NOTES ON GEOMETRIC LANGLANDS: CATEGORIES OVER THE RAN SPACE

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Introduction

The purpose of this paper is three-fold: (I) we'll introduce the notion of category over the Ran space; (II) provide Ran versions of the constructions from [GL:extWhit]; (III) prove a number of statements left unproved in *loc.cit.*

A terminological remark: by a "category" we'll mean a DG category in the conventions adopted in [GL:DG]. Limits and colimits are taken inside the $(\infty,1)$-category DGCat_{cont} in the notation of *loc.cit.*.

For the definitions of categories of D-modules on stacks such as Bun_G or $\overline{\mathrm{Bun}}_B$, see [DrGa1].

1. The Ran space

1.1. **Categories over the Ran space.** By a category $\mathcal{C}$ over the Ran space we'll mean the following data:

For a finite set I we have a category $\mathcal{C}_{X^I}$, tensored [1] over $D(X^I)$ (where $D(X^I)$ is regarded as a monoidal category w.r.t. $\overset{!}{\otimes}$), equipped with a compatible family of pairs of mutually adjoint functors

$$(\Delta_\phi)_! : \mathcal{C}_{X^J} \rightleftarrows \mathcal{C}_{X^I} : (\Delta_\phi)^!$$

for every surjection $\phi : I \twoheadrightarrow J$. We require that the right adjoint $(\Delta_\phi)^!$ be compatible with the eponymous monoidal functor $D(X^I) \to D(X^J)$, and that the induced functor

$$D(X^J) \underset{D(X^I)}{\otimes} \mathcal{C}_{X^I} \to \mathcal{C}_{X^J}$$

be an equivalence (as categories tensored over $D(X^J)$).

1.1.1. We define the category $\mathcal{C}_{\mathrm{Ran}_X}$ as

$$\underset{I,(\Delta_\phi)^!}{\underset{\longleftarrow}{lim}} \mathcal{C}_{X^I},$$

which is equivalent to

$$\underset{I,(\Delta_\phi)_!}{\underset{\longrightarrow}{lim}} \mathcal{C}_{X^I}.$$

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[1] For a monoidal category $\mathcal{A}$ we shall say that a category $\mathcal{C}$ is tensored over $\mathcal{A}$ if it is a module over in the terminology of [GL:DG].

1.1.2. The basic example is $\mathcal{C} := D(X^{\mathrm{fSet}})$, where $\mathcal{C}_I := D(X^I)$, in which case the corresponding category $\mathcal{C}_{\mathrm{Ran}_X}$ will be denoted $D(\mathrm{Ran}_X)$ and referred to as the category of D-modules on the Ran space. By construction, $D(\mathrm{Ran}_X)$ is a monoidal category (we refer to this structure on $D(\mathrm{Ran}_X)$ as the pointwise tensor product), and for every $\mathcal{C}$, $\mathcal{C}_{\mathrm{Ran}_X}$ is a module over it.

1.1.3. By a functor between two categories over the Ran space $F : \mathcal{C}^1 \to \mathcal{C}^2$ we'll mean a family of functors $F_{X^I} : \mathcal{C}^1_{X^I} \to \mathcal{C}^2_{X^I}$, over $D(X^I)$, compatible in the sense that for $\phi : I \twoheadrightarrow J$

$$F_J \circ \Delta^!_\phi \simeq \Delta^!_\phi \circ F_I.$$

Under above circumstances, we have a naturally defined functor $F_{\mathrm{Ran}_X} : \mathcal{C}^1_{\mathrm{Ran}_X} \to \mathcal{C}^2_{\mathrm{Ran}_X}$.

1.2. **The unital setting.** Assume now that $\mathcal{C}$ is unital. This means, by definition, that for non-empty finite sets I_1 and I_2 we are given a pair of adjoint functors

$$(unit_{I_1,I_2})_! : D(X^{I_1}) \otimes \mathcal{C}_{X^{I_2}} \rightleftarrows \mathcal{C}_{X^{I_1 \sqcup I_2}} : (unit_{I_1,I_2})^!,$$

respecting the action of $D(X^{I_1 \sqcup I_2})$, which satisfy a natural compatibility condition with the data of $\Delta^!_\phi$.

1.2.1. The latter compatibility reads as follows:

For $\phi : I \twoheadrightarrow J$, $J_2 := \phi(I_2)$, $J_1 = J - J_2$, and $\phi_2 : I_2 \twoheadrightarrow J_2$, the diagram

$$\begin{array}{ccc}
\mathcal{C}_{X^I} & \xrightarrow{(\Delta_\phi)_!} & \mathcal{C}_{X^J} \\
{\scriptstyle (unit_{I_1,I_2})_!}\uparrow & & \uparrow{\scriptstyle (unit_{J_1,J_2})_!} \\
\mathcal{C}_{X^{I_2}} & \xrightarrow{(\Delta_{\phi_2})_!} & \mathcal{C}_{X^{J_2}}
\end{array}$$

commutes.

Remark. Lurie points out that a unital structure on $\mathcal{C}$ can be viewed an an extension of the functor $I \mapsto \mathcal{C}_{X^I}$ from the category fSet^{op} where the arrows are surjections, to the category where we allow all maps between non-empty finite sets.

1.2.2. In particular a unital structure, gives rise to an adjoint pair of functors

$$(unit_{\mathrm{Ran},\mathrm{Ran}})_! : D(\mathrm{Ran}_X) \otimes \mathcal{C}_{\mathrm{Ran}_X} \rightleftarrows \mathcal{C}_{\mathrm{Ran}_X} : (unit_{\mathrm{Ran},\mathrm{Ran}})^!.$$

1.2.3. Applying this to $\mathcal{C} = D(X^{\mathrm{fSet}})$ with its natural unital structure, we obtain that $D(\mathrm{Ran}_X)$ acquires a structure of (symmetric) monoidal category. We will refer to this structure as convolution, and denote the corresponding pairs of adjoint functors

$$D(\mathrm{Ran}_X) \otimes D(\mathrm{Ran}_X) \rightleftarrows D(\mathrm{Ran}_X)$$

by $(\mathrm{add}_!, \mathrm{add}^!)$.

By the same token, for any unital $\mathcal{C}$, we obtain that $\mathcal{C}_{\mathrm{Ran}_X}$ is naturally a $D(\mathrm{Ran}_X)$-module.

1.3. **Unital version of the global category.** Note that we have a (symmetric) monoidal functor

$$H(\mathrm{Ran}_X, -) : D(\mathrm{Ran}_X) \to \mathrm{Vect}$$

corresponding to the functors $I \mapsto H(X^I, -)$, when we think of $D(\mathrm{Ran}_X)$ as a colimit.

This functor admits a right adjoint that sends $k \in \mathrm{Vect}$ to the dualizing complex

$$\omega_{\mathrm{Ran}_X} \in D(\mathrm{Ran}_X),$$

i.e., in the realization of $D(\mathrm{Ran}_X)$ as a limit, for every I, the value of ω_{Ran_X} in $D(X^I)$ is ω_{X^I}.

The following fundamental result has been established in [BD], and is referred to as "contractibility of the Ran space":

Theorem 1.3.1. *The adjunction map $H(\mathrm{Ran}_X, \omega_{\mathrm{Ran}_X}) \to k$ is an isomorphism. I.e.,, the above functor $k \mapsto \omega_{\mathrm{Ran}_X} : \mathrm{Vect} \to D(\mathrm{Ran}_X)$ is fully faithful.*

1.3.2. For a unital $\mathcal{C}$, we define $\mathcal{C}_{\mathrm{Ran}_X, un}$ as the tensor product

$$\mathcal{C}_{\mathrm{Ran}_X} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Vect}\,.$$

Equivalently, $\mathcal{C}_{\mathrm{Ran}_X, un}$ can be realized as the totalization of the cosimplicial category

$$\mathcal{C}_{\mathrm{Ran}_X} \rightrightarrows D(\mathrm{Ran}_X) \otimes \mathcal{C}_{\mathrm{Ran}_X} ...,$$

i.e., it consists of objects $\mathcal{F} \in \mathcal{C}_{\mathrm{Ran}_X}$ endowed with a compatible system of isomorphisms

$$(unit_{\mathrm{Ran},\mathrm{Ran}})^!(\mathcal{F}) \simeq \omega_{\mathrm{Ran}_X} \boxtimes \mathcal{F}.$$

Proposition 1.3.3. *The forgetful functor $\mathcal{C}_{\mathrm{Ran}_X, un} \to \mathcal{C}_{\mathrm{Ran}_X}$ is fully faithful.*

Remark. This proposition would have been an obvious corollary of Theorem 1.3.1 if $D(\mathrm{Ran}_X)$ had been unital as a monoidal category (with respect to the convolution monoidal structure). Since it is not, some care is necessary.

Proof. Let $\mathcal{C}'$ be an arbitrary module category over $D(\mathrm{Ran}_X)$. Let

$$\mathrm{act}_! : D(\mathrm{Ran}_X) \otimes \mathcal{C}' \to \mathcal{C}'$$

denote the action map. Let's assume that this map admits a right adjoint; we denote it by $\mathrm{act}^!$. Let $\mathcal{C}'_{un} \subset \mathcal{C}'$ be a full subcategory spanned by objects

$$\{\mathbf{c}' \in \mathcal{C}' \mid \mathrm{act}^!(\mathbf{c}') \in \mathcal{C}' \simeq \mathrm{Vect} \otimes \mathcal{C}' \subset D(\mathrm{Ran}_X) \otimes \mathcal{C}'\},$$

where $\mathrm{Vect} \to D(\mathrm{Ran}_X)$ is the above fully faithful functor corresponding to ω_{Ran_X}.

By the fully faithfulness mentioned above, the restriction of $\mathrm{act}^!$ to $\mathcal{C}'_{un}$ defines a functor $F : \mathcal{C}'_{un} \to \mathcal{C}'_{un}$ endowed with an isomorphism $F^2 \simeq F$, satisfying the natural associativity condition. In particular, we conclude that if the functor $F : \mathcal{C}'_{un} \to \mathcal{C}'_{un}$ happens to be invertible, then there exists a unique identification $F \simeq \mathrm{Id}_{\mathcal{C}'_{un}}$ compatible with $F^2 \simeq F$.

By definition, the category

$$\mathcal{C}' \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Vect}$$

can be thought of as the category of objects $\mathbf{c}' \in \mathcal{C}'_{un}$, endowed with an isomorphism $F(\mathbf{c}') \simeq \mathbf{c}'$, which also satisfies the natural associativity condition. Hence, we conclude that if F is invertible, the natural arrow

$$\mathcal{C}' \underset{D(\mathrm{Ran}_X)}{\otimes} \mathrm{Vect} \to \mathcal{C}'_{un}$$

is an equivalence.

Finally, we observe that for $\mathcal{C}'$ of the form $\mathcal{C}_{\mathrm{Ran}_X}$ for a category $\mathcal{C}$ over the Ran space, the corresponding functor F is isomorphic to the identity functor, and in particular is invertible.

□

1.4. **A basic example.** Let's consider an example of the above situation, when $\mathcal{C} := D(X^{\mathrm{fSet}})$. Denote $D(\mathrm{Ran}_X, un) := \mathcal{C}_{\mathrm{Ran}_X, un}$. We have a canonical functor

$$D(\mathrm{Ran}_X, un) \to \mathrm{Vect}$$

corresponding to the functor

$$D(\mathrm{Ran}_X) \otimes \mathrm{Vect} \simeq D(\mathrm{Ran}_X) \overset{H(\mathrm{Ran}_X,-)}{\longrightarrow} \mathrm{Vect}\,.$$

Lemma 1.4.1. *The above functor* $D(\mathrm{Ran}_X, un) \to \mathrm{Vect}$ *is an equivalence.*

Remark. Again, this proposition would have been obvious, had $D(\mathrm{Ran}_X)$ been unital.

Proof. We'll construct a functor in the opposite direction using Proposition 1.3.3. Indeed, by *loc. cit.*, $D(\mathrm{Ran}_X, un)$ is a full subcategory of $D(\mathrm{Ran}_X)$; moreover it contains the image of Vect. The composition

$$\mathrm{Vect} \to D(\mathrm{Ran}_X, un) \to \mathrm{Vect}$$

is easily seen to be the identity functor. Hence, it remains to show that the inclusion

$$\mathrm{Vect} \hookrightarrow D(\mathrm{Ran}_X, un)$$

is an equivalence.

By definition, on object $\mathcal{F} \in D(\mathrm{Ran}_X)$ belongs to $D(\mathrm{Ran}_X, un)$ if it is equipped with an isomorphism

$$\mathrm{add}^!(\mathcal{F}) \simeq \omega_{\mathrm{Ran}_X} \boxtimes \mathcal{F} \in D(\mathrm{Ran}_X) \boxtimes D(\mathrm{Ran}_X).$$

By symmetry, this implies that $\mathrm{add}^!(\mathcal{F}) \simeq \mathcal{F} \boxtimes \omega_{\mathrm{Ran}_X}$. The contractibility of the Ran space implies now that $\mathcal{F}$ is of the form $V \otimes \omega_{\mathrm{Ran}_X}$, where V is a vector space.

□

Corollary 1.4.2. *Let* $\mathcal{C}'$ *be any module category for* $D(\mathrm{Ran}_X)$ *on which the action factors through*

$$D(\mathrm{Ran}_X) \overset{H(\mathrm{Ran}_X,-)}{\longrightarrow} \mathrm{Vect}\,.$$

Then the natural arrow

$$\mathcal{C}' \underset{D(\mathrm{Ran}_X)}{\otimes} D(\mathrm{Ran}_X) \to \mathcal{C}'$$

is an equivalence.

1.5. **Functoriality.** Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two unital categories over the Ran space. We say that a functor $F : \mathcal{C}^1 \to \mathcal{C}^2$ is weakly compatible with the unital structures if we are given isomorphisms

$$(unit_{I_1,I_2})_! \circ (\mathrm{Id} \otimes F_{X^{I_2}}) \simeq F_{X^{I_1 \sqcup I_2}} \circ (unit_{I_1,I_2})_!$$

(compatible in the natural sense with other pieces of structure).

If this happens, the functor $\mathcal{C}^1_{\mathrm{Ran}_X} \to \mathcal{C}^2_{\mathrm{Ran}_X}$ is compatible with the actions of $D(\mathrm{Ran}_X)$, and hence induces a functor $\mathcal{C}^1_{\mathrm{Ran}_X,un} \to \mathcal{C}^2_{\mathrm{Ran}_X,un}$, which makes the diagram

$$\begin{array}{ccc} \mathcal{C}^1_{\mathrm{Ran}_X} & \longrightarrow & \mathcal{C}^2_{\mathrm{Ran}_X} \\ \downarrow & & \downarrow \\ \mathcal{C}^1_{\mathrm{Ran}_X,un} & \longrightarrow & \mathcal{C}^2_{\mathrm{Ran}_X,un} \end{array}$$

commute.

We shall say that F is strongly compatible with the unital structures, if the induced map

$$(\mathrm{Id} \otimes F_{X^{I_2}}) \circ (unit_{I_1,I_2})^! \to (unit_{I_1,I_2})^! \circ F_{X^{I_1 \sqcup I_2}}$$

is an isomorphism as well.

In this case, the diagram

$$\begin{array}{ccc} \mathcal{C}^1_{\mathrm{Ran}_X} & \longrightarrow & \mathcal{C}^2_{\mathrm{Ran}_X} \\ \uparrow & & \uparrow \\ \mathcal{C}^1_{\mathrm{Ran}_X,un} & \longrightarrow & \mathcal{C}^2_{\mathrm{Ran}_X,un} \end{array}$$

is also commutative.

1.6. **Augmentation.** Let $\mathcal{C}$ be a unital category over the Ran space. By definition, an augmentation on $\mathcal{C}$ is a category $\mathcal{C}_\emptyset$, endowed with a family of functors

$$(unit_{I,\emptyset})_! : D(X^I) \otimes \mathcal{C}_\emptyset \rightleftarrows \mathcal{C}_{X^I} : (unit_{I,\emptyset})^!,$$

both functors compatible with the monoidal action of $D(X^I)$, and satisfying a natural compatibility condition with the rest of the data. In particular, we obtain a functor

$$(unit_{\mathrm{Ran},\emptyset})^! : \mathcal{C}_{\mathrm{Ran}} \to D(\mathrm{Ran}_X) \otimes \mathcal{C}_\emptyset.$$

Remark. By the same observation of Lurie's, we can view augmentation as an extension of the functor from fSet^{op} (with arbitrary maps) to the category that includes also the empty set.

1.6.1. Note that $(unit_{\mathrm{Ran},\emptyset})^!$ induces a functor

$$\mathcal{C}_{\mathrm{Ran},un} \to D(\mathrm{Ran}_X, un) \otimes \mathcal{C}_\emptyset \simeq \mathcal{C}_\emptyset.$$

1.6.2. For a functor $F : \mathcal{C}^1 \to \mathcal{C}^2$ we define what it means to be weakly or strongly compatible with the augmentation following the pattern of Sect. 1.5.

2. A Ran version of $D(\mathrm{Bun}_B^{rat})$

In this section we'll explain two variants of the construction of the category $D(\mathrm{Bun}_B^{rat})$ (introduced in [GL:extWhit]) that use the Ran space.

2.1. **A polar version of $\overline{\mathrm{Bun}}_B$.** Consider the following ind-stack $\overline{\mathrm{Bun}}_B^{polar}$ that lives over Ran_X: for a finite set I, points of $(\overline{\mathrm{Bun}}_B^{polar})_{X^I}$ are quadruples

$$(\underline{x}, P_G, P_T, \kappa),$$

where $\underline{x} \in X^I$ is a finite collection of points of x, P_G is a principal G-bundle on X, P_T is a principal T-bundle on X, and κ is a Plücker data which is allowed to have poles at $\underline{x}$:

$$\kappa^{\check\lambda} : \check\lambda(P_T) \to V^{\check\lambda}_{P_G}|_{X-\underline{x}}.$$

Set $D(\overline{\mathrm{Bun}}_B^{polar})$ to be the category over Ran_X that assigns to a finite set I the category $D((\overline{\mathrm{Bun}}_B^{polar})_{X^I})$. It's naturally unital and augmented, by virtue of the tautological closed embedding

$$X^{I_1} \times (\overline{\mathrm{Bun}}_B^{polar})_{X^{I_2}} \to (\overline{\mathrm{Bun}}_B^{polar})_{X^{I_1 \sqcup I_2}}$$

(uncluding the case $I_2 = \emptyset$), with $D((\overline{\mathrm{Bun}}_B^{polar})_\emptyset) := D(\overline{\mathrm{Bun}}_B)$. We'll consider the corresponding categories

$$D((\overline{\mathrm{Bun}}_B^{polar})_{\mathrm{Ran}_X}) \text{ and } D((\overline{\mathrm{Bun}}_B^{polar})_{\mathrm{Ran}_X,un}).$$

2.1.1. Consider the groupoid $(\mathcal{H}_{G,B}^{polar,\text{b.g.}})_{X^I}$ that identifies two points

$$(\underline{x}, P_G, P_T, \kappa) \text{ and } (\underline{x}, P_G, P_T^1, \kappa^1),$$

with the same $(\underline{x}, P_G)$ whenever the two Plučker data agree at the generic point of X. [2]

We define the category $D(\text{Bun}_B^{rat,polar,\text{b.g.}})$ over the Ran space by setting

$$D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{X^I} := D((\overline{\text{Bun}}_B^{polar})_{X^I})^{(\mathcal{H}_{G,B}^{polar,\text{b.g.}})_{X^I}}.$$

It is also naturally unital and augmented with

$$D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{\emptyset} = D(\text{Bun}_B^{rat}).$$

2.1.2. Consider the corresponding categories

$$D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{\text{Ran}_X} \text{ and } D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{\text{Ran}_X,un}.$$

The first goal of this section is to prove the following:

Proposition 2.1.3. *The functor*

$$(unit_{\text{Ran},\emptyset})^! : D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{\text{Ran}_X,un} \to D(\text{Bun}_B^{rat})$$

is an equivalence.

Proof. In fact, a stronger assertion is true. Augmentation defines a functor

$$(unit_{\text{Ran},I})^! : D(\text{Bun}_B^{rat,polar,\text{b.g.}})_{X^I} \to D(X^I) \otimes D(\text{Bun}_B^{rat}),$$

and we claim that this map is already an equivalence. So, $D(\text{Bun}_B^{rat,polar,\text{b.g.}})$ is equivalent, as a category over the Ran space to

$$D(X^{\text{fSet}}) \otimes D(\text{Bun}_B^{rat}).$$

To prove the claim that $(unit_{\text{Ran},I})^!$ is an equivalence, in order to simplify the notation, let's assume that we are allowing poles only at a fixed point x. I.e., we want to show that pull-back with respect to the closed embedding

$$\overline{\text{Bun}}_B \hookrightarrow (\overline{\text{Bun}}_B^{polar})_x$$

induces an equivalence of the corresponding categories

$$D((\overline{\text{Bun}}_B^{polar})_x)^{(\mathcal{H}_{G,B}^{polar,\text{b.g.}})_x} \to D(\overline{\text{Bun}}_B)^{\mathcal{H}_{G,B}}.$$

The stack $D((\overline{\text{Bun}}_B^{polar})_x)$ is a union of its closed substacks $D((\overline{\text{Bun}}_B^{polar,\leq\lambda})_x)$, numbered by elements $\lambda \in \Lambda^{pos}$, where each such closed substack corresponds to triples (P_G, P_T, κ), where the pole of κ at x is bounded from above by λ.

It is enough to show that the closed embedding $\overline{\text{Bun}}_B \hookrightarrow (\overline{\text{Bun}}_B^{polar,\leq\lambda})_x$ induces an equivalence

$$D((\overline{\text{Bun}}_B^{polar,\leq\lambda})_x)^{(\mathcal{H}_{G,B}^{polar,\leq\lambda})_x} \to D(\overline{\text{Bun}}_B)^{\mathcal{H}_{G,B}}$$

for every λ.

We define a map (also a closed embedding) in the opposite direction

$$(\overline{\text{Bun}}_B^{polar,\leq\lambda})_x \to \overline{\text{Bun}}_B$$

[2]The notation "b.g." stands for "big groupoid" for the reasons to be explained later.

by sending (P_G, P_T, κ) to (P'_G, P'_T, κ'), where $P'_G := P_G$, $P'_T := P_T(-\lambda\cdot x)$, and κ' is set to be equal to κ on $X - x$. The assumption that (P_G, P_T, κ) belongs to $(\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x$ implies that (P'_G, P'_T, κ') belongs to $\overline{\mathrm{Bun}}_B$.

Now we claim that the two compositions

$$D((\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x)^{(\mathcal{H}_{G,B}^{polar,\leq\lambda})_x} \to D(\overline{\mathrm{Bun}}_B)^{\mathcal{H}_{G,B}} \to D((\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x)^{(\mathcal{H}_{G,B}^{polar,\leq\lambda})_x}$$

and

$$D(\overline{\mathrm{Bun}}_B)^{\mathcal{H}_{G,B}} \to D((\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x)^{(\mathcal{H}_{G,B}^{polar,\leq\lambda})_x} \to D(\overline{\mathrm{Bun}}_B)^{\mathcal{H}_{G,B}}$$

are naturally isomorphic to the identity functor. Indeed, the corresponding maps at the level of stacks

$$(\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x \to \overline{\mathrm{Bun}}_B \to (\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x \text{ and } \overline{\mathrm{Bun}}_B \to (\overline{\mathrm{Bun}}_B^{polar,\leq\lambda})_x \to \overline{\mathrm{Bun}}_B$$

are both given by the action of a section of the groupoids $(\mathcal{H}_{G,B}^{polar,\leq\lambda})_x$ and $\mathcal{H}_{G,B}$, respectively. □

2.1.4. By the same token as in [GL:extWhit], Sect. 2, on each $D(\mathrm{Bun}_B^{rat,polar,\text{b.g.}})_{X^I}$ we can impose an equivariance condition with respect to the adelic N. We obtain a category over Ran_X, which we will denote $D(\mathrm{Bun}_T^{rat,polar,\text{b.g.}})$, along with

$$D(\mathrm{Bun}_T^{rat,polar,\text{b.g.}})_{\mathrm{Ran}_X} \text{ and } D(\mathrm{Bun}_T^{rat,polar,\text{b.g.}})_{\mathrm{Ran}_X,un}.$$

Proposition 2.1.3 implies:

Corollary 2.1.5. *The augmentation functor*

$$(unit_{\mathrm{Ran},\emptyset})^! : D(\mathrm{Bun}_T^{rat,polar,\text{b.g.}})_{\mathrm{Ran}_X,un} \to D(\mathrm{Bun}_T^{rat})$$

is an equivalence.

2.2. **A version of Bun_B^{rat} with a smaller groupoid.** A disadvantage of the definition of $D(\mathrm{Bun}_B^{rat})$ given in [GL:extWhit], Sect. 2, or the description given above, is that the groupoid $\mathcal{H}_{G,B}$ isn't very explicit. In this subsection we'll discuss another way of approaching this category using a smaller groupoid.

2.2.1. For a finite set I consider the category $D(X^I \times \overline{\mathrm{Bun}}_B)$, but we'll consider a smaller groupoid $(\mathcal{H}_{G,B}^{\text{s.g.}})_{X^I} \subset X^I \times \mathcal{H}_{G,B}$ acting on it. Namely, we require our meromorphic isomorphisms between the corresponding T-bundles to be regular on $X - \underline{x}$.

For each I consider the category

$$D(\mathrm{Bun}_B^{rat,\text{s.g.}})_I := D(X^I \times \overline{\mathrm{Bun}}_B)^{(\mathcal{H}_{G,B}^{\text{s.g.}})_{X^I}}.$$

(Here "s. g." stands for "small groupoid".) We have a pair of adjoint functors:

$$\mathbf{ind}_{B,\text{s.g.},I} : D(X^I \times \overline{\mathrm{Bun}}_B) \rightleftarrows D(\mathrm{Bun}_B^{rat,\text{s.g.}})_I :: \mathbf{oblv}_{B,\text{s.g.},I}.$$

Consider the assigment

$$I \mapsto D(\mathrm{Bun}_B^{rat,\text{s.g.}})_I$$

as a category over the Ran space, which we denote by $D(\mathrm{Bun}_B^{rat,\text{s.g.}})$. This category is naturally unital and augmeted: the corresponding functors

$$(unit_{I_1,I_2})^! : D(\mathrm{Bun}_B^{rat,\text{s.g.}})_{I_1\sqcup I_2} \to D(X^{I_1}) \otimes D(\mathrm{Bun}_B^{rat,\text{s.g.}})_{I_2}$$

are given by restriction.

Consider the corresponding categories

$$D(\mathrm{Bun}_B^{rat,\mathrm{s.g.}})_{\mathrm{Ran}_X} \text{ and } D(\mathrm{Bun}_B^{rat,\mathrm{s.g.}})_{\mathrm{Ran}_X,un}.$$

2.2.2. Augmentation defines a functor

$$(unit_{\mathrm{Ran},\emptyset})^! : D(\mathrm{Bun}_B^{rat,\mathrm{s.g.}})_{\mathrm{Ran}_X,un} \to D(\overline{\mathrm{Bun}}_B).$$

The goal of this subsection is to prove the following:

Proposition 2.2.3. *The above functor* $(unit_{\mathrm{Ran},\emptyset})^! : D(\mathrm{Bun}_B^{rat,\mathrm{s.g.}})_{\mathrm{Ran}_X,un} \to D(\overline{\mathrm{Bun}}_B)$ *canonically factors as*

$$D(\mathrm{Bun}_B^{rat,\mathrm{s.g.}})_{\mathrm{Ran}_X,un} \to D(\mathrm{Bun}_B^{rat}) \overset{\mathbf{oblv}_B}{\longrightarrow} D(\overline{\mathrm{Bun}}_B),$$

and the first arrow is an equivalence.

Remark. An analogous assertion, with the same proof, remains valid for the correspoding polar version. I.e., instead of $D(X^I \times \overline{\mathrm{Bun}}_B)$ we can consider $D((\overline{\mathrm{Bun}}_B^{polar})_{X^I})$ and still consider the small groupoid $(\mathcal{H}_{G,B}^{polar,\mathrm{s.g.}})_{X^I}$, by requiring isomorphisms away from $\underline{x}$.

2.2.4. Before giving a proof of Proposition 2.2.3, let us consider the following general paradigm. Let $\mathcal{C}'$ be a category, and suppose that for every finite set I we are given a monad

$$M_I : D(X^I) \otimes \mathcal{C}' \to D(X^I) \otimes \mathcal{C}',$$

compatible with the monoidal action of $D(X^I)$. Assume that these monads satisfy the following compatibilities:

- For $I = \emptyset$, the monad $M_\emptyset : \mathcal{C}' \to \mathcal{C}'$ is the identity.

- For a surjection $\phi : I \twoheadrightarrow J$, the diagram of functors

$$\begin{array}{ccc} D(X^J) \otimes \mathcal{C}' & \xrightarrow{M_J} & D(X^J) \otimes \mathcal{C}' \\ {\scriptstyle (\Delta_\phi)^!}\uparrow & & \uparrow{\scriptstyle (\Delta_\phi)^!} \\ D(X^I) \otimes \mathcal{C}' & \xrightarrow{M_I} & D(X^I) \otimes \mathcal{C}' \end{array}$$

commutes. Moreover, the compatibility isomorphisms are compatible with compositions of maps.

-For $I = I_1 \sqcup I_2$ (including the case $I_2 = \emptyset$), we are given a homomorphism of monads

$$\mathrm{Id}_{D(X^{I_1})} \boxtimes M_{I_2} \to M_I.$$

These homomorphisms must be compatible with the commutative diagrams corresponding to restrictions.

Under such circumstances, we can consider the assignment

$$I \mapsto M_I\text{-mod}(D(X^I) \otimes \mathcal{C}')$$

as a category over the Ran space; we'll denote it $M_{\mathrm{fSet}}\text{-mod}(\mathcal{C}')$. It is naturally unital and augmented. Consider the functor

$$(unit_{\mathrm{Ran},\emptyset})^! : M_{\mathrm{fSet}}\text{-mod}(\mathcal{C}')_{\mathrm{Ran}_X,un} \to \mathcal{C}'.$$

By construction, this functor is conservative. We claim that it admits a left adjoint; let us describe it explicitly.

Note that for any $\mathbf{c}' \in \mathcal{C}'$, the object

$$\mathbf{c}'_1 := \underset{\overrightarrow{I}}{colim}\ \left(H(X^I, -) \boxtimes \mathrm{Id}_{\mathcal{C}'}\right) \circ M_I(\omega_{X^I} \boxtimes \mathbf{c}')$$

(colimit taken over the category of non-empty finite sets with surjective maps) has the property that for any $J \in \text{fSets}$,

$$\omega_{X^J} \boxtimes \mathbf{c}'_1 \in D(X^J) \otimes \mathcal{C}'$$

has a structure of module over M_J, in a way compatible with surjections of J's. Hence, the assignment $\mathbf{c}' \mapsto \omega_{\text{Ran}_X} \boxtimes \mathbf{c}'_1$ is a functor $\mathcal{C}' \to M_{\text{fSet}}\text{-mod}(\mathcal{C}')_{\text{Ran}_X, un}$. The next assertion follows from the constructions:

Lemma 2.2.5. *The above functor*

$$\mathbf{c}' \mapsto \omega_{\text{Ran}_X} \boxtimes \left(\underset{\overrightarrow{I}}{colim} \left(H(X^I, -) \boxtimes \text{Id}_{\mathcal{C}'} \right) \circ M_I(\omega_{X^I} \boxtimes \mathbf{c}') \right)$$

is the left adjoint to $(unit_{\text{Ran},\emptyset})^! : M_{\text{fSet}}\text{-mod}(\mathcal{C}')_{\text{Ran}_X, un} \to \mathcal{C}'$.

Let us denote by M the resulting monad on $\mathcal{C}'$. The above lemma implies that M is canonically isomorphic to

$$\mathbf{c}' \mapsto \underset{\overrightarrow{I}}{colim} \left(H(X^I, -) \boxtimes \text{Id}_{\mathcal{C}'} \right) \circ M_I(\omega_{X^I} \boxtimes \mathbf{c}') \tag{2.1}$$

with a natural monad structure on the latter.

2.2.6. We apply the above discussion to $\mathcal{C}' := D(\overline{\text{Bun}}_B)$ and $M_I := \mathbf{oblv}_{B,\text{s.g.},I} \circ \mathbf{ind}_{B,\text{s.g.},I}$. To prove Proposition 2.2.3, we have to show that the monad $\mathbf{oblv}_B \circ \mathbf{ind}_B$ is canonically isomorphic to the monad given by (2.1). This will boil down to a geometric property of the ind-stack $\mathcal{H}_{G,B}$ vis-a-vis the ind-stacks $(\mathcal{H}^{\text{s.g.}}_{G,B})_{X^I}$.

Consider again the following abstract situation. Let Z be an (ind)-stack, and let Z^I be (ind)-stacks over X^I, each equipped with a closed embedding $\psi_I : Z^I \hookrightarrow X^I \times Z$. Assume also that we are given a compatible family of isomorphisms

$$Z^J \simeq X^J \underset{X^I}{\times} Z^I$$

for every surjection $I \twoheadrightarrow J$, which are also compatible with the maps ψ_I, ψ_J in the natural sense.

Consider the following object

$$\underset{\overrightarrow{I}}{colim} \left(H(X^I, -) \boxtimes \text{Id}_{D(Z)} \right) \circ (\psi_I)_!(\omega_{Z^I}) \in D(Z), \tag{2.2}$$

where the colimit is again taken over the category of non-empty finite sets with surjective maps.

There is a natural map from the object in (2.2) to ω_Z.

Lemma 2.2.7. *Assume that the images of the compositions*

$$Z^I \overset{\psi_I}{\hookrightarrow} X^I \times Z \to Z$$

cover Z*. Then the above map*

$$\underset{\overrightarrow{I}}{colim} \left(H(X^I, -) \boxtimes \text{Id}_{D(Z)} \right) \circ (\psi_I)_!(\omega_{Z^I}) \to \omega_Z$$

is an isomorphism.

It is clear that Lemma 2.2.7 implies Proposition 2.2.3.

2.2.8. *Proof of Lemma 2.2.7.* It is enough to show that for any fixed finite set I^0, the map in question induces an isomorphism after applying $\psi^!_{I^0}$. For a finite set I, let Γ^{I,I^0} be a closed subscheme of $X^I \times X^{I^0}$ consisting of points $(\underline{x}, \underline{x}^0)$ such that $\underline{x}$ as a subset is contained in $\underline{x}^0$. By assumption, we have an isomorphism:

$$Z^{I^0} \underset{Z}{\times} Z^I \simeq Z^{I^0} \underset{X^{I^0}}{\times} \Gamma^{I,I^0}.$$

By properness, this reduces the assertion of the lemma to the case when $Z = X^{I^0}$ and $Z^I = \Gamma^{I,I^0}$, in which case it is easy.

□

3. Ran versions of the Whittaker category

3.1. **The three spaces.** For a finite set I consider the ind-stacks over X^I denoted

$$({}^{(a)}\mathrm{Bun}_N^{polar})_{X^I},\ ({}^{(b)}\mathrm{Bun}_N^{polar})_{X^I} \text{ and } ({}^{(c)}\mathrm{Bun}_N^{polar})_{X^I},$$

each classifying quintuples $(\underline{x}, P_G, P_T, \kappa, \gamma)$, where $(\underline{x}, P_G, P_T, \kappa)$ is as in the definition of $(\mathrm{Bun}_N^{polar})_{X^I}$ (see [GL:extWhit], Sect. 5), and γ is now an isomorphism

$$P_T \simeq \rho(\omega_X)|_{X-\underline{x}},$$

where superscripts (a), (b) and (c) correspond to the conditions that

(a) γ extends to an isomorphism over X,

(b) κ extends to a Plücker data over X,

(c) no condition

Note that $({}^{(a)}\mathrm{Bun}_N^{polar})_{X^I}$ is the old $(\mathrm{Bun}_N^{polar})_{X^I}$ from [GL:extWhit], Sect. 5.

In each of the above cases we have another groupoid acting, denoted $({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}$, that identifies two points $(\underline{x}, P_G, P_T, \kappa, \gamma)$ and $(\underline{x}, P_G, P_T^1, \kappa^1, \gamma^1)$ whenever we are given an isomorphism

$$P_T \simeq P_T^1|_{X-\underline{x}},$$

which intertwines between the data of κ and κ_1, and γ and γ_1. (Note that in case (a) the above isomorphism must necessarily extend to an isomorphism over the entire X.)

The assignment

$$I \mapsto D(({}^{?}\mathrm{Bun}_N^{polar})_{X^I})^{({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}}$$

is naturally a category over Ran_X, which we will denote by $D({}^{?}\mathrm{Bun}_N^{rat})$.

3.2. **Comparisons.** We have natural closed embeddings:

$$({}^{(a)}\mathrm{Bun}_N^{polar})_{X^I} \hookrightarrow ({}^{(c)}\mathrm{Bun}_N^{polar})_{X^I} \text{ and } ({}^{(b)}\mathrm{Bun}_N^{polar})_{X^I} \hookrightarrow ({}^{(c)}\mathrm{Bun}_N^{polar})_{X^I},$$

and the corresponding functors

$$D(({}^{(c)}\mathrm{Bun}_N^{polar})_{X^I})^{({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}} \to D(({}^{(a)}\mathrm{Bun}_N^{polar})_{X^I})^{({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}}$$

and

$$D(({}^{(c)}\mathrm{Bun}_N^{polar})_{X^I})^{({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}} \to D(({}^{(b)}\mathrm{Bun}_N^{polar})_{X^I})^{({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}}.$$

We claim, however, that the above functors are equivalences.

Indeed, for (c) → (a) this is a tautology: we are introducing an auxiliary data of (P_T, γ) and then modding out by it. For (c) → (b) this follows by the same argument as in the proof of Proposition 2.1.3.

Thus, we obtain equivalences of categories over the Ran space

$$D({}^{(a)}\mathrm{Bun}_N^{rat}) \simeq D({}^{(c)}\mathrm{Bun}_N^{rat}) \simeq D({}^{(b)}\mathrm{Bun}_N^{rat}).$$

3.3. **Whittaker subcategories.** In each of the above cases we have the corresponding groupoid $(\mathcal{H}_N)_{X^I}$ acting on $({}^{?}\mathrm{Bun}_N^{polar})_{X^I}$, endowed with a character $\chi : (\mathcal{H}_N)_{X^I} \to \mathbb{G}_a$, and we consider the corresponding full subcategory

$$D(({}^{?}\mathrm{Bun}_N^{polar})_{X^I})^{((\mathcal{H}_N)_{X^I},\chi)} \subset D(({}^{?}\mathrm{Bun}_N^{polar})_{X^I})$$

for $? = (a), (b)$ or (c).

The action of $({}^{?}\mathcal{H}_{G,N}^{polar})_{X^I}$ commutes with that of $(\mathcal{H}_N)_{X^I}$ in the natural sense, and we can consider the resulting full subcategory

$${}^{?}\mathrm{Whit}(G)_{X^I} := D(({}^{?}\mathrm{Bun}_N^{polar})_{X^I})^{((\mathcal{H}_N)_{X^I},\chi),{}^{?}(\mathcal{H}_{G,N}^{polar})_{X^I}} \subset D(({}^{?}\mathrm{Bun}_N^{polar})_{X^I})^{{}^{?}(\mathcal{H}_{G,N}^{polar})_{X^I}}.$$

However, from Sect. 3.2 we obtain that the natural restriction functors

$${}^{(c)}\mathrm{Whit}(G)_{X^I} \to {}^{(a)}\mathrm{Whit}(G)_{X^I} \text{ and } {}^{(c)}\mathrm{Whit}(G)_{X^I} \to {}^{(b)}\mathrm{Whit}(G)_{X^I}$$

are equivalences.

Hence, the assignment

$$I \mapsto {}^{?}\mathrm{Whit}(G)_{X^I}$$

leads to the same category over the Ran space, namely, $\mathrm{Whit}(G)$.

4. Ran versions of the reduced Whittaker category

In this section we'll be assuming that Z_G is connected. In what follows we'll discuss several (equivalent) variants of the construction of $\mathrm{Whit}(G)^{red}_{glob}$.

4.1. **Variant 1.** The discussion in this subsection will apply equally well to $\mathrm{Whit}(G)^{red,ext}_{glob}$.

4.1.1. For a finite set I consider the following ind-stack, denoted ${}^{?}\mathrm{Bun}_N^{polar,red,X^I}$ for $? = (a), (b), (c)$ or (d)

A point of $({}^{?}\mathrm{Bun}_N^{polar,red})_{X^I}$ is a quintuple $(\underline{x}, P_G, P_T, \kappa, \gamma^{red})$, where $(\underline{x}, P_G, P_T, \kappa)$ is as in the definition of $(\mathrm{Bun}_N^{polar})_{X^I}$, and where γ^{red} is a data of a *non-zero map*

$$\gamma_i^{red} : \check{\alpha}_i(P_T) \to \omega_X|_{X-\underline{x}}$$

for each simple root $\check{\alpha}_i$.

The variants $? = (a), (b), (c)$ or (d) correspond to the conditions that

(a) γ^{red} is defined over X,

(b) κ extends to a Plücker data over X,

(c) no condition

(d) (a) $\wedge$ (b)

Note that version (d) reproduces the ind-stack $\overline{\mathrm{Bun}}_N^{red} \times X^I$.

4.1.2. There are two versions of the groupod that acts on ${}^?\operatorname{Bun}_N^{polar,red,X^I}$:

We have the groupoid $({}^?\mathcal{H}_{G,N}^{polar,red,\text{b.g.}})_{X^I}$ that identifies two points

$$(\underline{x}, P_G, P_T, \kappa, \gamma^{red}) \text{ and } (\underline{x}, P_G, P_T^1, \kappa^1, \gamma^{red1})$$

whenever we are given an isomorphism between P_T and P_T^1 *at the generic point of* X, that intertwines the data of κ and κ^1, and γ^{red} and γ^{red1}.

There us a sub-groupoid

$$({}^?\mathcal{H}_{G,N}^{polar,red,\text{s.g.}})_{X^I} \subset ({}^?\mathcal{H}_{G,N}^{polar,red,\text{b.g.}})_{X^I},$$

obtained by the requirement that the isomorphism between P_T and P_T^1 be regular on $X - \underline{x}$.

In each if the above cases we obtain the category over the Ran space

$$I \mapsto D(({}^?\operatorname{Bun}_N^{polar,red})_{X^I})^{({}^?\mathcal{H}_{G,N}^{polar,red,\text{b.g.}})_{X^I}} \text{ and } I \mapsto D(({}^?\operatorname{Bun}_N^{polar,red})_{X^I})^{({}^?\mathcal{H}_{G,N}^{polar,red,\text{s.g.}})_{X^I}}.$$

We denote these categories by

$$D({}^?\operatorname{Bun}_N^{rat,red,\text{b.g.}}) \text{ and } D({}^?\operatorname{Bun}_N^{rat,red,\text{s.g.}}),$$

respectively.

By imposing equivariance with respect to the corresponding groupoid $I \mapsto (\mathcal{H}_N)_{X^I}$, we obtain categories over the Ran space, denoted ${}^?\text{Whit}(G)^{red,\text{b.g.}}$ and ${}^?\text{Whit}(G)^{red,\text{s.g.}}$ for $? =$ (a), (b), (c) or (d), respectively.

4.1.3. We have the natural closed embeddings:

$$({}^{(\text{a})}\operatorname{Bun}_N^{polar,red})_{X^I} \hookrightarrow ({}^{(\text{c})}\operatorname{Bun}_N^{polar,red})_{X^I},\ ({}^{(\text{b})}\operatorname{Bun}_N^{polar,red})_{X^I} \hookrightarrow ({}^{(\text{c})}\operatorname{Bun}_N^{polar,red})_{X^I},$$
$$({}^{(\text{d})}\operatorname{Bun}_N^{polar,red})_{X^I} \hookrightarrow ({}^{(\text{a})}\operatorname{Bun}_N^{polar,red})_{X^I},\ ({}^{(\text{d})}\operatorname{Bun}_N^{polar,red})_{X^I} \hookrightarrow ({}^{(\text{b})}\operatorname{Bun}_N^{polar,red})_{X^I}$$

and the restriction functors

$$D({}^{?'}\operatorname{Bun}_N^{rat,red,\text{b.g.}}) \to D({}^{?''}\operatorname{Bun}_N^{rat,red,\text{b.g.}}) \tag{4.1}$$

and

$$D({}^{?'}\operatorname{Bun}_N^{rat,red,\text{s.g.}}) \to D({}^{?''}\operatorname{Bun}_N^{rat,red,\text{s.g.}}) \tag{4.2}$$

for for the corresponding values of $?'$ and $?''$.

4.1.4. However, as in Proposition 2.1.3, we show that the functors in (4.1) and (4.2) are equivalences. In particular, we have well-defined categories over the Ran space

$$D(\operatorname{Bun}_N^{rat,red,\text{b.g.}}) \text{ and } D(\operatorname{Bun}_N^{rat,red,\text{s.g.}}), \tag{4.3}$$

where $D(\operatorname{Bun}_N^{rat,red,\text{b.g.}})$ is equivalent to

$$D(\operatorname{Bun}_N^{rat,red}) \otimes D(X^{\text{fSet}}),$$

where $D(\operatorname{Bun}_N^{rat,red}$ is as in [GL:extWhit], Sect. 6.

Furthermore, as in Sect. 2.2, we obtain that although the categories in (4.3) are *inequivalent* as categories over the Ran space, the corresponding categories

$$D(\operatorname{Bun}_N^{rat,red,\text{b.g.}})_{\operatorname{Ran}_X,un} \text{ and } D(\operatorname{Bun}_N^{rat,red,\text{s.g.}})_{\operatorname{Ran}_X,un}$$

are canonically equivalent to one another, and are equivalent to $D(\operatorname{Bun}_N^{rat,red})$ as defined in [GL:extWhit], Sect. 6.

4.1.5. Passing to the corresponding $((\mathcal{H}_N)_{X^I}, \chi)$-equivariant categories, we obtain that the categories over the Ran space ${}^?\mathrm{Whit}(G)^{red,\text{b.g.}}$ for $? = \text{(a)}, \text{(b)}, \text{(c)}$ or (d) are equivalent to each other, and so are ${}^?\mathrm{Whit}(G)^{red,\text{s.g.}}$. The resulting categories

$$\mathrm{Whit}(G)^{red,\text{b.g.}} \text{ and } \mathrm{Whit}(G)^{red,\text{s.g.}}$$

are inequivalent as categories over the Ran space, but the categories

$$\mathrm{Whit}(G)^{red,\text{b.g.}}_{\mathrm{Ran}_X,un} \text{ and } \mathrm{Whit}(G)^{red,\text{s.g.}}_{\mathrm{Ran}_X,un}$$

are equivalent to each other, and hence reproduce $\mathrm{Whit}(G)^{red}_{glob}$.

4.2. **Variant 2.** The discussion in this subsection will be specific to $\mathrm{Whit}(G)^{red}_{glob}$, i.e., it won't apply to $\mathrm{Whit}(G)^{red,ext}_{glob}$.

4.2.1. For a finite set I consider the following ind-stack, denoted $({}^?\mathrm{Bun}_N^{polar,red'})_{X^I}$ for $? = \text{(a)}, \text{(b)}, \text{(c)}$ or (d).

A point of $({}^?\mathrm{Bun}_N^{polar,red'})_{X^I}$ is a quintuple $(\underline{x}, P_G, P_T, \kappa, \gamma^{red'})$, where $(\underline{x}, P_G, P_T, \kappa)$ is as in the definition of $(\mathrm{Bun}_N^{polar})_{X^I}$, and where $\gamma^{red'}$ is a data of an *isomorphism*

$$\gamma_i^{red'} : \check{\alpha}_i(P_T) \simeq \omega_X|_{X-\underline{x}}$$

for each simple root $\check{\alpha}_i$.

The variants $? = \text{(a)}, \text{(b)}, \text{(c)}$ or (d) correspond to the conditions that

(a) γ^{red} extends to regular maps $\check{\alpha}_i(P_T) \to \omega_X$ defined over X,

(b) κ extends to a Plücker data over X,

(c) no condition

(d) (a) $\wedge$ (b)

4.2.2. Again, as before, $({}^?\mathrm{Bun}_N^{polar,red'})_{X^I}$ is acted on by a groupoid, denoted

$$({}^?\mathcal{H}_{G,N}^{polar,red'})_{X^I}$$

that identifies two points

$$(\underline{x}, P_G, P_T, \kappa, \gamma^{red'}) \text{ and } (\underline{x}, P_G, P_T^1, \kappa^1, \gamma^{red'1})$$

whenever we are given an isomorphism between P_T and P_T^1 away from $\underline{x}$ that intertwines the data of κ and κ_1, and $\gamma^{red'}$ and $\gamma_1^{red'}$.

Remark. Note that we could have considered a seemingly bigger groupoid which allows isomorphisms between P_T and P_T^1 at the generic point of X. However, any such isomorphism necessarily extends to $X - \underline{x}$: the data of $\gamma^{red'}$ takes care of the induced T/Z_G-bundles, whereas the data of κ takes care of the induced $G/[G,G]$-bundles.

The assignment

$$I \mapsto D(({}^?\mathrm{Bun}_N^{polar,red'})_{X^I})^{({}^?\mathcal{H}_{G,N}^{polar,red'})_{X^I}}$$

forms a category over the Ran space, which we denote $D({}^?\mathrm{Bun}_N^{rat,red'})$.

By imposing equivariance with respect to the corresponding groupoid $I \mapsto (\mathcal{H}_N)_{X^I}$, we obtain categories over the Ran space, denoted ${}^?\mathrm{Whit}(G)^{red'}$.

4.2.3. We have the tautological closed embeddings

$$({}^{(a)}\mathrm{Bun}_N^{polar,red'})_{X^I} \hookrightarrow ({}^{(c)}\mathrm{Bun}_N^{polar,red'})_{X^I}, ({}^{(b)}\mathrm{Bun}_N^{polar,red'})_{X^I} \hookrightarrow ({}^{(c)}\mathrm{Bun}_N^{polar,red'})_{X^I}$$

and

$$({}^{(d)}\mathrm{Bun}_N^{polar,red'})_{X^I} \hookrightarrow ({}^{(a)}\mathrm{Bun}_N^{polar,red'})_{X^I}, ({}^{(d)}\mathrm{Bun}_N^{polar,red'})_{X^I} \hookrightarrow ({}^{(b)}\mathrm{Bun}_N^{polar,red'})_{X^I}$$

and the restriction functors

$$D({}^{?'}\mathrm{Bun}_N^{rat,red'})_{X^I} \to D({}^{?''}\mathrm{Bun}_N^{rat,red'})_{X^I}$$

for the corresponding values of $?'$ and $?''$.

However, the same argument as in the proof of Proposition 2.1.3 shows that these arrows are equivalences. Thus, the categories over the Ran space $D({}^{?}\,\mathrm{Bun}_N^{rat,var})$ for $? = (a), (b)$, (c) and (d) are all equivalent. We'll denote the resulting category over the Ran space simply by $D(\mathrm{Bun}_N^{rat,red'})$.

As a corollary, we obtain that the categories ${}^{?}\mathrm{Whit}(G)^{red'}$ over the Ran space are all equivalent. We denote the resulting category by $\mathrm{Whit}(G)^{red'}$.

4.3. **Comparison of Variants 1 and 2.** Note that we have the maps

$$({}^{?}\mathrm{Bun}_N^{polar,red'})_{X^I} \hookrightarrow ({}^{?}\mathrm{Bun}_N^{polar,red})_{X^I}$$

for $? = (a), (b), (c)$ and (c) (these maps are in fact closed embeddings).

Pullback defines functors

$$D({}^{?'}\mathrm{Bun}_N^{rat,red,\mathrm{b.g.}})_{X^I} \to D({}^{?'}\mathrm{Bun}_N^{rat,red'})_{X^I}. \tag{4.4}$$

Proposition 4.3.1. *The functors in* (4.4) *are equivalences.*

The proposition implies that the analogues statement is true for the corresponding categories ${}^{?}\mathrm{Whit}(G)^{red'}$ and ${}^{?}\mathrm{Whit}(G)^{red,\mathrm{b.g.}}$ over the Ran space.

4.4. **Variant 3.** To prove Proposition 4.3.1 we'll introduce yet another variant of the construction, and compare it to $D({}^{?'}\mathrm{Bun}_N^{rat,red,\mathrm{b.g.}})_{X^I}$ and $D({}^{?'}\mathrm{Bun}_N^{rat,red'})_{X^I}$.

Consider the ind-stack $(\mathrm{Bun}_N^{polar,red''})_{X^I}$ that classifies quintuples $(\underline{x}, P_G, P_T, \kappa, \gamma^{red''})$ where $(\underline{x}, P_G, P_T, \kappa)$ are as in $(\mathrm{Bun}_N^{polar})_{X^I}$, and $\gamma^{red''}$ is a data of *isomorphism*

$$\gamma_i^{red''} : \check{\alpha}_i(P_T) \simeq \omega_X$$

for each simple root $\check{\alpha}_i$.

By definition, $(\mathrm{Bun}_N^{polar,red''})_{X^I}$ is naturally a closed substack of both $({}^{?}\mathrm{Bun}_N^{polar,red'})_{X^I}$ and $({}^{?}\mathrm{Bun}_N^{polar,red})_{X^I}$ for any of the values of ?.

4.4.1. We have a groupoid denoted $(\mathcal{H}^{polar,red''}_{G,N})_{X^I}$ acting on $(\mathrm{Bun}^{polar,red''}_N)_{X^I}$, obtained by restriction from $({}^?\mathcal{H}^{polar,red'}_{G,N})_{X^I}$, or, which is the same, $({}^?\mathcal{H}^{polar,red,\text{b.g.}}_{G,N})_{X^I}$ or $({}^?\mathcal{H}^{polar,red,\text{s.g.}}_{G,N})_{X^I}$, for any value of ?.

Remark. Note that $(\mathcal{H}^{polar,red''}_{G,N})_{X^I}$ is in fact given by a group-ind-scheme over X^I, namely, the affine Grassmannian Gr_{Z_G,X^I} of the torus Z_G.

Let $D(\mathrm{Bun}^{rat,red''}_N)_{X^I}$ denote the category $D((\mathrm{Bun}^{polar,red''}_N)_{X^I})^{(\mathcal{H}^{polar,red''}_{G,N})_{X^I}}$. Restriction defines the functors

$$D({}^?\mathrm{Bun}^{rat,red,\text{b.g.}}_N)_{X^I} \to D(\mathrm{Bun}^{rat,red''}_N)_{X^I} \leftarrow D({}^?\mathrm{Bun}^{rat,red'}_N)_{X^I}, \tag{4.5}$$

for any value of ?, and we claim that both arrows in (4.5) are equivalences. We'll prove it for the $\to$ arrow, as the other case is similar.

4.4.2. First, it is easy to see that the functor in question is conservative. Indeed, any point of $({}^?\mathrm{Bun}^{polar,red}_N)_{X^I}$ can be moved to inside $(\mathrm{Bun}^{polar,red''}_N)_{X^I}$ using the groupoid $({}^?\mathcal{H}^{polar,red,\text{b.g.}}_{G,N})_{X^I}$. Hence, by Barr-Beck, to show that the two categories are equivalent, we need to show that the corresponding map of monads acting on $D((\mathrm{Bun}^{polar,red''}_N)_{X^I})$ is an isomorphism. But this follows from the definition of $(\mathcal{H}^{polar,red''}_{G,N})_{X^I}$ since all the groupoids involved are ind-proper over the spaces on which they act.

5. Monoidal structures over the Ran space

5.1. **Tensoring two categories.** Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two categories over Ran_X. We can form their tensor product $\mathcal{C}^1 \otimes \mathcal{C}^2$:

$$(\mathcal{C}^1 \otimes \mathcal{C}^2)_{X^I} := \mathcal{C}^1_{X^I} \underset{D(X^I)}{\otimes} \mathcal{C}^2_{X^I}.$$

Lemma-Construction 5.1.1. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be endowed with a unital structure. Then there exists a canonically defined functor*

$$\mathcal{C}^1_{\mathrm{Ran}_X} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathcal{C}^1_{\mathrm{Ran}_X} \to (\mathcal{C}^1 \otimes \mathcal{C}^2)_{\mathrm{Ran}_X}.$$

Proof. To construct a functor in question we need to give a functor

$$\mathcal{C}^1_{\mathrm{Ran}_X} \underset{D(\mathrm{Ran}_X)}{\otimes} \mathcal{C}^1_{\mathrm{Ran}_X} \to (\mathcal{C}^1 \otimes \mathcal{C}^2)_{\mathrm{Ran}_X}$$

and check its invariance property with respect to the action of $D(\mathrm{Ran}_X)$ by convolution.

To write down such a functor it is sufficient to define a compatible family of functors

$$\mathcal{C}^1_{X^{I^1}} \otimes \mathcal{C}^2_{X^{I^2}} \to (\mathcal{C}^1 \otimes \mathcal{C}^2)_{X^{I^1 \sqcup I^2}}.$$

The latter are given by

$$\mathcal{C}^1_{X^{I^1}} \otimes \mathcal{C}^1_{X^{I^2}} \simeq \left(\mathcal{C}^1_{X^{I^1}} \otimes D(X^{I^2})\right) \underset{D(X^{I^1 \sqcup I^2})}{\otimes} \left(D(X^{I^1}) \otimes \mathcal{C}^2_{X^{I^2}}\right) \overset{unit^{\mathcal{C}^1} \otimes unit^{\mathcal{C}^2}}{\longrightarrow}$$
$$\to \mathcal{C}^1_{X^{I^1 \sqcup I^2}} \underset{D(X^{I^1 \sqcup I^2})}{\otimes} \mathcal{C}^2_{X^{I^1 \sqcup I^2}} =: (\mathcal{C}^1 \otimes \mathcal{C}^2)_{X^{I^1 \sqcup I^2}}.$$

The required invariance property follows from the construction.

□

5.1.2. Note that by tensoring both sides in Lemma 5.1.1 over $D(\mathrm{Ran}_X)$ with Vect, we obtain a functor

$$\mathcal{C}^1_{\mathrm{Ran}_X,un} \otimes \mathcal{C}^2_{\mathrm{Ran}_X,un} \to (\mathcal{C}^1 \otimes \mathcal{C}^2)_{\mathrm{Ran}_X,un}. \tag{5.1}$$

Lemma 5.1.3. *The functor* (5.1) *is an equivalence.*

The proof follows the argument of Theorem 4.3.6 of [BD].

Remark. The functor in (5.1) makes the following diagram of functors commute:

$$\begin{array}{ccc} \mathcal{C}^1_{\mathrm{Ran}_X} \otimes \mathcal{C}^2_{\mathrm{Ran}_X} & \longrightarrow & (\mathcal{C}^1 \otimes \mathcal{C}^2)_{\mathrm{Ran}_X,un} \\ \downarrow & & \downarrow \\ \mathcal{C}^1_{\mathrm{Ran}_X,un} \otimes \mathcal{C}^2_{\mathrm{Ran}_X,un} & \longrightarrow & (\mathcal{C}^1 \otimes \mathcal{C}^2)_{\mathrm{Ran}_X,un}, \end{array}$$

where the vertical arrows are the canonical localization functors $\mathcal{C}_{\mathrm{Ran}_X} \to \mathcal{C}_{\mathrm{Ran}_X,un}$. However, it is not true that the diagram with the vertical arrows given by $\mathcal{C}_{\mathrm{Ran}_X,un} \to \mathcal{C}_{\mathrm{Ran}_X}$ will commute: this fails in the unit example of $\mathcal{C}^1 = \mathcal{C}^2 = D(X^{\mathrm{fSet}})$.

5.2. **Monoidal categories over the Ran space.** Let now $\mathcal{C}$ be a category over the Ran space, which is, as such, endowed with a monoidal structure, i.e., we have an associative operation

$$\mathcal{C} \otimes \mathcal{C} \to \mathcal{C},$$

in a way compatible with the $\Delta^!_\phi$ functors.

Assume now that $\mathcal{C}$ is unital as a category over Ran_X, in a way compatible with the monoidal structure. I.e., the functors

$$unit_{I_1,I_2} : D(X^{I_1}) \otimes \mathcal{C}_{X^{I_2}} \to \mathcal{C}_{X^{I_1 \sqcup I_2}}$$

are required to be monoidal.

From Lemma-Construction 5.1.1 we obtain that the category $\mathcal{C}_{\mathrm{Ran}_X}$ acquires a monoidal structure. Furthermore, we obtain that $\mathcal{C}_{\mathrm{Ran}_X,un}$ acquires a monoidal structure, such that the localization functor $\mathcal{C}_{\mathrm{Ran}_X} \to \mathcal{C}_{\mathrm{Ran}_X,un}$ is monoidal.

Note, however, that $\mathcal{C}_{\mathrm{Ran}_X}$ is not, in general, unital as a monoidal category, e.g., in the example $\mathcal{C} = D(X^{\mathrm{fSet}})$.

5.2.1. Assume now that $\mathcal{C}$ is, in addition, augmented, also in a way compatible with the monoidal structure, i.e., the functor

$$unit_{I,\emptyset} : D(X^I) \otimes \mathcal{C}_\emptyset \to \mathcal{C}_{X^I}$$

is monoidal.

Assume in addition that $\mathcal{C}_\emptyset$ is unital as a monoidal category, and that for any I the composed map

$$unit_{I,\emptyset} : \mathrm{Vect} \overset{k \mapsto \omega_{X^I}}{\longrightarrow} D(X^I) \to D(X^I) \otimes \mathcal{C}_\emptyset \to C_{X^I}$$

defines a unit in $\mathcal{C}_{X^I}$ as a monoidal category.

In this case the composed functor

$$\mathrm{Vect} \to \mathcal{C}_\emptyset \simeq \mathcal{C}_\emptyset \otimes D(X^{\mathrm{fSet}})_{\mathrm{Ran}_X,un} \to \mathcal{C}_{\mathrm{Ran}_X,un}$$

defines a unit in $\mathcal{C}_{\mathrm{Ran}_X,un}$ as a monoidal category.

5.2.2. One defines in a similar way the notion of module category for a given monoidal category over Ran_X.

If $\mathcal{C}$ is a monoidal category over Ran_X, and $\mathcal{M}$ is a module over it, we obtain that $\mathcal{C}_{\mathrm{Ran}_X}$ acts on $\mathcal{M}_{\mathrm{Ran}_X}$, and $\mathcal{C}_{\mathrm{Ran}_X,un}$ acts on $\mathcal{M}_{\mathrm{Ran}_X,un}$. Moreover, we have:

$$\mathcal{M}_{\mathrm{Ran}_X,un} \simeq \mathcal{M}_{\mathrm{Ran}_X} \underset{\mathcal{C}_{\mathrm{Ran}_X}}{\otimes} \mathcal{C}_{\mathrm{Ran}_X,un}.$$

When $\mathcal{C}$ is as in Sect. 5.2.1, and the action of $\mathcal{C}$ on $\mathcal{M}$ is unital, then the action of $\mathcal{C}_{\mathrm{Ran}_X,un}$ on $\mathcal{M}_{\mathrm{Ran}_X,un}$ is unital.

5.3. **Rigidity.** Let's recall that a unital monoidal category $\mathcal{A}$ is rigid if

(1) The monoidal unit functor $\mathrm{Vect} \to \mathcal{A}$ admits a right adjoint,
(2) The right adjoint to the tensor operation, which is a functor $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, exists and is *strictly compatible* with the action of $\mathcal{A}$ by left and right multiplication (a priori, it's only lax compatible).

5.3.1. Rigidity is a very convenient property of a monoidal category. It follows from the definition that the composition

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \to \mathrm{Vect},$$

(where the first arrow is the monoidal operation, and the second arrow is the right adjoint to the unit) defines an identification of $\mathcal{A}$ with its own dual.

Moreover, for any module category $\mathcal{M}$, the action functor

$$\mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$$

admits a right adjoint $\mathcal{M} \to \mathcal{A} \otimes \mathcal{M}$, which is canonically isomorphic to the dual of the action map, under the above identification $\mathcal{A} \simeq \mathcal{A}^\vee$.

The above observation allows to identify, for an $\mathcal{A}$-bimodule category $\mathcal{M}$, the corresponding Hohschild homology and cohomology categories

$$HH_\bullet(\mathcal{A}, M) \simeq HH^\bullet(\mathcal{A}, M).$$

5.3.2. Let now $\mathcal{A}_0$ be another unital monoidal category, and let $\mathcal{A}_0 \to \mathcal{A}$ be a unital monoidal functor. In this case, it makes sense to talk about $\mathcal{A}$ being rigid relative to $\mathcal{A}_0$. This means that

(1) The unit functor $\mathcal{A}_0 \to \mathcal{A}$ admits a right adjoint,
(2) The right adjoint to the tensor operation $\mathcal{A} \underset{\mathcal{A}_0}{\otimes} \mathcal{A} \to \mathcal{A}$, which is a functor $\mathcal{A} \to \mathcal{A} \underset{\mathcal{A}_0}{\otimes} \mathcal{A}$, exists and is strictly compatible with the action of $\mathcal{A}$ by left and right multiplication.

If this happens, the functors

$$\mathrm{Vect} \to \mathcal{A} \to \mathcal{A} \underset{\mathcal{A}_0}{\otimes} \mathcal{A} \text{ and } \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \to \mathcal{A}_0$$

define a self-duality datum for $\mathcal{A}$, considered as a left (or right) $\mathcal{A}_0$-module.

5.3.3. Let $\mathcal{C}$ be as in Sect. 5.2.1. Assume that each $\mathcal{C}_{X^I}$ is rigid relative to $D(X^I)$.

Lemma 5.3.4. *Under the above circumstances, the monoidal category $\mathcal{C}_{\mathrm{Ran}_X,un}$ is rigid.*

5.4. **An example.** An example of a monoidal category over the Ran space is provided by the following construction.

Let T_0 be a torus. Consider the Beilinson-Drinfeld Grassmannian $I \mapsto \mathrm{Gr}_{T_0,X^I}$, and the corresponding category over the Ran space

$$I \mapsto D(\mathrm{Gr}_{T_0,X^I}).$$

We shall denote this category by $D(\mathrm{Gr}_{T_0,\mathrm{fSet}})$, and the corresponding global categories by $D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X})$ and $D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X,un})$, respectively.

Since $D(\mathrm{Gr}_{T_0,X^I})$ are rigid relative to $D(X^I)$, we obtain that $D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X,un})$ is rigid.

5.4.1. Note that we have a pair of adjoint functors

$$D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X}) \rightleftarrows D(\mathrm{Bun}_{T_0}),$$

which factor through

$$D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X,un}) \rightleftarrows D(\mathrm{Bun}_{T_0}).$$

The left adjoint has a natural monoidal structure given by convolution on Bun_{T_0}. We have:

Theorem 5.4.2. *The right adjoint functor*

$$D(\mathrm{Bun}_{T_0}) \to D(\mathrm{Gr}_{T_0,\mathrm{Ran}_X,un})$$

is fully faithful.

6. Reduced vs. non-reduced versions of the Whittaker category

The goal of this section is to prove Proposition 6.3.1 from [GL:extWhit].

6.1. **The relation between the two spaces.** We are going to use the approach to the category $D(\mathrm{Bun}_N^{rat,red})$ as in Sect. 4.4. Namely, we'll realize it as the category over the Ran space given by

$$I \mapsto D((\mathrm{Bun}_N^{polar,red''})_{X^I})^{(\mathcal{H}_{G,N}^{polar,red''})_{X^I}}.$$

We have a natural forgetful map

$$(\mathrm{Bun}_N^{polar})_{X^I} \to (\mathrm{Bun}_N^{polar,red''})_{X^I},$$

which remembers out of the data of γ the induced isomorphism of T/Z_G-bundles.

Consider the corresponding pull-back functor between categories over the Ran space:

$$D((\mathrm{Bun}_N^{rat,red''})_{X^I}) \to D((\mathrm{Bun}_N^{rat})_{X^I}).$$

6.2. **Action of the affine Grassmannian.** Note now that we have a natural action of $\mathrm{Gr}_{Z(G),X^I}$, considered as a ind-group-scheme over X^I, on $(\mathrm{Bun}_N^{polar})_{X^I}$ by modifying the G-bundle at $\underline{x}$.

This defines a monoidal action of $D(\mathrm{Gr}_{Z(G),X^I})$ on $(D(\mathrm{Bun}_N^{polar})_{X^I})$.

Lemma 6.2.1. *We have a canonical isomorphism*

$$D((\mathrm{Bun}_N^{rat,red''})_{X^I}) \simeq D((\mathrm{Bun}_N^{polar})_{X^I}) \underset{D(\mathrm{Gr}_{Z(G),X^I})}{\otimes} \left(D(\mathrm{Bun}_{Z_G}) \otimes D(X^I)\right).$$

Proof. By definition,

$$D(\mathrm{Bun}_N^{rat,red''})_{X^I} \simeq D((\mathrm{Bun}_N^{polar,red''})_{X^I}) \underset{D(\mathrm{Gr}_{Z(G),X^I})}{\otimes} D(X^I).$$

Note also that we have an isomorphism of stacks

$$(\mathrm{Bun}_N^{rat,red''})_{X^I} \simeq (\mathrm{Bun}_N^{polar})_{X^I} \times \mathrm{Bun}_{Z_G},$$

such that the action of $D(\mathrm{Gr}_{Z(G),X^I})$ on the LHS corresponds to the diagonal action on the RHS. Hence, the assertion of the lemma. □

By passing to the corresponding (Ran_X, un) categories, we obtain:

Corollary 6.2.2.

$$D(\mathrm{Bun}_N^{rat,red}) \simeq D(\mathrm{Bun}_N^{rat}) \underset{D(\mathrm{Gr}_{Z(G),\mathrm{Ran}_X,un})}{\otimes} D(\mathrm{Bun}_{Z_G}).$$

Finally, by taking $(\mathcal{H}_N, \chi)$-invariants, we arrive to the assertion of Proposition 6.3.1.

References

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[GL:extWhit] Notes on Geometric Langlands, *The extended Whittaker category.*