcategory of $(n+k)\text{-}\mathrm{Grpd} \subset \infty\text{-}\mathrm{Grpd}$ of $(n+k)$-groupoids. I.e., for every $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, the homotopy type

$$\mathcal{Y}(S) = \mathrm{Maps}(S, \mathcal{Y})$$

must have vanishing homotopy groups π_i for $i > n$

For example, an n-coconnected affine DG scheme is 0-truncated (for two connective and n-coconnective DG algebras A and B, the space $\mathrm{Maps}(A, B)$ has vanishing homotopy groups π_i for $i > n$).

In fact for any DG scheme (not necessarily affine), its restriction to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ is k-truncated as an object of ${}^{\leq n}\mathrm{PreStk}$. Furthermore, this is true for algebraic spaces. Part of the definition of a k-Artin stack is that for any n, its restriction to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ is k-truncated.

As another extreme example, to any homotopy type $K \in \infty\text{-}\mathrm{Grpd}$ one can associate the corresponding constant prestack $\underline{K}$:

$$\underline{K}(S) := K,$$

and $\underline{K}$ is k-truncated if and only if K is.

We have a fully faithful embedding of the $(n+k+1,1)$-category of n-coconnective and k-truncated prestacks into all n-coconnective prestacks. This functor admits a left adjoint, given by truncation at the $(n+k)$-level of the Postnikov tower. We shall denote the latter functor by

$$\mathcal{Y} \mapsto \mathsf{P}^{\leq k}(\mathcal{Y}).$$

When $n=0$ and $k=1$, we shall call objects of the resulting $(2,1)$-category "ordinary classical prestacks". I.e., this category is by definition that of functors from the category of classical affine schemes to that of ordinary groupoids.

When $n=0$ and $k=1$, we obtain the category of presheaves of sets on $\mathrm{Sch}^{\mathrm{aff}}$.

1.1.8. *The right Kan extension.* The restriction functor

$$\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y} : \mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$$

admits also a right adjoint, given by right Kan extension. This functor lacks a clear geometric meaning. However, it can be explicitly described: by adjunction we have

$$\mathrm{RKE}_{\leq m\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y})(S) \simeq \mathcal{Y}(\tau^{\leq n}(S)).$$

1.2. **Convergence.**

1.2.1. Let $\mathcal{Y}$ be a prestack. We say that $\mathcal{Y}$ is convergent if for $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the map

$$\mathcal{Y}(S) \to \underset{n}{lim}\, \mathcal{Y}(\tau^{\leq n}(S))$$

is an isomorphism.

Remark 1.2.2. As we shall see in the sequel, all prestacks "of geometric nature", such as DG schemes and Artin stacks (and also DG indschemes), are convergent. The idea of the notion of convergence is that if we perceive a connective DG algebra as built iteratively by adding higher and higher homotopy groups, the value of our prestack on such an algebra is determined by its values on the above sequence of truncations. In other words, the convergence condition on a prestack becomes necessary if we ever want to approach this prestack via deformation theory.

Here is, however, an example of a non-convergent prestack (with values in $\infty\text{-}\mathrm{Cat}$ rather than $\infty\text{-}\mathrm{Grpd}$): namely, one that associates to an affine scheme $S = \mathrm{Spec}(A)$ the category A-mod, i.e., this prestack is what is denoted QCoh in [GL:QCoh].

Tautologically, we have:

Lemma 1.2.3. *A prestack $\mathcal{Y}$ is convergent if and only if, as a functor* $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd}$ *is it the* right *Kan extension from the subcategory* ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}} \subset \mathrm{DGSch}^{\mathrm{aff}}$.

Since every DG algebra A maps isomorphically to $\underset{n}{lim}\, \tau^{\geq -n}(A)$, we have:

Lemma 1.2.4. *Any prestack representable by an affine DG scheme is convergent.*

1.2.5. Let ${}^{conv}\mathrm{PreStk} \subset \mathrm{PreStk}$ denote the full subcategory of convergent prestacks. This embedding admits a left adjoint, which we call convergent-completion and denote $\mathcal{Y} \mapsto \widehat{\mathcal{Y}}$, and which is given by

$$\widehat{\mathcal{Y}}(S) = \underset{n}{lim}\, \mathcal{Y}(\tau^{\leq n}(S)).$$

1.2.6. Consider the canonical map

$$\underset{n}{colim}\, \tau^{\leq n}(\mathcal{Y}) \to \mathcal{Y}.$$

Tautologically, $\mathcal{Y}_1 \in \text{PreStk}$ is convergent if and only if for every $\mathcal{Y}$, the map

$$\text{Maps}(\mathcal{Y}, \mathcal{Y}_1) \to \text{Maps}(\underset{n}{colim}\, \tau^{\leq n}(\mathcal{Y}), \mathcal{Y}_1) = \underset{n}{lim}\, \text{Maps}(\tau^{\leq n}(\mathcal{Y}), \mathcal{Y}_1)$$

is an isomorphism.

Remark 1.2.7. Note that the *left* Kan extension functor

$$\text{LKE}_{^{\leq n}\text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}} : {}^{\leq n}\text{PreStk} \to \text{PreStk}$$

does not map into ${}^{conv}\text{PreStk}$.

1.3. **The (almost) finite type condition.**

1.3.1. *Affine schemes almost of finite type.*

We say that an affine DG-scheme $\text{Spec}(A)$ is *almost of finite type* if $\pi_0(A)$ is of finite type, each $\pi_i(A)$ is f.g. as a module over $\pi_0(A)$.

Let $\text{DGSch}^{\text{aff}}_{\text{aft}}$ denote the full subcategory of $\text{DGSch}^{\text{aff}}$ consisting of affine schemes almost of finite type.

Denote by ${}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}}$ the intersection $\text{DGSch}^{\text{aff}}_{\text{aft}} \cap {}^{\leq n}\text{DGSch}^{\text{aff}}$. Denote also ${}^{<\infty}\text{DGSch}^{\text{aff}}_{\text{ft}} := {}^{<\infty}\text{DGSch}^{\text{aff}} \cap \text{DGSch}^{\text{aff}}_{\text{aft}}$.

1.3.2. Let $\mathcal{Y}$ be an object of ${}^{\leq n}\text{PreStk}$ for some n. We say that it is *locally of finite type* if it is the left Kan extension (of its own restriction) along the embedding

$${}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}} \hookrightarrow {}^{\leq n}\text{DGSch}^{\text{aff}}.$$

We denote the resulting full subcategory of ${}^{\leq n}\text{PreStk}$ by ${}^{\leq n}\text{PreStk}_{\text{lft}}$.

In other words, we can identify ${}^{\leq n}\text{PreStk}_{\text{lft}}$ with the category of functors

$$({}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}})^{op} \to \infty\text{-Grpd},$$

and we have a pair of mutually adjoint functors

$${}^{\leq n}\text{PreStk}_{\text{lft}} \rightleftarrows {}^{\leq n}\text{PreStk},$$

given by restriction and left Kan extension along ${}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}} \hookrightarrow {}^{\leq n}\text{DGSch}^{\text{aff}}$, respectively, where the latter functor is fully faithful.

Since the subcategory ${}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}} \subset {}^{\leq n}\text{DGSch}^{\text{aff}}$ is stable under retracts, general category theory implies:

Lemma 1.3.3. *Let S be an object of ${}^{\leq n}\text{DGSch}^{\text{aff}}$. It belongs to ${}^{\leq n}\text{DGSch}^{\text{aff}}_{\text{ft}}$ if and only if the prestack represented by it belongs to ${}^{\leq n}\text{PreStk}_{\text{lft}}$.*

1.3.4. We have the following basic fact:

Lemma 1.3.5.

(a) *The objects of* ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ *are co-compact in* ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$.

(b) *Every object of* ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ *is a filtered colimit of objects from* ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Since ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ is closed under retracts, we obtain that ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ equals the subcategory of co-compact objects of ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, and

$$ {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \simeq \mathrm{Ind}({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}). $$

Moreover, we have:

Corollary 1.3.6. *An object* $\mathcal{F} \in {}^{\leq n}\mathrm{PreStk}$ *belongs to* ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ *if and only if it takes filtered limits in* ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ *to colimits in* $\infty\text{-}\mathrm{Grpd}$.

1.3.7. Evidently, the restriction functor ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \leftarrow {}^{\leq n}\mathrm{PreStk}$ commutes with limits and colimits. The functor

$$ \mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}, $$

being a left adjoint commutes with colimits.

In addition, we have the following:

Lemma 1.3.8. *The functor* $\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ *commutes with finite limits.*

Proof. This follows from the fact that for every object $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, the category of objects of ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ *under* it is co-filtered (indeed, this category contains pullbacks). □

1.3.9. We say that $\mathcal{Y} \in \mathrm{PreStk}$ is *locally almost of finite type* if the following conditions hold:

(1) $\mathcal{Y}$ is convergent.
(2) For every n, we have ${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$

We denote the corresponding full subcategory by

$$ \mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}\,. $$

In particular, if $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, then ${}^{cl}\mathcal{Y}$ is an object of ${}^{cl}\mathrm{PreStk}$, which is locally of finite type, i.e., it is a classical prestack locally of finite type.

1.3.10. Note that by Remark 1.2.7, the left Kan extension functor does *not* send ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ to $\mathrm{PreStk}_{\mathrm{laft}}$: the resulting prestack will satisfy the second condition, but in general, not the first one.

However, if $\mathcal{Y} \in \mathrm{PreStk}$ is obtained as a left Kan extension functor of an object of ${}^{\leq n}\mathrm{PreStk}$, which belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, then its convergent completion $\widehat{\mathcal{Y}}$ will belong to $\mathrm{PreStk}_{\mathrm{laft}}$.

1.3.11. More generally, we can identify $\mathrm{PreStk}_{\mathrm{laft}}$ with the category

$$\mathrm{Funct}(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{op}, \infty\text{-}\mathrm{Grpd}).$$

Indeed, the functor in one direction is given by restriction along

$$ {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}. \tag{1}$$

The functor in the opposite direction is given by first applying the *left* Kan extension along

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}},$$

followed by the *right* Kan extension along

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}.$$

From Lemma 1.3.8 we obtain:

Corollary 1.3.12. *The subcategory* $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$ *is closed under finite limits.*

2. Descent and stacks

2.1. **Flat morphisms.**

2.1.1. Let us recall the notion of flatness for a morphism between affine DG schemes:

A map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ between affine DG schemes is said to be flat if $H^0(B)$ is flat as a module over $H^0(A)$, plus the following equivalent conditions hold:

- The natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(A) \to H^i(B)$$
is an isomorphism for every i.
- For any A-module M, the natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(M) \to H^i(B \underset{A}{\otimes} M)$$
is an isomorphism for every i.
- If an A-module N is concentrated in degree 0 then so is $B \underset{A}{\otimes} N$.

The above notion is easily seen to be local in the Zariski topology [1] in both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$.

2.1.2. *Smooth and pppf morphisms.* Let $f : S_1 \to S_2$ be a morphism of affine DG schemes. We shall say that it is *smooth* (resp., pppf[2]) if the following conditions hold:

(1) f is flat (in particular, the base-changed DG scheme ${}^{cl}S_2 \underset{S_2}{\times} S_1$ is classical), and

(2) the map of classical schemes ${}^{cl}S_2 \underset{S_2}{\times} S_1 \to {}^{cl}S_2$ is smooth (resp., of finite presentation).

Remark 2.1.3. The motivation for the above definition of smoothness is the following: we want to impose the condition that the relative cotangent complex $T^*_{S_1/S_2}$ be a vector bundle on S_2. However, the latter property can be checked via base change by classical schemes.

[1] By definition, the Zariski topology on an affine DG scheme $\mathrm{Spec}(A)$ is the Zariski topology on $\mathrm{Spec}({}^{cl}A)$.

[2] pppf=plat+de présentation presque finie. We shall write ppf when talking about eventually coconnective affine DG schemes.

We say that a morphism is $f : S_1 \to S_2$ is fpppf (surjective + ppppf) [3] if it is pppf and the corresponding morphism ${}^{cl}S_1 \to {}^{cl}S_2$ is surjective.

2.2. The descent condition.

2.2.1. Let $\mathcal{Y}$ be a prestack. We say that it satisfies fpppf descent if whenever

$$f : S_1 \to S_2 \in \mathrm{DGSch}^{\mathrm{aff}}$$

is an fpppf morphism, the map

$$\mathcal{Y}(S_2) \to \mathrm{Tot}(\mathcal{Y}(S_1^\bullet/S_2))$$

is an isomorphism, where $S_1^\bullet/S_2$ is the simplicial object of $\mathrm{DGSch}^{\mathrm{aff}}$ equal to the Čech nerve of the map f.

2.2.2. We shall call prestacks that satisfy the above descent condition *stacks*, and denote the corresponding full subcategory of PreStk by Stk.

2.3. **Equivalences for fpppf.** We say that a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is an *fpppf equivalence* if it induces an isomorphism

$$\mathrm{Maps}(\mathcal{Y}_2, \mathcal{Y}) \to \mathrm{Maps}(\mathcal{Y}_1, \mathcal{Y})$$

whenever $\mathcal{Y} \in \mathrm{Stk}$.

The inclusion $\mathrm{PreStk} \hookleftarrow \mathrm{Stk}$ admits a left adjoint making Stk a localization of PreStk. Concretely, the functor $\mathrm{PreStk} \to \mathrm{Stk}$ is universal among functors that turn fpppf equivalences into isomorphisms, see [Lu0], Sect. 6.2.1.

We shall denote by L the corresponding localization functor, i.e., the composition

$$\mathrm{PreStk} \to \mathrm{Stk} \to \mathrm{PreStk}\,.$$

By [Lu0], Corollary 6.2.1.7, the functor L is left exact, i.e., commutes with finite limits.

2.3.1. *Čech covers.* Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk.

We say that f is an fpppf surjection if for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and an object $y_2 \in \mathcal{Y}_2(S)$ there exists an fpppf cover $\phi : S' \to S$, such that $\phi^*(y_2) \in \mathcal{Y}_2(S')$ belongs to the essential image of $f(S') : \mathcal{Y}_1(S') \to \mathcal{Y}_2(S')$.

The following is [Lu0], Cor. 6.2.3.5:

Lemma 2.3.2. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be an fpppf surjection. Then the induced map*

$$|\mathcal{Y}_1^\bullet/\mathcal{Y}_2|_{\mathrm{PreStk}} \to \mathcal{Y}_2$$

is an fpppf equivalence, where $\mathcal{Y}_1^\bullet/\mathcal{Y}_2$ is the Čech simplicial object of PreStk*, and* $|-|_{\mathrm{PreStk}}$ *denotes geometric realization taken in the category* PreStk.

Corollary 2.3.3. *For $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ as above and $\mathcal{Y} \in \mathrm{Stk}$, the induced map*

$$\mathrm{Maps}(\mathcal{Y}_2, \mathcal{Y}) \to \mathrm{Tot}(\mathrm{Maps}(\mathcal{Y}_1^\bullet/\mathcal{Y}_2, \mathcal{Y}))$$

is an isomorphism.

2.4. Descent and n-coconnectivity.

[3] fpppf=fidèdelement plat+de présentation presque finie. Note, however, that most authors write just "fppf" instead of "fpppf". We shall use fppf when talking about eventually coconnective affine DG schemes.

2.4.1. Let us denote by ${}^{\leq n}\mathrm{Stk}$ the full subcategory of ${}^{\leq n}\mathrm{PreStk}$ consisting of objects that satisfy descent for fppf covers $S_1 \to S_2 \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$.

We obtain that ${}^{\leq n}\mathrm{Stk}$ is a localization of ${}^{\leq n}\mathrm{PreStk}$. Let ${}^{\leq n}L$ denote the corresponding localization functor

$$ {}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{Stk} \to {}^{\leq n}\mathrm{PreStk}\,. $$

The construction of the sheafification functor ${}^{\leq n}L$ implies:

Lemma 2.4.2. *The functor* ${}^{\leq n}L : {}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ *sends k-truncated objects to k-truncated ones.*

2.4.3. The following results from the definitions:

Lemma 2.4.4.
(a) *The restriction functor* $\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ *sends* Stk *to* ${}^{\leq n}\mathrm{Stk}$.
(b) *The functor*

$$ \mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk} $$

sends fppf equivalences to fpppf equivalences.

2.4.5. Note now that the *right* Kan extension functor along ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$:

$$ \mathrm{RKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk} $$

tautologically sends ${}^{\leq n}\mathrm{Stk}$ to Stk. This implies that the restriction functor $\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y}$ sends fpppf equivalences to fppf equivalences.

Thus, from Lemma 2.4.4 we obtain:

Corollary 2.4.6. *For* $\mathcal{Y} \in \mathrm{PreStk}$ *we have:*

$$ {}^{\leq n}L({}^{\leq n}\mathcal{Y}) \simeq {}^{\leq n}(L(\mathcal{Y})). $$

2.4.7. *The notion of n-coconnective stack.* Note that the functor

$$ \mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk} $$

does *not* send ${}^{\leq n}\mathrm{Stk}$ to Stk. Instead, the left adjoint to the restriction functor ${}^{\leq n}\mathrm{Stk} \leftarrow \mathrm{Stk}$ is given by the composition

$$ {}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} := {}^{\leq n}\mathrm{Stk} \hookrightarrow {}^{\leq n}\mathrm{PreStk} \overset{\mathrm{LKE}}{\longrightarrow} \mathrm{PreStk} \overset{L}{\longrightarrow} \mathrm{Stk}\,. $$

The above left adjoint is fully faithful. Hence, we can identify ${}^{\leq n}\mathrm{Stk}$ with a full subcategory of Stk. We shall denote by ${}^{L}\tau^{\leq n} : \mathrm{Stk} \to \mathrm{Stk}$ the resulting localization functor

$$ \mathcal{Y} \mapsto {}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{\leq n}\mathcal{Y}). $$

By definition, ${}^{L}\tau^{\leq n} \simeq L \circ \tau^{\leq n}$.

We shall call objects of Stk that belong to the essential image of ${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$ n-coconnective stacks. I.e., $\mathcal{Y} \in \mathrm{Stk}$ is n-coconnective as a stack if and only if the adjunction map

$$ {}^{L}\tau^{\leq n}(\mathcal{Y}) \to \mathcal{Y} $$

is an isomorphism.

I.e., the functor ${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$ identifies the category ${}^{\leq n}\mathrm{Stk}$ with the full subcategory of PreStk spanned by n-coconnective stacks.

Warning: We emphasize again that, as a subcategories PreStk, it is *not* true that ${}^{\leq n}\mathrm{Stk}$ is contained in ${}^{\leq n}\mathrm{PreStk}$. That is to say, that a n-coconnective stack is not necessarily n-coconnective as a prestack.

For example, it is not clear that $\mathrm{LKE}_{{}^{\leq 0}\mathrm{DGSch}^{\mathrm{aff}}\hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathbb{P}^1)$ satisfies even Zariski descent. If it does not, then ${}^{L}\mathrm{LKE}_{{}^{\leq 0}\mathrm{DGSch}^{\mathrm{aff}}\hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathbb{P}^1)$ is 0-connective as a stack, but not as a prestack.

Note, however, that we do have an inclusion

$$\mathrm{Stk}\cap {}^{\leq n}\mathrm{PreStk} \subset {}^{\leq n}\mathrm{Stk}$$

as subcategories of PreStk.

We shall say that a stack is eventually coconnective if it is n-coconnective for some n.

We shall refer to objects of ${}^{\leq 0}\mathrm{Stk} =: {}^{cl}\mathrm{Stk}$ as "classical stacks", and also denote ${}^{L}\tau^{\leq n} =: {}^{L}\tau^{cl}$.

2.4.8. *Right Kan extensions from* ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$. Let $\mathcal{Y}'$ be a functor

$$({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd},$$

which we can think of as a compatible family of objects $\mathcal{Y}'_n \in {}^{\leq n}\mathrm{PreStk}$. Let

$$\mathcal{Y} := \mathrm{RKE}_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}\hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y}') \in \mathrm{PreStk}\,.$$

Lemma 2.4.9. *Assume that for all n, $\mathcal{Y}'_n \in {}^{\leq n}\mathrm{Stk}$. Then $\mathcal{Y}$ belongs to* Stk.

From here we obtain:

Corollary 2.4.10. *Suppose that $\mathcal{Y} \in \mathrm{PreStk}$ belongs to* Stk. *Then so does $\widehat{\mathcal{Y}}$.*

2.5. Descent and the "almost of finite type" condition.

2.5.1. Let n be a fixed integer. We can consider the fppf topology on the category ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$. Thus, we obtain a localization of ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ that we denote ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$.

We shall denote by ${}^{\leq n}L_{\mathrm{ft}}$ the corresponding localization functor

$${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}\,.$$

As in Lemma 2.4.2, we have:

Lemma 2.5.2. *The functor ${}^{\leq n}L_{\mathrm{ft}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ sends k-truncated objects to k-truncated ones.*

2.5.3. Consider the restriction functor for ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, i.e.,

$${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \leftarrow {}^{\leq n}\mathrm{PreStk}\,.$$

It is clear that it sends ${}^{\leq n}\mathrm{Stk}$ to ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$. By adjunction, the functor of left Kan extension

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}\hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}$$

sends fppf equivalences to fppf equivalences.

Moreover, the functor of *right* Kan extension

$$\mathrm{RKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}\hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}$$

sends ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$ to ${}^{\leq n}\mathrm{Stk}$. This implies:

Lemma 2.5.4. *For $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ we have:*

$${}^{\leq n}L(\mathcal{Y})|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}} \simeq {}^{\leq n}L_{\mathrm{ft}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}}).$$

2.5.5. Let us return to the functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}\,.$$

It is not clear, and probably not true, that this functor sends ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$ to ${}^{\leq n}\mathrm{Stk}$. However, as we have learned from J. Lurie, there is the following partial result:

Proposition 2.5.6. *Suppose that an object $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ is k-truncated for some k (see Sect. 1.1.7), and that $\mathcal{Y} \in {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$. Then the object*

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y})$$

of ${}^{\leq n}\mathrm{PreStk}$ belongs to ${}^{\leq n}\mathrm{Stk}$.

In what follows we shall use the notation

$${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} := {}^{\leq n}\mathrm{Stk} \cap {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}\,.$$

We shall refer to objects of the subcategory ${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$ of ${}^{\leq n}\mathrm{Stk}$ as "n-coconnective stacks locally of finite type".

We have the inclusion

$${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} \subset {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}\,.$$

Thus, the above proposition reads as saying that the essential image of this inclusion contains all truncated objects.

As a corollary of Proposition 2.5.6 and Lemma 2.5.2, we obtain:

Corollary 2.5.7. *For $\mathcal{Y} \in {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$, which is truncated, the natural map*

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}\left({}^{\leq n}L_{\mathrm{ft}}(\mathcal{Y})\right) \to {}^{\leq n}L\left(\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y})\right)$$

is an isomorphism.

2.5.8. *Proof of Proposition 2.5.6.* The proof will use the following assertion in algebraic geometry:

Let $f : S_1 \to S_2$ be an fppf morphism in ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$. Consider the category of Cartesian diagrams

$$\begin{array}{ccc} S_1 & \longrightarrow & S_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S_2 & \longrightarrow & S_2' \end{array}$$

with $S_2', S_1' \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$, and f' is fppf. Denote this category by f_{ft}. We have the natural forgetful functors

$$\{S_2 \to S_2', S_2' \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}\} \leftarrow f_{\mathrm{ft}} \to \{S_1 \to S_1', S_1' \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}\}. \tag{2}$$

Lemma 2.5.9. *Both functors in* (2) *are cofinal.*

Thus, let $\mathcal{Y}'$ be an object of ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$, and let $\mathcal{Y}$ be its left Kan extension to an object of ${}^{\leq n}\mathrm{PreStk}$. Let $f : S_1 \to S_2$ be a fppf cover. We need to check that the map

$$\mathcal{Y}(S_2) \to \mathrm{Tot}(\mathcal{Y}(S_1^\bullet/S_2)) \tag{3}$$

is an isomorphism.

First, we observe that the functor $\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ sends k-truncated objects to k-truncated ones. Hence, we can replace Tot, which is a limit in ∞-Grpd over the index category $\mathbf{\Delta}$, by the the corresponding limit in the category $(n+k)$-Grpd of $(k+n)$-groupoids over the index category $\mathbf{\Delta}^{\leq k+n}$. For the remainder of the proof, limits and colimits will be understood as taking place in $(n+k)$-Grpd.

We rewrite the left-hand side in (3) as

$$\underset{S_2\to S_2', S_2'\in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}}{colim} \mathcal{Y}'(S_2').$$

Applying Lemma 2.5.9 for the $\to$ functor, we rewrite the right-hand side in (3) as

$$\mathrm{Tot}\left(\underset{f_{\mathrm{ft}}}{colim}\, \mathcal{Y}(S_1'^{\bullet}/S_2')\right).$$

The category f_{ft} is filtered, as it contains push-outs. Since the limit Tot can be replaced by the finite limit $\mathrm{Tot}^{\leq n+k}$, we can commute the limit and the colimit in the above expression, and therefore it can be rewritten as

$$\underset{f_{\mathrm{ft}}}{colim}\, (\mathrm{Tot}(\mathcal{Y}(S_1'^{\bullet}/S_2'))).$$

By the descent condition for $\mathcal{Y}'$, the latter expression is isomorphic to $\underset{f_{\mathrm{ft}}}{colim}\, \mathcal{Y}(S_2')$. Applying Lemma 2.5.9 for the $\leftarrow$ functor, we obtain that

$$\underset{f_{\mathrm{ft}}}{colim}\, \mathcal{Y}(S_2') \simeq \underset{S_2\to S_2', S_2'\in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}}{colim} \mathcal{Y}'(S_2'),$$

as required.

□

2.6. **Stacks locally almost of finite type.**

2.6.1. By a similar token, we can consider the fppf topology on the category ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$, so that

$$\mathrm{PreStk}_{\mathrm{laft}} = \mathrm{Funct}\left(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})^{op}, \infty\text{-}\mathrm{Grpd}\right)$$

(see Sect. 1.3.11). Thus, we obtain a localization of $\mathrm{PreStk}_{\mathrm{laft}}$ that we denote $\mathrm{NearStk}_{\mathrm{laft}}$.

Let us denote by L_{laft} the corresponding localization functor

$$\mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{NearStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}.$$

2.6.2. Consider the functor

$$\mathrm{PreStk} \to \mathrm{PreStk}_{\mathrm{laft}}$$

given by restriction along

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}.$$

It is clear that this functor sends Stk to $\mathrm{NearStk}_{\mathrm{laft}}$. Moreover, as in Lemma 2.5.4 and Corollary 2.4.6, we obtain:

Lemma 2.6.3. *For $\mathcal{Y}\in \mathrm{PreStk}$ we have:*

$$L(\mathcal{Y})|_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}} \simeq L_{\mathrm{laft}}(\mathcal{Y}|_{{}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}}).$$

From Proposition 2.5.6 and Lemma 2.4.9 we obtain:

Corollary 2.6.4. *Let $\mathcal{Y}$ be an object of $\mathrm{NearStk}_{\mathrm{laft}}$, thought of as a functor*

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \to \infty\text{-}\mathrm{Grpd}$$

(see Sect. 1.3.11). Suppose that for each n, the restriction ${}^{\leq n}\mathcal{Y}$ of $\mathcal{Y}$ to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ is k_n-truncated for some $k_n\in\mathbb{N}$. Then $\mathcal{Y}\in\mathrm{Stk}$.

2.6.5. In what follows, we will denote the intersection

$$\mathrm{Stk} \cap \mathrm{PreStk}_{\mathrm{laft}}$$

by $\mathrm{Stk}_{\mathrm{laft}}$. We shall refer to objects of the subcategory $\mathrm{Stk}_{\mathrm{laft}} \subset \mathrm{Stk}$ as "stacks locally almost of finite type".

We have an evident inclusion

$$\mathrm{Stk}_{\mathrm{laft}} \subset \mathrm{NearStk}_{\mathrm{laft}}\,.$$

The above corollary reads that the essential image of $\mathrm{Stk}_{\mathrm{laft}}$ in $\mathrm{NearStk}_{\mathrm{laft}}$ contains all objects $\mathcal{Y}$, such that for every n, the restriction ${}^{\leq n}\mathcal{Y}$ of $\mathcal{Y}$ to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ is truncated.

3. DG schemes

3.1. Definitions.

3.1.1. Let S be an affine DG scheme. By an open DG subscheme $\overset{\circ}{S}$ of S we shall mean an object of PreStk equipped with a map $\overset{\circ}{S} \to S$ such that there exists an open subscheme ${}^{cl}\overset{\circ}{S} \subset {}^{cl}S$ with the property that for $T \in \mathrm{DGSch}^{\mathrm{aff}}$,

$$\overset{\circ}{S}(T) \subset S(T)$$

is the connected component consisting of those maps $T \to S$, for which the corresponding maps ${}^{cl}T \to {}^{cl}S$ factors through ${}^{cl}\overset{\circ}{S}$.

Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk. We say that it is an open embedding if for every $S \to \mathcal{Y}_2$ where $S \in \mathrm{DGSch}^{\mathrm{aff}}$, the map

$$S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$$

is the embedding of an open subscheme in the above sense.

3.1.2. We say that an object $Z \in \mathrm{PreStk}$ is a DG scheme if:

(1) $\mathcal{Y}$ satisfies fpppf descent.
(2) There exist a collection of affine DG schemes S_i, $i \in I$ and maps $f_i : S_i \to Z$, such that each f_i is an open embedding, and for every $T \in \mathrm{DGSch}^{\mathrm{aff}}$ and a map $T \to Z$, the open subsets ${}^{cl}(T \underset{Z}{\times} S_i) \subset {}^{cl}T$, $i \in I$ cover T.

Remark 3.1.3. One can show that condition (1) above can be replaced by a weaker one: namely, it is sufficient to require that Z satisfy Zariski descent. The latter means that instead of all fpppf morphisms $f : S' \to S$ we only consider Zariski covers, i.e., we require that S' be isomorphic to a disjoint union $\underset{i}{\cup} S'_i$, where each of the maps $f_i : S'_i \to S$ is an open embedding.

We shall denote the full subcategory of Stk spanned by DG schemes by DGSch.

3.1.4. The following assertion results from the definitions:

Lemma 3.1.5. *A DG scheme, regarded as an object of* PreStk*, is convergent.*

3.2. DG schemes and n-coconnectivity.

3.2.1. Replacing the category PreStk by ${}^{\leq n}\mathrm{PreStk}$, we obtain a category of ${}^{\leq n}$DG schemes that we denote by ${}^{\leq n}\mathrm{DGSch}$.

We shall also use the terminology "classical scheme" for a ${}^{\leq 0}$DG scheme.

It is easy to see that the restriction functor for ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ sends DGSch to ${}^{\leq n}\mathrm{DGSch}$ (replace the original Zariski cover S_i by ${}^{\leq n}S_i$).

In particular, if X is a DG scheme, the corresponding classical prestack ${}^{cl}X$ is a classical scheme.

We shall say that a DG scheme X is quasi-compact if the classical scheme ${}^{cl}X$ is. Equivalently, this means that X admits a Zariski cover by a finite collection of affine schemes.

3.2.2. The functor

$$ {}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{Stk} \hookrightarrow \mathrm{Stk} $$

sends ${}^{\leq n}\mathrm{DGSch}$ to DGSch, providing a left adjoint to the above functor ${}^{\leq n}\mathrm{DGSch} \leftarrow \mathrm{DGSch}$. This left adjoint is fully faithful, because this is so on the bigger category ${}^{\leq n}\mathrm{Stk}$.

We shall call a DG scheme "n-coconnective" if it is n-coconnective as an object of Stk. We obtain that the functor ${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$ identifies the category ${}^{\leq n}\mathrm{DGSch}$ with that of n-coconnective DG schemes.

Lemma 3.2.3. *For $X \in \mathrm{DGSch}$ the following conditions are equivalent:*

(1) *X is n-coconnective.*
(2) *For every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ equipped with an open embedding $S \to X$, we have $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$.*
(3) *X admits a Zariski cover by affine DG schemes belonging to ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$.*

Remark 3.2.4. We emphasize that an n-coconnective DG scheme is *not* necessarily n-connected as as a prestack, but it is n-coconnective as a stack.

3.2.5. We shall say that $X \in \mathrm{DGSch}$ is eventually coconnective if it is n-coconnective as a DG scheme for some n.

We shall say that $X \in \mathrm{DGSch}$ is locally eventually coconnective if for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ equipped with an open embedding $S \hookrightarrow X$, the affine DG scheme is eventually coconecctive. Equivalently, X is locally eventually coconnective if some/any affine cover of X consists of eventually coconnective affine DG schemes. (The point being that in the non-quasi-compact case, different members of the cover may be n-coconnective for different n's with no uniform bound.)

3.3. DG schemes locally almost of finite type.

3.3.1. We shall denote by ${}^{\leq n}\mathrm{DGSch}_{\mathrm{lft}}$ and $\mathrm{DGSch}_{\mathrm{laft}}$ the full subcategories of Stk

$$ \mathrm{DGSch} \cap {}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} \text{ and } \mathrm{DGSch} \cap \mathrm{Stk}_{\mathrm{laft}}, $$

respectively.

We have:

Lemma 3.3.2. *For $X \in {}^{\leq n}\mathrm{DGSch}$ (resp., $X \in \mathrm{DGSch}$) the following conditions are equivalent:*

(1) *$X \in {}^{\leq n}\mathrm{DGSch}_{\mathrm{lft}}$ (resp., $X \in \mathrm{DGSch}_{\mathrm{laft}}$)*
(2) *For every $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ (resp., $S \in \mathrm{DGSch}^{\mathrm{aff}}$) and an open embedding $S \to X$, we have $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ (resp., $S \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$).*
(3) *X admits a Zariski cover by affine DG schemes from ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ (resp., $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$).*

3.4. **Properties of morphisms.**

3.4.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk. We say that f is schematic if for any $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and $S \to \mathcal{Y}_2$, the Cartesian product

$$S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

is representable by an object of DGSch.

3.4.2. Since the properties of a morphism in $\mathrm{DGSch}^{\mathrm{aff}}$ of being flat/smooth/ppf/fpppf are local in the Zariski topology of the source, they transfer to the corresponding notions for morphisms in DGSch. We will refer to the latter to notions as plppf and fplppf, respectively, reserving ppf and fpppf to the case when the morphism in question is quasi-compact. [4]

Thus, by base change, we obtain the notion of a schematic-flat/schematic-smooth/schematic-plppf/ schematic-fplppf morphism in PreStk.

The next assertion also follows from the definitions:

Lemma 3.4.3. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map in ${}^{conv}\mathrm{PreStk}$. To test the property of f of being schematic (resp., schematic-flat, schematic-smooth, schematic-plppf, schematic-fplppf) it is enough to do so on affine schemes S belonging to ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}$. If, moreover, $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{laft}}$, then it is enough to take $S \in {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$.*

3.4.4. For a future use, let us give the following definition: a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is called a closed embedding if the corresponding map of classical prestacks ${}^{cl}\mathcal{Y}_1 \to {}^{cl}\mathcal{Y}_2$ is a closed embedding, i.e., becomes a closed embedding after a base change by an affine classical scheme.

4. Artin stacks

4.1. **Algebraic spaces.**

4.1.1. By definition, $\mathcal{Y} \in \mathrm{Stk}$ is an algebraic space if the following conditions hold:

(1) For every n, the restriction ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is 0-truncated (see Sect. 1.1.7).
(2) The diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic.
(3) There exists $Z \in \mathrm{DGSch}$ and a map $f : Z \to \mathcal{Y}$ (which is itself schematic by the previous point) which is smooth and surjective.

We shall call the pair (Z, f) an atlas for $\mathcal{Y}$.

One can show that if $\mathcal{Y}$ is an algebraic space, then the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is separated.

We shall say that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is representable if for every $S \to \mathcal{Y}_2$ with $S \in \mathrm{DGSch}^{\mathrm{aff}}$ the Cartesian product $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is an algebraic space in the above sense.

4.2. **k-Artin stacks.**

[4] plppf=plat+localement de présentation presque finie; fplppf=fidèlement plat+localement de présentation presque finie. We shall use plpf and fplpf for schemes that are locally eventually coconnective.

4.2.1. For an integer k, we define a full subcategory of Stk spanned by objects that we refer to as k-Artin stacks inductively. [5]

Along with this notion, we will define what it means for a morphism in PreStk to be k-representable, and for a k-representable morphism what it means to be flat (resp., surjective, fpppf, smooth).

4.2.2. We start with $k = 0$. By definition, $\mathcal{Y} \in \text{Stk}$ is a 0-Artin stack if it is an algebraic space.

We shall say that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is 0-representable if it is representable in the sense of Sect. 4.1.1.

We shall say that a representable morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is flat (resp., surjective, fpppf, smooth) if for every $S \to \mathcal{Y}_2$ with $S \in \text{DGSch}_{\text{aft}}$, and *any* atlas $(Z_S; f_S : Z_S \to S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1)$, the composed map

$$Z_S \xrightarrow{f_S} S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$$

is a flat (resp., surjective, fpppf, smooth) map of DG schemes.

Remark 4.2.3. We will show in Lemma 4.3.2 that the condition of a representable map to be flat/smooth/surjective does not depend on the choice of an atlas of $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$, i.e., it can be tested by using just one given atlas.

Since the functor $L : \text{PreStk} \to \text{PreStk}$ is left exact, we obtain:

Lemma 4.2.4. *If a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is k-representable, then so is the morphism $L(f) : L(\mathcal{Y}_1) \to L(\mathcal{Y}_2)$.*

4.2.5. Suppose the above notions have been defined for $k' = 0, ..., k - 1$.

We say that $\mathcal{Y} \in \text{Stk}$ is a k-Artin stack if the following conditions hold:

(1) For every n, the restriction ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\text{DGSch}^{\text{aff}}}$ is k-truncated (see Sect. 1.1.7).
(2) The diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is $(k-1)$-representable.
(3) There exists $Z \in \text{DGSch}$ and a map $f : Z \to \mathcal{Y}$ (which is a $(k-1)$-representable by the previous point) which is smooth and surjective.

We shall call the pair $f : Z \to \mathcal{Y}$ an atlas for $\mathcal{Y}$.

Remark 4.2.6. A fundamental theorem of Toën ([To]) says that condition (3) above can be relaxed: it is sufficient that there exist a scheme Z with an fplppf morphism $f : Z \to \mathcal{Y}$.

4.2.7. We say that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is k-representable if for every $S \to \mathcal{Y}_2$ with $S \in \text{DGSch}^{\text{aff}}$ the Cartesian product $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a k-Artin stack in the above sense.

We shall say that a k-representable morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is flat (resp., surjective, fpppf, smooth) if for every $S \to \mathcal{Y}_2$ with $S \in \text{DGSch}_{\text{aft}}$, and *any* atlas $(Z_S; f_S : Z_S \to S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1)$, the composed map

$$Z_S \xrightarrow{f_S} S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$$

is a flat (resp., surjective, fpppf, smooth) map of DG schemes.

[5] Our terminology is slightly different from that of [TV1] and [TV2]: what we call "k-Artin stacks" in *loc.cit.* is called "Artin k-stacks".

4.2.8. *Terminology.* We shall say that $\mathcal{Y} \in \mathrm{Stk}$ is an Artin stack if it is a k-Artin stack for some k.

We shall call k-Artin stacks for $k = 1$ "algebraic stacks".

4.3. **Some basic properties.**

4.3.1. By induction one shows:

Lemma 4.3.2. *Let $\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2 \xrightarrow{g} \mathcal{Y}_3$ be k-representable morphisms in* PreStk *with f surjective and flat (resp., smooth). Then g is flat (resp., smooth) if and only if $g \circ f$ is.*

This lemma implies that the definition of flat (resp., fpppf, smooth) morphisms given above does not depend on the choice of an atlas of $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$, i.e., it can be tested any given atlas.

4.3.3. Inductively one shows the following:

Lemma 4.3.4.
(1) *If $f : Z \to \mathcal{Y}$ is an atlas of a k-Artin stack, then it is an fpppf surjection.*
(2) *If $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a k-representable morphism, which is fpppf and surjective, then it is an fpppf surjection.*

4.4. **Alegbraic stacks and groupoids.**

4.4.1. From Lemmas 4.3.4 and 2.3.2, we obtain:

Lemma 4.4.2. *Let $\mathcal{Y}$ be a k-Artin stack and let $f : Z \to \mathcal{Y}$ be an atlas. Then the natural map*
$$|Z^\bullet/\mathcal{Y}|_{\mathrm{Stk}} \to \mathcal{Y}$$
*is an isomorphism, where the subscript "*Stk*" indicates that the geometric realization is taken in this category, which is equivalent to $L(|Z^\bullet/\mathcal{Y}|_{\mathrm{PreStk}})$.*

4.4.3. We shall now supply an (amplified) inverse to Lemma 4.4.2. Let
$$\mathcal{Z}^1 \rightrightarrows \mathcal{Z}^0$$
be a groupoid object in Stk (see [Lu0], Sect. 6.1.2). Let $\mathcal{Z}^\bullet$ be the corresponding simplicial object in Stk, and let
$$\mathcal{Y} := |\mathcal{Z}^\bullet|_{\mathrm{Stk}} \simeq L(|\mathcal{Z}^\bullet|_{\mathrm{PreStk}})$$
be its geometric realization. By [Lu0], Prop. 6.1.3, we have:
$$\mathcal{Z}^\bullet \simeq \mathcal{Z}^\bullet/\mathcal{Y}.$$

Using Lemma 2.4.2, by induction on k, one establishes the following descent property of k-Artin stacks and k-representable morphisms:

Lemma 4.4.4.
(a) *Assume that in the above situation $\mathcal{Z}^1$ and $\mathcal{Z}^0$ are $(k-1)$-Artin stacks, and both maps $\mathcal{Z}^1 \rightrightarrows \mathcal{Z}^0$, which are automatically $(k-1)$-representable, are smooth. Then $\mathcal{Y}$ is a k-Artin stack.*
(b) *Let*
$$\begin{array}{ccc} \mathcal{Z}_1 & \xleftarrow{g_1} & \mathcal{Y}_1 \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{Z}_2 & \xleftarrow{g_2} & \mathcal{Y}_2 \end{array}$$
be a Cartesian square in PreStk*. Assume that the morphism f is k-representable, smooth and surjective. If the morphism g_1 is k-representable, then g_2 is k-representable as well.*

(A key observation for the proof of point (a) is that for a groupoid object in ∞ - Grpd, whose terms are k-truncated, its geometric realization is at most $(k+1)$-truncated.)

Remark 4.4.5. By Remark 4.2.6, statement (b) the above lemma can be strengthened: one can relax the condition that the morphism f be fpppf instead of smooth. I.e., Artin stacks satisfy fpppf descent, and not just smooth descent. Statement (a) can be strengthened accordingly, by requiring that the maps $\mathcal{Z}^1 \rightrightarrows \mathcal{Z}^0$ be just plppf.

Remark 4.4.6. Lemma 4.4.4 allows to construct the familiar examples of algebraic stacks. For example, if G is a smooth group-scheme acting on a scheme X, we consider $\mathcal{Z}^1 := G \times X$ as a groupoid acting on $\mathcal{Z}^0 := X$, and the resulting 1-Artin stack $\mathcal{Y}$ is what we usually refer to as X/G.

4.4.7. *Reshuffling the axioms.* Let us briefly return to the definition of k-Artin stacks for $k \geq 1$.

Let $\mathcal{Y}$ be an object of Stk, and let $f : \mathcal{Z} \to \mathcal{Y}$ be a $(k-1)$-representable, smooth and surjective morphism, where $\mathcal{Z}$ is a $(k-1)$-Artin stack.

From Lemma 4.4.4(b) we obtain that the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is $(k-1)$-representable. From Lemma 4.3.4, we obtain that

$$\mathcal{Y} \simeq |\mathcal{Z}^\bullet/\mathcal{Y}|_{\mathrm{Stk}},$$

in particular, for any n, the restriction ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is at most k-truncated. Thus, we deduce:

Corollary 4.4.8. *Under the above circumstances, $\mathcal{Y}$ is a k-Artin stack.*

4.4.9. One can similarly characterize algebraic spaces:

Let $\mathcal{Y}$ be an object of Stk, and let $f : Z \to \mathcal{Y}$ be a schematic smooth and surjective morphism, where Z is a DG scheme.

Corollary 4.4.10. *Under the above circumstances, $\mathcal{Y}$ is an algebraic space if and only if the underlying presheaf of ∞-groupoids is 0-truncated (i.e., takes values in sets).*

4.5. **Algebraic stacks and convergence.**

4.5.1. We shall now establish:

Proposition 4.5.2. *Any k-Artin stack, viewed as an object of* PreStk*, is convergent.*

4.5.3. *Proof of Proposition 4.5.2.* We proceed by induction on k. Let

$$f : \mathcal{Z} \to \mathcal{Y}$$

be a faithfully flat cover of $\mathcal{Y}$, where $\mathcal{Z}$ is a $(k-1)$-Artin stack (for $k = 0$, $\mathcal{Z}$ is a scheme). By Lemma 4.3.4, we have:

$$\mathcal{Y} \simeq |\mathcal{Z}^\bullet/\mathcal{Y}|_{\mathrm{Stk}}.$$

Consider the induced map $\widehat{\mathcal{Z}} \to \widehat{\mathcal{Y}}$. By Lemma 3.4.3, this map is still an fpppf surjection. So,

$$\widehat{\mathcal{Y}} \simeq |\widehat{\mathcal{Z}}^\bullet/\widehat{\mathcal{Y}}|_{\mathrm{Stk}}.$$

However, we claim that the corresponding simplicial objects of Stk, namely, $\mathcal{Z}^\bullet/\mathcal{Y}$ and $\widehat{\mathcal{Z}}^\bullet/\widehat{\mathcal{Y}}$ are isomorphic. Since both are groupoid objects, it is enough to check this on 0- and 1-simplices.

For 0-simplices, this is clear from the induction hypothesis: $\mathcal{Z} \simeq \widehat{\mathcal{Z}}$.

To check for 1-simplices, we note that the functor of convergent-completion commutes with limits. Hence,

$$\widehat{\mathcal{Z}}^1/\widehat{\mathcal{Y}} = \widehat{\mathcal{Z}} \underset{\widehat{\mathcal{Y}}}{\times} \widehat{\mathcal{Z}} \simeq \widehat{\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Z}} = \widehat{\mathcal{Z}^1/\mathcal{Y}},$$

and the assertion follows by the induction hypothesis from the fact that $\mathcal{Z}^1/\mathcal{Y}$ is a $(k-1)$-Artin stack.

□

4.6. **Artin stacks and n-coconnectivity.**

4.6.1. First, we observe the following:

Proposition 4.6.2.
(a) If $\mathcal{Y}$ is an Artin stack, then so is ${}^L\tau^{\leq n}(\mathcal{Y})$ for any n.
(b) If $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a morphism between Artin stacks, which is flat, then the diagram

$$\begin{array}{ccc} {}^L\tau^{\leq n}(\mathcal{Y}_1) & \longrightarrow & \mathcal{Y}_1 \\ \downarrow & & \downarrow \\ {}^L\tau^{\leq n}(\mathcal{Y}_2) & \longrightarrow & \mathcal{Y}_2 \end{array}$$

is Cartesian.

Proof. We will prove both statements by induction, assuming that both statements of the proposition hold for k'-Artin stacks with $k' < k$. Note that point (b) formally implies that if we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}'_1 & \longrightarrow & \mathcal{Y}_1 \\ \downarrow & & \downarrow \\ \mathcal{Y}'_2 & \longrightarrow & \mathcal{Y}_2 \end{array} \tag{4}$$

of k'-Artin stacks with $\mathcal{Y}_1 \to \mathcal{Y}_2$ flat, then the diagram

$$\begin{array}{ccc} {}^L\tau^{\leq n}(\mathcal{Y}'_1) & \longrightarrow & {}^L\tau^{\leq n}(\mathcal{Y}_1) \\ \downarrow & & \downarrow \\ {}^L\tau^{\leq n}(\mathcal{Y}'_2) & \longrightarrow & {}^L\tau^{\leq n}(\mathcal{Y}_2) \end{array} \tag{5}$$

is Cartesian as well.

Let us prove point (a) for k-Artin stacks.

By Lemma 4.4.2, we can represent $\mathcal{Y}$ as $|\mathcal{Z}^\bullet|_{\mathrm{Stk}}|$, for a groupoid object $\mathcal{Z}^1 \rightrightarrows \mathcal{Z}^0$, where $\mathcal{Z}^i$'s are k'-Artin stacks with $k' < k$ (for $k = 0$, $\mathcal{Z}^i$'s are DG schemes). Consider the groupoid

$$ {}^L\tau^{\leq n}(\mathcal{Z}^1) \rightrightarrows {}^L\tau^{\leq n}(\mathcal{Z}^0), \tag{6}$$

where the maps are still smooth by the induction hypothesis of point (b). Let $\mathcal{Y}'$ be the k-Artin stack attached to the groupoid (6) by Lemma 4.4.4(a). We claim that $\mathcal{Y}' \simeq {}^L\tau^{\leq n}(\mathcal{Y})$.

Indeed, by (5), the simplicial object ${}^L\tau^{\leq n}(\mathcal{Z}^\bullet)$ is isomorphic to the simplicial object corresponding to the groupoid (6), and since the functor $\tau^{\leq n}$ commutes with geometric realizations, we have:

$$\mathcal{Y}' \simeq |{}^L\tau^{\leq n}(\mathcal{Z}^\bullet)|_{\mathrm{Stk}} = |L \circ \tau^{\leq n}(\mathcal{Z}^\bullet)|_{\mathrm{Stk}} \simeq L\left(|\tau^{\leq n}(\mathcal{Z}^\bullet)|_{\mathrm{PreStk}}\right) \simeq L \circ \tau^{\leq n}\left(|\mathcal{Z}^\bullet|_{\mathrm{PreStk}}\right),$$

which by 2.4.4 and Corollary 2.4.6 is isomorphic to

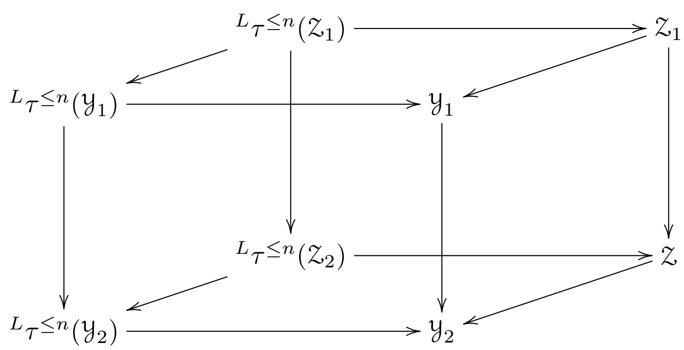
$$L \circ \tau^{\leq n} \circ L\left(|\mathcal{Z}^{\bullet}|_{\mathrm{PreStk}}\right) \simeq L \circ \tau^{\leq n}(\mathcal{Y}) = {}^L\tau^{\leq n}(\mathcal{Y}).$$

Let us now prove point (b) for k-Artin stacks. The statement holds for DG schemes, by definition.

Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between k-Artin stacks. However, by covering $\mathcal{Y}_1$ by an atlas, we can reduce the assertion to the case when $\mathcal{Y}_1$ is a k'-Artin stack with $k' < k$ (in fact, a DG scheme).

Let $\mathcal{Z}_2 \to \mathcal{Y}_2$ be an atlas of $\mathcal{Y}_2$, where $\mathcal{Z}_2$ is a k'-Artin stack with $k' < k$ (again, it can be chosen to be a DG scheme). Set $\mathcal{Z}_1 := \mathcal{Z}_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$. Consider the 3-dimensional commutative diagram

(7)



To show that the front facet square is Cartesian, it suffices to show that the remaining five square are Cartesian.

However, the right facet square is Cartesian by definition; the top lid and the back square are Cartesian by the induction hypothesis; finally, the bottom lid and the left square are Cartesian by the construction of ${}^L\tau^{\leq n}(\mathcal{Y}_2)$ in the course of the proof of point (a).

□

4.6.3. We shall now characterize k-Artin stacks that have the form ${}^L\tau^{\leq n}(\mathcal{Y})$ for some k-Artin stack $\mathcal{Y}$:

Proposition 4.6.4. *Let $\mathcal{Y}$ be a k-Artin stack. The following conditions are equivalent:*

(a) *$\mathcal{Y}$ is n-coconnective as a stack.*

(b) *There exists an atlas $f : Z \to \mathcal{Y}$, where $Z \in {}^{\leq n}\mathrm{DGSch}$.*

(c) *If S is an affine scheme mapping smoothly to $\mathcal{Y}$, then S is n-coconnective.*

Proof. This has been implicitly established in the course of the proof of the previous proposition. □

We shall refer to k-Artin stacks satisfying the equivalent conditions of Proposition 4.6.4 as "n-coconnective k-Artin stacks".

Warning: We emphasize again that being n-coconnective as a stack does *not* imply being n-coconnective as a prestack.

4.7. **Quasi-compactness and quasi-separatedness.**

4.7.1. *Quasi-compactness.* Let $\mathcal{Y}$ be a k-Artin stack. We say that $\mathcal{Y}$ is *quasi-compact* if there exists an atlas $f : Z \to \mathcal{Y}$ with $Z \in \text{DGSch}$ being quasi-compact.

For a k-representable morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk, we say that it is quasi-compact, if its base change by an affine scheme yields a quasi-compact k-Artin stack.

4.7.2. *Quasi-separatedness.* For $0 \leq k' \leq k$, we define the notion of k'-quasi-separatedness of a k-Artin stack or a k-representable morphism inductively on k'.

We say that a k-Artin stack $\mathcal{Y}$ is 0-quasi-separated if the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is quasi-compact, as a $(k-1)$-representable map. We say that a k-representable map is 0-quasi-separated if its base change by an affine scheme yields a 0-quasi-separated k-Artin stack.

For $k > 0$, we say that a k-Artin stack $\mathcal{Y}$ is k'-quasi-separated if the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is $(k'-1)$-quasi-separated, as a $(k-1)$-representable map. We shall say that a k-representable map is k'-quasi-separated if its base change by an affine scheme yields a k'-quasi-separated k-Artin stack.

We shall say that a k-Artin stack is quasi-separated if is k'-quasi-separated for all k', $0 \leq k' \leq k$. We shall say that a k-representable map is quasi-separated if its base change by an affine scheme yields a quasi-separated k-Artin stack.

4.8. **Algebraic spaces, revisited.**

4.8.1. Let $\mathcal{X}$ be an algebraic space, such that the underlying classical algebraic space ${}^{cl}\mathcal{X}$ is representable by a scheme.

Deformation theory implies:

Lemma 4.8.2. *Under the above circumstances $\mathcal{X}$ itself is representable by a DG scheme.*

4.8.3. Let $\mathcal{X}_0$ be a classical algebraic space, i.e., an algebraic space which is classical as an object of Stk (see Sect. 2.4.7 where the terminology is introduced).

From Lemma 4.4.4(2) and Lemma 4.8.2, we obtain:

Lemma 4.8.4. *Let $f_0 : Z_0 \to \mathcal{X}_0$ be an étale map, where Z_0 is a (classical) scheme. The groupoid of algebraic spaces $\mathcal{X}$ equipped with an isomorphism ${}^{L}\tau^{cl}(\mathcal{X}) \simeq \mathcal{X}_0$ is canonically equivalent to the groupoid of Cartesian diagrams*

$$\begin{array}{ccc} Z_0 & \longrightarrow & Z \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}_0 & \longrightarrow & \mathcal{X}, \end{array}$$

where Z is a DG scheme, and the map f is étale.

As a result, we obtain:

Corollary 4.8.5. *An algebraic space admits an étale atlas.*

Proof. This follows from Lemma 4.8.4 by Artin's theorem (see [LM], Cor. 8.1.1) that says that a classical algebraic space admits an étale atlas. □

4.9. **Artin stacks locally almost of finite type.**

4.9.1. The goal of this subsection is to establish the following:

Proposition 4.9.2. *Let $\mathcal{Y}$ be a k-Artin stack. The following conditions are equivalent:*

(1) $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{laft}}$.

(2) $\mathcal{Y}$ *admits an atlas* $f : Z \to \mathcal{Y}$ *with* $Z \in \mathrm{DGSch}_{\mathrm{laft}}$.

(3) *If S is an affine scheme mapping smoothly to $\mathcal{Y}$, then S is almost of finite type.*

We shall call k-Artin stacks satisfying the equivalent conditions of the above proposition "k-Artin stacks locally almost of finite type".

The proof of the proposition proceeds by induction, so we are assume that all three conditions are equivalent for $k' < k$.

4.9.3. *Implication* (3) ⇒ (2)*.* This is tautological.

4.9.4. *Implication* (2) ⇒ (1)*.* First, we claim that the Cartesian product $Z \underset{\mathcal{Y}}{\times} Z$, which is a $(k-1)$-Artin stack, is locally almost of finite type. Indeed, let $W \to Z \underset{\mathcal{Y}}{\times} Z$ be an atlas, where $W \in \mathrm{DGSch}$. Then the composition

$$W \to Z \underset{\mathcal{Y}}{\times} Z \overset{p_1}{\to} Z$$

is smooth. Since Z is locally almost of finite type, this implies that so is W. So, $Z \underset{\mathcal{Y}}{\times} Z$ satisfies condition (2). Using Corollary 1.3.12 we deduce that all the terms of simplicial object $Z^\bullet/\mathcal{Y}$ are locally almost of finite type.

By Proposition 4.5.2, $\mathcal{Y}$ is convergent. Hence, it remains to show that for any n, the restriction ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ is locally of finite type as an object of ${}^{\leq n}\mathrm{PreStk}$, i.e., that it lies in the essential image of the functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}\,.$$

Recall the notation

$${}^{\leq n}Z := Z|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} \in {}^{\leq n}\mathrm{DGSch} \subset {}^{\leq n}\mathrm{PreStk}\,.$$

We have:

$$(8) \qquad {}^{\leq n}\mathcal{Y} \simeq {}^{\leq n}L\left(|{}^{\leq n}Z^\bullet/{}^{\leq n}\mathcal{Y}|_{{}^{\leq n}\mathrm{PreStk}} \right).$$

Let us analyze the right-hand side of the expression in (8). By the above, the terms of the simplicial object ${}^{\leq n}Z^\bullet/{}^{\leq n}\mathcal{Y}$ are all left Kan extensions from ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$. Since the functor $\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$ commutes with colimits, we obtain that $|{}^{\leq n}Z^\bullet/{}^{\leq n}\mathcal{Y}|_{{}^{\leq n}\mathrm{PreStk}}$ is the left Kan extension of a canonically defined object $\mathcal{Y}_n \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$. By construction, $\mathcal{Y}_n$ is k-truncated. Therefore, by Corollary 2.5.7, we obtain that

$${}^{\leq n}\mathcal{Y} \simeq {}^{\leq n}L\left(\mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y}_n)\right) \simeq \mathrm{LKE}_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}({}^{\leq n}L_{\mathrm{ft}}(\mathcal{Y}_n)),$$

which is what we had to show.

□

4.9.5. *Implication* (1) $\Rightarrow$ (3). (J. Lurie)

Let $f : S \to \mathcal{Y}$ be a smooth map. We need to show that for all n, the affine DG scheme $\tau^{\leq n}(S)$ is of finite type. Denote as above ${}^{\leq n}S := S|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$, ${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$.

Consider the map ${}^{\leq n}f : {}^{\leq n}S \to {}^{\leq n}\mathcal{Y}$. Since ${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, there exists $T \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$, such that ${}^{\leq n}f$ factors as

$$ {}^{\leq n}S \xrightarrow{h} T \xrightarrow{g} {}^{\leq n}\mathcal{Y}. $$

Consider the Cartesian square:

$$ \begin{array}{ccc} T \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}S & \xrightarrow{g'} & {}^{\leq n}S \\ {\scriptstyle {}^{\leq n}f'}\downarrow & & \downarrow{\scriptstyle {}^{\leq n}f} \\ T & \xrightarrow{g} & {}^{\leq n}\mathcal{Y}. \end{array} $$

Since the map ${}^{\leq n}f$ was smooth, so is ${}^{\leq n}f'$. Since T was of finite type, we obtain that $T \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}S$ is also of finite type.

Consider now the maps

$$ {}^{\leq n}S \xrightarrow{\Delta} {}^{\leq n}S \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}S \xrightarrow{h\times\mathrm{id}} T \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}S \to {}^{\leq n}S, $$

where the last map if the projection on the second factor. The composition is the identity map on ${}^{\leq n}S$. Hence, ${}^{\leq n}S$ is a retract of $T \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}S$. Since the subcategory ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ is stable under retracts, the assertion follows.

□

4.10. **Finite presentation.**

4.10.1. *Morphisms almost of finite presentation.* Let is recall that a map

$$ \phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A) $$

between affine DG schemes is said to be *of finite presentation* if B is a compact object in the category of connective A-algebras.

We say that ϕ is *almost of finite presentation* if for any n we have that $\tau^{\geq -n}(B)$ is a compact object in the category of n-coconnective $\tau^{\geq -n}(A)$-algebras.

Lemma 4.10.2.

(a) *An affine DG scheme S is almost of finite presentation over* $\mathrm{pt} := \mathrm{Spec}(k)$ *if and only if it is almost of finite type.*

(b) *If a map $S_1 \to S_2$ of affine DG schemes if flat, then it is almost of finite presentation if and only if the corresponding map of classical affine schemes ${}^{cl}S_1 \to {}^{cl}S_2$ is of finite presentation.* [6]

(c) *If a map $S_1 \to S_2$ of affine DG schemes if smooth, then it is of finite presentation.*

---

[6] I.e., if a map is flat, then it is almost of finite presentation if and only if it is pppf; see Sect. 2.1.2 for the terminology.

4.10.3. Let $\mathcal{Y}$ be a k-Artin stack mapping to an affine scheme $S = \text{Spec}(A)$. We shall say that $\mathcal{Y}$ is *locally almost of finite presentation over* S if for every affine scheme $T = \text{Spec}(B)$ that maps smoothly to $\mathcal{Y}$, we have that B is almost of finite presentation over A.

Note that for S is itself almost of finite type (e.g., $S = \text{pt}$), Proposition 4.9.2 implies that the above definition is equivalent to $\mathcal{Y}$ being a k-Artin stack locally almost of finite type (see definition in Sect. 4.9.1).

4.10.4. Finally, let now $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk. We say that it is k-representable and locally almost of finite presentation if the following conditions hold:

(1) f is k-representable.

(2) For every affine scheme S mapping to $\mathcal{Y}_2$, the k-Artin stack $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is locally almost of finite presentation over S.

Thus, the above notion is a relative version of the notion of being a k-Artin stack locally almost of finite type.

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