A CONJECTURAL EXTENSION OF THE KAZHDAN-LUSZTIG EQUIVALENCE

DENNIS GAITSGORY

To Masaki Kashiwara, with admiration

Abstract. A theorem of Kazhdan and Lusztig establishes an equivalence between the category of $G(\mathcal{O})$-integrable representations of the Kac-Moody algebra $\widehat{\mathfrak{g}}_{-\kappa}$ at a negative level $-\kappa$ and the category $\operatorname{Rep}_q(G)$ of (algebraic) representations of the "big" (a.k.a. Lusztig's) quantum group. In this paper we propose a conjecture that describes the category of Iwahori-integrable Kac-Moody modules. The corresponding object on the quantum group side, denoted $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$, involves Lusztig's version of the quantum group for the Borel and the De Concini-Kac version for the negative Borel.

Contents

Date: November 6, 2018.

Introduction

0.1. **What is this paper about?**

0.1.1. In their series of papers [KL], D. Kazhdan and G. Lusztig established an equivalence between the (abelian) category

$$\mathrm{KL}(G,-\kappa) := \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{G(\mathcal{O})}$$

of $G(\mathcal{O})$-integrable modules over the affine Kac-Moody Lie algebra at a negative level $-\kappa$ and the (abelian) category $\mathrm{Rep}_q(G)$ of integrable (=algebraic) representations of the "big" quantum group, whose quantum parameter q is related to κ via formula (0.7).

This paper addresses the following natural question: if we enlarge the category $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{G(\mathcal{O})}$ to $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{I}$, i.e., if we relax $G(\mathcal{O})$-integrability to Iwahori-integrability, what would the corresponsing category on the quantum group side be?

0.1.2. We propose a conjectural answer to this question: namely, we define a version of the category of modules over the quantum group, denoted $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$, where we let the positive quantum Borel be the Lusztig version, and the negative quantum Borel be the De Concini-Kac version, see Sect. 5.3 for the actual definition. Our Conjecture 9.2.2 says that there supposed to be an equivalence

$$\mathsf{F}_{-\kappa} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{I} \simeq \mathrm{Rep}_q^{\mathrm{mxd}}(G). \tag{0.1}$$

0.1.3. Here is one caveat: as was just mentioned, the original equivalence of [KL]

$$\mathsf{F}_{-\kappa} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{G(\mathcal{O})} \simeq \mathrm{Rep}_q(G) \tag{0.2}$$

is an exact equivalence of abelian categories. Hence, it induces to an equivalence of their derived categories, preserving the t-structures.

From now on, when we say "category" we will mean a triangulated category, or even more precisely, a DG category. If $\mathcal{C}$ is such a category, and if it is equipped with a t-structure, we will write $\mathcal{C}^{\heartsuit}$ for the heart of the t-structure. So let us read (0.2) as an equivalence of derived categories; the original equivalence at the abelian level is

$$(\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{G(\mathcal{O})})^{\heartsuit} \simeq (\mathrm{Rep}_q(G))^{\heartsuit}.$$

Now, the point here is that the extended equivalence (0.1) only holds at the level of DG categories. I.e., it is *not* t-exact with respect to the natural t-structures that exist on both sides.

Remark 0.1.4. That said, one can try to mimic the construction of [FG2, Sect. 2] to define a new t-structure on $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{I}$ so that the equivalence (0.1) becomes t-exact. This is an interesting problem, but we will not pursue it in this paper. See, however, Sect. 0.2.4 below.

0.1.5. Let us mention one curious feature of the equivalence (0.1).

Recall that the Kazhdan-Lusztig equivalence (0.2) sends the standard objects

$$\mathbb{V}^{\check{\lambda}}_{-\kappa} \in \widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa}, \quad \check{\lambda} \in \check{\Lambda}^+$$

(affine Weyl modules) to the standard objects

$$\mathcal{V}^{\check{\lambda}}_q \in \text{Rep}_q(G), \quad \check{\lambda} \in \check{\Lambda}^+$$

(quantum Weyl modules).

Now, the category $\text{Rep}^{\text{mxd}}_q(G)$ contains a naturally defined family of standard objects, denoted $\mathbb{M}^{\check{\lambda}}_{q,\text{mixed}}$, where $\check{\lambda}$ is a weight of G. These are modules induced from characters of the quantum Borel.

Under the equivalence (0.1), the objects $\mathbb{M}^{\check{\lambda}}_{q,\text{mixed}}$ *do not* correspond to the affine Verma modules. Rather, they go over the *Wakimoto modules*[1], denoted $\mathbb{W}^{\check{\lambda}}_{-\kappa}$.

Let us mention that if κ is irrational, then $\mathbb{W}^{\check{\lambda}}_{-\kappa}$ is actually isomorphic to the affine Verma module $\mathbb{M}^{\check{\lambda}}_{-\kappa}$. If κ is rational, the isomorphism still holds for $\check{\lambda}$ dominant, but not otherwise. For example, if $\check{\lambda}$ is sufficiently anti-dominant, the Wakimoto module $\mathbb{W}^{\check{\lambda}}_{-\kappa}$ is isomorphic to the *dual affine Verma module* $\mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}$.

0.2. **Where did the motivation come from?** The motivation for guessing the equivalence (0.1) came from multiple sources.

One source of motivation is the author's desire to re-prove the original Kazhdan-Lusztig equivalence (0.2) by a "more algebraic method".

0.2.1. We recall that the statement of the equivalence (0.2) in [KL] is not as mere abelian categories, but as *braided monoidal* abelian categories. In fact, the proof of the equivalence in [KL] uses this additional structure in a most essential way.

One can interpret the braided monoidal structure on $\text{KL}(G, -\kappa)$ as a structure of *de Rham factorization category*, and the braided monoidal structure on $\text{Rep}_q(G)$ as a structure of *Betti factorization factorization category*. From this point of view, the equivalence (0.2) should read that these two structures match up under the (appropriately defined) Riemann-Hilbert functor that maps Betti factorization categories to de Rham factorization categories.

0.2.2. Now, a structure of factorization category in either of the two contexts is a complicated piece of data. However, there one case when a factorization category can be described succinctly: namely, when a factorization category in question is that of *factorization modules* for a *factorization algebra* (say, within another factorization category, but one which is easier to understand).

The "trouble with" the original Kazhdan-Lusztig equivalence (0.2) is that the categories involved are *not* of this form.

0.2.3. By contrast, the factorization category corresponding to $\text{Rep}^{\text{mxd}}_q(G)$ is (more or less tautologically) equivalent to that of factorization modules.

The ambient factorization category in question is the factorization category corresponding to the braided monoidal category $\text{Rep}_q(T)$ of representations of the quantum torus. The factorization algebra in question, denoted Ω^{Lus}_q, is the Koszul dual of the Hopf algebra $U^{\text{Lus}}_q(N) \in \text{Rep}_q(T)$. This actually explains the appearance of the De Concini-Kac version: it enters as the dual Hopf algebra of $U^{\text{Lus}}_q(N)$.

[1] Our conventions regarding Wakimoto modules are different from those in most places in the literature such as [Fr2] abd [FG2], see Remark 2.4.5

0.2.4. Let us now look at the left-hand side of the proposed equivalence (0.1).

The Iwahori subgroup I is not a factorizable object. However, the following result was proved by S. Raskin (see, e.g., [Ga2, Sect. 5] for a proof): for a category $\mathcal{C}$ equipped with an action of $G(\mathcal{K})$ (see Sect. 1.2.2 for what this means), there is a canonical equivalence

$$\mathcal{C}^I \simeq \mathcal{C}^{N(\mathcal{K})\cdot T(\mathcal{O})};$$

here the superscript indicates taking the equivariant category with respect to the corresponding subgroup.

Hence, we can interpret the category $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ as

$$\widehat{\mathfrak{g}}\text{-mod}^{N(\mathcal{K})\cdot T(\mathcal{O})}_{-\kappa}. \tag{0.3}$$

This paves a way to using factorization methods, as the category (0.3) admits a natural factorization structure.

If we could prove that a certain explicit functor from $\widehat{\mathfrak{g}}\text{-mod}^{N(\mathcal{K})\cdot T(\mathcal{O})}_{-\kappa}$ to the category of factorization modules over a De Rham version $\Omega^{\text{Lus}}_{-\kappa}$ of Ω^{Lus}_q was an equivalence, that would establish the equivalence (0.1). And having (0.1), one can hope to be able to extract the original equivalence (0.2).

0.2.5. Now, an equivalence between $\widehat{\mathfrak{g}}\text{-mod}^{N(\mathcal{K})\cdot T(\mathcal{O})}_{-\kappa}$ and $\Omega^{\text{Lus}}_{-\kappa}\text{-mod}^{\text{Fact}}$ as factorization categories may be too much to hope for (i.e., it is likely that it *does not* hold "as-is").

However, such an equivalence is supposed to take place at the level of fibers (this is what our Conjecture (0.1) says) and also on sufficiently large subcategories, which should be enough to deduce the original equivalence (0.2).

0.3. **Motivation from local geometric Langlands.** There is yet another aspect to the above story, which has to do with local geometric Langlands.

0.3.1. One of the key conjectures (proposed in 2008 by J. Lurie and the author) is that the category $\text{KL}(G, -\kappa)$ is supposed to be equivalent (as a factorization category) to the twisted Whittaker category $\text{Whit}_{-\check{\kappa}}(\text{Gr}_{\check{G}})$ of the affine Grassmannian of the Langlands dual group (here $\check{\kappa}$ is the level for $\check{G}$ dual to the level κ for G, see Sect. 0.5.2):

$$\widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa} \simeq \text{Whit}_{-\check{\kappa}}(\text{Gr}_{\check{G}}) \tag{0.4}$$

This conjecture is called the *Fundamental Local Equivalence*, or the FLE.

The trouble with proving the FLE is essentially the same one as with the original Kazhdan-Lusztig equivalence (0.2): the two sides are some complicated factorization categories, yet we must relate them, based just on the combinatorial information that the groups G and $\check{G}$ are mutually dual.

0.3.2. However, just as in Sect. 0.2, one can have a better chance to first prove the Iwahori version of the FLE, namely, an equivalence

$$\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} \simeq \text{Whit}_{-\check{\kappa}}(\text{Fl}^{\text{aff}}_{\check{G}}), \tag{0.5}$$

and then bootstrap from it the original FLE (0.4).

0.3.3. In a subsequent publication, the author is planning to record the properties of the conjectural equivalence (0.5): the behavior of the standard and costandard objects, compatibility with duality, etc.

0.4. **Representation-theoretic motivation.** Finally, there is a purely representation-theoretic piece of motivation for the conjectural equivalence (0.1)[2].

---

[2]It originated in discussions between S. Arkhipov, R. Bezrukavnikov, M. Finkelberg, I. Miković and the author some 20 years ago.

0.4.1. We have the equivalence established in the paper [ABG] that says that a regular block $\mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml}}(G))$ of the category of modules over the *small* quantum group is equivalent to the category

$$\mathrm{IndCoh}(\{0\} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})$$

of ind-coherent sheaves on the *derived Spring fiber* over 0 for the Langlands dual Lie algebra.

0.4.2. Let now χ be a point of the spectrum of the "q-center" of the De Concini-Kac algebra. We can form the corresponding category

$$\mathrm{Rep}_q^{\chi}(G)$$

(so that for $\chi = 0$ we recover $\mathrm{Rep}_q^{\mathrm{sml}}(G)$), and consider its regular block $\mathsf{Bl}(\mathrm{Rep}_q^{\chi}(G))$.

By analogy with representations in positive characteristic, one conjectured an equivalence

$$\mathsf{Bl}(\mathrm{Rep}_q^{\chi}(G)) \simeq \mathrm{IndCoh}(\{\chi\} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}), \tag{0.6}$$

generalizing the equivalence of [ABG].

0.4.3. Now, as we shall see in Sect. 13, our conjectural equivalence (0.1), when restricted to a regular block gives an equivalence

$$\mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{mxd}}(G)) \simeq \mathrm{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}),$$

which is a version of (0.6) in the family $\chi \in \check{\mathfrak{n}}/\operatorname{Ad}(\check{B})$.

0.5. **What is done in this paper?** We now proceed to review the actual mathematical contents of the paper.

0.5.1. This paper centers around Conjecture 9.2.2, which proclaims the existence of an equivalence (0.1). However, the statement of the conjecture is both preceded and followed by some 45 pages of mathematical text.

In fact, this paper is divided into 3 parts. In the first part, we recall some facts pertaining to the affine category $\mathcal{O}$. In the second part, we review various versions of the category of modules over the quantum group. Neither Part I nor Part II contain substantial original results. In the third part, after stating Conjecture 9.2.2, we run some consistency checks and derive some consequences.

0.5.2. Part 3 is the core of this paper, in which study the affine algebra vs quantum group relationship. Here we need to take our field of coefficients k to be $\mathbb{C}$.

Our quantum parameter is the quadratic form q on $\check{\Lambda}$ with coefficients in $k^{\times}$, related with κ by the formula

$$q(\check{\lambda}) := \exp(2 \cdot \pi \cdot i \cdot \frac{\check{\kappa}(\check{\lambda}, \check{\lambda})}{2}), \tag{0.7}$$

where $\check{\kappa}$ is as in Sect. 0.7.3 below.

Note that the quadratic form q comes as restriction to the diagonal of the symmetric bilinear form b' on $\check{\Lambda}$ with coefficients in $k^{\times}$ equal to

$$b'(\check{\lambda}_1, \check{\lambda}_2) := \exp(2 \cdot \pi \cdot i \cdot \frac{\check{\kappa}(\check{\lambda}_1, \check{\lambda}_2)}{2}). \tag{0.8}$$

0.5.3. After stating Conjecture 9.2.2, we do the following:

–We note that Conjecture 9.2.2 implies that original Kazhdan-Lusztig equivalence (0.2) satisfies

$$\mathsf{F}_{-\kappa}(\mathbb{V}^{\check\lambda}_{-\kappa}) := \mathcal{V}^{\check\lambda}_q, \quad \check\lambda \in \check\Lambda \tag{0.9}$$

where $\mathbb{V}^{\check\lambda}_{-\kappa} \in \mathrm{KL}(G, -\kappa)$ is the (derived) Weyl module, defined by

$$\mathbb{V}^{\check\lambda}_{-\kappa} := \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\check\lambda}_{-\kappa}),$$

and $\mathcal{V}^{\check\lambda}_q \in \mathrm{Rep}_q(G)$ is the (derived) Weyl module, defined by

$$\mathcal{V}^{\check\lambda}_{-\kappa} := \mathbf{ind}^{\mathrm{Rep}_q(G)}_{\mathrm{Rep}_q(B)}(k^{\check\lambda}).$$

–We prove the isomorphism (0.9) unconditionally (i.e., without assuming Conjecture 9.2.2). This occupies most of Sect. 10. We should mention that this isomorphism is essentially equivalent to the main result of [Liu].

–We give an expression for the functor

$$\mathrm{Rep}_q(G) \to \mathrm{Vect}, \quad \mathcal{M} \mapsto \mathrm{C}^\cdot(u_q(N), \mathcal{M})^{\check\lambda}, \quad \check\lambda \in \check\Lambda \tag{0.10}$$

in terms of the original Kazhdan-Lusztig equivalence (0.2). This is done in Sect. 10.9.

–Assuming Conjecture 9.2.2, we show that the functor

$$\mathrm{Rep}^{\mathrm{mixed}}_q(G) \to \mathrm{Vect}, \quad \mathcal{M} \mapsto \mathrm{C}^\cdot(U^{\mathrm{DK}}_q(N^-), \mathcal{M})^{\check\lambda}, \quad \check\lambda \in \check\Lambda \tag{0.11}$$

corresponds under the equivalence (0.1) to the functor

$$\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} \to \mathrm{Vect}, \quad \mathcal{M} \mapsto \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathcal{M})^{\check\lambda},$$

see Sect. 2.1.1 for the notation. This is done in Sect. 12.

–Assuming Conjecture 9.2.2, we show that under the equivalence

$$\mathsf{Bl}(\mathrm{Rep}^{\mathrm{mixed}}_q(G)) \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check G)$$

(normalized as in Sect. 13.2.7), the functor (0.11) for $\check\lambda = \check\lambda_0$ corresponds to the functor of restriction to the big Schubert cell

$$\mathrm{pt}/\check T \simeq (\check G/\check B \times \check G/\check B)^o/\check G \subset (\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check G,$$

followed by the functor of $\check T$-invariants. This is done in Sect. 13.4.

–Finally, we show that the duality equivalences

$$(\mathrm{Rep}_q(G))^\vee \simeq \mathrm{Rep}_{q^{-1}}(G) \text{ and } (\mathrm{Rep}^{\mathrm{mxd}}_q(G))^\vee \simeq \mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G)$$

and

$$\mathrm{KL}(G, -\kappa)^\vee \simeq \mathrm{KL}(G, \kappa) \text{ and } (\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa})^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}^I_\kappa$$

combined with the equivalence(s) $\mathsf{F}_{-\kappa}$ induce an equivalence

$$\mathsf{F}_\kappa : \mathrm{KL}(G, \kappa) \simeq \mathrm{Rep}_{q^{-1}}(G),$$

and, assuming Conjecture 9.2.2, also an equivalence

$$\mathsf{F}_\kappa : \widehat{\mathfrak{g}}\text{-mod}^I_\kappa \simeq \mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G).$$

We study properties of these dual equivalences parallel to ones at the negative level, listed above.

0.5.4. *What is* not *done in this paper?* There are two main themes that could have been part of this paper but that are not:

One is the discussion of factorization (see Sect. 0.2). The other is the relationship of the categories appearing in (0.1) to the Whittaker category $\mathrm{Whit}_{-\check\kappa}(\mathrm{Fl}^{\mathrm{aff}}_{\check G})$.

0.6. **Structure of the paper.** We now proceed to describing the contents of the paper section-by-section.

0.6.1. In Sect. 1 we recollect some basic facts pertaining to the category of Kac-Moody modules: the definition is not completely straightforward as it involves a *renormalization procedure* designed to make the category compactly generated. We explain that the categories at opposite levels are in relation of *duality*, see Sect. 0.7.7 for what this means.

In Sect. 2 we introduce Wakimoto modules. We first introduce "the true", i.e., semi-infinite Wakimoto modules $\mathbb{W}_\kappa^{\check\lambda,\frac{\infty}{2}}$; they are as induced from the loop subalgebra $\mathfrak{b}(\mathcal{K}) \subset \mathfrak{g}(\mathcal{K})$. The key feature of $\mathbb{W}_\kappa^{\check\lambda,\frac{\infty}{2}}$ is that it belongs to the category $\widehat{\mathfrak{g}}\text{-mod}_\kappa^{N(\mathcal{K})\cdot T(\mathcal{O})}$, see Sect. 0.2.4. We then show that the usual Wakimoto module $\mathbb{W}_\kappa^{\check\lambda}$ can be obtained from $\mathbb{W}_\kappa^{\check\lambda,\frac{\infty}{2}}$ by the procedure of averaging with respect to the Iwahori group. We study the pattern of convolution of Wakimoto modules with the standard objects J_μ in the category $\text{D-mod}_{-\kappa}(\text{Fl}_G^{\text{aff}})^I$. We show that

$$J_\mu \star \mathbb{W}_\kappa^{\check\lambda} \simeq \mathbb{W}_\kappa^{\check\lambda+\mu}.$$

(This result had been previously obtained in [FG2].)

In Sect. 3 we study the relationship between Wakimoto modules and Verma modules. We recall that these two classes of modules coincide when the level κ is irrational. When the level κ is negative, we re-prove the theorem of [Fr1] which says that the Wakimoto module $\mathbb{W}_\kappa^{\check\lambda}$ is isomorphic to the affine Verma module $\mathbb{M}_\kappa^{\check\lambda}$ if $\check\lambda$ is dominant, and to the dual affine Verma module $\mathbb{M}_\kappa^{\vee,\check\lambda}$ if $\check\lambda$ is sufficiently anti-dominant. Whereas the original proof in [Fr1] relies on the analysis of singular vectors, our proof uses the Kashiwara-Tanisaki localization theorem.

0.6.2. In Sect. 4 we introduce the general framework that most versions of the category of modules over the quantum group fit in. Namely, we start with the datum of quadratic form q on the weight lattice with values in $k^\times$ and attach to it the braided monoidal category $\text{Rep}_q(T)$ that we think of as the category of representations of the quantum torus. Given a Hopf algebra A in $\text{Rep}_q(T)$, we consider the category $A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}}$ of locally nilpotent A-modules and its (relative to $\text{Rep}_q(T)$) Drinfeld's center, denoted $Z_{\text{Dr},\text{Rep}_q(T)}(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}})$. We study the basic properties of such categories: t-structures, standard and costandard objects, etc.

In Sect. 5 we specialize to the case of Hopf algebras A relevant to quantum groups. The main example is $A = U_q^{\text{Lus}}(N)$. The resulting category $Z_{\text{Dr},\text{Rep}_q(T)}(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}})$ is our $\text{Rep}_q^{\text{mixed}}(G)$. In this section we also recall the definition of the category of algebraic (a.k.a. locally finite or integrable) modules of Lusztig's quantum group, denoted $\text{Rep}_q(G)$. We note that this category does *not* fit into the pattern of $Z_{\text{Dr},\text{Rep}_q(T)}(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}})$, which makes it more difficult and more interesting to study.

In Sect. 6 we specialize to the case when q takes values in the group of roots of unity, and consider the case of the Hopf algebra $A = u_q(N)$, the positive part of the "small" quantum group. We introduce the corresponding category $\text{Rep}_q^{\text{sml,grd}}(G) := Z_{\text{Dr},\text{Rep}_q(T)}(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}})$, and study how it is related to the category $\text{Rep}_q(G)$. We note that there are three versions of $\text{Rep}_q^{\text{sml,grd}}(G)$ that differ from each other by renormalization (i.e., which objects are declared compact); it is important to keep track of these distinctions, for otherwise various desired equivalences would not hold "as-is".

In Sect. 7 we introduce yet another *non-standard* version of the category of modules over the quantum group, denoted $\text{Rep}_q^{\frac{1}{2}}(G)$: it corresponds to having as the positive part Lusztig's algebra $U_q^{\text{Lus}}(N)$ and as the negative part the "small" version $u_q(N^-)$. There are three versions of this category that differ from each other by renormalization, and they serve as intermediaries between $\text{Rep}_q^{\text{sml,grd}}(G)$ and $\text{Rep}_q^{\text{mxd}}(G)$.

In Sect. 8 we prove that the categories $\text{Rep}_q^{\text{mixed}}(G)$ and $\text{Rep}_{q^{-1}}^{\text{mixed}}(G)$ are each other's duals. The corresponding duality functor $\mathbb{D}^{\text{can}}$ sends the standard object $\mathbb{M}_{q,\text{mixed}}^{\check\lambda}$ to $\mathbb{M}_{q^{-1},\text{mixed}}^{-\check\lambda-2\check\rho}[d]$. In additon, we introduce a contragredient duality functor $\mathbb{D}^{\text{contr}}$ from (a certain subcategory containing all compact objects of) $\mathrm{Rep}_q^{\mathrm{mixed}}(G)$ to the category $\mathrm{Rep}_{q^{-1}}^{\widetilde{\mathrm{mixed}}}(G)$ (in which the roles of Lusztig's version and the De Concini-Kac version are swapped). We show that the functors $\mathbb{D}^{\mathrm{can}}$ and $\mathbb{D}^{\mathrm{contr}}$ differ by a "long intertwining functor"

$$\Upsilon : \mathrm{Rep}_q^{\mathrm{mixed}}(G) \to \mathrm{Rep}_q^{\widetilde{\mathrm{mixed}}}(G).$$

0.6.3. In Sect. 9 we formulate our Conjecture 9.2.2, which states the existence of the equivalence satisfying certain properties. We then run some initial consistency checks. We also formulate a version of Conjecture 9.2.2 for the positive level, obtained from the initial one by duality.

In Sect. 10 we prove Theorem 9.2.8, which states the existence of the isomorphism (0.9). The key idea in the proof is to identify the object in the category

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G, -\kappa) \tag{0.12}$$

that corresponds to the *baby Verma module*, considered as an object of the category

$$\mathrm{Rep}_q^{\frac{1}{2}}(G) \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G).$$

It turns out that such a description is essentially equivalent to the description of the functor

$$\mathrm{KL}(G, -\kappa) \to \mathrm{Vect}$$

corresponding to (0.10) via the Kazhdan-Lusztig equivalence (0.2). The description of the sought-for object in (0.12) is closely related to a certain geometric object which was recently introduced in [Ga2] under the name *semi-infinite intersection cohomology sheaf.*

In Sect. 11 we introduce an additional requirement that the conjectural equivalence (0.1) is supposed to satisfy. namely, it is supposed to be compatible with the actions on the two sides of the monoidal category $\mathrm{QCoh}(\mathfrak{n}/\mathrm{Ad}(B_H)$, where $\mathrm{QCoh}(\mathfrak{n}/\mathrm{Ad}(B_H)$ acts on $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ via the Arkhipov-Bezrukavnikov functor

$$\mathrm{QCoh}(\mathfrak{n}/\mathrm{Ad}(B_H) \to \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I.$$

We will see that this compatibility gives a conceptual explanation of the identification of the object in (0.12) corresponding to the baby Verma module.

In Sect. 12 we will show that the functor $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} \to \mathrm{Vect}$ corresponding under the conjectural equivalence (0.1) to the functor (0.11), is given by *semi-infinite cohomology* with respect to $\mathfrak{n}^-(\mathcal{K})$.

Finally, in Sect. 13, we will explain how the equivalence (0.1) leads to the identification of a regular block $\mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{mixed}}(G))$ of $\mathrm{Rep}_q^{\mathrm{mixed}}(G)$ with the category $\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G})$ of ind-coherent sheaves on the Steinberg stack $(\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}$.

0.7. **Conventions.**

0.7.1. *Ground field.* In this paper we will be working over a ground field k, assumed algebraically closed and of characteristic 0. All algebro-geometric objects in this paper will be schemes (or more generally, prestacks) over k.

0.7.2. *Reductive groups.* We let G be a reductive group over k. We denote by B a (chosen) Borel subgroup in G and by T its Cartan quotient. We let

$$\mathfrak{g} \supset \mathfrak{b} \twoheadrightarrow \mathfrak{t}$$

denote their respective Lie algebras.

Deviating slightly from the usual conventions, we will denote by Λ the *coweight* lattice of T, and by $\check{\Lambda}$ the weight lattice.

0.7.3. *The level.* By a *level* we will mean a W-invariant symmetric bilinear form κ on $\mathfrak{t}$, or which is the same, a W-invariant symmetric bilinear form on the coweight lattice Λ with coefficients in k.

We will assume that κ is non-degenerate, i.e., it defines an isomorphism $\mathfrak{t} \to \mathfrak{t}^\vee =: \check{\mathfrak{t}}$. We let $\check{\kappa}$ denote the resulting symmetric bilinear form on $\mathfrak{t}^\vee$, or which is the same, a W-invariant symmetric bilinear form on the weight lattice $\check{\Lambda}$ with coefficients in k.

To κ we attach an $\mathrm{Ad}(G)$-invariant symmetric bilinear form on $\mathfrak{g}$ so that its restriction to $\mathfrak{t}$ equals

$$\kappa + \frac{\kappa_{\mathrm{Kil}}}{2},$$

where κ_{Kil} is the Killing form of the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$.

Remark 0.7.4. Note that the level $\kappa = 0$ corresponds to the form $-\frac{\kappa_{\mathrm{Kil}}}{2}$ on $\mathfrak{g}$; it is called the critical level. Our convention of shifting the level for $\mathfrak{g}$ by the critical is an affine version of the "ρ-shift" in the usual representation theory of $\mathfrak{g}$.

0.7.5. *DG categories.* The object of study of this paper is DG categories over k. This automatically puts us in the context of higher algebra, developed in [Lu]. We refer the reader to [GR, Chapter 1] for a user guide.

There are a few things we need to mention about DG categories, viewed both intrinsically and extrinsically.

When we say "DG category" we will assume it to be *cocomplete*, unless explicitly stated otherwise. When talking about a functor between two DG categories, we will always mean an *exact* functor (i.e., a functor preserving finite colimits). When the DG categories in question are cocomplete, we will assume our functor to be *continuous* (i.e., preserving infinite direct sums, which, given exactness, is equivalent to preserving filtered colimits, and in fact all colimits).

Most of the DG categories we will encounter (and ones that we will end up working with) are *compactly generated*. If $\mathcal{C}$ is such a category, we will denote by $\mathcal{C}_c$ its full (but *not* cocomplete) subcategory consisting of compact objects.

Vice versa, starting from a non-cocomplete category $\mathcal{C}_0$, one can produce a cocomplete DG category by the procedure of *ind-completion*; the resulting cocomplete category $\mathcal{C}$ will be denoted $\mathrm{IndCompl}(\mathcal{C}_0)$. The category $\mathcal{C}$ is compactly generated and $\mathcal{C}_0 \subset \mathcal{C}_c$. For any compactly generated $\mathcal{C}$, we have $\mathcal{C} \simeq \mathrm{IndCompl}(\mathcal{C}_c)$.

Given a DG category $\mathcal{C}$ one can talk about a t-structure on $\mathcal{C}$. We will denote by $\mathcal{C}^{\leq 0}$ (resp., $\mathcal{C}^{\geq 0}$) the full subcategory of connective (resp., coconnetive) objects. We let $\mathcal{C}^\heartsuit := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ denote the heart of the t-structure; this is an abelian category. We let $\mathcal{C}^{<\infty}$ (resp., $\mathcal{C}^{>\infty}$) the full subcategory consisting of *eventually connective* (resp., *eventually coconnective*) objects.

0.7.6. *Tensor product of DG categories.* The $(\infty, 1)$-category DGCat of DG categories carries a symmetric monoidal structure, denoted $\mathcal{C}_1, \mathcal{C}_2 \mapsto \mathcal{C}_1 \otimes \mathcal{C}_2$.

The unit object for this symmetric monoidal structure is the category Vect of chain complexes of vector spaces.

As a basic example, if $\mathcal{C}_i = A_i$-mod for an associative algebra A_i, then

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \simeq (A_1 \otimes A_2)\text{-mod}.$$

0.7.7. *Duality for DG categories.* In particular, it makes sense to talk about a DG category $\mathcal{C}$ being *dualizable*. For a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two dualizable DG categories we let $F^\vee : \mathcal{C}_2^\vee \to \mathcal{C}_1^\vee$ denote the dual functor.

If $\mathcal{C}$ is compactly generated, it is dualizable and we have

$$\mathcal{C}^\vee \simeq \mathrm{IndCompl}((\mathcal{C}_c)^{\mathrm{op}}).$$

0.7.8. *Algebra in DG categories.* The symmetric monoidal structure on DGCat allows "to do algebra" in the world of DG categories. In particular, given a monoidal DG category $\mathcal{C}$ (i.e., an associative algebra object in DGCat) and its right and left module categories $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, it makes sense to talk about

$$\mathcal{C}_1 \underset{\mathcal{C}}{\otimes} \mathcal{C}_2 \in \mathrm{DGCat}\,.$$

0.7.9. *(Weak) actions of groups on categories.* We have the symmetric monoidal functor

$$(\mathrm{Sch}_{\mathrm{f.t.}})_{/k} \to \mathrm{DGCat}, \quad S \mapsto \mathrm{QCoh}(S), \quad (S_1 \overset{f}{\to} S_2) \mapsto \mathrm{QCoh}(S_1) \overset{f_*}{\longrightarrow} \mathrm{QCoh}(S_2).$$

In particular, if H is an algebraic group, the DG category $\mathrm{QCoh}(H)$ a structure of monoidal category under convolution. We a category acted on *weakly* by H we will mean a module category over $\mathrm{QCoh}(H)$.

For such a module category $\mathcal{C}$, we set

$$\mathbf{inv}_H(\mathcal{C}) := \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathcal{C}).$$

For $\mathcal{C} = \mathrm{Vect}$, we have

$$\mathbf{inv}_H(\mathrm{Vect}) \simeq \mathrm{Rep}(H),$$

and for any $\mathcal{C}$, the category $\mathbf{inv}_H(\mathcal{C})$ is naturally a module category over $\mathrm{Rep}(H)$.

It is shown in [Ga3, Theorem 2.5.5], the above functor

$$H\text{-}\mathbf{mod} \to \mathrm{Rep}(H)\text{-}\mathbf{mod}, \quad \mathcal{C} \mapsto \mathbf{inv}_H(\mathcal{C})$$

is an equivalence, with the inverse given by

$$\mathcal{C}' \mapsto \mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathcal{C}'.$$

Remark 0.7.10. In this paper we will also encounter the notion of *strong* action of an algebraic group (rather, group ind-scheme) on a DG category. We defer the discussion of this notion until Sect. 1.2.

0.8. **Acknowledgements.** It is an honor to dedicate this paper to Masaki Kashiwara, especially as this paper deals with areas where he had made crucial contributions: quantum groups and representations of affine Kac-Moody algebras.

I am grateful to S. Arkhipov, R. Bezrukavnikov, M. Finkelberg, E. Frenkel, D. Kazhdan and I. Mirković, discussions with whom have shaped my thinking about the subjects treated in this paper over the years.

I am grateful to J. Lurie for introducing me to the world of higher categories, without which this project could never take off.

I am grateful to P. Etingof for providing me with some very useful references.

The author's research is supported by NSF grant DMS-1063470.

Part I: Iwahori-intergrable Kac-Moody representations

1. Modules over the Kac-Moody algebra: recollections

1.1. **Definition of the category of modules.** One of the two primary objects of study in this paper is the (DG) category of modules over the Kac-Moody algebra at a given level κ, denoted $\widehat{\mathfrak{g}}\text{-mod}_\kappa$. In this subsection we recall its definition and discuss some basic properties.

1.1.1. The version of the DG category $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ that we will use was defined in [FG3, Sect. 23.1]. It involves a renormalization procedure.

Namely, we start with the abelian category $(\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit$ and consider the usual derived category $D((\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit)$ (by which we mean the corresponding DG category, following [Lu, Sect. 1.3.2]).

For every congruence subgroup $K_i \subset G(\mathcal{O})$, we consider the corresponding induced module

$$\text{Ind}_{\mathfrak{k}_i}^{\widehat{\mathfrak{g}}_\kappa}(k) \in (\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit. \tag{1.1}$$

We let $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ be the ind-completion of the full (but not cocomplete) subcategory of $D((\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit)$ generated under finite colimits by the objects (1.1). By construction, the objects (1.1) form a set of compact generators of $\widehat{\mathfrak{g}}\text{-mod}_\kappa$.

Ind-completing the tautological embedding, we obtain a functor

$$\mathfrak{s} : \widehat{\mathfrak{g}}\text{-mod}_\kappa \to D((\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit). \tag{1.2}$$

It is shown in *loc.cit.* that $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ carries a unique t-structure, for which the functor (1.2) is t-exact and defines an equivalence of the corresponding eventually coconnective subcategories, i.e.,

$$(\widehat{\mathfrak{g}}\text{-mod}_\kappa)^{>\infty} \to D^+((\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit).$$

Note, however, that the functor $\mathfrak{s}$ is *not* conservative. In particular, the category $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ is *not* left-separated in its t-structure.

1.1.2. Let K be a subgroup of finite-codimension in $G(\mathcal{O})$. The category $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ has a version, denoted $\widehat{\mathfrak{g}}\text{-mod}_\kappa^K$ and defined as follows.

We start with the abelian category $(\widehat{\mathfrak{g}}\text{-mod}_\kappa^K)^\heartsuit$ of modules for the Harish-Chandra pair $(\widehat{\mathfrak{g}}_\kappa, K)$, and consider its derived category $D((\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\heartsuit)$.

The renormalization procedure is done with respect to modules of the form

$$\text{Ind}_{\mathfrak{k}}^{\widehat{\mathfrak{g}}_\kappa}(V), \quad V \in (\text{Rep}(K)_{\text{f.d.}})^\heartsuit.$$

When K is pro-unipotent, the object $\text{Ind}_{\mathfrak{k}}^{\widehat{\mathfrak{g}}_\kappa}(k)$ is a compact generator of $\widehat{\mathfrak{g}}\text{-mod}_\kappa^K$.

1.1.3. The cases of particular interest for us are when $K = G(\mathcal{O})$ and $K = I$, the Iwahori subgroup.

We denote

$$\mathbb{V}_\kappa^{\check{\lambda}} := \text{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_\kappa}(V^{\check{\lambda}}) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^{G(\mathcal{O})}, \quad \check{\lambda} \in \check{\Lambda}^+, \tag{1.3}$$

where $V^{\check{\lambda}}$ denote the irreducible representation of G with highest weight $\check{\lambda}$, viewed as a representation of $G(\mathcal{O})$ via the evaluation map $G(\mathcal{O}) \to G$.

Denote also

$$\mathbb{M}_\kappa^{\check{\lambda}} := \text{Ind}_{\text{Lie}}^{\widehat{\mathfrak{g}}_\kappa}(k^{\check{\lambda}}) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^I, \quad \check{\lambda} \in \check{\Lambda},$$

where $k^{\check{\lambda}}$ is one-dimensional representation of T corresponding to character $\check{\lambda}$, viewed as a representation of I via $I \to T$.

The Weyl modules $\mathbb{V}_\kappa^{\check{\lambda}}$ (resp., the Verma modules $\mathbb{M}_\kappa^{\check{\lambda}}$) form a set of compact generators for $\widehat{\mathfrak{g}}\text{-mod}_\kappa^{G(\mathcal{O})}$ (resp., $\widehat{\mathfrak{g}}\text{-mod}_\kappa^I$).

1.1.4. Let us be given a pair of subgroups $K' \subset K''$. In this case we have the (obvious) forgetful functor

$$\mathbf{oblv}_{K''/K'} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^{K''} \to \widehat{\mathfrak{g}}\text{-mod}_\kappa^{K'},$$

which admits a right adjoint, denoted

$$\text{Av}_*^{K''/K'} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^{K'} \to \widehat{\mathfrak{g}}\text{-mod}_\kappa^{K''}.$$

If the quotient K''/K' is homologically contractible, i.e., if $k \simeq H_{\text{dR}}(K''/K')$ (e.g., both K' and K'' are pro-unipotent), then the functor $\mathbf{oblv}_{K''/K'}$ is fully faithful.

1.1.5. One shows that the naturally defined functor

$$\underset{i}{\operatorname{colim}}\, \widehat{\mathfrak{g}}\text{-mod}_\kappa^{K_i} \to \widehat{\mathfrak{g}}\text{-mod}_\kappa$$

is an equivalence.

From here we obtain that for any $K \subset G(\mathcal{O})$ as above, there is an adjoint pair

$$\mathbf{oblv}_K : \widehat{\mathfrak{g}}\text{-mod}_\kappa^K \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_\kappa : \mathrm{Av}_*^K .$$

1.1.6. Assume for a moment that K''/K' is a proper scheme (e.g., $K'' = G(\mathcal{O})$ and $K' = I$). Then the functor $\mathbf{oblv}_{K''/K'}$ also admits a *left* adjoint, denoted $\mathrm{Av}_!^{K''/K'}$.

However, Verdier duality on K''/K' implies that we have a canonical isomorphism

$$\mathrm{Av}_!^{K''/K'} \simeq \mathrm{Av}_*^{K''/K'}[2(\dim(K''/K')],$$

see [FG2, Sect. 22.10].

1.2. **Categories with Kac-Moody group actions.** In certain places in this paper, it will be convenient to place the pattern described in Sect. 1.1 in the more general context of *categories equipped with an action of* $G(\mathcal{K})$.

The material in this subsection is drawn from [FG2, Sect. 22].

1.2.1. We introduce the category $\text{D-mod}_\kappa(G(\mathcal{K}))$ of D-modules on the loop group $G(\mathcal{K})$ as follows.

We start with the abelian category $(\text{D-mod}_\kappa(G(\mathcal{K})))^\heartsuit$ defined in [FG2, Sect. 21], and apply the renormalization procedure with respect to the following class of objects:

$$\mathrm{Ind}_{\mathrm{IndCoh}(G(\mathcal{K}))}^{\text{D-mod}_\kappa(G(\mathcal{K}))}(\mathcal{F}), \quad \mathcal{F} \in (\mathrm{Coh}(G(\mathcal{K})))^\heartsuit,$$

where $\mathrm{Ind}_{\mathrm{IndCoh}(G(\mathcal{K}))}^{\text{D-mod}_\kappa(G(\mathcal{K}))}$ is the left adjoint of the forgetful functor

$$(\text{D-mod}_\kappa(G(\mathcal{K})))^\heartsuit \to (\mathrm{IndCoh}(G(\mathcal{K})))^\heartsuit$$

(one can show that the above functor $\mathrm{Ind}_{\mathrm{IndCoh}(G(\mathcal{K}))}^{\text{D-mod}_\kappa(G(\mathcal{K}))}$ is exact).

Convolution defines on $\text{D-mod}_\kappa(G(\mathcal{K}))$ a structure of monoidal DG category, see [FG2, Sect. 22.3].

1.2.2. By a (DG) category $\mathcal{C}$ acted on (strongly) by $G(\mathcal{K})$ at level κ we will mean a module category over $\text{D-mod}_\kappa(G(\mathcal{K}))$.

In particular, such $\mathcal{C}$ acquires an action of the group-scheme $G(\mathcal{O})$. For every subgroup $K \subset G(\mathcal{O})$ of finite codimension, we can consider the corresponding equivariant category $\mathcal{C}^K$. It is acted on by the corresponding Hecke category $\text{D-mod}_\kappa(G(\mathcal{K})/K))^K$.

An example of a category acted on (strongly) by $G(\mathcal{K})$ at level κ at level κ is $\mathcal{C} := \widehat{\mathfrak{g}}\text{-mod}_\kappa$. The corresponding categories $\mathcal{C}^K$ are what we have denoted earlier by $\widehat{\mathfrak{g}}\text{-mod}_\kappa^K$

The categories $\mathcal{C}^K$ for different K are related by the functors $(\mathbf{oblv}, \mathrm{Av}_*)$ as in Sects. 1.1.4 and 1.1.6.

1.2.3. Let $\mathcal{C}$ be a category acted (strongly) on by $G(\mathcal{K})$ at level κ, and let $H \subset G(\mathcal{K})$ be a group indsubscheme, such that the Kac-Moody extension $\widehat{\mathfrak{g}}_\kappa$ splits over its Lie algebra.

Then it makes sense to consider the equivariant category $\mathcal{C}^H$.

A case of particular interest for us is when $H = N(\mathcal{K}) \cdot T(\mathcal{O})$. Explicitly, if we write $N(\mathcal{K})$ as a union of group subschemes N_α that are invariant under conjugation by $T(\mathcal{O})$, then each $\mathcal{C}^{N_\alpha \cdot T(\mathcal{O})}$ is a full subcategory in $\mathcal{C}^{T(\mathcal{O})}$ (due to pro-unipotence), and we have

$$\mathcal{C}^{N(\mathcal{K})\cdot T(\mathcal{O})} = \bigcap_\alpha \mathcal{C}^{N_\alpha \cdot T(\mathcal{O})},$$

which is also a full subcategory in $\mathcal{C}^{T(\mathcal{O})}$.

1.3. **Duality.** It turns out that the categories of Kac-Moody modules at opposite levels are related to each other by the procedure of *duality of DG categories*, see Sect. 0.7.7. This often provides a convenient tool for the study of their representation theory.

The material in this subsection does not admit adequate references in the published literature.

1.3.1. We start with the discussion of the finite-dimensional $\mathfrak{g}$. Let A be an associative algebra and let A^{rev} be the algebra with reversed multiplication. Then the categories

$$A\text{-mod and } A^{\mathrm{rev}}\text{-mod}$$

are related by

$$(A\text{-mod})^\vee \simeq A^{\mathrm{rev}}\text{-mod}.$$

The pairing

$$A\text{-mod} \otimes A^{\mathrm{rev}}\text{-mod} \to \mathrm{Vect}$$

is given by

$$M_1, M_2 \mapsto M_1 \underset{A}{\otimes} M_2.$$

The dualizing object in A-mod $\otimes$ A^{rev}-mod is provided by A, viewed as a bimodule.

Taking $A = U(\mathfrak{g})$, and identifying $U(\mathfrak{g})^{\mathrm{rev}} \simeq U(\mathfrak{g})$ via the antipode map, we obtain a canonical identification

$$(\mathfrak{g}\text{-mod})^\vee \simeq \mathfrak{g}\text{-mod}. \tag{1.4}$$

The corresponding pairing

$$\mathfrak{g}\text{-mod} \otimes \mathfrak{g}\text{-mod} \to \mathrm{Vect}$$

is given by

$$M_1, M_2 \mapsto \mathcal{H}om_{\mathfrak{g}\text{-mod}}(k, \mathcal{M}_1 \otimes \mathcal{M}_2).$$

The duality (1.4) induces a duality

$$(\mathfrak{g}\text{-mod}^K)^\vee \simeq \mathfrak{g}\text{-mod}^K$$

for any subgroup $K \subset G$, so that the functors

$$\mathbf{oblv}_K : \mathfrak{g}\text{-mod}^K \rightarrow \mathfrak{g}\text{-mod} : \mathrm{Av}_*^K$$

satisfy

$$(\mathbf{oblv}_K)^\vee \simeq \mathrm{Av}_K^* \text{ and } (\mathrm{Av}_K^*)^\vee \simeq \mathbf{oblv}_K.$$

Let $\mathbb{D}$ denote the resulting contravariant auto-equivalence

$$(\mathfrak{g}\text{-mod}^K)_c \to (\mathfrak{g}\text{-mod}^K)_c.$$

Taking $K = G$, the corresponding duality on $\mathfrak{g}\text{-mod}^G = \mathrm{Rep}(G)$ is just contragredient duality, i.e.,

$$\mathbb{D}(V) \simeq V^\vee.$$

For $K = B$, we have

$$\mathbb{D}(M^{\check{\lambda}}) \simeq M^{-\check{\lambda}-2\check{\rho}}[d].$$

1.3.2. We now discuss the Kac-Moody case.

Let $-\kappa$ be the opposite level of κ. A key feature of the category $\text{D-mod}_\kappa(G(\mathcal{K}))$ is that it admits a global sections functor

$$\Gamma(G(\mathcal{K}), -) : \text{D-mod}_\kappa(G(\mathcal{K})) \to \widehat{\mathfrak{g}}\text{-mod}_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}_{-\kappa},$$

see [FG2, Sect. 21].

For a subgroup $K \subset G(\mathcal{O})$ of finite codimension, consider the object

$$\delta_{G(\mathcal{K}),K} := \text{Ind}^{\text{D-mod}_\kappa(G(\mathcal{K}))}_{\text{IndCoh}(G(\mathcal{K}))}(\mathcal{O}_K) \in \text{D-mod}_\kappa(G(\mathcal{K})),$$

where $\mathcal{O}_K \in \text{Coh}(G(\mathcal{K}))$ is the structure sheaf of K.

Consider the object

$$\Gamma(G(\mathcal{K}), \delta_{G(\mathcal{K}),K}) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}.$$

It naturally upgrades to an object of $\widehat{\mathfrak{g}}\text{-mod}^K_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}^K_{-\kappa}$. Denote

$$\text{unit}_K := \Gamma(G(\mathcal{K}), \delta_{G(\mathcal{K}),K})[-\dim(G(\mathcal{O})/K)].$$

1.3.3. The following assertion is proved in the same way as [AG2, Theorem 6.2]:

Proposition 1.3.4. *The object* unit_K *defines the co-unit of a duality.*

Thus, we obtain that the categories $\widehat{\mathfrak{g}}\text{-mod}^K_\kappa$ are and $\widehat{\mathfrak{g}}\text{-mod}^K_{-\kappa}$ are canonically mutually dual. We let

$$\mathbb{D} : (\widehat{\mathfrak{g}}\text{-mod}^K_\kappa)_c \to (\widehat{\mathfrak{g}}\text{-mod}^K_{-\kappa})_c$$

denote the resulting contravariant equivalence.

We will denote the resulting pairing

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}_\kappa \to \text{Vect}$$

by $\langle -, - \rangle_{\widehat{\mathfrak{g}}\text{-mod}^K_\kappa}$.

Remark 1.3.5. The pairing $\langle -, - \rangle_{\widehat{\mathfrak{g}}\text{-mod}^K_\kappa}$ is explicitly given by the operation *semi-infinite Tor* relative to K. Alternatively, we can take $\langle -, - \rangle_{\widehat{\mathfrak{g}}\text{-mod}^K_\kappa}$ as the definition of this semi-infinite Tor functor.

1.3.6. By unwinding the definitions, we obtain the following:

Lemma 1.3.7. *For* $V \in \text{Rep}(K)_{\text{fin.dim}}$, *we have a canonical isomorphism*

$$\mathbb{D}(\text{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{k}}(V)) \simeq \text{Ind}^{\widehat{\mathfrak{g}}_{-\kappa}}_{\mathfrak{k}}(V^\vee \otimes \det(\mathfrak{g}(\mathcal{O})/\mathfrak{k})).$$

In the above formula, the line $\det(\mathfrak{g}(\mathcal{O})/\mathfrak{k})$ is, according to our conventions, placed in cohomological degree $-\dim(G(\mathcal{O})/K)$, and is regarded as a one-dimensional representation of K.

In particular, for $K = G(\mathcal{O})$, we have

$$\mathbb{D}(\text{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}(\mathcal{O})}(V)) \simeq \text{Ind}^{\widehat{\mathfrak{g}}_{-\kappa}}_{\mathfrak{g}(\mathcal{O})}(V^\vee)$$

and for $K = I$, we have

$$\mathbb{D}(\mathbb{M}^{\check{\lambda}}_\kappa) \simeq \mathbb{M}^{-\check{\lambda}-2\check{\rho}}_{-\kappa}[d]. \tag{1.5}$$

1.3.8. Let $K' \subset K''$ be a pair of subgroups. Then the functors

$$\textbf{oblv}_{K''/K'} : \widehat{\mathfrak{g}}\text{-mod}^{K''}_\kappa \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}^{K'}_\kappa : \text{Av}^{K''/K'}_*$$

and

$$\textbf{oblv}_{K''/K'} : \widehat{\mathfrak{g}}\text{-mod}^{K''}_{-\kappa} \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}^{K'}_{-\kappa} : \text{Av}^{K''/K'}_*$$

are related by

$$(\textbf{oblv}_{K''/K'})^\vee \simeq \text{Av}^{K''/K'}_* \text{ and } (\text{Av}^{K''/K'}_*)^\vee \simeq \textbf{oblv}_{K''/K'},$$

with respect to the identifications

$$(\widehat{\mathfrak{g}}\text{-mod}^{K''}_\kappa)^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}^{K''}_{-\kappa} \text{ and } (\widehat{\mathfrak{g}}\text{-mod}^{K'}_\kappa)^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}^{K'}_{-\kappa}.$$

1.3.9. The compatible family of equivalences

$$(\widehat{\mathfrak{g}}\text{-mod}_\kappa^{K_i})^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{K_i}$$

for congruence subgroups induces an equivalence

$$(\widehat{\mathfrak{g}}\text{-mod}_\kappa)^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}_{-\kappa},$$

uniquely characterized by the property that for every K as above, the functors

$$\mathbf{oblv}_K : \widehat{\mathfrak{g}}\text{-mod}_\kappa^K \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_\kappa : \mathrm{Av}_*^K$$

and

$$\mathbf{oblv}_K : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^K \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_{-\kappa} : \mathrm{Av}_*^K$$

are obtained from each other as duals.

We denote the corresponding pairing

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}_{-\kappa} \to \mathrm{Vect}$$

by $\langle -,- \rangle_{\widehat{\mathfrak{g}}\text{-mod}_\kappa}$.

For any *group-scheme*, $H \subset G(\mathcal{K})$, the pairing $\langle -,- \rangle_{\widehat{\mathfrak{g}}\text{-mod}_\kappa}$ induces a perfect pairing

$$\langle -,- \rangle_{\widehat{\mathfrak{g}}\text{-mod}_\kappa^H} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^H \otimes \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^H \to \mathrm{Vect},$$

so that the functors $\mathbf{oblv}_H$ and Av_*^H are mutually dual as before.

1.4. **BRST.** We will now discuss the functor of BRST reduction (a.k.a. semi-infinite cohomology) with respect to the loop algebra $\mathfrak{n}(\mathcal{K})$. It appears prominently in this paper as it is closely related to Wakimoto modules.

1.4.1. Consider the topological Lie algebra $\mathfrak{n}(\mathcal{K})$, and consider the category $\mathfrak{n}(\mathcal{K})$-mod, defined in the same manner as in Sect. 1.1.1.

We define the functor

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-) : \widehat{\mathfrak{g}}\text{-mod}_\kappa \to \mathrm{Vect}$$

as follows.

Write $\mathfrak{n}(\mathcal{K}) = \underset{\alpha}{\cup} N_\alpha$. Note that for $N_{\alpha_1} \subset N_{\alpha_2}$ we we have a canonically defined natural transformation

$$\mathrm{C}^\cdot(\mathfrak{n}_{\alpha_1},-) \to \mathrm{C}^\cdot(\mathfrak{n}_{\alpha_2}, - \otimes \det(\mathfrak{n}_{\alpha_2}/\mathfrak{n}_{\alpha_1})).$$

Assume without loss of generality that $\mathfrak{n}(\mathcal{O}) \subset \mathfrak{n}_\alpha$ for all α. The assignment

$$\alpha \mapsto \mathrm{C}^\cdot(\mathfrak{n}_\alpha, - \otimes \det(n_\alpha/\mathfrak{n}(\mathcal{O}))$$

is a functor from the category of indices α to that of functors $\widehat{\mathfrak{g}}\text{-mod}_\kappa \to \mathrm{Vect}$.

We set

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-) := \underset{\alpha}{\mathrm{colim}}\ \mathrm{C}^\cdot(\mathfrak{n}_\alpha, - \otimes \det(n_\alpha/\mathfrak{n}(\mathcal{O})).$$

1.4.2. Consider now the topological Lie algebra $\mathfrak{b}(\mathcal{K})$. According to [BD, Sect. 2.7.7], there exists a canonically defined central extension

$$0 \to k \to \widehat{\mathfrak{t}}_{\mathrm{Tate}} \to \mathfrak{t}(\mathcal{K}) \to 0 \tag{1.6}$$

with the following property:

The composite functor

$$\mathfrak{b}(\mathcal{K})\text{-mod} \to \mathfrak{n}(\mathcal{K})\text{-mod} \overset{C^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-)}{\longrightarrow} \mathrm{Vect}$$

canonically lifts to a functor

$$\mathrm{BRST}_{\mathfrak{n}} : \mathfrak{b}(\mathcal{K})\text{-mod} \to \widehat{\mathfrak{t}}_{\mathrm{Tate}}\text{-mod} \tag{1.7}$$

Moreover, the above functor (1.7) has a structure of functor between categories acted on by $B(\mathcal{O})$.

Remark 1.4.3. A remarkable feature of the extension (1.6) is that it *does* not admit a canonical slitting at the level of vector space. In fact, it does not admit a splitting which is $\mathrm{Aut}(\mathcal{O})$-invariant.

1.4.4. Since (1.6) is a central extension of the abelian Lie algebra $\mathfrak{t}(\mathcal{K})$, it corresponds to a skew symmetric form on $\mathfrak{t}(\mathcal{K})$. It is shown in [BD, Sect. 2.8.17] that this form equals the usual Kac-Moody cocycle corresponding to the symmetric bilinear form on $\mathfrak{t}$ equal to

$$\frac{\kappa_{\mathrm{Kil}}}{2}|_{\mathfrak{t}} = -\kappa_{\mathrm{crit}}|_{\mathfrak{t}}.$$

Given a symmetric bilinear form κ on $\mathfrak{t}$, let $\widehat{\mathfrak{t}}'_\kappa$ be the central extension

$$0 \to k \to \widehat{\mathfrak{t}}'_\kappa \to \mathfrak{t}(\mathcal{K}) \to 0$$

equal to the Baer sum of the restriction of

$$\widehat{\mathfrak{g}}_\kappa|_{\mathfrak{t}(\mathcal{K})} := \widehat{\mathfrak{g}}_\kappa \underset{\mathfrak{g}(\mathcal{K})}{\times} \widehat{\mathfrak{t}}(\mathcal{K})$$

and $\widehat{\mathfrak{h}}_{\mathrm{Tate}}$.

Note that due to out conventions of shifting the level for $\mathfrak{g}$ by the critical value, the extension $\widehat{\mathfrak{t}}'_\kappa$ is the Baer sum of $\widehat{\mathfrak{t}}_\kappa$ and an abelian (but not canonically split) extension of $\mathfrak{t}(\mathcal{K})$.

We obtain that the composite functor

$$\widehat{\mathfrak{g}}_\kappa\text{-mod} \to \widehat{\mathfrak{g}}_\kappa|_{\mathfrak{b}(\mathcal{K})}\text{-mod} \to \mathfrak{n}(\mathcal{K})\text{-mod} \to \mathrm{Vect}$$

can be canonically lifted to a functor

$$\mathrm{BRST}_{\mathfrak{n}} : \widehat{\mathfrak{g}}_\kappa\text{-mod} \to \widehat{\mathfrak{t}}'\text{-mod}_\kappa. \tag{1.8}$$

2. Semi-infinite cohomology and Wakimoto modules

Wakimoto modules for the Kac-Moody Lie algebra $\widehat{\mathfrak{g}}_\kappa$, first introduced in [FF], play a central role in this paper.

In this section we will reinterpret the construction of Wakimoto modules by first defining *semi-infinite* Wakimoto modules (as modules in a certain precise sense (co)induced from loops into the Borel subalgebra), and then applying the functor of Iwahori-averaging.

2.1. **The functor of semi-infinite cohomology.** In this subsection we will discuss a "hands-on" way to describe a certain variant of the functor of semi-infinite cohomology

$$\mathrm{BRST}_{\mathfrak{n}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa \to \widehat{\mathfrak{t}}'\text{-mod}_\kappa,$$

see (1.8).

2.1.1. The above functor $\mathrm{BRST}_{\mathfrak{n}}$ is a functor between categories acted on by $B(\mathcal{O})$. In particular, it gives rise to the (same-named) functor

$$\mathrm{BRST}_{\mathfrak{n}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})} \to \widehat{\mathfrak{t}}'\text{-mod}_\kappa^{T(\mathcal{O})}. \tag{2.1}$$

Fix $\check{\lambda} \in \check{\Lambda}$, and let

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check{\lambda}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})} \to \mathrm{Vect}$$

be the functor equal to the composite of $\mathrm{BRST}_{\mathfrak{n}}$ of (2.1) and the functor

$$\widehat{\mathfrak{t}}\text{-mod}_\kappa^{T(\mathcal{O})} \to \mathrm{Vect}, \quad \mathcal{M} \mapsto \mathcal{H}om_{\mathrm{Rep}(T(\mathcal{O}))}(k^{\check{\lambda}}, \mathcal{M}). \tag{2.2}$$

We will now describe the functor $\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check{\lambda}}$ explicitly as a colimit.

Remark 2.1.2. Recall that when we write $\widehat{\mathfrak{g}}\text{-mod}_\kappa$, we take into account the critical shift (so that the symmetric bilinear form on $\mathfrak{g}$ corresponding to κ is one whose restriction to $\mathfrak{t}$ is $\kappa - \kappa_{\mathrm{crit}}$). Recall that we are assuming that κ is non-degenerate. In this case, the category

$$\mathrm{KL}(T)'_\kappa := \widehat{\mathfrak{t}}'\text{-mod}_\kappa^{T(\mathcal{O})}$$

is semi-simple and is equivalent to $\mathrm{Rep}(T) \simeq \mathrm{Vect}^{\check{\Lambda}}$ by means of

$$k^{\check{\lambda}} \mapsto \mathrm{Ind}_{\mathfrak{t}(\mathcal{O})}^{\widehat{\mathfrak{t}}'_\kappa}(k^{\check{\lambda}}),$$

with the functors (2.2) providing the inverse. Hence, in this case, the collection of the functors $\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check{\lambda}}$ recovers the functor (2.1).

2.1.3. Let A be a filtered set, and let

$$\alpha \in I_\alpha$$

be a family of pro-solvable subgroups of $G(\mathcal{K})$ with the following properties:

- $T(\mathcal{O}) \subset I_\alpha$ and each I_α admits a triangular decomposition in the sense that the multiplication map defines an isomorphism of schemes
$$I_\alpha^+ \times T(\mathcal{O}) \times I_\alpha^- \to I_\alpha,$$
where
$$I_\alpha^+ := I_\alpha \cap N(\mathcal{K}), \quad I_\alpha^- := I_\alpha \cap N^-(\mathcal{K}).$$
- For each α, we have $N(\mathcal{O}) \subset I_\alpha^+$, for $\alpha_1 < \alpha_2$, we have $I_{\alpha_1}^+ \subset I_{\alpha_2}^+$ and
$$\underset{\alpha \in A}{\cup} I_\alpha^+ = N(\mathcal{K}).$$
- For each α, we have $N^-(\mathcal{O}) \supset I_\alpha^-$, for $\alpha_1 < \alpha_2$, we have $I_{\alpha_1}^- \supset I_{\alpha_2}^-$ and
$$\underset{\alpha \in A}{\cap} I_\alpha^- = \{1\}.$$

2.1.4. An example of such a family of subgroups is provided by taking $A = \Lambda^+$ with the order relation given by

$$\mu_1 < \mu_2 \Leftrightarrow \mu_2 - \mu_1 \in \Lambda^+,$$

and setting

$$I_\mu := \mathrm{Ad}_{t^{-\mu}}(I).$$

2.1.5. It follows that for every α we have a canonical (surjective) homomorphism $I_\alpha \to T$, whose kernel, denoted $\overset{\circ}{I}_\alpha$, is the pro-unipotent radical of I_α, i.e.,

$$I_\alpha \simeq T \ltimes \overset{\circ}{I}_\alpha.$$

2.1.6. Note the restriction of the Kac-Moody extension $\widehat{\mathfrak{g}}_\kappa$ to each $\mathrm{Lie}(I_\alpha)$ admits a canonical splitting fixed by the condition that it agrees with the given splitting over $\mathfrak{t}(\mathcal{O}) \subset \mathfrak{g}(\mathcal{O})$. These splittings agree under the inclusions

$$\mathrm{Lie}(I_{\alpha_1}) \supset \mathrm{Lie}(I_{\alpha_1} \cap I_{\alpha_2}) \subset \mathrm{Lie}(I_{\alpha_2}), \quad \alpha_1, \alpha_2 \in A.$$

In particular, for every $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa$ and $\alpha \in A$, it makes sense to consider

$$\mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M}) \in \mathfrak{t}\text{-mod},$$

and for a pair of indices $\alpha_1, \alpha_2 \in A$ we have the natural maps of $\mathfrak{t}$-modules

$$\mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_1}), \mathcal{M}) \to \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_1} \cap \overset{\circ}{I}_{\alpha_2}), \mathcal{M}) \leftarrow \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_2}), \mathcal{M}).$$

In what follows, for $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^T$ and $\check{\lambda} \in \check{\Lambda}$, we will write

$$\mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M})^{\check{\lambda}} := \mathcal{H}om_{\mathrm{Rep}(T)}\left(k^{\check{\lambda}}, \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M})\right).$$

2.1.7. Note that for every pair of indices $\alpha_1 < \alpha_2$ and $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa$, we have the canonical maps

$$\mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_1}), \mathcal{M}) \to \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_1} \cap \overset{\circ}{I}_{\alpha_2}), \mathcal{M}) \to \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_2}), \mathcal{M} \otimes \ell_{\alpha_1,\alpha_2}),$$

where ℓ_{α_1,α_2} is the graded line

$$\det(\mathrm{Lie}(I^+_{\alpha_2}) / \mathrm{Lie}(I^+_{\alpha_1})),$$

equipped with the natural T-action.

Here and thereafter, for a finite-dimensional vector space V, its determinant line $\det(V)$ will be placed in cohomological degree $-\dim(V)$.

2.1.8. Let ℓ_α be the graded line $\det(\mathrm{Lie}(I^+_\alpha)/\mathfrak{n}(\mathcal{O}))$, equipped with the natural action of T.

Thus, for $\alpha_1 < \alpha_2$ and $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa$, we have a canonical map

$$\mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_1}), \mathcal{M} \otimes \ell_{\alpha_1}) \to \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_{\alpha_2}), \mathcal{M} \otimes \ell_{\alpha_2})$$

of $\mathfrak{t}$-modules. The assignment

$$\alpha \mapsto \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M} \otimes \ell_\alpha)^{\check{\lambda}}$$

upgrades to a functor from the category of indices A to Vect. Set

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa^T \to \mathrm{Vect}, \quad \mathcal{M} \mapsto \underset{\alpha}{\mathrm{colim}}\ \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M} \otimes \ell_\alpha)^{\check{\lambda}}. \tag{2.3}$$

It is easy to see that the above construction is canonically independent of the choice of the family $\alpha \mapsto I_\alpha$.

The following results by unfolding the definitions:

Proposition 2.1.9. *For $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}$ there exists a canonical isomorphism*

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check{\lambda}} \simeq \underset{\alpha}{\mathrm{colim}}\ \mathrm{C}^\cdot(\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M} \otimes \ell_\alpha)^{\check{\lambda}}.$$

2.2. **The semi-infinite Wakimoto module.** In this subsection we will define the *semi-infinite* Wakimoto modules, denoted $\mathbb{W}_\kappa^{\check{\lambda}, \frac{\infty}{2}}$ for $\check{\lambda} \in \check{\Lambda}$.

2.2.1. Let $\alpha \mapsto I_\alpha$ be as in Sect. 2.1.3. Let ℓ^-_α denote the graded line

$$\det(\mathrm{Lie}(I^-)/\,\mathrm{Lie}(I^-_\alpha)).$$

For each α consider the object

$$\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_\alpha)}(k^{\check{\lambda}} \otimes \ell^-_\alpha) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa.$$

Note that it naturally belongs to

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa^{I_\alpha} \subset \widehat{\mathfrak{g}}\text{-mod}_\kappa^{I^+_\alpha \cdot T(\mathcal{O})} \subset \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}.$$

Note also that it is concentrated in cohomological degree $-\dim(I^-/I^-_\alpha)$, which is the cohomological degree of ℓ^-_α.

2.2.2. We claim that for $\alpha_1 \leq \alpha_2$ we have a naturally defined map

$$\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_1})}(k^{\check{\lambda}} \otimes \ell^-_{\alpha_1}) \to \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_2})}(k^{\check{\lambda}} \otimes \ell^-_{\alpha_2}).$$

Indeed, this map equals the composition

$$\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_1})}(k^{\check{\lambda}} \otimes \ell^-_{\alpha_1}) \to \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_1}\cap I_{\alpha_2})}(k^{\check{\lambda}} \otimes \ell^-_{\alpha_2}) \to \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_2})}(k^{\check{\lambda}} \otimes \ell^-_{\alpha_2}),$$

where the second arrow comes by applying $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_2})}$ to the counit of the adjunction

$$\mathrm{Ind}^{\mathrm{Lie}(I_{\alpha_2})}_{\mathrm{Lie}(I_{\alpha_1}\cap I_{\alpha_2})} \circ \mathrm{Res}^{\mathrm{Lie}(I_{\alpha_2})}_{\mathrm{Lie}(I_{\alpha_1}\cap I_{\alpha_2})} \to \mathrm{Id},$$

and the first arrow comes by applying $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_{\alpha_1})}$ to the natural transformation

$$\mathrm{Id} \otimes \ell^-_{\alpha_1,\alpha_2} \to \mathrm{Ind}^{\mathrm{Lie}(I_{\alpha_1})}_{\mathrm{Lie}(I_{\alpha_1}\cap I_{\alpha_2})} \circ \mathrm{Res}^{\mathrm{Lie}(I_{\alpha_1})}_{\mathrm{Lie}(I_{\alpha_1}\cap I_{\alpha_2})},$$

where

$$\ell^-_{\alpha_1,\alpha_2} := \det(\mathrm{Lie}(I^-_{\alpha_1})/\,\mathrm{Lie}(I^-_{\alpha_1} \cap I^-_{\alpha_2}))^{\otimes -1}.$$

2.2.3. We define

$$\mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}} := \underset{\alpha}{\mathrm{colim}}\ \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_\alpha)}(k^{\check{\lambda}} \otimes \ell^-_\alpha) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}.$$

By construction, $\mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}}$ belongs to the full subcategory $\widehat{\mathfrak{g}}\text{-mod}_\kappa^{N(\mathcal{K})\cdot T(\mathcal{O})} \subset \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}$, i.e., it is equivariant with respect to any group-subscheme of $N(\mathcal{K})$.

It is easy to see that $\mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}}$ is canonically independent of the choice of the family of the subgroups $\alpha \mapsto I_\alpha$.

Remark 2.2.4. A feature of $\mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}}$ is that, when viewed as an object of $\widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}$, it is *infinitely connective*, i.e., is cohomologically $\leq -n$ for any n with respect to the natural t-structure on $\widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}$. In other words, all of its cohomologies with respect to this t-structure are zero.

2.3. **Relation to duality.** As we shall presently see, the semi-infinite Wakimoto modules defined above are precisely the objects that represent the BRST functor.

2.3.1. Recall that we have a canonical pairing

$$\langle -,-\rangle_{\widehat{\mathfrak{g}}\text{-mod}^{T(\mathcal{O})}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})} \otimes \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{T(\mathcal{O})} \to \mathrm{Vect},$$

see Sect. 1.3.9.

We claim:

Proposition 2.3.2. *The functor*

$$\langle \mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}}, \mathcal{M}\rangle_{\widehat{\mathfrak{g}}\text{-mod}^{T(\mathcal{O})}} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^{T(\mathcal{O})} \to \mathrm{Vect}$$

is canonically isomorphic to the functor

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M} \otimes \det(\mathfrak{n}^-)^{\otimes -1})^{-\check{\lambda}}.$$

Note that in the above formula, the graded line $\det(\mathfrak{n}^-)^{\otimes -1}$ is placed in the cohomological degree $\dim(\mathfrak{n}^-) =: d := \dim(\mathfrak{n})$, and as a character of T, it corresponds to 2ρ. In other words, of we trivialize $\det(\mathfrak{n}^-)$ as a line, we have:

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M} \otimes \det(\mathfrak{n}^-)^{\otimes -1})^{-\check{\lambda}} \simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{-\check{\lambda}-2\check{\rho}}[-d].$$

Remark 2.3.3. Note that Proposition 2.3.2 is an affine analog of the following isomorphism of functors for the category $\mathfrak{g}\text{-mod}^T$:

$$\langle M^{\check{\lambda}}, \mathcal{M}\rangle_{\mathfrak{g}\text{-mod}^T} \simeq \mathrm{C}^{\cdot}(\mathfrak{n}, \mathcal{M} \otimes \det(\mathfrak{n}^-)^{\otimes -1})^{-\check{\lambda}} \simeq \mathrm{C}^{\cdot}(\mathfrak{n}, \mathcal{M})^{-\check{\lambda}-2\check{\rho}}[-d],$$

where $M^{\check{\lambda}} \in \mathfrak{g}\text{-mod}^B$ is the Verma module, i.e., $M^{\check{\lambda}} = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(k^{\check{\lambda}})$.

Proof of Proposition 2.3.2. The proof is tautological from the following canonical identification:

Let I' be a solvable group sub-scheme of $G(\mathcal{K})$ that contains $T \subset T(\mathcal{O})$ as a maximal torus. Let $\overset{\circ}{I}{}'$ denote the unipotent radical of I', so that $I' \simeq T \ltimes \overset{\circ}{I}{}'$. Note that the Kac-Moody extension $\widehat{\mathfrak{g}}_\kappa$, as well as $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}$, is equipped with splittings over $\mathrm{Lie}(I')$, compatible with the given splitting over $T(\mathcal{O})$.

Note also that the splitting of the extension $\widehat{\mathfrak{g}}_{\mathrm{Kil}}$ endows the determinant line

$$\det.\mathrm{rel}.(\mathfrak{g}(\mathcal{O}), \mathrm{Lie}(I')) := \det(\mathfrak{g}(\mathcal{O})/\mathfrak{g}(\mathcal{O}) \cap \mathrm{Lie}(I')) \otimes \det(\mathrm{Lie}(I')/\mathfrak{g}(\mathcal{O}) \cap \mathrm{Lie}(I'))^{\otimes -1}$$

with an action of I'. The corresponding character of T corresponds to the natural action of T on $\det.\mathrm{rel}.(\mathfrak{g}(\mathcal{O}), \mathrm{Lie}(I'))$ (note T is contained in $G(\mathcal{O}) \cap I'$).

Let $\mathcal{M}$ be an object of $\widehat{\mathfrak{g}}\text{-mod}^T_{-\kappa}$, and let $\mathcal{N}$ be an object of $\mathrm{Lie}(I')\text{-mod}^T$. Then we have a canonical isomorphism:

$$\langle \mathcal{N} \otimes \det.\mathrm{rel}.(\mathfrak{g}(\mathcal{O}), \mathrm{Lie}(I')), \mathcal{M}\rangle_{\mathrm{Lie}(I')\text{-mod}^T} \simeq \langle \mathrm{Ind}_{I'}^{\widehat{\mathfrak{g}}\text{-mod}_\kappa}(\mathcal{N}), \mathcal{M}\rangle_{\widehat{\mathfrak{g}}\text{-mod}^{T(\mathcal{O})}}.$$

□

2.4. **The usual Wakimoto modules.** In this section we will define the "usual" Wakimoto modules and relate them to the semi-infinite Wakimoto modules defined earlier.

2.4.1. Recall the averaging functor

$$\mathrm{Av}_*^{I/T} : \widehat{\mathfrak{g}}\text{-mod}_\kappa^T \to \widehat{\mathfrak{g}}\text{-mod}_\kappa^I.$$

We set

$$\mathbb{W}_\kappa^{\check{\lambda}} := \mathrm{Av}_*^{I/T}(\mathbb{W}_\kappa^{\check{\lambda},\frac{\infty}{2}}). \tag{2.4}$$

Remark 2.4.2. A remarkable feature of $\mathbb{W}_\kappa^{\check{\lambda}}$, not obvious from the above definition, is that it belongs to the heart of the t-structure on $\widehat{\mathfrak{g}}\text{-mod}_\kappa^I$.

2.4.3. From Proposition 2.3.2 and Sect. 1.3.8 we obtain:

Corollary 2.4.4. *There exists a canonical isomorphism of functors* $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I \to \mathrm{Vect}$

$$\langle \mathbb{W}_\kappa^{\check{\lambda}}, \mathcal{M}\rangle_{\widehat{\mathfrak{g}}\text{-mod}^I} \simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M} \otimes \det(\mathfrak{n}^-)^{\otimes -1})^{-\check{\lambda}}.$$

This corollary shows that $\mathbb{W}_\kappa^{\check{\lambda}}$, defined by formula (2.4), indeed agrees with the usual definition of Wakimoto modules, defined e.g. in [FG2, Sect. 11].

Remark 2.4.5. Our conventions regarding what we call "the usual" Wakimoto modules are slightly different from those in [FG2].

Namely, the most standard Wakimoto module is what in *loc.cit.* was denoted $\mathbb{W}^{w_0}_{\kappa,\check{\lambda}}$; for example for $\check{\lambda} = 0$, it has a structure of chiral/vertex operator algebra. Our $\mathbb{W}_\kappa^{\check{\lambda}}$ is what in [FG2] is denoted $\mathbb{W}^1_{\kappa,\check{\lambda}}$.

Note, however, that $\mathbb{W}^{w_0}_{\kappa,\check{\lambda}}$ and $\mathbb{W}^1_{\kappa,\check{\lambda}}$ can be easily related:

$$\mathbb{W}^{w_0}_{\kappa,\check{\lambda}} \simeq j_{w_0,*} \star \mathbb{W}^1_{\kappa,\check{\lambda}} \text{ and } \mathbb{W}^1_{\kappa,\check{\lambda}} \simeq j_{w_0,!} \star \mathbb{W}^{w_0}_{\kappa,\check{\lambda}}.$$

2.4.6. For completeness, we will show:

Proposition 2.4.7. *The semi-infinite cohomology* $\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathbb{W}^{\check{\lambda}}_\kappa)^{\check{\lambda}'}$ *is given by:*

$$\begin{cases} \det(\mathfrak{n}^-)^{\otimes -1} \text{ for } \check{\lambda}' = \check{\lambda} + 2\check{\rho} \\ 0 \text{ otherwise.} \end{cases}$$

Proof. For any $\mathcal{M} \in \mathfrak{g}\text{-mod}_\kappa^{T(\mathcal{O})}$, we have

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathcal{M})^{\check{\lambda}'} \simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathrm{Av}_*^{I^-}(\mathcal{M}))^{\check{\lambda}'}.$$

If $\mathcal{M}$ was I^+-equivariant, the latter further identifies with

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathrm{Av}_*^{I/T}(\mathcal{M}))^{\check{\lambda}'}.$$

Hence, it suffices to show that

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathbb{W}^{\check{\lambda},\frac{\infty}{2}}_\kappa)^{\check{\lambda}'} = \begin{cases} \det(\mathfrak{n}^-)^{\otimes -1} \text{ for } \check{\lambda}' = \check{\lambda} + 2\check{\rho} \\ 0 \text{ otherwise.} \end{cases}$$

Let ${}^-\mathrm{Lie}(I_\alpha)$ be the opposite subalgebra to I_α, i.e.,

$${}^-\mathrm{Lie}(I_\alpha) \cap \mathrm{Lie}(I_\alpha) = \mathfrak{t}(\mathcal{O}) \text{ and } {}^-\mathrm{Lie}(I_\alpha) \oplus \mathrm{Lie}(\overset{\circ}{I}_\alpha) = \mathfrak{g}(\mathcal{O}).$$

Parallel to Proposition 2.1.9, for $\mathcal{M} \in \mathfrak{g}\text{-mod}_\kappa^{T(\mathcal{O})}$, we have:

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathcal{M})^{\check{\lambda}'} \simeq \underset{\alpha}{\mathrm{colim}}\ \mathrm{C}_\cdot({}^-\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathcal{M} \otimes \det({}^-\mathrm{Lie}(I^-_\alpha)/{}^-\mathrm{Lie}(I^-))^{\otimes -1} \otimes \det(\mathfrak{n}^-)^{\otimes -1})^{\check{\lambda}'}.$$

Taking

$$\mathcal{M} = \mathbb{W}^{\check{\lambda},\frac{\infty}{2}}_\kappa := \underset{\alpha}{\mathrm{colim}}\ \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_\alpha)}(k^{\check{\lambda}} \otimes \ell^-_\alpha) \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})},$$

and contracting the double colimit, it suffices to show that for each index α, we have

$$\mathrm{C}_\cdot({}^-\mathrm{Lie}(\overset{\circ}{I}_\alpha), \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_\alpha)}(k^{\check{\lambda}} \otimes \det({}^-\mathrm{Lie}(I^-_\alpha)/{}^-\mathrm{Lie}(I^-))^{\otimes -1} \otimes \ell^-_\alpha)^{\check{\lambda}'} = \begin{cases} k \text{ for } \check{\lambda}' = \check{\lambda}' \\ 0 \text{ otherwise.} \end{cases}$$

However, this follows from the fact that $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I_\alpha)}(-)$ is free over ${}^-\mathrm{Lie}(\overset{\circ}{I}_\alpha)$, while the lines $\det({}^-\mathrm{Lie}(I^-_\alpha)/{}^-\mathrm{Lie}(I^-))$ and ℓ^-_α are canonically isomorphic.

□

2.5. **Convolution action on Wakimoto modules.** In this subsection we will assume that our level is integral; we will denote it by $-\kappa$. That said, the discussion will go through verbatim for a rational level, see Remark 2.5.2.

We will study the behavior of Wakimoto modules with respect to convolution with certain standard D-modules on the affine flag scheme.

2.5.1. Let us view κ (i.e., the opposite of the given level) as a pairing

$$\kappa : \Lambda \otimes \Lambda \to \mathbb{Z}.$$

Hence, κ defines an embedding

$$\Lambda \to \check{\Lambda}.$$

Remark 2.5.2. If κ is rational, rather than integral, in what follows the sublattice $\Lambda \subset \check{\Lambda}$ should be replaced by

$$\{\check{\mu} \in \check{\Lambda} \,|\, \kappa(\check{\mu}, \check{\lambda}) \in \mathbb{Z},\ \forall \check{\lambda} \in \check{\Lambda}\}.$$

2.5.3. Let t be a uniformizer in $\operatorname{Spec}(\mathcal{O})$, so that for $\mu \in \Lambda$, we can view t^μ as a point in $T(\mathcal{K})$. Note, however, that for $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa^{T(\mathcal{O})}$, the object

$$t^\mu \cdot \mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}_\kappa$$

is canonically independent of the choice of t (this is because the image of t^μ modulo $T(\mathcal{O})$ is independent of the choice of t).

Let ω_x denote the cotangent fiber of the formal disc $\operatorname{Spec}(\mathcal{O})$. A choice of t trivializes this line, and a change $t \mapsto f(t) \cdot t$ results the scaling this trivialization by $f(0)$.

For $\mu \in \Lambda^-$ let $\mathfrak{l}_\mu$ denote the line $\det(\operatorname{Lie}(I^-)/\operatorname{Lie}(\operatorname{Ad}_{t^\mu}(I^-)))$, considered without the grading or T-action.

We claim:

Proposition 2.5.4. *For $\mu \in \Lambda^-$ we have a canonical isomorphism.*

$$(t^\mu \cdot \mathbb{W}_{-\kappa}^{\check{\lambda},\frac{\infty}{2}}) \otimes \mathfrak{l}_\mu[-\langle \mu, 2\check{\rho}\rangle] \simeq \mathbb{W}_{-\kappa}^{\check{\lambda}+\mu,\frac{\infty}{2}} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle}.$$

Proof. Follows from the observation that for $\mu \in \Lambda^-$, the action of T on the determinant line of $\operatorname{Lie}(I^-)/\operatorname{Lie}(\operatorname{Ad}_{t^\mu}(I^-))$ is given by the character $\kappa_{\text{crit}}(\mu, -)$.

□

2.5.5. The (monoidal) category $\text{D-mod}_{-\kappa}(\text{Fl}_G^{\text{aff}})^I$ of $-\kappa$-twisted D-modules on the affine flag scheme $\text{Fl}_G^{\text{aff}} = G(\mathcal{K})/I$ acts on $\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I$ by convolutions.

(As κ was assumed integral, we can identity $\text{D-mod}_{-\kappa}(\text{Fl}_G^{\text{aff}})^I$ with the non-twisted version $\text{D-mod}(\text{Fl}_G^{\text{aff}})^I$, but it is more convenient not to resort to this identification.)

For $\mu \in \Lambda^-$ we let

$$j_{\mu,!}, j_{\mu,*} \in (\text{D-mod}_{-\kappa}(\text{Fl}_G^{\text{aff}})^I)^\heartsuit$$

denote the corresponding standard (resp., standard object) corresponding to the coset $I \cdot t^\mu \in G(\mathcal{K})/I$, normalized so that its !-fiber at t^μ is the line $\mathfrak{l}_\mu$ placed in degree $\langle -\mu, 2\check{\rho}\rangle$.

For $\mu_1, \mu_2 \in \Lambda^-$ we have a canonical isomorphism

$$j_{\mu_1,*} \star j_{\mu_2,*} \simeq j_{\mu_1+\mu_2,*} \otimes \omega_x^{\kappa(\mu_1,\mu_2)}. \tag{2.5}$$

2.5.6. It is known after [AB] that the assignment

$$\mu \mapsto j_{\mu,*}, \quad \mu \in \Lambda^-$$

can be extended to an assignment

$$\mu \mapsto J_\mu, \quad \mu \in \Lambda,$$

uniquely determined by the requirement that

$$J_{\mu_1} \star J_{\mu_2} \simeq J_{\mu_1+\mu_2} \otimes \omega_x^{\kappa(\mu_1,\mu_2)}. \tag{2.6}$$

It is also known that for $\mu \in \Lambda^+$, the corresponding object J_μ is *a* standard object object on the orbit $I \cdot t^\mu \subset \text{Fl}_G^{\text{aff}}$; denote it by $j_{\mu,!}$.

The !-fiber of $j_{\mu,!}$ at t^μ is the line

$$\mathfrak{l}_\mu := \det(\operatorname{Lie}(I^+)/\operatorname{Lie}(\operatorname{Ad}_{t^\mu}(I^+))) \simeq \mathfrak{l}_{-\mu}^{\otimes -1} \otimes \omega_x^{-\kappa_{\text{crit}}(\mu,\mu)},$$

placed in degree $\langle \mu, 2\check{\rho}\rangle$. Note that for $\mu \in \Lambda^+$, we have

$$j_{-\mu,*} \star j_{\mu,!} \simeq \delta_{1,\text{Fl}} \otimes \omega_x^{-\kappa(\mu,\mu)} \simeq j_{\mu,!} \star j_{-\mu,*}.$$

For future reference we introduce the corresponding costandard object $j_{\mu,*}$ normalized so that its !-fiber at t^μ is the above line $\mathfrak{l}_\mu$.

By construction

$$\begin{cases} J_\mu \simeq \operatorname{Av}_*^{I/T}(t^\mu \cdot \delta_{\text{Fl}}) \otimes \mathfrak{l}_\mu[-\langle \mu, 2\check{\rho}\rangle], & \mu \in \Lambda^-; \\ J_\mu \simeq \operatorname{Av}_!^{I/T}(t^\mu \cdot \delta_{1,\text{Fl}}) \otimes \mathfrak{l}_\mu[-\langle \mu, 2\check{\rho}\rangle], & \mu \in \Lambda^+. \end{cases}$$

From here, for $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}^I_\kappa$, we have:

$$\begin{cases} J_\mu \star \mathcal{M} \simeq \mathrm{Av}_*^{I/T}(t^\mu \cdot \mathcal{M}) \otimes \mathfrak{l}_\mu[-\langle \mu, 2\check{\rho}\rangle], & \mu \in \Lambda^-; \\ J_\mu \star \mathcal{M} \simeq \mathrm{Av}_!^{I/T}(t^\mu \cdot \mathcal{M}) \otimes \mathfrak{l}_\mu[-\langle \mu, 2\check{\rho}\rangle], & \mu \in \Lambda^+. \end{cases} \tag{2.7}$$

2.5.7. From Proposition 2.5.4 (and Corollary 2.4.4) we obtain:

Corollary 2.5.8. *Let $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$.*
(a) *For $\mu \in \Lambda^-$ there exists the canonical isomorphism*

$$j_{\mu,*} \star \mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq \mathbb{W}^{\check{\lambda}+\mu}_{-\kappa} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle}.$$

(b) *For $\mu \in \Lambda^+$ there exists the canonical isomorphism*

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), j_{\mu,*} \star \mathcal{M})^{\check{\lambda}} \simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check{\lambda}-\mu} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle - \kappa(\mu,\mu)}.$$

From Corollary 2.5.8(a) we also obtain:

Corollary 2.5.9. *For any $\mu \in \Lambda$ and any $\check{\lambda} \in \check{\Lambda}$, there exists a canonical isomorphism*

$$J_\mu \star \mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq \mathbb{W}^{\check{\lambda}+\mu}_{-\kappa} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle}.$$

Remark 2.5.10. We can define a new family of twisted D-modules

$$'J_{\mu,*} := J_{\mu,*} \otimes \omega_x^{\frac{\kappa(\mu,\mu+2\check{\rho})}{2}}, \quad \mu \in \Lambda^-,$$

and for this family we will have

$$'J_{\mu_1,*} \star {}'J_{\mu_2,*} \simeq {}'J_{\mu_1+\mu_2,*}.$$

Similarly, choosing a compatible family of powers ω_x^c with $c \in \mathbb{Q}$, we can define

$$'\mathbb{W}^{\check{\lambda}}_{-\kappa} := \mathbb{W}^{\check{\lambda}}_{-\kappa} \otimes \omega_x^{\frac{\kappa^{-1}(\check{\lambda},\check{\lambda}+2\check{\rho})}{2}},$$

where κ^{-1} is the resulting form $\check{\Lambda} \otimes \check{\Lambda} \to \mathbb{Q}$. In this case we have

$$'J_\mu \star {}'\mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq {}'\mathbb{W}^{\check{\lambda}+\mu}_{-\kappa}.$$

In fact, the module $'\mathbb{W}^{\check{\lambda}}_{-\kappa}$ defined in the above way carries a unique action of (a finite cover of) the group ind-scheme

$$\mathrm{Aut}(\mathcal{O}), \quad \mathrm{Lie}(\mathrm{Aut}(\mathcal{O})) = \mathrm{Span}(t^i \partial_t, i \geq 0)$$

compatible with the $\mathrm{Aut}(\mathcal{O})$-action on $\widehat{\mathfrak{g}}_\kappa$, in such a way that its highest weight line, i.e., $\omega_x^{\frac{\kappa^{-1}(\check{\lambda},\check{\lambda}+2\check{\rho})}{2}}$, is acted on by (a finite cover of) the group-scheme

$$\mathrm{Aut}_0(\mathrm{Spec}(\mathcal{O})), \quad \mathrm{Lie}(\mathrm{Aut}_0(\mathrm{Spec}(\mathcal{O}))) = \mathrm{Span}(t^i \partial_t, i \geq 1)$$

by character $\frac{\kappa^{-1}(\check{\lambda},\check{\lambda}+2\check{\rho})}{2}$.

3. Wakimoto modules via Verma modules

In this section we will give a more explicit description of Wakimoto modules within the affine category $\mathcal{O}$, i.e., $\widehat{\mathfrak{g}}\text{-mod}^I_\kappa$. Namely, we will express them via affine Verma modules.

3.1. **Affine Verma modules.**

3.1.1. The affine Verma module $\mathbb{M}^{\check{\lambda}}_\kappa \in \widehat{\mathfrak{g}}\text{-mod}^I_\kappa$ is defined to be

$$\mathbb{M}^{\check{\lambda}}_\kappa := \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I)}(k^{\check{\lambda}}).$$

Note that $\mathbb{M}^{\check{\lambda}}_\kappa \in \widehat{\mathfrak{g}}\text{-mod}^I_\kappa$ co-represents the functor

$$\mathrm{C}^\cdot(\overset{\circ}{I}, \mathcal{M})^{\check{\lambda}} \simeq \mathcal{H}om_{\mathrm{Rep}(I)}(k^{\check{\lambda}}, \mathcal{M})$$

3.1.2. Note that by taking $I_\alpha = I$, we obtain a canonical map

$$\mathbb{M}_\kappa^{\check\lambda} \to \mathbb{W}_\kappa^{\check\lambda, \frac{\infty}{2}}.$$

From here, by adjunction, we obtain a map in $\widehat{\mathfrak{g}}\text{-mod}_\kappa^I$

$$\mathbb{M}_\kappa^{\check\lambda} \to \mathbb{W}_\kappa^{\check\lambda}. \tag{3.1}$$

3.2. **Irrational level.**

3.2.1. We have:

Proposition 3.2.2. *Let κ be irrational. Then the map* (3.1) *is an isomorphism for any $\check\lambda$.*

Proof. Follows from the fact that for κ irrational, the induction functor

$$\mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_\kappa} : \mathfrak{g}\text{-mod}^B \to \widehat{\mathfrak{g}}\text{-mod}_\kappa^I$$

is an equivalence.

□

3.2.3. As a corollary, combining Corollary 2.4.4 and (1.5), with we obtain:

Corollary 3.2.4. *For irrational κ and any $\check\lambda \in \check\Lambda$, the functors* $\widehat{\mathfrak{g}}\text{-mod}_\kappa^I \to \mathrm{Vect}$

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check\lambda} \text{ and } \mathrm{C}^{\cdot}(\overset{\circ}{I}, \mathcal{M})^{\lambda}$$

are canonically isomorphic.

3.3. **Convolution action on Verma modules.** From now on until the end of this section we will assume that our level is integral; we will denote it by $-\kappa$. That said, the discussion will go through verbatim for a negative-rational level, see Remark 2.5.2.

3.3.1. As in Sect. 2.2.2, for any $\check\lambda \in \check\Lambda$ and $\mu \in \Lambda^+$ there exists a canonical map in $\widehat{\mathfrak{g}}_{-\kappa}\text{-mod}^T$.

$$\mathbb{M}_{-\kappa}^{\check\lambda-\mu} \otimes \omega_x^{\langle -\mu, \check\lambda\rangle} \to (t^{-\mu} \cdot \mathbb{M}_{-\kappa}^{\check\lambda}) \otimes \mathfrak{l}_{-\mu}[\langle \mu, 2\check\rho\rangle].$$

From here, we obtain a map

$$\mathbb{M}_{-\kappa}^{\check\lambda-\mu} \otimes \omega_x^{\langle -\mu, \check\lambda\rangle} \to j_{-\mu,*} \star \mathbb{M}_{-\kappa}^{\check\lambda}, \quad \mu \in \Lambda^+. \tag{3.2}$$

We claim:

Proposition 3.3.2. *There exists a canonical isomorphism*

$$\mathbb{W}_{-\kappa}^{\check\lambda} \simeq \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, j_{-\mu,*} \star \mathbb{M}_{-\kappa}^{\check\lambda+\mu} \otimes \omega_x^{\langle\check\lambda,\mu\rangle + \kappa(\mu,\mu)}.$$

Proof. By construction

$$\mathbb{W}_{-\kappa}^{\check\lambda,\frac{\infty}{2}} \simeq \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, (t^{-\mu} \cdot \mathbb{M}_{-\kappa}^{\check\lambda+\mu}) \otimes \mathfrak{l}_{-\mu} \otimes \omega_x^{\langle\check\lambda,\mu\rangle+\kappa(\mu,\mu)}[\langle\mu, 2\check\rho\rangle].$$

The required isomorphism follows from (2.7) by applying $\mathrm{Av}_*^{I/T}$.

□

3.3.3. Combining with Corollary 2.4.4, we obtain:

Corollary 3.3.4. *For $\mathcal{M} \in \widehat{\mathfrak{g}}_{-\kappa}\text{-mod}^I$ and any $\check\lambda \in \check\Lambda$, there exists a canonical equivalence*

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check\lambda} \simeq \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, \mathcal{H}om_{\widehat{\mathfrak{g}}_{-\kappa}\text{-mod}^I}(\mathbb{M}_{-\kappa}^{\check\lambda+\mu}, j_{\mu,*} \star \mathcal{M}) \otimes \omega_x^{-\langle\mu,\check\lambda\rangle}.$$

3.4. **The negative level case.** In turns out that when the level is rational, the precise relation between Verma modules and Wakimoto modules drastically depends on the sign of the level. In this subsection we will specialize to the case when the level is negative.

3.4.1. We have the following assertion that follows from Kashiwara-Tanisaki localization, proved in Sect. 3.6:

Theorem 3.4.2. *If $\check{\lambda} - \mu \in \check{\Lambda}^+$, then the map (3.2) is an isomorphism.*

From here, using Proposition 3.3.2, we obtain:

Corollary 3.4.3. *For $\check{\lambda} \in \check{\Lambda}^+$, the map*

$$\mathbb{M}^{\check{\lambda}}_{-\kappa} \to \mathbb{W}^{\check{\lambda}}_{-\kappa}$$

is an isomorphism.

Remark 3.4.4. Corollary 3.4.3 was originally proved in [Fr1, Theorem 2], by a different method. Note also that Theorem 3.4.2 is logically equivalent to Corollary 3.4.3.

3.4.5. Let us now give an expression for $\mathbb{W}^{\check{\lambda}}_{-\kappa}$ for $\check{\lambda}$ not necessarily dominant. Namely, let $\mu \in \Lambda^+$ be such that $\check{\lambda} + \mu \in \check{\Lambda}^+$. We have:

$$\mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq j_{-\mu,*} \star \mathbb{W}^{\check{\lambda}+\mu}_{-\kappa} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle + \kappa(\mu,\mu)},$$

and combining with Theorem 3.4.2, we obtain:

$$\mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq j_{-\mu,*} \star \mathbb{M}^{\check{\lambda}+\mu}_{-\kappa} \otimes \omega_x^{\langle \mu, \check{\lambda}\rangle + \kappa(\mu,\mu)}. \tag{3.3}$$

3.4.6. We have just seen that Wakimoto modules at the negative level are expressible via affine Verma modules by a "finite" procedure. By contrast, the expression for functor of semi-infinite cohomology $\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check{\lambda}}$ involves a colimit. Indeed, by combining Theorem 3.4.2 and Corollary 3.3.4, we obtain:

Corollary 3.4.7. *For any $\check{\lambda} \in \check{\Lambda}$ and $\mathcal{M} \in \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$, there exists a canonical isomorphism*

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check{\lambda}} \simeq \underset{\mu \in \Lambda^+}{\mathrm{colim}}\, \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check{\lambda}}_{-\kappa}, j_{-\mu,*} \star j_{\mu,*} \star \mathcal{M} \otimes \omega_x^{\kappa(\mu,\mu)}),$$

where the transition maps are given by the maps

$$j_{-\mu,*} \star j_{\mu,*} \otimes \omega_x^{\kappa(\mu,\mu)} \to j_{-\mu-\mu',*} \star j_{\mu+\mu',*} \otimes \omega_x^{\kappa(\mu+\mu',\mu+\mu')}, \quad \mu' \in \Lambda^+$$

that come from the canonical maps $j_{\mu',!} \to j_{\mu',*}$.

Remark 3.4.8. Let us also note that the object

$$\underset{\mu \in \Lambda^+}{\mathrm{colim}}\, j_{-\mu,*} \star j_{\mu,*} \star \omega_x^{\kappa(\mu,\mu)} \in \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I$$

identifies with

$$\mathrm{Av}^{I/T}_*(\iota_*(\omega_{N(\mathcal{K})\cdot 1})),$$

where

$$N(\mathcal{K}) \cdot 1 \overset{\iota}{\hookrightarrow} \mathrm{Fl}^{\mathrm{aff}}_G$$

is the embedding of the $N(\mathcal{K})$-orbit of the point $1 \in \mathrm{Fl}^{\mathrm{aff}}_G$; here we are using the fact that the twisting corresponding to κ is canonically trivialized on $N(\mathcal{K}) \cdot 1$ so that

$$\iota_*(\omega_{N(\mathcal{K})\cdot 1}) \in \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{N(\mathcal{K})\cdot T(\mathcal{O})}$$

makes sense.

So the conclusion of Corollary 3.4.7 can we rewritten as

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M})^{\check{\lambda}} \simeq \mathcal{H}om_{\mathfrak{g}\text{-mod}^T_{-\kappa}}(\mathbb{W}^{\check{\lambda}}_{-\kappa}, \iota_*(\omega_{N(\mathcal{K})\cdot 1}) \star \mathcal{M}),$$

where we can further rewrite

$$\iota_*(\omega_{N(\mathcal{K})\cdot 1}) \star \mathcal{M} \simeq \omega_{N(\mathcal{K})/N(\mathcal{O})} \star \mathcal{M}$$

in terms of the action of the group $N(\mathcal{K})$ on $\widehat{\mathfrak{g}}\text{-mod}_\kappa$.

3.4.9. We can now exhibit the collection of objects that are right-orthogonal to the Wakimito modules:

Corollary 3.4.10. *The collection of modules*

$$\check{\lambda}' \mapsto \underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*}\star j_{\mu,*}\star j_{w_0,*}\star \mathbb{W}_{-\kappa}^{w_0(\check{\lambda}')-2\check{\rho}}\otimes\omega_x^{\kappa(\mu,\mu)}$$

is right-orthogonal to the collection

$$\check{\lambda}\mapsto \mathbb{W}_{-\kappa}^{\check{\lambda}}.$$

Proof. Follows by combining Proposition 2.4.7 and Corollary 3.4.7. □

3.5. **The positive level case.** In this subsection we will take our level κ to be positive integral.

3.5.1. At the positive level, the behavior of Wakimoto modules and the functor of semi-infinite cohomology will be in a certain sense opposite to that of the negative level: the latter will be co-representable by a compact object, while the former will be (infinite) colimits of Verma modules.

Indeed, the latter assertion amounts to the isomorphism

$$\mathbb{W}_\kappa^{\check{\lambda}}\simeq \underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*}\star\mathbb{M}_\kappa^{\check{\lambda}-\mu}\otimes\omega_x^{\langle\check{\lambda},\mu\rangle-\kappa(\mu,\mu)},$$

given by Proposition 3.3.2.

3.5.2. For the expression of semi-infinite cohomology we have the following consequence of Corollaries 2.4.4 and 3.4.3:

Corollary 3.5.3. *For $\mathcal{M}\in\widehat{\mathfrak{g}}\text{-mod}_\kappa^I$ and $\check{\lambda}\in\check{\Lambda}$ such that $-\check{\lambda}-2\check{\rho}\in\check{\Lambda}^+$, we have a canonical isomorphism:*

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\mathcal{M})^{\check{\lambda}}\simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}_\kappa^I}(\mathbb{M}_\kappa^{\check{\lambda}},\mathcal{M}).$$

For $\mu\in\Lambda^+$, we have a canonical isomorphism

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\mathcal{M})^{\check{\lambda}+\mu}\simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),j_{\mu,*}\star\mathcal{M})^{\check{\lambda}}\otimes\omega_x^{-\langle\mu,\check{\lambda}\rangle-\kappa(\mu,\mu)}.$$

Remark 3.5.4. Let $\check{\lambda}\in\check{\Lambda}$ be such that $-\check{\lambda}-2\check{\rho}\in\check{\Lambda}^+$. Then from the above corollary we obtain an isomorphism

$$j_{\mu,*}\star\mathbb{M}_\kappa^{\check{\lambda}}\simeq\mathbb{M}_\kappa^{\check{\lambda}-\mu}\otimes\omega_x^{\langle\mu,\check{\lambda}\rangle},\quad \mu\in\Lambda^+$$

The latter isomorphism can be obtained from the Kashiwara-Tanisaki localization at the positive level.

3.6. **Proof of Theorem 3.4.2.** For the proof of Theorem 3.4.2 we will choose a uniformizer $t\in\mathcal{O}$; so we will trivialize the line ω_x.

3.6.1. First, let us recall the version of the Kasiwara-Tanisaki localization theorem that we will need.

Let us call a weight $\check{\lambda}_0\in\check{\Lambda}$ *admissible* for κ is the following conditions hold:

$$\begin{cases}\langle\alpha,\check{\lambda}_0+\check{\rho}\rangle\notin\mathbb{Z}^{>0} \text{ for all positive coroots } \alpha,\\ \pm\langle\alpha,\check{\lambda}_0+\check{\rho}\rangle+n\cdot\frac{-\kappa(\alpha,\alpha)}{2}\notin\mathbb{Z}^{>0} \text{ for all positive coroots } \alpha \text{ and all } n\in\mathbb{Z}^{>0}.\end{cases}\tag{3.4}$$

We consider the twisting $(-\kappa,\check{\lambda}_0)$ on the affine flag scheme $\mathrm{Fl}^{\mathrm{aff}_G}$ such that

$$\Gamma(\mathrm{Fl}^{\mathrm{aff}_G},\delta_{1,\mathrm{Fl}})\simeq\mathbb{M}_{-\kappa}^{\check{\lambda}_0}.$$

The Kasiwara-Tanisaki theorem of [KT] says that in this case the functor

$$\Gamma(\mathrm{Fl}^{\mathrm{aff}_G},-):\text{D-mod}_{(-\kappa,\check{\lambda}_0)}(\mathrm{Fl}^{\mathrm{aff}_G})^I\to\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I$$

is t-exact, sends standard objects to standard objects, and costandard objects to costandard objects.

3.6.2. For an element $\check{\lambda} \in \check{\Lambda}$ we can always find an element

$$\mu \cdot w \in W^{\mathrm{aff}} := W \ltimes \Lambda,$$

so that

$$\check{\lambda} = \mu + w(\check{\lambda}_0 + \check{\rho}) - \check{\rho}$$

with $\check{\lambda}_0$ being κ-admissible.

Hence,

$$\mathbb{M}^{\check{\lambda}}_{-\kappa} \simeq \Gamma(\mathrm{Fl}^{\mathrm{aff}}{}_G, j_{\mu\cdot w,!}), \quad j_{\mu\cdot w,!} \in \text{D-mod}_{(-\kappa,\check{\lambda}_0)}(\mathrm{Fl}^{\mathrm{aff}}{}_G)^I.$$

Similarly, for any $\nu \in \Lambda$,

$$\mathbb{M}^{\check{\lambda}+\nu}_{-\kappa} \simeq \Gamma(\mathrm{Fl}^{\mathrm{aff}}{}_G, j_{(\nu+\mu)\cdot w,!}).$$

Thus, to prove Theorem 3.4.2 it suffices to show that if $\check{\lambda} \in \check{\Lambda}^+$ and $\nu \in \Lambda^+$, we have

$$j_{\nu,!} \star j_{\mu\cdot w,!} \simeq j_{(\nu+\mu)\cdot w,!}.$$

The latter is a combinatorial condition that translates as

$$\ell(\nu) + \ell(\mu \cdot w) = \ell((\nu + \mu) \cdot w), \quad \nu \in \Lambda^+.$$

It is equivalent to

$$\ell(\mu \cdot w) = \langle \mu, 2\check{\rho}\rangle + \ell(w). \tag{3.5}$$

3.6.3. In order to prove (3.5) it is easy to see that it suffices to show that the element $\mu \cdot w$ satisfies the following condition for every positive root $\check{\alpha}$:

$$\begin{cases} \langle \mu, \check{\alpha}\rangle > 0 \text{ if } w^{-1}(\check{\alpha}) \text{ is positive,} \\ \langle \mu, \check{\alpha}\rangle \geq 0 \text{ if } w^{-1}(\check{\alpha}) \text{ is negative.} \end{cases} \tag{3.6}$$

We claim that (3.6) is forced by the condition that $\check{\lambda}$ is dominant.

Suppose that there exists a positive root $\check{\alpha}$ for which $\langle \mu, \check{\alpha}\rangle \leq 0$ and $w^{-1}(\check{\alpha}) =: \beta$ is positive. We have

$$0 < \langle \alpha, \check{\lambda} + \check{\rho}\rangle = \langle \alpha, w(\check{\lambda}_0 + \check{\rho})\rangle + \langle \alpha, \mu\rangle = \langle \beta, \check{\lambda}_0 + \check{\rho}\rangle + \langle \mu, \check{\alpha}\rangle \cdot \frac{\kappa(\alpha,\alpha)}{2}.$$

However, this violates (3.4).

Similarly, suppose there exists a positive root $\check{\alpha}$ for which $\langle \mu, \check{\alpha}\rangle < 0$ and $w^{-1}(\check{\alpha}) =: \beta$ is negative. We have

$$0 < \langle \alpha, \check{\lambda} + \check{\rho}\rangle = \langle \alpha, w(\check{\lambda}_0 + \check{\rho})\rangle + \langle \alpha, \mu\rangle = -\langle -\beta, \check{\lambda}_0 + \check{\rho}\rangle + \langle \mu, \check{\alpha}\rangle \cdot \frac{\kappa(\alpha,\alpha)}{2},$$

and this again violates (3.4).

□[Theorem 3.4.2]

3.6.4. For $\check{\lambda} \in \check{\Lambda}$, let $\mathbb{M}^{\vee,\check{\lambda}}_{-\kappa} \in \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ be the dual Verma module, i.e.,

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{M}^{\check{\lambda}'}_{-\kappa}, \mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}) = \begin{cases} k \text{ if } \check{\lambda}' = \check{\lambda}, \\ 0 \text{ otherwise.} \end{cases}$$

Let us note that a similar argument to the one proving Theorem 3.4.2, proves the following assertion:

Theorem 3.6.5. *Let $\check{\lambda}$ be dominant and $\mu \in \Lambda^+$ be such that $-2\check{\rho} - \check{\lambda} + \mu \in \check{\Lambda}^+$. Then*

$$j_{\mu,!} \star \mathbb{M}^{\vee,-2\check{\rho}-\check{\lambda}}_{-\kappa} \simeq \mathbb{M}^{-2\check{\rho}-\check{\lambda}+\mu}_{-\kappa}.$$

From here, we obtain the following result originally proved in [Fr2, Proposition 6.3 and Remark 6.4] (see also [FG2, Proposition 13.2.1] for a proof):

Corollary 3.6.6. *For $\check{\lambda} \in \check{\Lambda}^+$ we have a canonical isomorphism*

$$\mathbb{W}^{-2\check{\rho}-\check{\lambda}}_{-\kappa} \simeq \mathbb{M}^{\vee,-2\check{\rho}-\check{\lambda}}_{-\kappa}.$$

Part II: Representations of the quantum group

4. Quantum algebras

This section is devoted to the review of the basic setting in which quantum groups are constructed. It can be summarized as follows: we start with the quantum torus, take a Hopf algebra A in the corresponding braided monoidal category, and then take the (relative) Drinfeld center of the category of the category of A-modules.

4.1. **The quantum torus.** Our approach to the definition of categories of modules for the various versions of the quantum group is to build them starting from the basic case, and that being the case of the quantum torus.

4.1.1. We start with the lattice $\check{\Lambda}$ and a symmetric bilinear W-invariant form, denoted b',

$$\check{\Lambda} \otimes \check{\Lambda} \to k^\times.$$

We denote by q the associated quadratic form $q(\check{\lambda}) = b'(\check{\lambda}, \check{\lambda})$, and by b the symmetric bilinear form associated to q, i.e.,

$$b(\check{\lambda}_1, \check{\lambda}_2) = q(\check{\lambda}_1 + \check{\lambda}_2) \cdot q(\check{\lambda}_1)^{-1} \cdot q(\check{\lambda}_2)^{-1} = b'(\check{\lambda}_1, \check{\lambda}_2)^2.$$

We regard $\operatorname{Rep}(T)$ as a monoidal category, and the form b' defines on $\operatorname{Rep}(T)$ a new braided structure, obtained by multiplying the tautological one by b': the new braiding

$$R_{k^{\check{\lambda}_1}, k^{\check{\lambda}_2}} : k^{\check{\lambda}_1} \otimes k^{\check{\lambda}_2} \to k^{\check{\lambda}_2} \otimes k^{\check{\lambda}_1}$$

is the identity map times $b'(\check{\lambda}_1, \check{\lambda}_2)$.

Denote the resulting braided monoidal category by $\operatorname{Rep}_q(T)$.

4.1.2. Although we will not use this extensively in the current paper, we remark that the braided monoidal category $\operatorname{Rep}_q(T)$ carries a ribbon structure, where the ribbon automorphism of the object $k^{\check{\lambda}} \in \operatorname{Rep}_q(T)$ equals

$$b'(\check{\lambda}, \check{\lambda} + 2\check{\rho}).$$

The 2ρ-shift in this formula is necessary in order to make the quantum Hopf algebras in Sect. 5.1 *ribbon-equivariant*.

4.1.3. Let c be a compact object of $\operatorname{Rep}_q(T)$. To it we can attach its left monoidal and right monoidal duals, which are objects equipped with perfect pairings

$$c^{\vee,L} \otimes c \to k \text{ and } c \otimes c^{\vee,R} \to k,$$

respectively.

The braided structure on $\operatorname{Rep}_q(T)$ allows to identify $c^{\vee,L} \simeq c^{\vee,R}$. Multiplying this identification by the ribbon twist, we can make it compatible with the monoidal structure

$$\begin{array}{ccc} (c_1 \otimes c_2)^{\vee,L} & \longrightarrow & (c_2 \otimes c_1)^{\vee,R} \\ \downarrow & & \downarrow \\ c_2^{\vee,L} \otimes c_1^{\vee,L} & \longrightarrow & c_2^{\vee,R} \otimes c_1^{\vee,R} \end{array}$$

Henceforth, we will use the above identification between $c^{\vee,L}$ and $c^{\vee,R}$ and simply write $c^\vee$.

4.1.4. Let $\operatorname{Rep}_{q^{-1}}(T)$ denote the braided monoidal category with the inverted b'. Note that it can be identified with $(\operatorname{Rep}_q(T))^{\text{rev-br}}$, the braided monoidal category obtained from $\operatorname{Rep}_q(T)$ by reversing the braiding.

4.2. **Quantum Hopf algebras.** The Hopf algebras that appear in this subsection should be thought of as "positive" parts of the corresponding versions of the quantum group.

4.2.1. Note that given a braided monoidal category, it makes sense to talk about Hopf algebras in it. Our interest will be Hopf algebras A in $(\mathrm{Rep}_q(T))^\heartsuit$ such that their weight components $A^{\check\lambda}$ satisfy

$$A^{\check\lambda} = \begin{cases} 0 \text{ for } \check\lambda \notin \check\Lambda^+ \\ k \text{ for } \check\lambda = 0 \\ \text{is finite-dimensional for } \check\lambda \in \check\Lambda^+. \end{cases}$$

We will also consider Hopf algebras that satisfy the opposite condition (i.e., replace $\check\Lambda^+$ by $-\check\Lambda^+$).

For a given A, we will consider the DG category $A\text{-mod}(\mathrm{Rep}_q(T))$, equipped with the forgetful functor $\mathbf{oblv}_A$ to $\mathrm{Rep}_q(T)$. Let $\mathbf{ind}_A : \mathrm{Rep}_q(T) \to A\text{-mod}(\mathrm{Rep}_q(T))$ denote its left adjoint.

The datum of Hopf algebra structure on A is equivalent to a structure of monoidal category on $A\text{-mod}(\mathrm{Rep}_q(T))$ so that the functor $\mathbf{oblv}_A$ is monoidal.

4.2.2. Let $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}} \subset A\text{-mod}(\mathrm{Rep}_q(T))$ denote the full (but not cocomplete) subcategory of finite-dimensional A-modules. We let

$$A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$$

denote the ind-completion of $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}}$.

The monoidal structure on $A\text{-mod}(\mathrm{Rep}_q(T))$ restricts to one on $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}}$, which, in turn, ind-extends to one on $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$.

Below we will give a different interpretation of $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$.

4.2.3. By ind-extending the tautological embedding $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}} \hookrightarrow A\text{-mod}(\mathrm{Rep}_q(T))$ we obtain a functor

$$A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \to A\text{-mod}(\mathrm{Rep}_q(T)).$$

Warning: in general, this functor does not send compacts to compacts (as $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}}$ is not necessarily contained in $A\text{-mod}(\mathrm{Rep}_q(T))_c)$), nor is it fully faithful. In fact, this functor may fail to even be conservative (this is the case if k is non-compact as an object of $A\text{-mod}(\mathrm{Rep}_q(T))$).

By a slight abuse of notation, we will denote by the same symbol $\mathbf{oblv}_A$ the composite functor

$$A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \to A\text{-mod}(\mathrm{Rep}_q(T)) \overset{\mathbf{oblv}_A}{\to} \mathrm{Rep}_q(T).$$

Note that the resulting functor

$$\mathbf{oblv}_A : A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \to \mathrm{Rep}_q(T) \tag{4.1}$$

is not necessarily conservative.

4.2.4. Since A was assumed connective, the category $A\text{-mod}(\mathrm{Rep}_q(T))$ carries a t-structure, for which the forgetful functor $\mathbf{oblv}_A$ is t-exact. The subcategory $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}}$ is compatible with this t-structure, and that in turn induces a t-structure on $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$. The functor (4.1) is t-exact, and an object of $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$ is connective if and only if its image under (4.1) is connective.

Note, however, that the above t-structure on $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}}$ is in general not left separated.

Since A was assumed co-connective (recall that we are assuming that A is actually in the heart), it follows (e.g., using Sect. 4.3.6 below) that $(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})^\heartsuit$ contains enough injective objects $\mathcal{I}$, such that

$$\mathcal{H}om_{A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}}(\mathcal{M}, \mathcal{I})$$

is acyclic off degree 0 for any $\mathcal{M} \in (A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})^\heartsuit$.

This implies that the bounded part of $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$, i.e., $(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty}$ can be recovered from the abelian category $(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})^\heartsuit$ as its *bounded derived category* in the sense of [Lu, Sect. 1.3.2].

All of $A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}})$ can be recovered as the ind-completion of $(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}}))_c$, which is a full subcategory in $(A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}}))^{\leq\infty,\geq\infty}$.

4.3. **Coalgebras and comodules.**

4.3.1. Let $A^\vee \in \text{Rep}_q(T)$ be the component-wise monoidal dual of A. The Hopf algebra structure on A induces one on $A^\vee$ so that the diagrams

$$\begin{array}{ccccc}
A\otimes A^\vee\otimes A^\vee & & \xrightarrow{\text{id}_A\otimes \text{mult}_{A^\vee}} & & A\otimes A^\vee \\
\Big\downarrow{\scriptstyle \text{comult}_A\otimes\text{id}} & & & & \Big\downarrow{\scriptstyle \text{pairing}} \\
A\otimes A\otimes A^\vee\otimes A^\vee & \xrightarrow{\text{id}_A\otimes\text{pairing}\otimes\text{id}_{A^\vee}} & A\otimes A^\vee & \xrightarrow{\text{pairing}} & k
\end{array}$$

and

$$\begin{array}{ccccc}
A\otimes A\otimes A^\vee & & \xrightarrow{\text{mult}_A\otimes \text{id}_{A^\vee}} & & A\otimes A^\vee \\
\Big\downarrow{\scriptstyle \text{id}_{A\otimes A}\otimes\text{comult}_{A^\vee}} & & & & \Big\downarrow{\scriptstyle \text{pairing}} \\
A\otimes A\otimes A^\vee\otimes A^\vee & \xrightarrow{\text{id}_A\otimes\text{pairing}\otimes\text{id}_{A^\vee}} & A\otimes A^\vee & \xrightarrow{\text{pairing}} & k
\end{array}$$

are commutative.

4.3.2. Note also that if B is a Hopf algebra in a braided monoidal category $\mathcal{C}$, we can attach to it a Hopf algebra $B^{\text{rev-comult}}$ in the braided monoidal category $\mathcal{C}^{\text{rev-br}}$.

Namely, we let $B^{\text{rev-comult}}$ be the same as B as an associative algebra, but we set the comultiplication to be

$$B \overset{\text{comult}}{\longrightarrow} B\otimes B \overset{R^{-1}_{B,B}}{\longrightarrow} B\otimes B.$$

Similarly, we can consider the Hopf algebra $B^{\text{rev-mult}}$ in $\mathcal{C}^{\text{rev-br}}$. It is the same as B as a co-associative coalgebra, and the multiplication is defined by

$$B\otimes B \overset{R^{-1}_{B,B}}{\longrightarrow} B\otimes B \overset{\text{mult}}{\to} B.$$

We note, however, that the antipode on B identifies $B^{\text{rev-comult}}$ and $B^{\text{rev-mult}}$ as Hopf algebras.

4.3.3. If B is a coalgebra as in Sect. 4.2.1, we let

$$B\text{-comod}(\text{Rep}_q(T))_{\text{fin.dim}}$$

denote the category of finite-dimensional B-comodules. We let $B\text{-comod}(\text{Rep}_q(T))$ denote the ind-completion of $B\text{-comod}(\text{Rep}_q(T))_{\text{fin.dim}}$.

We have the forgetful functor

$$\mathbf{oblv}_B : B\text{-comod}(\text{Rep}_q(T)) \to \text{Rep}_q(T),$$

which admits a continuous *right* adjoint, denoted $\mathbf{coind}_B$. The corresponding comonad on $\text{Rep}_q(T)$ is given by tensor product with B.

Note, however, that the functor $\mathbf{oblv}_B$ is *not* necessarily conservative.

4.3.4. We have:

Lemma 4.3.5. *There exists a canonical equivalence of monoidal categories*

$$A\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}} \simeq (A^\vee)^{\text{rev-mult}}\text{-comod}(\text{Rep}_{q^{-1}}(T))$$

that commutes with the forgetful functors to

$$\text{Rep}_q(T) \simeq (\text{Rep}_{q^{-1}}(T))^{\text{rev-br}}.$$

4.3.6. As a consequence, we obtain that the forgetful functor $\mathbf{oblv}_A$ of (4.1) admits a continuous *right adjoint*, denoted $\mathbf{coind}_A$.

The corresponding co-monad on $\text{Rep}_q(T)$ is given by tensor product with $A^\vee$.

4.4. **Formation of the relative Drinfeld double.** We will now perform the crucial step in the construction of quantum groups: we will add the "negative part" of the quantum group by considering the (relative) Drinfeld center.

4.4.1. Note that tensor product by objects of $\mathrm{Rep}_q(T)$ *on the right* makes the monoidal category $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$ into a *right module category* for the braided monoidal category $\mathrm{Rep}_q(T)$.

In this case, it makes sense to form the *relative Drinfeld double*

$$Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}).$$

This is a braided monoidal category, universal with respect to the property that it acts on $A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$ *on the left*, in a way commuting with the right action of $\mathrm{Rep}_q(T)$.

4.4.2. Explicitly, objects of $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ are objects $z \in A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}$ equipped with a system of identifications

$$z \otimes \mathcal{M} \to \mathcal{M} \otimes z, \quad \forall\, \mathcal{M} \in A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}, \tag{4.2}$$

compatible with tensor products of the $\mathcal{M}$'s, and such that for $\mathcal{M}$ coming via augmentation from an object $c \in \mathrm{Rep}_q(T)$, the resulting map

$$z \otimes c \to c \otimes z$$

is the braiding $R_{z,c}$ in $\mathrm{Rep}_q(T)$.

4.4.3. We have the evident (monoidal) conservative forgetful functor

$$\mathbf{oblv}_{\mathrm{Dr}\to A} : Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) \to A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}},$$

which admits a *left* adjoint, denoted $\mathbf{ind}_{A\to\mathrm{Dr}}$.

Via the equivalence of Lemma 4.3.5, we obtain that $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ admits also a (monoidal) forgetful functor

$$\mathbf{oblv}_{\mathrm{Dr}\to A^\vee} : Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) \to (A^\vee)^{\text{rev-mult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp}},$$

which we further identify with

$$(A^\vee)^{\text{rev-comult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp}},$$

via the antipode map.

If we disregard the monoidal structures, we can view the latter functor as

$$Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) \to A^\vee\text{-mod}(\mathrm{Rep}_q(T)).$$

4.5. **Basic structures on the category of modules.**

4.5.1. We have the following commutative diagram

$$\begin{array}{ccc} Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Dr}\to A}} & A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \\ {\scriptstyle \mathbf{oblv}_{\mathrm{Dr}\to A^\vee}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_A} \\ A^\vee\text{-mod}(\mathrm{Rep}_q(T)) & \xrightarrow{\mathbf{oblv}_{A^\vee}} & \mathrm{Rep}_q(T), \end{array} \tag{4.3}$$

and the following two commutative diagrams, obtained by passing to left adjoints along the horizontal arrows (resp., right adjoints along the vertical arrows):

$$\begin{array}{ccc} Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) & \xleftarrow{\mathbf{ind}_{\mathrm{Dr}\to A}} & A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \\ {\scriptstyle \mathbf{oblv}_{\mathrm{Dr}\to A^\vee}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_A} \\ A^\vee\text{-mod}(\mathrm{Rep}_q(T)) & \xleftarrow{\mathbf{ind}_{A^\vee}} & \mathrm{Rep}_q(T), \end{array} \tag{4.4}$$

and

$$
\begin{array}{ccc}
Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Dr}\to A}} & A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \\
{\scriptstyle \mathbf{coind}_{\mathrm{Dr}\to A^\vee}}\uparrow & & \uparrow{\scriptstyle \mathbf{coind}_A} \\
A^\vee\text{-mod}(\mathrm{Rep}_q(T)) & \xrightarrow{\mathbf{oblv}_{A^\vee}} & \mathrm{Rep}_q(T).
\end{array}
\tag{4.5}
$$

4.5.2. The category $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ carries a t-structure for which the functor $\mathbf{oblv}_{\mathrm{Dr}\to A}$ is t-exact.

It follows from the diagrams (4.3), (4.4) and (4.5) that the functors $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$, $\mathbf{ind}_{\mathrm{Dr}\to A}$ and $\mathbf{coind}_{\mathrm{Dr}\to A^\vee}$ are also t-exact.

Note, however, that the above t-structure on $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ is in general *not separated*. In particular, the functor $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$ is in general *not* conservative.

4.5.3. For every $\check{\lambda}\in\check{\Lambda}$, we have the standard and the costandard objects

$$\mathbb{M}_A^{\check{\lambda}} := \mathbf{ind}_{\mathrm{Dr}\to A}(k^{\check{\lambda}}) \text{ and } \mathbb{M}_A^{\vee,\check{\lambda}} := \mathbf{coind}_{\mathrm{Dr}\to A^\vee}(k^{\check{\lambda}})$$

that both lie in the heart of $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$.

From the commutative diagrams (4.4) and (4.5) we obtain that

$$\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}(\mathbb{M}_A^{\check{\lambda}}) \simeq \mathbf{ind}_{A^\vee}(k^{\check{\lambda}}) \text{ and } \mathbf{oblv}_{\mathrm{Dr}\to A^\vee}(\mathbb{M}_A^{\vee,\check{\lambda}}) \simeq \mathbf{coind}_A(k^{\check{\lambda}}).$$

From either of those we obtain:

$$\mathcal{H}om_{Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})}(\mathbb{M}_A^{\check{\lambda}}, \mathbb{M}_A^{\vee,\check{\lambda}'}) = \begin{cases} k \text{ if } \check{\lambda}' = \lambda \\ 0 \text{ otherwise.} \end{cases}$$

4.5.4. Note that, by construction, the objects $\mathbb{M}_A^{\check{\lambda}} \in Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ are compact and generate $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$. From here it follows that

$$(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))_c \subset (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty}.$$

We will now give an explicit description of all compact objects in $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$:

Proposition 4.5.5. *An object* $\mathcal{M} \in (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty}$ *is compact if and only if* $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}(\mathcal{M}) \in A^\vee\text{-mod}(\mathrm{Rep}_q(T))$ *is compact.*

Warning: if in the statement of the proposition we omited the condition that $\mathcal{M}$ be cohomologically bounded, the assertion would be false: indeed, the functor $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$ is not necessarily conservative.

Proof. The "only if" direction is clear, since the functor $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$ preserves compactness, being the left adjoint of a continuous functor.

For the "if" direction, let us start with an object in

$$\mathcal{M} \in (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty}$$

whose image in $A^\vee\text{-mod}(\mathrm{Rep}_q(T))$ along $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$ is compact. A standard argument shows that there exists a fiber sequence

$$\mathcal{M}' \to \mathcal{M}'' \to \mathcal{M}',$$

where $\mathcal{M}'' \in (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))_c$ and $\mathcal{M}' \in (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^\heartsuit$ is such that its image in $A^\vee\text{-mod}(\mathrm{Rep}_q(T))$ along $\mathbf{oblv}_{\mathrm{Dr}\to A^\vee}$ is finitely generated and free.

It is easy to see that such $\mathcal{M}'$ admits a filtration

$$0 = \mathcal{M}'_0 \subset \mathcal{M}'_1 \subset ... \subset \mathcal{M}'_n = \mathcal{M}',$$

where each successive quotient is generated under the action of $A^\vee$ by an element of a single degree. However, such as element is necessarily annihilated by the action of A. Hence, we obtain that each successive quotient is isomorphic to $\mathbb{M}_A^{\check{\lambda}}$ for some $\check{\lambda}$.

□

4.5.6. From Sect. 4.2.4 it follows that the abelian category $(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\heartsuit}$ has enough injectives, which are also acyclic in $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$.

This implies that the cohomologically bounded part of $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$, i.e.,

$$(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty},$$

can be recovered from the abelian category $(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\heartsuit}$ as its bounded derived category. The entire $Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}})$ can be recovered as the ind-completion of $(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))_c \subset (Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}))^{\leq\infty,\geq\infty}$.

5. Modules over the quantum group

We continue to review the basics of quantum groups. In this section we define the categories of primary interest, $\mathrm{Rep}_q(G)$ and $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$. The former is (the usual) category of modules over Lusztig's quantum group. The latter category is less known: it combines Lusztig's version for the positive part and the De Concini-Kac version for the negative part.

5.1. **The various version of $U_q(N)$.** In this subsection we will introduce some particular Hopf algebras in $\mathrm{Rep}_q(T)$ that correspond to the several versions of the quantum group that we will consider.

5.1.1. Let $U_q^{\mathrm{free}}(N)$ be the free associative algebra in $\mathrm{Rep}_q(T)$ on the generators e_i, each in degree the simple root $\check{\alpha}_i$. It has a canonical Hopf algebra structure, defined by the condition that the comultiplication sends

$$e_i \mapsto e_1 \otimes 1 + 1 \otimes e_i.$$

Remark 5.1.2. We emphasize that $U_q^{\mathrm{free}}(N)$ is a Hopf algebra in $\mathrm{Rep}_q(T)$, and not in Vect. The usual formula

$$\nabla(e_i) = e_i \otimes 1 + K_i \cdot e_i \otimes 1$$

arises from the braiding on $\mathrm{Rep}_q(T)$, where K_i acts on the $\check{\lambda}$-weight space as $b'(\check{\alpha}_i, \check{\lambda})$.

5.1.3. We introduce the De Concini-Kac version of $U_q(N)$, denoted $U_q^{\mathrm{DK}}(N)$, as the quotient of $U_q^{\mathrm{free}}(N)$ by the quantum Serre relations.

It is known that the ideal generated by the quantum Serre relations is a Hopf ideal; hence $U_q^{\mathrm{DK}}(N)$ has a unique structure of Hopf algebra, compatible with the projection

$$U_q^{\mathrm{free}}(N) \twoheadrightarrow U_q^{\mathrm{DK}}(N).$$

Swapping the roles of $\check{\Lambda}^+$ and $\check{\Lambda}^-$, we obtain the Hopf algebras in $\mathrm{Rep}_q(T)$:

$$U_q^{\mathrm{free}}(N^-) \twoheadrightarrow U_q^{\mathrm{DK}}(N^-).$$

5.1.4. We define

$$U_q^{\mathrm{Lus}}(N)$$

to be the Hopf algebra in $\mathrm{Rep}_q(T)$ equal to the graded dual of $U_q^{\mathrm{DK}}(N^-)$.

5.1.5. Let $U_q^{\mathrm{cofree}}(N)$ be the graded dual of $U_q^{\mathrm{free}}(N^-)$. By duality, we have an injection

$$U_q^{\mathrm{Lus}}(N) \hookrightarrow U_q^{\mathrm{cofree}}(N).$$

It is known that $U_q^{\mathrm{Lus}}(N)$ is Lusztig's (i.e., quantum divided power) version of the quantum group. In what follows we will use the notation

$$\mathrm{Rep}_q(B) := U_q^{\mathrm{Lus}}(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}.$$

5.1.6. Note that we have a canonical identification

$$(U_q^{\text{free}}(N^-))^{\text{rev-comult}} \simeq U_{q^{-1}}^{\text{free}}(N^-),$$

which induces an identification

$$(U_q^{\text{DK}}(N^-))^{\text{rev-comult}} \simeq U_{q^{-1}}^{\text{DK}}(N^-).$$

This implies that we can think of $U_{q^{-1}}^{\text{DK}}(N^-)$ as obtained from $U_q^{\text{Lus}}(N)$ by the procedure of Lemma 4.3.5.

5.1.7. We have a canonical map of Hopf algebras

$$U_q^{\text{free}}(N) \to U_q^{\text{cofree}}(N), \tag{5.1}$$

obtained by extending the identity map on the generators.

It is known that the quantum Serre relations belong to the kernel of (5.1). Hence, (5.1) factors as

$$U_q^{\text{free}}(N) \twoheadrightarrow U_q^{\text{DK}}(N) \to U_q^{\text{cofree}}(N).$$

By duality, (5.1) also factors as

$$U_q^{\text{free}}(N) \to U_q^{\text{Lus}}(N) \hookrightarrow U_q^{\text{cofree}}(N).$$

Hence, the map (5.1) actually factors as

$$U_q^{\text{free}}(N) \twoheadrightarrow U_q^{\text{DK}}(N) \to U_q^{\text{Lus}}(N) \hookrightarrow U_q^{\text{cofree}}(N). \tag{5.2}$$

5.1.8. We let $u_q(N)$ denote the image of the above map $U_q^{\text{DK}}(N) \to U_q^{\text{Lus}}(N)$, so that we have

$$U_q^{\text{free}}(N) \twoheadrightarrow U_q^{\text{DK}}(N) \twoheadrightarrow u_q(N) \hookrightarrow U_q^{\text{Lus}}(N) \hookrightarrow U_q^{\text{cofree}}(N). \tag{5.3}$$

The Hopf algebra $u_q(N)$ is the positive part of the "small" quantum group.

5.1.9. Let $u_q(N^-)$ be defined similarly. We have:

$$(u_q(N))^\vee \simeq u_q(N^-).$$

In what follows we will denote

$$\text{Rep}_q^{\text{sml,grd}}(B) := u_q(N)\text{-mod}(\text{Rep}_q(T))_{\text{loc.nilp}}.$$

5.1.10. It is known that away from the case of the root of unity (i.e., when $q(\check{\alpha}_i)$ is not a root of unity for any simple root $\check{\alpha}_i$), the maps

$$U_q^{\text{DK}}(N) \twoheadrightarrow u_q(N) \hookrightarrow U_q^{\text{Lus}}(N)$$

are isomorphisms.

By contrast, in the root of unity case (i.e., when $q(\check{\alpha}_i)$ is a root of unity for all simple roots), it is known that $u_q(N)$ is finite-dimensional. In fact, its generators e_i satisfy $e_i^{d_i} = 0$ for $d_i = \text{ord}(q(\check{\alpha}_i))$.

5.2. **Cohomology of the De Concini-Kac algebra.** For what follows we will need to review some cohomological properties of $U_q^{\text{DK}}(N^-)$, viewed as an associative algebra. They can be summarized by saying that it behaves like the classical universal enveloping $U(\mathfrak{n}^-)$.

5.2.1. The following assertion is established in [Geo, Theorem 6.4.1]:

Theorem 5.2.2. *We have*

$$\text{Tor}_i^{U_q^{\text{DK}}(N^-)}(k,k)^{\check{\lambda}} = \begin{cases} k, & \check{\lambda} = w(\check{\rho}) - \rho \text{ for } w \in W \text{ and } i = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, this theorem says that the homology of $U_q^{\text{DK}}(N^-)$ looks exactly the same as that of the classical universal enveloping $U(\mathfrak{n}^-)$.

5.2.3. In particular from Theorem 5.2.2, we obtain:

Corollary 5.2.4. *The groups* $\mathrm{Tor}_i^{U_q^{\mathrm{DK}}(N^-)}(k,k)$ *are non-zero only for finitely many* i.

Note that a statement analogous to Corollary 5.2.4 would be completely false for the other two versions of the quantum group: $U_q^{\mathrm{Lus}}(N^-)$ and $u_q(N^-)$.

5.2.5. We have the following general assertion:

Lemma 5.2.6. *Let an an associative algebra* A *is graded by monoid isomorphic to* $(\mathbb{Z}^+)^n$ *and such that* $k \to A^0$ *is an isomorphism. Then the following conditions are equivalent:*

(i) $\mathrm{Tor}_i^A(k,k) \neq 0$ *for finitely many* i.

(ii) A *has a finite cohomological dimension.*

Combining this with Corollary 5.2.4, we obtain:

Corollary 5.2.7. *The algebra* $U_q^{\mathrm{DK}}(N^-)$ *has a finite cohomological dimension.*

In particular:

Corollary 5.2.8. *The augmentation module* k *is perfect as a* $U_q^{\mathrm{DK}}(N^-)$*-module.*

5.3. **The mixed quantum group.** In this subsection we will define the second principal actor for this paper: the category of modules over the mixed quantum group. The terminology "mixed" comes from the fact that this version has Lusztig's quantum group as its positive part, and the De Concini-Kac one as the negative part.

5.3.1. The basic object of study in this paper is the category

$$\mathrm{Rep}_q^{\mathrm{mxd}}(G) := Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(\mathrm{Rep}_q(B)).$$

By Sect. 4.4.3, this category is equipped with a pair of adjoint functors

$$\mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}} : \mathrm{Rep}_q(B) \rightleftarrows \mathrm{Rep}_q^{\mathrm{mxd}}(G) : \mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}.$$

In addition, we have the following adjoint pair

$$\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-} : \mathrm{Rep}_q^{\mathrm{mxd}}(G) \rightleftarrows U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T)) : \mathbf{coind}_{\mathrm{DK}^-\to\mathrm{mxd}}.$$

5.3.2. For $\check\lambda \in \check\Lambda$ we let

$$\mathbb{M}_{q,\mathrm{mxd}}^{\check\lambda} := \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}}(k^{\check\lambda}) \in \mathrm{Rep}_q^{\mathrm{mxd}}(G) \text{ and } \mathbb{M}_{q,\mathrm{mxd}}^{\vee,\check\lambda} := \mathbf{coind}_{\mathrm{DK}^-\to\mathrm{mxd}}(k^{\check\lambda}) \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)$$

be the corresponding standard and contandard objects, respectively.

From the commutative diagrams (4.4) and (4.5), we obtain that $\mathbb{M}_{q,\mathrm{mxd}}^{\check\lambda}$ is free over $U_q^{\mathrm{DK}}(N^-)$ and $\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}(\mathbb{M}_{q,\mathrm{mxd}}^{\vee,\check\lambda})$ is cofree as an object of $\mathrm{Rep}_q(B)$, and we have

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(\mathbb{M}_{q,\mathrm{mxd}}^{\check\lambda}, \mathbb{M}_{q,\mathrm{mxd}}^{\vee,\check\lambda'}) = \begin{cases} k \text{ if } \check\lambda = \check\lambda'; \\ 0 \text{ otherwise.} \end{cases}$$

5.3.3. The objects $\mathbb{M}_{q,\mathrm{mxd}}^{\check\lambda}$ are compact and generate $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$.

Recall (see Sect. 4.5.4) that an object of $(\mathrm{Rep}_q^{\mathrm{mxd}}(G))^{\leq\infty,\geq\infty}$ is compact if and only if its image under $\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-}$ is compact in $U_q^{\mathrm{DK}}(N^-)$-mod$(\mathrm{Rep}_q(T))$. Note that according to Corollary 5.2.7, the compactness condition in $U_q^{\mathrm{DK}}(\mathfrak{n}^-)$-mod$(\mathrm{Rep}_q(T))$ is equivalent to finite generation.

5.3.4. According to Sect. 4.5.2, the category $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ carries a unique t-structure for which the functors $\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}$ and $\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-}$ are t-exact. Moreover, the functors $\mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}}$ and $\mathbf{coind}_{\mathrm{DK}^-\to\mathrm{mxd}}$ are also t-exact, and in particular, $\mathbb{M}^{\check\lambda}_{q,\mathrm{mxd}}$ and $\mathbb{M}^{\vee,\check\lambda}_{q,\mathrm{mxd}}$ belong to $(\operatorname{Rep}_q^{\mathrm{mxd}}(G))^\heartsuit$.

Remark 5.3.5. As was explained in Sect. 4.5.6, the abelian category $(\operatorname{Rep}_q^{\mathrm{mxd}}(G))^\heartsuit$ can be described explicitly as the category of objects of $(\operatorname{Rep}_q(T))^\heartsuit$, endowed with a locally nilpotent action of $U_q^{\mathrm{Lus}}(N)$ and a compatible action of $U_q^{\mathrm{DK}}(N^-)$. Furthermore, $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ can be recovered from $(\operatorname{Rep}_q^{\mathrm{mxd}}(G))^\heartsuit$ by the following procedure:

First, we note that the cohomologically bounded part of $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$, i.e., $\left(\operatorname{Rep}_q^{\mathrm{mxd}}(G)\right)^{<\infty,>\infty}$, is recovered as

$$\left(\operatorname{Rep}_q^{\mathrm{mxd}}(G)\right)^{<\infty,>\infty} \simeq D^b\left((\operatorname{Rep}_q^{\mathrm{mxd}}(G))^\heartsuit\right).$$

Now, all of $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ is recovered as the ind-completion of the full subcategory of $\left(\operatorname{Rep}_q^{\mathrm{mxd}}(G)\right)^{<\infty,>\infty}$ generated under finite colimits by the objects $\mathbb{M}^{\check\lambda}_{q,\mathrm{mxd}}$.

5.4. **The category of modules over the "big" quantum group.** In this subsection we recall the definition of another object of primary interest: the category of (algebraic=a.k.a. locally finite) modules over Lusztig's quantum group, denoted $\operatorname{Rep}_q(G)$.

A salient feature of this category is that it does *not* arise as Drinfeld's center. The only way we know how to construct $\operatorname{Rep}_q(G)$ is via the underlying abelian category.

5.4.1. Let $\operatorname{Rep}_q(G)_{\mathrm{fin.dim}}$ be the category of algebraic representations of Lusztig's quantum group. By definition, this is the bounded derived category of the abelian category $(\operatorname{Rep}_q(G)_{\mathrm{fin.dim}})^\heartsuit$ that consists of finite-dimensional objects of $(\operatorname{Rep}_q(T))^\heartsuit$, endowed with actions of $U_q^{\mathrm{Lus}}(N)$ and $U_q^{\mathrm{Lus}}(N^-)$ that satisfy the usual relations.

The abelian category $(\operatorname{Rep}_q(G)_{\mathrm{fin.dim}})^\heartsuit$ has enough projectives. We let

$$\operatorname{Rep}_q(G)_{\mathrm{perf}} \subset \operatorname{Rep}_q(G)_{\mathrm{fin.dim}} \tag{5.4}$$

be the full subcategory consisting of perfect objects (i.e., those represented by finite complexes of projectives).

We set

$$\operatorname{Rep}_q(G)_{\mathrm{ren}} := \operatorname{IndCompl}\left(\operatorname{Rep}_q(G)_{\mathrm{fin.dim}}\right) \text{ and } \operatorname{Rep}_q(G) := \operatorname{IndCompl}\left(\operatorname{Rep}_q(G)_{\mathrm{perf}}\right).$$

Both the categories $\operatorname{Rep}_q(G)_{\mathrm{ren}}$ and $\operatorname{Rep}_q(G)$ carry naturally defined t-structures.

5.4.2. The inclusion (5.4) extends to a fully faithful functor

$$\mathfrak{r}{:}\operatorname{Rep}_q(G) \to \operatorname{Rep}_q(G)_{\mathrm{ren}}.$$

The above functor $\mathfrak{r}$ admits a right adjoint, denoted $\mathfrak{s}$, given by ind-extending the inclusion

$$\operatorname{Rep}_q(G)_{\mathrm{fin.dim}} \hookrightarrow \operatorname{Rep}_q(G).$$

Note, however, that the functor $\mathfrak{s}$ is *not* fully faithful (even though its restriction to the subcategory of compact objects is.)

5.4.3. The situation of the adjoint pair

$$\mathfrak{r} : \operatorname{Rep}_q(G) \rightleftarrows \operatorname{Rep}_q(G)_{\mathrm{ren}} : \mathfrak{s}$$

is completely parallel to that of

$$\operatorname{QCoh}(X) \rightleftarrows \operatorname{IndCoh}(X) \tag{5.5}$$

of [Ga1, Sect. 1] for a finite type scheme X. In particular, both $\operatorname{Rep}_q(G)_{\mathrm{ren}}$ and $\operatorname{Rep}_q(G)$ have t-structures such that the following properties hold:

- The functor $\mathfrak{s}$ is t-exact and induces an equivalence on eventually coconnective subcategories (in particular, on the hearts);
- The category $\operatorname{Rep}_q(G)$ is left-complete in its t-structure;

- The kernel of $\mathfrak{s}$ consists of objects that are infinitely connective (i.e., those objects all of whose cohomologies are zero).

5.4.4. We have the obvious forgetful functor

$$\mathbf{oblv}_{\text{big}\to\text{Lus}^+} : \text{Rep}_q(G)_{\text{ren}} \to \text{Rep}_q(B).$$

It admits both a left and a right adjoints, denoted $\mathbf{ind}_{\text{Lus}^+\to\text{big}}$ and $\mathbf{coind}_{\text{Lus}^+\to\text{big}}$, respectively. For $\check{\lambda} \in \check{\Lambda}$, set

$$\mathcal{V}_q^{\check{\lambda}} := \mathbf{ind}_{\text{Lus}^+\to\text{big}}(k^{\check{\lambda}}) \in \text{Rep}_q(G)_{\text{ren}}.$$

It is known that for $\check{\lambda} \in \check{\Lambda}^+$, the object $\mathcal{V}_q^{\check{\lambda}}$ belongs to $\text{Rep}_q(G)^\heartsuit$; it is called the Weyl module of highest weight $\check{\lambda}$.

Denote:

$$\mathcal{V}_q^{\vee,\check{\lambda}} := \mathbf{coind}_{\text{Lus}^+\to\text{big}}(k^{w_0(\check{\lambda})}) \in \text{Rep}_q(G)_{\text{ren}}.$$

It is known that for $\check{\lambda} \in \check{\Lambda}^+$, the object $\mathcal{V}_q^{\vee,\check{\lambda}}$ belongs to $\text{Rep}_q(G)^\heartsuit$; it is called the dual Weyl module of highest weight $\check{\lambda}$.

5.5. "Big" vs "mixed".

5.5.1. We have a canonically defined (braided monoidal) functor

$$(\text{Rep}_q(G)_{\text{fin.dim}})^\heartsuit \to (Z_{\text{Dr},\text{Rep}_q(T)}(\text{Rep}_q(B)))^\heartsuit = (\text{Rep}_q^{\text{mxd}}(G))^\heartsuit,$$

which extends to a (braided monoidal) functor

$$\mathbf{oblv}_{\text{big}\to\text{mxd}} : \text{Rep}_q(G)_{\text{ren}} \to \text{Rep}_q^{\text{mxd}}(G).$$

Remark 5.5.2. Note that the functor $\mathbf{oblv}_{\text{big}\to\text{mxd}}$ does *not* factor through the projection

$$\mathfrak{s}: \text{Rep}_q(G)_{\text{ren}} \to \text{Rep}_q(G).$$

This is due to the fact that $\text{Rep}_q^{\text{mxd}}(G)$ is not separated in its t-structure.

5.5.3. We claim:

Proposition 5.5.4. *The functor* $\mathbf{oblv}_{\text{big}\to\text{mxd}}$ *sends compacts to compacts.*

Proof. First, by Sect. 5.3.3, the image of $k \in (\text{Rep}_q(G))^\heartsuit$ under $\mathbf{oblv}_{\text{big}\to\text{mxd}}(k)$, viewed as an object of $(\text{Rep}_q^{\text{mxd}}(G))^\heartsuit \subset \text{Rep}_q^{\text{mxd}}(G)$, is compact.

For any $\mathcal{M} \in \text{Rep}_q(G)_{\text{fin.dim}}$, we have

$$\mathbf{oblv}_{\text{big}\to\text{mxd}}(\mathcal{M}) \simeq \mathcal{M} \otimes \mathbf{oblv}_{\text{big}\to\text{mxd}}(k).$$

However, it is easy to see that the operation of tensor product by $\mathcal{M} \in \text{Rep}_q(G)_{\text{fin.dim}}$ on $\text{Rep}_q^{\text{mxd}}(G)$ preserves the subcategory generated under finite colimits by objects $\mathbb{M}_{q,\text{mxd}}^{\check{\lambda}}$. Indeed, every

$$\mathcal{M} \otimes \mathbb{M}_{q,\text{mxd}}^{\check{\lambda}}$$

admits a finite filtration with subquotients $\mathbb{M}_{q,\text{mxd}}^{\check{\lambda}+\check{\lambda}'}$, where $\check{\lambda}'$ runs through the set of weights of $\mathcal{M}$ (with multiplicities).

□

5.5.5. By construction, the composite functor

$$\mathrm{Rep}_q(G)_{\mathrm{ren}} \overset{\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}}{\longrightarrow} \mathrm{Rep}_q^{\mathrm{mxd}}(G) \overset{\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}}{\longrightarrow} \mathrm{Rep}_q(B)$$

identifies with the functor $\mathbf{oblv}_{\mathrm{big}\to\mathrm{Lus}^+}$.

By adjunction, we obtain an identification

$$\mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}} \circ \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}} \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}.$$

In particular, we have a canonical isomorphism

$$\mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}) \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}}) =: \mathcal{V}^{\check{\lambda}}_q \tag{5.6}$$

for *any* $\check{\lambda} \in \check{\Lambda}$.

6. The small quantum group

In this section we will specialize to the case when q takes values in the group of roots of unity in k^*. We will study the category of representations of the *small quantum group* and its relation to that of Lusztig's version.

6.1. **Modules for the graded small quantum group.** In this subsection we define the (one of the three versions of the) category of modules for the $\check{\Lambda}$-graded version of the small quantum group. We will have two more versions versions of this category that differ from each by *renormalization* (i.e., which objects are declared to be compact).

6.1.1. We set

$$\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}} := Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(B)).$$

We have the usual adjunctions

$$\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}} : \mathrm{Rep}_q^{\mathrm{sml,grd}}(B) \rightleftarrows \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}} : \mathbf{oblv}_{\mathrm{sml}^+\to\mathrm{sml}}$$

and

$$\mathbf{oblv}_{\mathrm{sml}\to\mathrm{sml}^-} : \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}} \rightleftarrows u_q(N^-)\text{-mod}(\mathrm{Rep}_q(T)) : \mathbf{coind}_{\mathrm{sml}^-\to\mathrm{sml}}.$$

6.1.2. For $\check{\lambda} \in \check{\Lambda}$, we let

$$\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}} := \mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check{\lambda}}) \text{ and } \mathbb{M}^{\vee,\check{\lambda}}_{q,\mathrm{sml}} := \mathbf{coind}_{\mathrm{sml}^-\to\mathrm{sml}}(k^{\check{\lambda}})$$

be the corresponding standard and costandard objects.

We have

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}}}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}}, \mathbb{M}^{\vee,\check{\lambda}'}_{q,\mathrm{sml}}) = \begin{cases} k \text{ if } \check{\lambda} = \check{\lambda}'; \\ 0 \text{ otherwise.} \end{cases}$$

Remark 6.1.3. The objects $\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}}$ and $\mathbb{M}^{\vee,\check{\lambda}}_{q,\mathrm{sml}}$ are sometimes called the *baby Verma* and *dual baby Verma* modules, respectively. This is the origin of the notation "baby-ren" in the subscript.

6.2. **Renormalized categories.** We will now introduce two more versions of the category of modules over the small quantum group, denoted $\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}$ and $\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)$, respectively, that differ from the original one by a renormalization procedure (i.e., by redefining the class of compact objects).

6.2.1. Note that since $u_q(N)$ is finite-dimensional, we have a fully faithful embedding

$$u_q(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{perf}} \hookrightarrow u_q(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}},$$

and by ind-extension a fully faithful embedding

$$\mathfrak{r}_{\mathrm{baby}} : u_q(N)\text{-mod}(\mathrm{Rep}_q(T)) \hookrightarrow u_q(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}.$$

The latter admits a right adjoint, denoted $\mathfrak{s}_{\mathrm{baby}}$; it is the ind-extension of the fully faithful embedding

$$u_q(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{fin.dim}} \hookrightarrow u_q(N)\text{-mod}(\mathrm{Rep}_q(T)).$$

The adjoint pair $(\mathfrak{r}, \mathfrak{s})$ has the same properties as the pair (5.5).

6.2.2. Note that the category $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$ can be thought of as modules for the monad

$$\mathbf{oblv}_{\mathrm{sml}^+\to\mathrm{sml}}\circ\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}$$

acting on $u_q(N)\text{-mod}(\operatorname{Rep}_q(T))_{\text{loc.nilp}}$. We observe that the action of this monad preserves the subcategory $u_q(N)\text{-mod}(\operatorname{Rep}_q(T))_{\mathrm{perf}}$, and hence also $u_q(N)\text{-mod}(\operatorname{Rep}_q(T))$.

We define the category $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$ to be

$$\mathbf{oblv}_{\mathrm{sml}^+\to\mathrm{sml}}\circ\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}\text{-mod}(u_q(N)\text{-mod}(\operatorname{Rep}_q(T))).$$

6.2.3. By construction, we have an adjoint pair

$$\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}: u_q(N)\text{-mod}(\operatorname{Rep}_q(T))\rightleftarrows \operatorname{Rep}_q^{\mathrm{sml,grd}}(G):\mathbf{oblv}_{\mathrm{sml}^+\to\mathrm{sml}},$$

and a pair of adjoint functors

$$\mathfrak{r}_{\mathrm{baby}}:\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\rightleftarrows \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}:\mathfrak{s}_{\mathrm{baby}}$$

that makes all circuits in the following diagram commute:

$$\begin{array}{ccc} u_q(N)\text{-mod}(\operatorname{Rep}_q(T)) & \rightleftarrows & \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)^{\text{non-ren}} \\ {\scriptstyle\mathfrak{s}_{\mathrm{baby}}}\uparrow\downarrow{\scriptstyle\mathfrak{r}_{\mathrm{baby}}} & & {\scriptstyle\mathfrak{s}_{\mathrm{baby}}}\uparrow\downarrow{\scriptstyle\mathfrak{r}_{\mathrm{baby}}} \\ u_q(N)\text{-mod}(\operatorname{Rep}_q(T))_{\text{loc.nilp}} & \rightleftarrows & \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}} \end{array} \tag{6.1}$$

The adjoint pair $(\mathfrak{r}_{\mathrm{baby}},\mathfrak{s}_{\mathrm{baby}})$ has the same properties as the pair (5.5), specified in Sect. 5.4.3. In particular, $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$ is left-complete in its t-structure.

6.2.4. By a slight abuse of notation, we will denote by the same symbol $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}$ the image of $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}\in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$ under the functor

$$\mathfrak{s}_{\mathrm{baby}}:\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}\to \operatorname{Rep}_q^{\mathrm{sml,grd}}(G).$$

Note, however, that $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}\in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$ is *not* compact. Moreover, it is *not* true that the image of $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}\in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$ under the functor

$$\mathfrak{r}_{\mathrm{baby}}:\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\rightleftarrows \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$$

gives back $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}\in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$ (but we do have a map from the former to the latter).

Similarly, we define $\mathbb{M}^{\vee,\check\lambda}_{q,\mathrm{sml}}\in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$, and the above remarks apply. Since the functor $\mathfrak{s}_{\mathrm{baby}}$ is an equivalence on the eventually coconnective subcategories, we have:

$$\mathcal{H}om_{\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)}(\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}},\mathbb{M}^{\vee,\check\lambda'}_{q,\mathrm{sml}})=\begin{cases} k \text{ if } \check\lambda=\check\lambda'; \\ 0 \text{ otherwise.}\end{cases}$$

6.2.5. Let

$$\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{fin.dim}}\subset \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$$

be the full (but not cocomplete) subcategory consisting of finite-dimensional objects (i.e., those objects that have non-zero cohomology only in finitely many cohomological degrees, and each of these cohomologies is finite-dimensional).

We define

$$\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}:=\operatorname{IndCompl}(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{fin.dim}}).$$

We have the adjoint pair

$$\mathfrak{r}:\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\rightleftarrows \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}:\mathfrak{s}$$

that has the same properties as the pair (5.5), specified in Sect. 5.4.3.

6.2.6. Consider the category $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{c,\text{baby-ren}}$ of compact objects in $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$. Note that the functor $\mathfrak{r}_{\text{baby}}$ induces a fully faithful embedding

$$\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{c,\text{baby-ren}} \hookrightarrow \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{fin.dim}}.$$

Ind-extending, we obtain a fully faithful functor

$$\mathfrak{r}_{\text{baby-ren}\to\text{ren}} : \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}} \to \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}$$

so that

$$\mathfrak{r} \simeq \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathfrak{r}_{\text{baby}}.$$

Since the functor $\mathfrak{r}_{\text{baby-ren}\to\text{ren}}$ sends compacts to compacts, it admits a continuous right adjoint, which we will denote by

$$\mathfrak{s}_{\text{ren}\to\text{baby-ren}} : \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}} \to \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}.$$

By adjunction, we have:

$$\mathfrak{s} \simeq \mathfrak{s}_{\text{baby}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}}.$$

6.2.7. As in [AriG, Corollary 4.4.3], we obtain that the functor $\mathfrak{s}_{\text{ren}\to\text{baby-ren}}$ is t-exact and induces an equivalence on the eventually coconnective subcategories.

To summarize, we have the following sequence of fully faithful embeddings

$$\operatorname{Rep}_q^{\mathrm{sml,grd}}(G) \overset{\mathfrak{r}_{\text{baby}}}{\hookrightarrow} \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}} \overset{\mathfrak{r}_{\text{baby-ren}\to\text{ren}}}{\hookrightarrow} \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}$$

and their right adjoints

$$\operatorname{Rep}_q^{\mathrm{sml,grd}}(G) \overset{\mathfrak{s}_{\text{baby}}}{\twoheadleftarrow} \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}} \overset{\mathfrak{s}_{\text{ren}\to\text{baby-ren}}}{\twoheadleftarrow} \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}.$$

6.2.8. By a slight abuse of notation, for $\check{\lambda} \in \check{\Lambda}$ set

$$\mathbb{M}^{\check{\lambda}}_{q,\text{sml}} := \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathbb{M}^{\check{\lambda}}_{q,\text{sml}}) \in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}.$$

By construction, $\mathbb{M}^{\check{\lambda}}_{q,\text{sml}}$ lies in the heart of $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}$, and when we apply the functor $\mathfrak{s}_{\text{ren}\to\text{baby-ren}}$ to it we recover the original $\mathbb{M}^{\check{\lambda}}_{q,\text{sml}} \in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$.

We let

$$\mathbb{M}^{\vee,\check{\lambda}}_{q,\text{sml}} \in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}$$

be the unique object in the heart that gets sent to $\mathbb{M}^{\vee,\check{\lambda}}_{q,\text{sml}} \in \operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}$ by the functor $\mathfrak{r}_{\text{ren}\to\text{baby-ren}}$.

By adjunction, we have:

$$\mathcal{H}om_{\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}}(\mathbb{M}^{\check{\lambda}}_{q,\text{sml}}, \mathbb{M}^{\vee,\check{\lambda}'}_{q,\text{sml}}) = \begin{cases} k \text{ if } \check{\lambda} = \check{\lambda}'; \\ 0 \text{ otherwise.} \end{cases}$$

Remark 6.2.9. A variant of the remark 5.3.5 applies to the above three versions of modules over the small quantum group as well. Namely, in all three cases, the corresponding abelian category

$$\left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\right)^{\heartsuit} \simeq \left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}\right)^{\heartsuit} \simeq \left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}\right)^{\heartsuit}$$

is that of objects of $\operatorname{Rep}_q(T)^{\heartsuit}$ equipped with an action of $u_q(N)$ (which is automatically locally nilpotent as $u_q(N)$ is finite-dimensional) and a compatible action of $u_q(N^-)$ (which also automatically happens to be locally nilpotent).

Each of the categories

$$\left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\right)^{<\infty,>\infty} \simeq \left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}}\right)^{<\infty,>\infty} \simeq \left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}\right)^{<\infty,>\infty}$$

identifies with $D^b\left(\left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\right)^{\heartsuit}\right)$.

Now, $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}}$ (resp., $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$) identifies with the ind-completion of the category generated under finite colimits by objects of the form $\mathbb{M}^{\check\lambda}_{q,\mathrm{sml}}$ (resp., $\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check\lambda}\otimes u_q(N))$).

For $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}$ we take as generators all finite-dimensional objects of $\left(\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)\right)^{\heartsuit}$.

6.3. **The ungraded small quantum group.**

6.3.1. Let H be the reductive group that is the recipient of Lusztig's quantum Frobenius. Let

$$T_H\subset B_H\subset H$$

be the Cartan and Borel subgroups of H, respectively.

Denote by Λ_H the weight lattice of T_H. Explicitly,

$$\Lambda_H=\{\check\lambda\in\Lambda\,|\,b(\check\lambda,\check\lambda')=1 \text{ for all } \check\lambda'\in\check\Lambda'\}.$$

6.3.2. Quantum Frobenius for tori defines a map from the category $\operatorname{Rep}(T_H)$ to the E_3-center of $\operatorname{Rep}_q(T)$. In particular, we obtain an action of $\operatorname{Rep}(T_H)$ on any of the categories of the form

$$Z_{\mathrm{Dr},\operatorname{Rep}_q(T)}(A\text{-mod}(\operatorname{Rep}_q(T))_{\mathrm{loc.nilp}})$$

of Sect. 4.2.1, and in particular on $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ and $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}}$.

In addition, by unwinding the constructions, we obtain that we also have an action of $\operatorname{Rep}(T_H)$ on the categories $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)$ and $\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}$.

Set

$$\operatorname{Rep}_q^{\mathrm{sml}}(G):=\mathrm{Vect}\underset{\operatorname{Rep}(T_H)}{\otimes}\operatorname{Rep}_q^{\mathrm{sml,grd}}(G),$$

$$\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{baby\text{-}ren}}:=\mathrm{Vect}\underset{\operatorname{Rep}(T_H)}{\otimes}\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}},$$

$$\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}:=\mathrm{Vect}\underset{\operatorname{Rep}(T_H)}{\otimes}\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}},$$

where $\operatorname{Rep}(T_H)\to\mathrm{Vect}$ is the forgetful functor.

These are the three versions of the category of representation of the *ungraded* small quantum group. Each of these categories carries a t-structure, uniquely characterized by the property that the forgetful functor from the corresponding graded version is t-exact.

6.3.3. Note that the identification

$$\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}:=\mathrm{Vect}\underset{\operatorname{Rep}(T_H)}{\otimes}\operatorname{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}$$

gives rise to an action of T_H on the category $\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}$ so that

$$\mathbf{inv}_{T_H}\left(\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}\right)\simeq\operatorname{Rep}_q^{\mathrm{sm,grd}}(G)_{\mathrm{ren}},$$

and similarly for the two other versions.

6.3.4. We have the fully faithful embeddings

$$\operatorname{Rep}_q^{\mathrm{sml}}(G)\overset{\mathfrak{r}_{\mathrm{baby}}}{\hookrightarrow}\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{baby\text{-}ren}}\overset{\mathfrak{r}_{\mathrm{baby\text{-}ren}\to\mathrm{ren}}}{\hookrightarrow}\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}} \tag{6.2}$$

and their right adjoints

$$\operatorname{Rep}_q^{\mathrm{sml}}(G)\overset{\mathfrak{s}_{\mathrm{baby}}}{\twoheadleftarrow}\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{baby\text{-}ren}}\overset{\mathfrak{s}_{\mathrm{ren}\to\mathrm{baby\text{-}ren}}}{\twoheadleftarrow}\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}, \tag{6.3}$$

which are t-exact and induce equivalences on eventually coconnective parts. The adjoint pairs

$$\mathfrak{r}_{\mathrm{baby}}:\operatorname{Rep}_q^{\mathrm{sml}}(G)\rightleftarrows\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{baby\text{-}ren}}:\mathfrak{s}_{\mathrm{baby}}$$

and

$$\mathfrak{r}:\operatorname{Rep}_q^{\mathrm{sml}}(G)\rightleftarrows\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}:\mathfrak{s}$$

have the same properties as the pair (5.5), specified in Sect. 5.4.3.

6.3.5. By a slight abuse of notation we will denote by

$$\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}} \in \mathrm{Rep}^{\mathrm{sml}}_q(G)_?$$

(for the above three options for the value of ?) the image of the standard object $\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}} \in \mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_?$ under the forgetful functor

$$\mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_? \to \mathrm{Rep}^{\mathrm{sml}}_q(G)_?.$$

Note, however, that these objects of the ungraded category only depend on $\check{\lambda}$ as an element of the quotient group $\check{\Lambda}/\Lambda_H$.

Remark 6.3.6. As in the case of the graded version, each of the above three categories can be recovered from its heart. First off, the abelian category

$$\left(\mathrm{Rep}^{\mathrm{sml}}_q(G)\right)^{\heartsuit} \simeq \left(\mathrm{Rep}^{\mathrm{sml}}_q(G)_{\text{baby-ren}}\right)^{\heartsuit} \simeq \left(\mathrm{Rep}^{\mathrm{sml}}_q(G)_{\mathrm{ren}}\right)^{\heartsuit}$$

identifies with modules over the usual small quantum universal enveloping algebra $u_q(G)$.

The category $\mathrm{Rep}^{\mathrm{sml}}_q(G)$, as defined above, identifies with the (derived) category of modules over $u_q(G)$. This is the most commonly used version of the category of modules over the small quantum group.

The categories

$$\mathrm{Rep}^{\mathrm{sml}}_q(G)_{\text{baby-ren}} \text{ and } \mathrm{Rep}^{\mathrm{sml}}_q(G)_{\mathrm{ren}}$$

are obtained as ind-completions of the full (but not cocomplete subcategory) of $\mathrm{Rep}^{\mathrm{sml}}_q(G)$, generated under finite colimits by modules of the form $\mathbb{M}^{\check{\lambda}}_{q,\mathrm{sml}}$ (for the "baby" version) and all finite-dimensional modules (for the "ren" version), respectively.

6.4. **Relation between the "big" and the "small" quantum groups.**

6.4.1. As in Sect. 5.5.1, we have a canonically defined braided monoidal functor

$$\mathbf{oblv}_{\mathrm{big}\to\mathrm{sml}} : \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_{\mathrm{ren}}. \tag{6.4}$$

By construction, this functor sends compacts to compacts.

By a slight abuse of notation, we will denote by the same symbol $\mathbf{oblv}_{\mathrm{big}\to\mathrm{sml}}$ the composition of the above functor with the forgetful functor

$$\mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}^{\mathrm{sml}}_q(G)_{\mathrm{ren}}.$$

6.4.2. Let

$$\mathrm{Frob}^*_q : \mathrm{Rep}(H) \to \mathrm{Rep}_q(G)_{\mathrm{ren}}$$

denote Lusztig's quantum Frobenius. This functor is in fact an E_3-functor from $\mathrm{Rep}(H)$, viewed as an E_3-category, to the E_3-center of $\mathrm{Rep}_q(G)_{\mathrm{ren}}$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(H) & \xrightarrow{\mathrm{Frob}^*_q} & \mathrm{Rep}_q(G)_{\mathrm{ren}} \\ {\scriptstyle \mathbf{oblv}_{H\to T_H}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{big}\to\mathrm{sml}}} \\ \mathrm{Rep}(T_H) & \xrightarrow{\mathrm{Frob}^*_q} & \mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_{\mathrm{ren}}. \end{array}$$

In particular, the functor $\mathbf{oblv}_{\mathrm{big}\to\mathrm{sml}}$ of (6.4) canonically factors as

$$\mathrm{Rep}(T_H) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}^{\mathrm{sml,grd}}_q(G)_{\mathrm{ren}}. \tag{6.5}$$

The following theorem was established in [AG1]:

Theorem 6.4.3. *The functor* (6.5) *is an equivalence.*

6.4.4. By tensoring (6.5) with Vect over $\mathrm{Rep}(T_H)$, we obtain a functor

$$\mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}. \tag{6.6}$$

From Theorem 6.4.3 we obtain:

Corollary 6.4.5. *The functor* (6.6) *is an equivalence. In particular, the category* $\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}$ *carries an action of the group* H*, and we have an identification*

$$\mathbf{inv}_H\left(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}\right) \simeq \mathrm{Rep}_q(G)_{\mathrm{ren}}.$$

Note that, by construction, we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{inv}_H\left(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}\right) & \xrightarrow{\sim} & \mathrm{Rep}_q(G)_{\mathrm{ren}} \\
{\scriptstyle \mathbf{oblv}_{H\to T_H}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{big}\to\mathrm{sml}}} \\
\mathbf{inv}_{T_H}\left(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}\right) & \xrightarrow{\sim} & \mathrm{Rep}_q^{\mathrm{sml},\mathrm{grd}}(G)_{\mathrm{ren}} \\
{\scriptstyle \mathbf{oblv}_{H}}\downarrow & & \downarrow \\
\mathbf{inv}_1\left(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}\right) & \xrightarrow{\sim} & \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}.
\end{array}$$

6.4.6. The action of H on $\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}$ given by Corollary 6.4.5 respects the t-structure. Thus, by viewing $\mathrm{Rep}_q^{\mathrm{sml}}(G)$ (resp., $\mathrm{Rep}_q(G)$) as a quotient of $\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}$ (resp., $\mathrm{Rep}_q(G)_{\mathrm{ren}}$) by the subcategory of infinitely connective objects, we obtain that the category $\mathrm{Rep}_q^{\mathrm{sml}}(G)$ also carries an action of H, compatible with the inclusion $\mathfrak{r}$, and we have a canonical identification

$$\mathrm{Inv}_H(\mathrm{Rep}_q^{\mathrm{sml}}(G)) \simeq \mathrm{Rep}_q(G),$$

and as a consequence

$$\mathrm{Rep}_q^{\mathrm{sml}}(G) \simeq \mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G).$$

These identifications are compatible with the functors $\mathfrak{r}$ and $\mathfrak{s}$.

6.5. **Digression: "big" vs "small" for the quantum Borel.** The material in this subsection seems not to have references in the literature.

6.5.1. Let $\mathfrak{n}_H$ be the Lie algebra of the unipotent radical of B_H. Quantum Frobenius for B is the map of Hopf algebras

$$\mathrm{Frob}_q : U_q^{\mathrm{Lus}}(N) \to U(\mathfrak{n}_H), \tag{6.7}$$

where $U(\mathfrak{n}_H)$ is regarded as a Hopf algebra in $\mathrm{Rep}_q(T)$ via the

$$\mathrm{Frob}_q : \mathrm{Rep}(T_H) \to \mathrm{Rep}_q(T).$$

We have a "short exact sequence" of Hopf algebras

$$0 \to u_q(N) \to U_q^{\mathrm{Lus}}(N) \to U(\mathfrak{n}_H) \to 0 \tag{6.8}$$

A key property of (6.7) is that it is *co-central.*

Dually, we have a Hopf algebra homomorphism

$$\mathrm{Sym}(\mathfrak{n}_H^-) \to U_q^{\mathrm{DK}}(N^-),$$

which is *central*, and a short exact sequence of Hopf algebras

$$0 \to \mathrm{Sym}(\mathfrak{n}^-) \to U_q^{\mathrm{DK}}(N^-) \to u_q(N^-) \to 0. \tag{6.9}$$

6.5.2. Pullback with respect to (6.7) defines a monoidal functor

$$\mathrm{Frob}_q^* : \mathrm{Rep}(B_H) \to \mathrm{Rep}_q(B)$$

so that the diagram

$$\begin{array}{ccc} \mathrm{Rep}(B_H) & \xrightarrow{\mathrm{Frob}_q^*} & \mathrm{Rep}_q(B) \\ {\scriptstyle \mathbf{oblv}_{B_H \to T_H}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Lus}^+ \to \mathrm{sml}^+}} \\ \mathrm{Rep}(T_H) & \xrightarrow{\mathrm{Frob}_q^*} & \mathrm{Rep}_q^{\mathrm{sml,grd}}(B) \end{array}$$

commutes.

In particular, we obtain that the functor

$$\mathbf{oblv}_{\mathrm{Lus}^+ \to \mathrm{sml}^+} : \mathrm{Rep}_q(B) \to \mathrm{Rep}_q^{\mathrm{sml,grd}}(B)$$

canonically factors as

$$\mathrm{Rep}(T_H) \underset{\mathrm{Rep}(B_H)}{\otimes} \mathrm{Rep}_q(B) \to \mathrm{Rep}_q^{\mathrm{sml,grd}}(B). \tag{6.10}$$

In the same way as one proves the equivalence in Theorem 6.4.3, one also establishes:

Theorem 6.5.3. *The functor* (6.10) *is an equivalence.*

6.5.4. By tensoring (6.11) with Vect over $\mathrm{Rep}(T_H)$, we obtain a functor

$$\mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(B) \to \mathrm{Vect} \underset{\mathrm{Rep}(T_H)}{\otimes} \mathrm{Rep}_q^{\mathrm{sml,grd}}(B) =: \mathrm{Rep}_q^{\mathrm{sml}}(B). \tag{6.11}$$

From Theorem 6.5.3 we obtain:

Corollary 6.5.5. *The functor* (6.11) *is an equivalence. In particular, the category* $\mathrm{Rep}_q^{\mathrm{sml}}(B)$ *carries an action of the group* B_H*, and we have an identification*

$$\mathbf{inv}_{B_H}(\mathrm{Rep}_q^{\mathrm{sml}}(B)) \simeq \mathrm{Rep}_q(B).$$

6.5.6. The restriction functor

$$\mathbf{oblv}_{\mathrm{big} \to \mathrm{Lus}^+} : \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}_q(B)$$

is compatible with the actions of $\mathrm{Rep}(H)$ and $\mathrm{Rep}(B_H)$ via the commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(H) & \xrightarrow{\mathbf{oblv}_{H \to B_H}} & \mathrm{Rep}(B_H) \\ {\scriptstyle \mathrm{Frob}_q^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}_q^*} \\ \mathrm{Rep}_q(G)_{\mathrm{ren}} & \xrightarrow[\mathbf{oblv}_{\mathrm{big} \to \mathrm{Lus}^+}]{} & \mathrm{Rep}_q(B). \end{array}$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_q(G)_{\mathrm{ren}} & \xrightarrow{\mathbf{oblv}_{\mathrm{big} \to \mathrm{Lus}^+}} & \mathrm{Rep}_q(B) \\ {\scriptstyle \mathbf{oblv}_{\mathrm{big} \to \mathrm{sml}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Lus}^+ \to \mathrm{sml}^+}} \\ \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}} & \xrightarrow{\mathbf{oblv}_{\mathrm{sm} \to \mathrm{sm}^+}} & \mathrm{Rep}_q^{\mathrm{sml,grd}}(B), \end{array}$$

which gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(T_H) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} & \longrightarrow & \mathrm{Rep}(T_H) \underset{\mathrm{Rep}(B_H)}{\otimes} \mathrm{Rep}_q(B) \\ \downarrow & & \downarrow \\ \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}} & \xrightarrow{\mathbf{oblv}_{\mathrm{sm} \to \mathrm{sm}^+}} & \mathrm{Rep}_q^{\mathrm{sml,grd}}(B), \end{array} \tag{6.12}$$

in which the vertical arrows are equivalences, by Theorems 6.4.3 and 6.5.3.

7. The "$\frac{1}{2}$" version of the quantum group

In this section we continue to assume that q takes values in roots of unity.

We will introduce and study yet one more version of the quantum group: it is one that has Lusztig's version as its positive part and the small quantum group as its negative part. The resulting category of modules, denoted, $\mathrm{Rep}_q^{\frac{1}{2}}(G)$ will play the role of intermediary between $\mathrm{Rep}_q(G)$ and $\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)$.

7.1. **Definition of the "$\frac{1}{2}$" version.**

7.1.1. We define the categories

$$\mathrm{Rep}_q^{\frac{1}{2}}(G) \text{ and } \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}}$$

to be

$$\mathrm{Inv}_{B_H}(\mathrm{Rep}_q^{\mathrm{sml}}(G)) \text{ and } \mathrm{Inv}_{B_H}(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}),$$

respectively.

Both categories carry a t-structure, uniquely characterized by the condition that the corresponding forgetful functors

$$\mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}} : \mathrm{Rep}_q^{\frac{1}{2}}(G) \to \mathrm{Rep}_q^{\mathrm{sml}}(G) \text{ and } \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}} : \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \to \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}$$

are t-exact. We have an adjoint pair

$$\mathfrak{r} : \mathrm{Rep}_q^{\frac{1}{2}}(G) \rightleftarrows \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} : \mathfrak{s}$$

with $\mathfrak{s}$ t-exact and inducing an equivalence on eventually coconnective parts.

7.1.2. By construction, we have:

$$\mathrm{Rep}_q^{\frac{1}{2}}(G) \simeq \mathrm{Rep}(B_H) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G) \text{ and } \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \simeq \mathrm{Rep}(B_H) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}}.$$

We have the obvious forgetful functors

$$\mathbf{oblv}_{\mathrm{big}\to\frac{1}{2}} : \mathrm{Rep}_q(G) \to \mathrm{Rep}_q^{\frac{1}{2}}(G) \text{ and } \mathbf{oblv}_{\mathrm{big}\to\frac{1}{2}} : \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}}$$

that are *fully faithful* because the forgetful functor

$$\mathbf{oblv}_{H\to B_H} : \mathrm{Rep}(H) \to \mathrm{Rep}(B_H) \tag{7.1}$$

is fully faithful.

The functor $\mathbf{oblv}_{\mathrm{big}\to\frac{1}{2}}$ admits a left and a right adjoints, denoted $\mathbf{ind}_{\frac{1}{2}\to\mathrm{big}}$ and $\mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}$, respectively. They are related by the formula

$$\mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}(-) \simeq \mathbf{ind}_{\frac{1}{2}\to\mathrm{big}}(- \otimes k^{-2\rho_H})[-d], \tag{7.2}$$

because this is the case for the corresponding left and right adjoints of the forgetful functor $\mathbf{oblv}_{H\to B_H}$ of (7.1) i.e.,

$$\mathbf{coind}_{B_H\to H}(-) \simeq \mathbf{ind}_{B_H\to H}(- \otimes k^{-2\rho_H})[-d]$$

as functors $\mathrm{Rep}(B_H) \to \mathrm{Rep}(H)$.

7.1.3. Recall that the restriction functor

$$\mathbf{oblv}_{\text{big}\to\text{Lus}^+} : \text{Rep}_q(G)_{\text{ren}} \to \text{Rep}_q(B)$$

is compatible with the actions of $\text{Rep}(H)$ and $\text{Rep}(B_H)$, respectively, via the forgetful functor $\mathbf{oblv}_{H\to B_H} : \text{Rep}(H) \to \text{Rep}(B_H)$.

From here we obtain a functor

$$\mathbf{oblv}_{\frac{1}{2}\to\text{Lus}^+} : \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} := \text{Rep}(B_H) \underset{\text{Rep}(H)}{\otimes} \text{Rep}_q(G)_{\text{ren}} \to \text{Rep}_q(B).$$

From the diagram (6.12), we obtain a commutative diagram

$$\begin{array}{ccc} \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} & \xrightarrow{\mathbf{oblv}_{\frac{1}{2}\to\text{Lus}^+}} & \text{Rep}_q(B) \\ {\scriptstyle \mathbf{oblv}_{B_H\to T_H}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\text{Lus}^+\to\text{sml}^+}} \\ \text{Rep}_q^{\text{sml,grd}}(G)_{\text{ren}} & \xrightarrow{\mathbf{oblv}_{\text{sml}\to\text{sml}^+}} & \text{Rep}_q^{\text{sml,grd}}(B), \end{array}$$

in which the bottom row is obtained from the top row by $\text{Rep}(T_H) \underset{\text{Rep}(B_H)}{\otimes} -$.

Hence, by passing to left adjoint along the horizontal arrows, we obtain another commutative diagram

$$\begin{array}{ccc} \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} & \xleftarrow{\mathbf{ind}_{\text{Lus}^+\to\frac{1}{2}}} & \text{Rep}_q(B) \\ {\scriptstyle \mathbf{oblv}_{B_H\to T_H}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\text{Lus}^+\to\text{sml}^+}} \\ \text{Rep}_q^{\text{sml,grd}}(G)_{\text{ren}} & \xleftarrow{\mathbf{ind}_{\text{sml}^+\to\text{sml}}} & \text{Rep}_q^{\text{sml,grd}}(B). \end{array}$$

7.1.4. For $\check{\lambda} \in \check{\Lambda}$, denote

$$\mathbb{M}^{\check{\lambda}}_{q,\frac{1}{2}} := \mathbf{ind}_{\text{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \in \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}.$$

Note that the restriction of $\mathbb{M}^{\check{\lambda}}_{q,\frac{1}{2}}$ along the functor

$$\mathbf{oblv}_{B_H\to T_H} : \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \to \text{Rep}_q^{\text{sml,grd}}(G)_{\text{ren}}$$

is the standard object (a.k.a. baby Verma module) $\mathbb{M}^{\check{\lambda}}_{q,\text{sml}}$.

7.1.5. We now claim:

Proposition 7.1.6. *The action of B_H on $\text{Rep}_q^{\text{sml}}(G)_{\text{ren}}$ preserves the full subcategory*

$$\text{Rep}_q^{\text{sml}}(G)_{\text{baby-ren}} \subset \text{Rep}_q^{\text{sml}}(G)_{\text{ren}}.$$

Proof. To prove the proposition, it would suffice to show that the objects $\mathbb{M}^{\check{\lambda}}_{q,\text{sml}} \in \text{Rep}_q^{\text{sml}}(G)_{\text{ren}}$ could be lifted to $\text{Inv}_{B_H}(\text{Rep}_q^{\text{sml}}(G)_{\text{ren}})$. However, we just saw that $\mathbb{M}^{\check{\lambda}}_{q,\text{sml}}$ lifts to an object denoted $\mathbb{M}^{\check{\lambda}}_{q,\frac{1}{2}} \in \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$.

□

7.1.7. It follows from Proposition 7.1.6 that the category $\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\text{baby-ren}}$ acquires a B_H-action, and the functors (6.2), and hence also the functors (6.3), are compatible with the B_H-actions.

We define

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} := \operatorname{Inv}_{B_H}(\operatorname{Rep}_q^{\mathrm{sml}}(G)_{\text{baby-ren}}).$$

We have the corresponding fully faithful embeddings

$$\operatorname{Rep}_q^{\frac{1}{2}}(G) \overset{\mathfrak{r}_{\text{baby}}}{\hookrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \overset{\mathfrak{r}_{\text{baby-ren}\to\text{ren}}}{\hookrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \tag{7.3}$$

and their right adjoints

$$\operatorname{Rep}_q^{\frac{1}{2}}(G) \overset{\mathfrak{s}_{\text{baby}}}{\twoheadleftarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \overset{\mathfrak{s}_{\text{ren}\to\text{baby-ren}}}{\twoheadleftarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G). \tag{7.4}$$

Moreover, $\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}$ also carries a t-structure, so that the forgetful functor

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \to \operatorname{Rep}_q^{\mathrm{sml}}(G)_{\text{baby-ren}}$$

is t-exact. The functors in (7.4) are t-exact and induce equivalences on eventually coconnective parts.

7.1.8. By construction, the objects $\mathbb{M}^{\check{\lambda}}_{q,\frac{1}{2}}$ lie in the essential image of $\mathfrak{r}_{\text{baby-ren}\to\text{ren}}$. This implies that the functor $\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}$ factors as

$$\operatorname{Rep}_q(B) \to \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \overset{\mathfrak{r}_{\text{baby-ren}\to\text{ren}}}{\hookrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}.$$

By adjunction, this implies that the functor

$$\mathbf{oblv}_{\frac{1}{2}\to\mathrm{Lus}^+} : \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \to \operatorname{Rep}_q(B)$$

factors as

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \overset{\mathfrak{s}_{\text{ren}\to\text{baby-ren}}}{\twoheadrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \to \operatorname{Rep}_q(B).$$

By a slight abuse of notation, we will denote the resulting pair of adjoint functors

$$\operatorname{Rep}_q(B) \rightleftarrows \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}$$

by the same symbols $(\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}, \mathbf{oblv}_{\frac{1}{2}\to\mathrm{Lus}^+})$.

Remark 7.1.9. The categories

$$\operatorname{Rep}_q^{\frac{1}{2}}(G),\ \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \text{ and } \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$$

can be recovered from their common heart by the same procedure as in Remark 6.3.6.

The abelian category $(\operatorname{Rep}_q^{\frac{1}{2}}(G))^{\heartsuit}$ can be explicitly described as objects of $(\operatorname{Rep}_q(T))^{\heartsuit}$, equipped with a locally nilpotent action of $U_q^{\mathrm{Lus}}(N)$ and a compatible action of $u_q(N^-)$.

7.2. "Mixed" vs "$\frac{1}{2}$".

7.2.1. Recall the co-central homomorphism

$$U_q^{\mathrm{Lus}}(N) \to U(\mathfrak{n}_H)$$

of (6.7).

It induces a functor from $\operatorname{Rep}(B_H)$ to the E_3-center of $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$. In particular, we obtain a monoidal action of $\operatorname{Rep}(B_H)$ on $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$.

7.2.2. The following conjecture seems to be within easy reach, and in what follows we will assume its validity:

Conjecture 7.2.3. *The above action of* $\operatorname{Rep}(B_H)$ *on* $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ *can be promoted to an action of the monoidal category* $\operatorname{QCoh}(\mathfrak{n}_H/\operatorname{Ad}(B_H))$. *Furthermore, we have a canonical equivalence*

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \simeq \operatorname{Rep}(B_H) \underset{\operatorname{QCoh}(\mathfrak{n}_H/\operatorname{Ad}(B_H))}{\otimes} \operatorname{Rep}_q^{\mathrm{mxd}}(G),$$

where the functor $\operatorname{QCoh}(\mathfrak{n}_H/\operatorname{Ad}(B_H)) \to \operatorname{Rep}(B_H)$ *is given by evaluation at the point* $0 \overset{\iota}{\hookrightarrow} \mathfrak{n}_H$.

Remark 7.2.4. The intuitive meaning behind the second statement in Conjecture 7.2.3 is the short exact sequence (6.9) of Hopf algebras in $\operatorname{Rep}_q(T)$:

$$0 \to \operatorname{Sym}(\mathfrak{n}_H^-) \to U_q^{\mathrm{DK}}(N^-) \to u_q(N^-) \to 0.$$

Namely, the passage from $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ to $\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}$ is obtained by imposing that the augmentation ideal in $\operatorname{Sym}(\mathfrak{n}_H^-)$ should act by 0. Here we identify $\mathfrak{n}_H^-$ with the dual vector space of $\mathfrak{n}_H$, so that

$$\mathfrak{n}_H \simeq \operatorname{Spec}(\operatorname{Sym}(\mathfrak{n}_H^-)).$$

7.2.5. Assuming Conjecture 7.2.3, we obtain that there exists a forgetful functor

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \to \operatorname{Rep}_q^{\mathrm{mxd}}(G),$$

denoted

$$\iota_* \simeq \mathbf{oblv}_{\frac{1}{2}\to\mathrm{mxd}},$$

which admits a left and a right adjoints, denoted

$$\iota^* \simeq \mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}} \text{ and } \iota^! \simeq \mathbf{coind}_{\mathrm{mxd}\to\frac{1}{2}},$$

respectively.

In addition, we have

$$\mathbf{coind}_{\mathrm{mxd}\to\frac{1}{2}}(-) \simeq \mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}(-) \otimes k^{2\rho_H}[-d], \tag{7.5}$$

because relationship holds for the functors

$$\iota^*, \iota^! : \operatorname{QCoh}(\mathfrak{n}_H/\operatorname{Ad}(B_H)) \to \operatorname{QCoh}(\mathrm{pt}/B_H).$$

7.2.6. The statement of Conjecture 7.2.3 should be complemented by the following two additional ones:

One is that the composite functor

$$\operatorname{Rep}_q(G)_{\mathrm{ren}} \overset{\mathbf{oblv}_{\mathrm{big}\to\frac{1}{2}}}{\longrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \overset{\mathfrak{s}_{\mathrm{ren}\to\text{baby-ren}}}{\longrightarrow} \operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \overset{\mathbf{oblv}_{\frac{1}{2}\to\mathrm{mxd}}}{\longrightarrow} \operatorname{Rep}_q^{\mathrm{mxd}}(G)$$

identifies with the functor $\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}$.

The other is that the composite functor

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \overset{\mathbf{oblv}_{\frac{1}{2}\to\mathrm{mxd}}}{\longrightarrow} \operatorname{Rep}_q^{\mathrm{mxd}}(G) \overset{\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}}{\longrightarrow} \operatorname{Rep}_q(B)$$

identifies with the functor $\mathbf{oblv}_{\frac{1}{2}\to\mathrm{Lus}^+}$.

These two identifications of functors must be compatible with the identification

$$\mathbf{oblv}_{\frac{1}{2}\to\mathrm{Lus}^+} \circ \mathbf{oblv}_{\mathrm{big}\to\frac{1}{2}} \simeq \mathbf{oblv}_{\mathrm{big}\to\mathrm{Lus}^+}.$$

7.2.7. Note that, by adjunction, the second of the above compatibilities implies an isomorphism

$$\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}} \circ \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}} \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}.$$

In particular, for $\check{\lambda} \in \check{\Lambda}$, we obtain an isomorphism

$$\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathbb{M}_{q,\mathrm{mxd}}) \simeq \mathbb{M}_{q,\frac{1}{2}}.$$

7.3. More on mixed vs $\frac{1}{2}$.

7.3.1. We claim:

Proposition 7.3.2. *The functor* $\mathbf{oblv}_{\frac{1}{2}\to\text{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}}$ *sends compacts to compacts.*

Proof. The category $\text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$ is generated by objects of the form

$$k^\mu \otimes \mathbf{oblv}_{\text{big}\to\frac{1}{2}}(\mathcal{M}), \quad \mathcal{M} \in \text{Rep}_q(G)_{\text{ren}},\ \mu \in \Lambda.$$

So it suffices to show that the functor $\mathbf{oblv}_{\frac{1}{2}\to\text{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}}$ sends such objects to compact objects in $\text{Rep}_q^{\text{mxd}}(G)$.

Now, since the operation $k^\mu \otimes -$ on $\text{Rep}_q^{\text{mxd}}(G)$ preserves compacts, the assertion follows from Proposition 5.5.4.

□

Remark 7.3.3. Note that the above proposition implies the following relationship between the categories $\text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$ and $\text{Rep}_q^{\text{mxd}}(G)$:

The functor

$$\text{Rep}(B_H) \underset{\text{QCoh}(\mathfrak{n}_H/\text{Ad}(B_H))}{\otimes} \text{Rep}_q^{\text{mxd}}(G) \to \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$$

is fully faithful, but not an equivalence; rather it is an equivalence onto the full subcategory $\text{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \subset \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$.

Instead, $\text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$ can be obtained from $\text{Rep}(B_H) \underset{\text{QCoh}(\mathfrak{n}_H/\text{Ad}(B_H))}{\otimes} \text{Rep}_q^{\text{mxd}}(G)$ as follows: it is the ind-completion of the full (but not cocomplete) subcategory of $\text{Rep}(B_H) \underset{\text{QCoh}(\mathfrak{n}_H/\text{Ad}(B_H))}{\otimes} \text{Rep}_q^{\text{mxd}}(G)$ spanned by objects that become compact after applying the forgetful functor

$$\iota_* : \text{Rep}(B_H) \underset{\text{QCoh}(\mathfrak{n}_H/\text{Ad}(B_H))}{\otimes} \text{Rep}_q^{\text{mxd}}(G) \to \text{Rep}_q^{\text{mxd}}(G).$$

This is analogous to [FG3, Theorem 11.4.2].

Remark 7.3.4. To summarize, we have the following diagram of stacks

$$\begin{array}{ccccc} \text{pt}/T_H & \longrightarrow & \text{pt}/B_H & \xrightarrow{\iota} & \mathfrak{n}_H/\text{Ad}(B_H) \\ & & \downarrow & & \\ & & \text{pt}/H. & & \end{array}$$

And we have the following diagram of categories over these stacks

$$\begin{array}{ccccccc} \text{Rep}_q^{\text{sml,grd}}(G)_{\text{ren}} & \longleftarrow & \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} & \xleftarrow{\mathfrak{r}_{\text{baby-ren}\to\text{ren}}} & \text{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} & \xleftarrow{\mathbf{ind}_{\text{mxd}\to\frac{1}{2}}} & \text{Rep}_q^{\text{mxd}}(G) \\ & & \uparrow & & & & \\ & & \text{Rep}_q(G)_{\text{ren}} & & & & \end{array}$$

This diagram of categories is *almost* compatible with pullbacks along the above diagram of stacks: the only wrinkle is the functor $\mathfrak{r}_{\text{baby-ren}\to\text{ren}}$, which is fully faithful but not an equivalence.

7.3.5. From Proposition 7.3.2 we obtain:

Corollary 7.3.6. *The functors*

$$\mathbf{oblv}_{\frac{1}{2}\to\text{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}} : \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \rightleftarrows \text{Rep}_q^{\text{mxd}}(G) : \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}})$$

form an adjoint pair.

Proof. Given $\mathcal{M}_1 \in \text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}$ and $\mathcal{M}_2 \in \text{Rep}_q^{\text{mxd}}(G)$ we need to construct a canonical isomorphism

$$\mathcal{H}om_{\text{Rep}_q^{\text{mxd}}(G)}(\mathbf{oblv}_{\frac{1}{2}\to\text{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}}(\mathcal{M}_1), \mathcal{M}_2) \simeq$$
$$\simeq \mathcal{H}om_{\text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}}(\mathcal{M}_1, \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}_2)).$$

With no restriction of generality, we can assume that $\mathcal{M}_1$ is compact. However, Proposition 7.3.2 implies that we can assume that $\mathcal{M}_2$ is compact as well. Note that since the functor $\mathbf{coind}_{\text{mxd}\to\frac{1}{2}}$ differs from $\mathbf{ind}_{\text{mxd}\to\frac{1}{2}}$ by a twist, we obtain that $\mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}_2)$ is then also compact.

Since the functor $\mathfrak{s}_{\text{ren}\to\text{baby-ren}}$ is fully faithful on compact objects, we have

$$\mathcal{H}om_{\text{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}}(\mathcal{M}_1, \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}_2)) \simeq$$
$$\simeq \mathcal{H}om_{\text{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}}(\mathfrak{s}_{\text{ren}\to\text{baby-ren}}(\mathcal{M}_1), \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}_2)) \simeq$$
$$\simeq \mathcal{H}om_{\text{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}}(\mathfrak{s}_{\text{ren}\to\text{baby-ren}}(\mathcal{M}_1), \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}_2))$$

and the assertion follows from the $(\mathbf{oblv}_{\frac{1}{2}\to\text{mxd}}, \mathbf{coind}_{\text{mxd}\to\frac{1}{2}})$-adjunction.

□

From here we obtain:

Corollary 7.3.7. *The left and right adjoints to* $\mathbf{oblv}_{\text{big}\to\text{mxd}}$ *are related by the formula*

$$\mathbf{coind}_{\text{mxd}\to\text{big}} \simeq \mathbf{ind}_{\text{mxd}\to\text{big}}[-2d].$$

Proof. We have:

$$\mathbf{ind}_{\text{mxd}\to\text{big}} \simeq \mathbf{ind}_{\frac{1}{2}\to\text{big}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{ind}_{\text{mxd}\to\frac{1}{2}}$$

and by Corollary 7.3.6, we have

$$\mathbf{coind}_{\text{mxd}\to\text{big}} \simeq \mathbf{coind}_{\frac{1}{2}\to\text{big}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}$$

The assertion follows now from (7.5) and (7.2).

□

Remark 7.3.8. The assertion of Corollary 7.3.7 holds as-is in the non-root of unity case. This follows from the fact that in this case the pair of categories

$$\text{Rep}_q(G) \simeq \text{Rep}_q(G)_{\text{ren}} \overset{\mathbf{oblv}_{\text{big}\to\text{mxd}}}{\longrightarrow} \text{Rep}_q^{\text{mxd}}(G)$$

is equivalent to

$$\text{Rep}(G) \to \mathfrak{g}\text{-mod}^B.$$

7.4. **Some consequences of the Frobenius algebra property.**

7.4.1. A remarkable property of $u_q(N^-)$ is that it is a *Frobenius algebra* (this is the case of any finite-dimensional Hopf algebra, see, e.g., [Et, Corollary 6.4.5]). We will use it in the following guise:

Lemma 7.4.2. *The object of* $u_q(N^-)\text{-mod}(\text{Rep}_q(T))$ *given by* $u_q(N^-)$ *itself is cofree:*

$$\mathcal{H}om_{u_q(N^-)\text{-mod}(\text{Rep}_q(T))}(\mathcal{M}, u_q(N^-)) \simeq \mathcal{H}om_{\text{Rep}_q(T)}(\mathcal{M}, k^{2\check{\rho}-2\rho_H}).$$

7.4.3. From Lemma 7.4.2 we will deduce:

Corollary 7.4.4. *For $\check\lambda \in \check\Lambda$, we have*

$$\mathcal{H}om_{U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))}(k^{\check\lambda}, U_q^{\mathrm{DK}}(N^-)) = \begin{cases} k[-d] \text{ for } \check\lambda = 2\check\rho; \\ 0 \text{ otherwise.} \end{cases}$$

Remark 7.4.5. One can show that the assertion of Corollary 7.4.4 holds verbatim also in the non-root of unity case.

Proof. For $\mathcal{M} \in U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))$, we calculate

$$\mathcal{H}om_{U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))}(k^{\check\lambda}, \mathcal{M}) \simeq \mathcal{H}om_{u_q(N^-)\text{-mod}(\mathrm{Rep}_q(T))}(k^{\check\lambda}, \mathcal{H}om_{\mathrm{Sym}(\mathfrak{n}_H^-)}(k, \mathcal{M})).$$

We note

$$\mathcal{H}om_{\mathrm{Sym}(\mathfrak{n}_H^-)}(k, \mathcal{M}) \simeq (k \underset{\mathrm{Sym}(\mathfrak{n}_H^-)}{\otimes} \mathcal{M}) \otimes k^{2\rho_H}[-d].$$

We take $\mathcal{M} = U_q^{\mathrm{DK}}(N^-)$, and we note that

$$k \underset{\mathrm{Sym}(\mathfrak{n}_H^-)}{\otimes} U_q^{\mathrm{DK}}(N^-) \simeq u_q(N^-).$$

Now the assertion follows from Lemma 7.4.2.

□

Remark 7.4.6. In fact, Lemma 7.4.2 implies that the two functors

$$u_q(N^-)\text{-mod}(\mathrm{Rep}_q(T)) \rightrightarrows \mathrm{Rep}_q(T),$$

given by

$$\mathcal{M} \mapsto k \underset{u_q(N^-)}{\otimes} \mathcal{M} \text{ and } \mathcal{H}om_{u_q(N^-)}(k, \mathcal{M}) \otimes k^{2\check\rho - 2\rho_H}$$

are canonically isomorphic when evaluated on $u_q(N^-)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{perf}}$.

This implies that the two functors

$$U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T)) \rightrightarrows \mathrm{Rep}_q(T),$$

given by

$$\mathcal{M} \mapsto k \underset{U_q^{\mathrm{DK}}(N^-)}{\otimes} \mathcal{M} \text{ and } \mathcal{H}om_{U_q^{\mathrm{DK}}(N^-)}(k, \mathcal{M}) \otimes k^{2\check\rho}[-d] \tag{7.6}$$

are canonically isomorphic when evaluated on $U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{perf}}$.

Note, however that Corollary 5.2.7 implies that the isomorphism between the functors (7.6) holds on all of $U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))$.

7.4.7. Another corollary of Lemma 7.4.2 that we will use is the following:

Corollary 7.4.8. *The functor*

$$\mathbf{oblv}_{\mathrm{sml}\to\mathrm{sml}^+} : \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)\text{-mod}_{\mathrm{ren}} \to \mathrm{Rep}_q^{\mathrm{sml,grd}}(B)\text{-mod}$$

admits a right *adjoint, denoted* $\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{sml}}$, *and for* $\check\lambda \in \check\Lambda$ *we have:*

$$\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check\lambda}) \simeq \mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check\lambda + 2\rho_H - 2\check\rho}).$$

Proof. We need to show that $\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check\lambda})$ is free over $u_q(N^-)$ on one generator of weight equal to $\check\lambda + 2\rho_H - 2\check\rho$. By construction, $\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check\lambda})$ is cofree over $u_q(N^-)$ on one generator of weight $\check\lambda$. Now the assertion follows from Lemma 7.4.2.

□

The same argument proves also the *ungraded* version of Corollary 7.4.8, as well as the following statement:

Corollary 7.4.9. *The functor*

$$\mathbf{oblv}_{\frac{1}{2}\to\mathrm{Lus}^+} : \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \to \mathrm{Rep}_q(B)$$

admits a right *adjoint, denoted* $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}$*, and for* $\check{\lambda}\in\check{\Lambda}$ *we have:*

$$\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}+2\rho_H-2\check{\rho}}).$$

7.4.10. One can use Corollary 7.4.9 to reproduce the result of [APW, Theorem 7.3]:

Corollary 7.4.11. *The objects*

$$\mathcal{V}_q^{\check{\lambda}} := \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}}) \text{ and } \mathcal{V}_q^{\vee,w_0(\check{\lambda})} := \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}})$$

are related by the formula

$$\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}}) \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-2\check{\rho}})[-d].$$

Proof. We have:

$$\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}}) \simeq \mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}\circ\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \simeq$$
$$\simeq \mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}\circ\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}+2\rho_H-2\check{\rho}}) \simeq \mathbf{ind}_{\frac{1}{2}\to\mathrm{big}}\circ\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}-2\check{\rho}})[-d].$$

□

Remark 7.4.12. The assertion of Corollary 7.4.11 is also valid in the non-root of unity case.

8. Duality for quantum groups

In this section we will show that the effect of replacing q by q^{-1} is *duality for DG categories* for the various versions of the category of modules over the quantum group.

8.1. **Framework for duality.**

8.1.1. Let $\mathcal{C}$ be a monoidal DG category, which is compactly generated, and such that the following conditions are satisfied:

- The monoidal operation preserves compactness;
- Every compact object admits a (right) monoidal dual.

Let $\mathcal{C}^{\mathrm{rev}}$ denoted the monoidal category obtained by reversing the monoidal operation. Then the functor

$$\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}} \to \mathrm{Vect} \tag{8.1}$$

given by ind-extending

$$c_1, c_2 \mapsto \mathcal{H}om_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, c_1\otimes c_2), \quad c_1\in\mathcal{C}_c,\, c_2\in\mathcal{C}_c^{\mathrm{rev}}$$

defines a perfect pairing and in itself is a right-lax monoidal functor.

In particular, we obtain an identification of DG categories

$$\mathcal{C}^\vee \simeq \mathcal{C}^{\mathrm{rev}}.$$

8.1.2. Assume now that $\mathcal{C}$ is *braided monoidal*. We define a braiding on $\mathcal{C}^{\mathrm{rev}}$ by letting the map

$$c_1 \overset{\mathrm{rev}}{\otimes} c_2 \to c_2 \overset{\mathrm{rev}}{\otimes} c_1$$

be the map

$$c_2\otimes c_1 \xrightarrow{R^{-1}_{c_1,c_2}} c_1\otimes c_2.$$

Then the above functor (8.1) has a natural right-lax braided monoidal structure.

8.1.3. Let $\mathcal{C}$ be braided monoidal, and let A be a Hopf algebra in $\mathcal{C}$. Let $A^{\text{rev-mult}}$ be the Hopf algebra in $\mathcal{C}^{\text{rev}}$ defined as follows:

As an object, $A^{\text{rev-mult}}$ is the same as A. The co-multiplication map

$$A^{\text{rev-mult}} \to A^{\text{rev-mult}} \overset{\text{rev}}{\otimes} A^{\text{rev-mult}}$$

is the initial comultiplication map

$$A \to A \otimes A.$$

Let the multiplication map

$$A^{\text{rev-mult}} \overset{\text{rev}}{\otimes} A^{\text{rev-mult}} \to A^{\text{rev-mult}}$$

is set to be the map

$$A \otimes A \overset{R^{-1}_{A,A}}{\longrightarrow} A \otimes A \to A.$$

In a similar way we define the Hopf algebra $A^{\text{rev-comult}}$ in $\mathcal{C}^{\text{rev}}$. We note, however, that the antipode defines an isomorphism of Hopf algebras

$$A^{\text{rev-mult}} \simeq A^{\text{rev-comult}}.$$

Remark 8.1.4. Recall that in Sect. 4.3.2 the notation $A^{\text{rev-mult}}$ (resp., $A^{\text{rev-comult}}$) had a different meaning: in *loc.cit.* they denoted Hopf algebras in $\mathcal{C}^{\text{rev-br}}$.

The two notions are related as follows: we have a canonical defined functor

$$\mathcal{C}^{\text{rev}} \to \mathcal{C}^{\text{rev-br}} \tag{8.2}$$

that *reverses* the product but preserves the braiding. Hence this functor maps Hopf algebras to Hopf algebras.

Now, the functor (8.2) sends $A^{\text{rev-mult}}$ (resp., $A^{\text{rev-comult}}$) in $\mathcal{C}^{\text{rev}}$ to $A^{\text{rev-mult}}$ (resp., $A^{\text{rev-comult}}$) in $\mathcal{C}^{\text{rev-br}}$.

8.1.5. Let us be in the setting of Sect. 8.1.3. We have a natural monoidal equivalence

$$(A\text{-mod}(\mathcal{C}))^{\text{rev}} \simeq A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\text{rev}}). \tag{8.3}$$

It sends an A-module M in $\mathcal{C}$ to an $A^{\text{rev-mult}}$-module in $\mathcal{C}^{\text{rev}}$, whose underlying object of $\mathcal{C}^{\text{rev}}$ is the same M, and where the action map

$$A^{\text{rev-mult}} \overset{\text{rev}}{\otimes} M \to M$$

is set to be

$$M \otimes A \overset{R^{-1}_{A,M}}{\longrightarrow} A \otimes M \to M.$$

8.1.6. The equivalence (8.3) induces a *braided monoidal* equivalence

$$(Z_{\text{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})))^{\text{rev}} \simeq Z_{\text{Dr},\mathcal{C}^{\text{rev}}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\text{rev}})). \tag{8.4}$$

At the level of the underlying DG categories, we obtain an equivalence

$$Z_{\text{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})) \simeq Z_{\text{Dr},\mathcal{C}^{\text{rev}}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\text{rev}})),$$

which makes the following diagrams commute:

$$\begin{array}{ccc} Z_{\text{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})) & \overset{\sim}{\longrightarrow} & Z_{\text{Dr},\mathcal{C}^{\text{rev}}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\text{rev}})) \\ \downarrow & & \downarrow \\ A\text{-mod}(\mathcal{C}) & \overset{\sim}{\longrightarrow} & A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\text{rev}}) \end{array} \tag{8.5}$$

and

$$
\begin{array}{ccc}
Z_{\mathrm{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})) & \xrightarrow{\sim} & Z_{\mathrm{Dr},\mathcal{C}^{\mathrm{rev}}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\mathrm{rev}})) \\
\downarrow & & \downarrow \\
A\text{-comod}(\mathcal{C}) & \xrightarrow{\sim} & A^{\text{rev-comult}}\text{-comod}(\mathcal{C}^{\mathrm{rev}}) \simeq A^{\text{rev-mult}}\text{-comod}(\mathcal{C}^{\mathrm{rev}}).
\end{array} \tag{8.6}
$$

In addition, the diagram

$$
\begin{array}{ccc}
Z_{\mathrm{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})) & \xrightarrow{\sim} & Z_{\mathrm{Dr},\mathcal{C}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\mathrm{rev}})) \\
\uparrow & & \uparrow \\
A\text{-mod}(\mathcal{C}) & \xrightarrow{\sim} & A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\mathrm{rev}}),
\end{array}
$$

(obtained from (8.5) by passing to *right* adjoints along the vertical arrows) and

$$
\begin{array}{ccc}
Z_{\mathrm{Dr},\mathcal{C}}(A\text{-mod}(\mathcal{C})) & \xrightarrow{\sim} & Z_{\mathrm{Dr},\mathcal{C}}(A^{\text{rev-mult}}\text{-mod}(\mathcal{C}^{\mathrm{rev}})) \\
\uparrow & & \uparrow \\
A\text{-comod}(\mathcal{C}) & \xrightarrow{\sim} & A^{\text{rev-comult}}\text{-comod}(\mathcal{C}^{\mathrm{rev}}) \simeq A^{\text{rev-mult}}\text{-comod}(\mathcal{C}^{\mathrm{rev}}).
\end{array}
$$

(obtained from (8.6) by passing to *left* adjoints along the vertical arrows) also commute.

8.2. **The case of quantum groups.**

8.2.1. We start by discussing duality for the big quantum group. We note that the braided monoidal abelian categories $(\mathrm{Rep}_q(G))^{\heartsuit}$ and $(\mathrm{Rep}_{q^{-1}}(G))^{\heartsuit}$ are related by

$$((\mathrm{Rep}_q(G))^{\heartsuit})^{\mathrm{rev}} \simeq (\mathrm{Rep}_{q^{-1}}(G))^{\heartsuit}, \quad \mathcal{M} \mapsto \mathcal{M}^{\sigma},$$

induced by the canonical algebra isomorphism

$$\sigma : U_q^{\mathrm{Lus}}(G) \to U_{q^{-1}}^{\mathrm{Lus}}(G),$$

which reverses the comultiplication.

The above equivalence induces an equivalence

$$(\mathrm{Rep}_q(G)_{\mathrm{ren}})^{\mathrm{rev}} \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}. \tag{8.7}$$

By Sect. 8.1.1, the equivalence (8.7) induces an identification

$$(\mathrm{Rep}_q(G)_{\mathrm{ren}})^{\vee} \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}, \tag{8.8}$$

with the pairing

$$\mathrm{Rep}_q(G)_{\mathrm{ren}} \otimes \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}} \to \mathrm{Vect}$$

given by

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{H}om_{\mathrm{Rep}_q(G)_{\mathrm{ren}}}(k, \mathcal{M}_1 \otimes \mathcal{M}_2^{\sigma}).$$

The corresponding contravariant functor on compact objects

$$\mathbb{D} : (\mathrm{Rep}_{q^{-1}}(G)_{\mathrm{fin.dim}}) \to (\mathrm{Rep}_q(G)_{\mathrm{ren}})_{\mathrm{fin.dim}}$$

is

$$\mathcal{M} \mapsto (\mathcal{M}^{\sigma})^{\vee}, \tag{8.9}$$

where $(-)^{\vee}$ is monoidal dualization.

Remark 8.2.2. The same discussion applies to $\mathrm{Rep}_q(G)$. Here we use the fact that contragredient duality on $(\mathrm{Rep}_q(G)_{\mathrm{fin.dim}})^{\heartsuit}$ sends projective to projectives; in other words, in the abelian category $(\mathrm{Rep}_q(G)_{\mathrm{fin.dim}})^{\heartsuit}$, the classes of projective and injective objects coincide (this is the case for any monoidal abelian category which is rigid).

8.2.3. We now consider the braided monoidal category $\mathrm{Rep}_q(T)$. Note that the corresponding category $(\mathrm{Rep}_q(T))^{\mathrm{rev}}$ identifies with $\mathrm{Rep}_{q^{-1}}(T)$.

If A is a Hopf algebra as in Sect. 4.2.1, we obtain a canonical identification

$$\Big(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}\Big)^{\mathrm{rev}} \simeq Z_{\mathrm{Dr},\mathrm{Rep}_{q^{-1}}(T)}(A^{\text{rev-mult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp}},$$

denoted $\mathcal{M}\mapsto\mathcal{M}^\sigma$, and in particular, an identification

$$\Big(Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}\Big)^{\vee} \simeq Z_{\mathrm{Dr},\mathrm{Rep}_{q^{-1}}(T)}(A^{\text{rev-mult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp}},$$

with the pairing

$$Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}\otimes Z_{\mathrm{Dr},\mathrm{Rep}_{q^{-1}}(T)}(A^{\text{rev-mult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp}}\to\mathrm{Vect}$$

given by ind-extending

$$\mathcal{M}_1,\mathcal{M}_2\mapsto \mathcal{H}om_{Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}}(k,\mathcal{M}_1\otimes\mathcal{M}_2^\sigma),$$
$$\mathcal{M}_1\in Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(A\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp,c}},\ \mathcal{M}_2\in Z_{\mathrm{Dr},\mathrm{Rep}_{q^{-1}}(T)}(A^{\text{rev-mult}}\text{-mod}(\mathrm{Rep}_{q^{-1}}(T))_{\mathrm{loc.nilp,c}}.$$

8.2.4. Let us take $A=u_q(N)$. Note that by construction, we can identify

$$(u_q(N))^{\text{rev-mult}}\simeq u_{q^{-1}}(N)$$

as Hopf algebras in $\mathrm{Rep}_{q^{-1}}(T)$.

From here we obtain an equivalence

$$(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{baby-ren}})^{\mathrm{rev}}\simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\text{baby-ren}},\quad \mathcal{M}\mapsto\mathcal{M}^\sigma$$

This equivalence induces an equivalence

$$(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G))^{\mathrm{rev}}\simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G),$$

and further, by restriction an ind-extension, an equivalence

$$(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}})^{\mathrm{rev}}\simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}.$$

8.2.5. Thus, we obtain an equivalence

$$(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}})^{\vee}\simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}, \tag{8.10}$$

with the pairing

$$\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}\otimes \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}\to\mathrm{Vect} \tag{8.11}$$

given by

$$\mathcal{M}_1,\mathcal{M}_2\mapsto \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}}(k,\mathcal{M}_1\otimes\mathcal{M}_2^\sigma),\quad \mathcal{M}_1\in \mathrm{Rep}_q^{\mathrm{sml.grd}}(G)_{\mathrm{fin.dim}},\ \mathcal{M}_2\in \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{fin.dim}} \tag{8.12}$$

Note, however, that formula (8.12) with *any* $\mathcal{M}_1\in \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)$ and $\mathcal{M}_2\in \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)$ defines the pairing (8.15) because $1\in \mathrm{Rep}_q^{\mathrm{sml,grd}}(G)$ is compact.

8.2.6. The corresponding contravariant functor

$$\mathbb{D}:\mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{fin.dim}}\to \mathrm{Rep}_q^{\mathrm{sml.grd}}(G)_{\mathrm{fin.dim}} \tag{8.13}$$

is

$$\mathcal{M}\mapsto(\mathcal{M}^\sigma)^\vee,$$

where $(-)^\vee$ is monoidal dualization.

8.2.7. The duality (8.10) induces the dualities

$$(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}})^\vee \simeq (\mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren}})^{\mathrm{rev}} \text{ and } (\mathrm{Rep}_q^{\mathrm{sml,grd}}(G))^\vee \simeq (\mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G))^{\mathrm{rev}},$$

with the corresponding contravariant functors

$$\mathbb{D}: \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_{\mathrm{baby\text{-}ren},c} \to \mathrm{Rep}_q^{\mathrm{sml.grd}}(G)_{\mathrm{baby\text{-}ren},c} \text{ and } \mathbb{D}: \mathrm{Rep}_{q^{-1}}^{\mathrm{sml,grd}}(G)_c \to \mathrm{Rep}_q^{\mathrm{sml.grd}}(G)_c$$

obtained by restriction.

8.3. **Cohomological duality for the mixed category.**

8.3.1. In the context of Sect. 8.2.3, let us take $A = U_q^{\mathrm{Lus}}(N)$. Note that by construction, we can identify

$$\left(U_q^{\mathrm{Lus}}(N)\right)^{\mathrm{rev\text{-}mult}} \simeq U_{q^{-1}}^{\mathrm{Lus}}(N)$$

as Hopf algebras in $\mathrm{Rep}_{q^{-1}}(T)$.

From here we obtain an equivalence

$$((\mathrm{Rep}_q^{\mathrm{mxd}}(G))^{\mathrm{rev}} \simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G), \quad \mathcal{M} \mapsto \mathcal{M}^\sigma,$$

which induces an equivalence

$$(\mathrm{Rep}_q^{\mathrm{mxd}}(G))^\vee \simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G), \tag{8.14}$$

with the pairing

$$\mathrm{Rep}_q^{\mathrm{mxd}}(G) \otimes \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G) \to \mathrm{Vect} \tag{8.15}$$

given by ind-extending

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(k, \mathcal{M}_1 \otimes \mathcal{M}_2^\sigma), \quad \mathcal{M}_1 \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)_c,\ \mathcal{M}_2 \in \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)_c. \tag{8.16}$$

Note, however, that formula (8.16) with *any* $\mathcal{M}_1 \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)$ and $\mathcal{M}_2 \in \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)$ defines the pairing (8.15) because $1 \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)$ is compact.

Let $\mathbb{D}^{\mathrm{can}}$ denote corresponding contravariant dualization functor

$$\mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)_c \to \mathrm{Rep}_q^{\mathrm{mxd}}(G)_c$$

8.3.2. Note that the resulting identifications

$$(\mathrm{Rep}_q(G)_{\mathrm{ren}})^\vee \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}$$

and

$$(\mathrm{Rep}_q^{\mathrm{mxd}}(G))^\vee \simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)$$

are compatible in the following way:

Proposition 8.3.3. *The following diagram, in which the vertical arrows are contravariant functors, commutes:*

$$\begin{array}{ccc} \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{fin.dim}} & \xrightarrow{\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} & (\mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G))_c \\ {\scriptstyle \mathbb{D}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}^{\mathrm{can}}} \\ (\mathrm{Rep}_q(G)_{\mathrm{ren}})_{\mathrm{fin.dim}} & \xrightarrow{\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} & (\mathrm{Rep}_q^{\mathrm{mxd}}(G))_c \end{array}$$

commutes.

Proof. We need to show that for $\mathcal{M}_1 \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)_c$ and $\mathcal{M}_2 \in \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{fin.dim}}$, we have a canonical isomorphism

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}(\mathbb{D}(\mathcal{M}_2)), \mathcal{M}_1) \simeq \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(k, \mathcal{M}_1 \otimes (\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}(\mathcal{M}_2))^\sigma).$$

This follows from the commutativity of the next diagram, which in turn follows from the construction

$$\begin{array}{ccc} \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{fin.dim}} & \xrightarrow{\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} & \mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G) \\ \sigma\downarrow & & \downarrow\sigma \\ (\mathrm{Rep}_q(G)_{\mathrm{fin.dim}})^{\mathrm{rev}} & \xrightarrow{\mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} & (\mathrm{Rep}^{\mathrm{mxd}}_q(G)))^{\mathrm{rev}}. \end{array}$$

□

8.3.4. The goal of this section is to prove the following result:

Theorem 8.3.5. *For $\check\lambda \in \check\Lambda$, we have*

$$\mathbb{D}^{\mathrm{can}}(\mathbb{M}^{\check\lambda}_{q^{-1},\mathrm{mxd}}) \simeq \mathbb{M}^{-\check\lambda-2\check\rho}_{q,\mathrm{mxd}}[d].$$

The proof of Theorem 8.3.5 will use a tool which will be of independent interest: the long intertwining functor.

Remark 8.3.6. The definition of the pairing (8.16) and Theorem 8.3.5 are direct quantum analogs of the corresponding assertions in the classical situation: we have the self-duality of $\mathfrak{g}\text{-mod}^B$ defined by the formula

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{H}om_{\mathfrak{g}\text{-mod}^B}(k, \mathcal{M}_1 \otimes \mathcal{M}_2),$$

and the corresponding contravraiant dualization functor

$$\mathbb{D}^{\mathrm{can}} : \mathfrak{g}\text{-mod}^B_c \to \mathfrak{g}\text{-mod}^B_c$$

is known to send the Verma module $M^{\check\lambda}$ to $M^{-\check\lambda-2\rho}[d]$.

8.4. **The long intertwining functor.**

8.4.1. Let us consider the following "opposite" version of the category $\mathrm{Rep}^{\mathrm{mxd}}_q(G)$.

Namely, set:

$$\mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G) := Z_{\mathrm{Dr},\mathrm{Rep}_q(T)}(U^{\mathrm{DK}}_q(N^-)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}}).$$

equipped with pairs of adjoint functors

$$\mathbf{ind}_{\mathrm{DK}^-\to\widetilde{\mathrm{mxd}}} : U^{\mathrm{DK}}_q(N^-)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \rightleftarrows \mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G) : \mathbf{oblv}_{\widetilde{\mathrm{mxd}}\to\mathrm{DK}^-}$$

and

$$\mathbf{oblv}_{\widetilde{\mathrm{mxd}}\to\mathrm{Lus}^+} : \mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G) \rightleftarrows U^{\mathrm{DK}}_q(N^-)\text{-mod}(\mathrm{Rep}_q(T)) : \mathbf{coind}_{\mathrm{Lus}^+\to\widetilde{\mathrm{mxd}}}.$$

For $\check\lambda \in \check\Lambda$ we will denote by

$$\mathbb{M}^{\check\lambda}_{q,\widetilde{\mathrm{mxd}}}, \mathbb{M}^{\vee,\check\lambda}_{q,\widetilde{\mathrm{mxd}}} \in \mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G)$$

the corresponding standard and costandard objects, respectively.

The category $\mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G)$ carries a t-structure and can be recovered from its heart by the procedure of Sect. 4.5.6.

Remark 8.4.2. Note that when q is not a root of unity, the action of $w_0 \in \mathrm{Aut}(\check\Lambda)$ defines an equivalence

$$\mathrm{Rep}^{\mathrm{mxd}}_q(G) \to \mathrm{Rep}^{\widetilde{\mathrm{mxd}}}_q(G).$$

However, for q a root of unity, the two versions are truly different: in the former the Lusztig algebra is supposed to act locally nilpotently, and in the latter the De Concini-Kac one.

8.4.3. Let $U_q(G)^{\mathrm{mxd}}$-mod be the derived category of the abelian category $(U_q(G)^{\mathrm{mxd}}\text{-mod})^{\heartsuit}$, where the latter consists of objects of $\mathrm{Rep}_q(T)$, equipped with an action of $U_q^{\mathrm{Lus}}(N)$ and a compatible action of $U_q^{\mathrm{DK}}(N^-)$. We have the natural forgetful functors

$$U_q^{\mathrm{Lus}}(N)\text{-mod}(\mathrm{Rep}_q(T)) \overset{\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{Lus}^+}}{\longleftarrow} U_q(G)^{\mathrm{mxd}}\text{-mod} \overset{\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-}}{\longrightarrow} U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T)),$$

which admit left adjoints, denoted $\mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{mxd}}$ and $\mathbf{ind}_{\mathrm{DK}^-\to\mathrm{mxd}}$, respectively.

We have the natural forgetful functors

$$\mathrm{Rep}_q^{\mathrm{mxd}}(G) \overset{j^+}{\longrightarrow} U_q(G)^{\mathrm{mxd}}\text{-mod} \overset{j^-}{\longleftarrow} \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)$$

that make the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Rep}_q^{\mathrm{mxd}}(G) & \longrightarrow & U_q(G)^{\mathrm{mxd}}\text{-mod} \\ \uparrow\downarrow & & \uparrow\downarrow \\ U_q^{\mathrm{Lus}}(N)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} & \longrightarrow & U_q^{\mathrm{Lus}}(N)\text{-mod}(\mathrm{Rep}_q(T)) \end{array}$$

and

$$\begin{array}{ccc} U_q(G)^{\mathrm{mxd}}\text{-mod} & \longleftarrow & \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G) \\ \uparrow\downarrow & & \uparrow\downarrow \\ U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T)) & \longleftarrow & U_q^{\mathrm{DK}}(N^-)\text{-mod}(\mathrm{Rep}_q(T))_{\mathrm{loc.nilp}} \end{array}$$

8.4.4. Note that it follows from Corollary 5.2.8 that the functor j^- sends compacts to compacts. Let $(j^-)^R$ denote its right adjoint.

We define the functor

$$\Upsilon : \mathrm{Rep}_q^{\mathrm{mxd}}(G) \to \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)$$

as

$$\Upsilon := (j^-)^R \circ j^+.$$

Remark 8.4.5. We will refer to Υ as the *long intertwining functor*. It is a direct analog of the functor in the classical situation given by

$$\mathrm{Av}_*^{N^-} : \mathfrak{g}\text{-mod}^B \to \mathfrak{g}\text{-mod}^{B^-}.$$

The analog of Proposition 8.4.7 below in the classical situation holds as well: the proof given below applies. Alternatively, one can deduce it from the Beilinson-Bernstein localization theory.

8.4.6. We claim:

Proposition 8.4.7. *For* $\check{\lambda} \in \check{\Lambda}$,

$$\Upsilon(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}) \simeq \mathbb{M}^{\vee,\check{\lambda}+2\check{\rho}}_{q,\widetilde{\mathrm{mxd}}}[-d].$$

Proof. It suffices to show that

$$\mathcal{H}om_{\mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)}(\mathbb{M}^{\check{\lambda}'}_{q,\widetilde{\mathrm{mxd}}}, \Upsilon(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})) \simeq \begin{cases} k[-d] \text{ if } \check{\lambda}' = \check{\lambda} + 2\check{\rho} \\ 0 \text{ otherwise.} \end{cases}$$

By definition,

$$\mathcal{H}om_{\mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)}(\mathbb{M}^{\check{\lambda}'}_{q,\widetilde{\mathrm{mxd}}}, \Upsilon(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})) \simeq \mathcal{H}om_{U_q(G)^{\mathrm{mxd}}\text{-mod}}(j^-(\mathbb{M}^{\check{\lambda}'}_{q,\widetilde{\mathrm{mxd}}}), j^+(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})),$$

which we further rewrite as

$$\mathcal{H}om_{U_q^{\mathrm{DK}}(N^-)\text{-mod}}(k^{\check{\lambda}'}, \mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})).$$

Now, since $\mathbf{oblv}_{\mathrm{mxd}\to\mathrm{DK}^-}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})$ is free over $U_q^{\mathrm{DK}}(N^-)$, the assertion follows from Corollary 7.4.4.

□

8.5. **Cohomological vs contragredient duality.**

8.5.1. Let
$$\mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{loc.fin.dim}} \subset \mathrm{Rep}_q^{\mathrm{mxd}}(G) \text{ and } \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)_{\mathrm{loc.fin.dim}} \subset \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)$$
be the full subcategories corresponding to the condition that in each cohomological degree each graded component is finite-dimensional. Note that the standard and costandard objects belong to the corresponding subcategories; hence so do all compact objects.

Component-wise dualization, combined with the identifications
$$U_q^{\mathrm{Lus}}(N) \simeq (U_q^{\mathrm{DK}}(N^-))^\vee \text{ and } U_q^{\mathrm{DK}}(N) \simeq (U_q^{\mathrm{Lus}}(N^-))^\vee$$
defines a contravariant equivalence
$$\mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{loc.fin.dim}} \to \mathrm{Rep}_{q^{-1}}^{\widetilde{\mathrm{mxd}}}(G)_{\mathrm{loc.fin.dim}}$$
that we will refer to as *contragredient duality* and denote by $\mathbb{D}^{\mathrm{contr}}$.

By construction, we have
$$\mathbb{D}^{\mathrm{contr}}(\mathbb{M}^{\check\lambda}_{q^{-1},\mathrm{mxd}}) \simeq \mathbb{M}^{\vee,-\check\lambda}_{q,\widetilde{\mathrm{mxd}}} \text{ and } \mathbb{D}^{\mathrm{contr}}(\mathbb{M}^{\vee,\check\lambda}_{q^{-1},\mathrm{mxd}}) \simeq \mathbb{M}^{-\check\lambda}_{q,\widetilde{\mathrm{mxd}}}. \tag{8.17}$$

8.5.2. We claim:

Proposition 8.5.3. *There is a canonical isomorphism*
$$\mathbb{D}^{\mathrm{can}} \simeq (\mathbb{D}^{\mathrm{contr}})^{-1} \circ \Upsilon.$$

Remark 8.5.4. Proposition 8.5.3 is a direct quantum analog of a similar assertion in the classical situation, see [GY, Corollary 3.2.3].

Proof. First, we note that Υ sends $(\mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G))_c$ to $(\mathrm{Rep}_{q^{-1}}^{\widetilde{\mathrm{mxd}}}(G))_{\mathrm{loc.fin.dim}}$: indeed, it suffices to check this for the standard objects, and the result follows from Proposition 8.4.7.

To prove the proposition, it suffices to show that for $\mathcal{M}_1 \in \mathrm{Rep}_q^{\mathrm{mxd}}(G)_c$ and $\mathcal{M}_2 \in \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)_c$, we have a canonical identification
$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(k, \mathcal{M}_1 \otimes \mathcal{M}_2^\sigma) \simeq \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}((\mathbb{D}^{\mathrm{contr}})^{-1} \circ \Upsilon(\mathcal{M}_2), \mathcal{M}_1).$$

We rewrite the RHS as
$$\mathcal{H}om_{\mathrm{Rep}_{q^{-1}}^{\widetilde{\mathrm{mxd}}}(G)}(\mathbb{D}^{\mathrm{contr}}(\mathcal{M}_1), \Upsilon(\mathcal{M}_2)),$$
and further as
$$\mathcal{H}om_{U_{q^{-1}}(G)^{\mathrm{mxd}}\text{-mod}}(j_-(\mathbb{D}^{\mathrm{contr}}(\mathcal{M}_1)), j_+(\mathcal{M}_2)).$$

How, the assertion follows from the (tautological) identification
$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}(k, \mathcal{M}_1 \otimes \mathcal{M}_2^\sigma) \simeq \mathcal{H}om_{U_{q^{-1}}(G)^{\mathrm{mxd}}\text{-mod}}(j_-(\mathbb{D}^{\mathrm{contr}}(\mathcal{M}_1)), j_+(\mathcal{M}_2)).$$

□

8.5.5. From Proposition 8.5.3 we obtain:

Corollary 8.5.6. *There are canonical isomorphisms of functors*
$$\Upsilon \simeq \mathbb{D}^{\mathrm{contr}} \circ \mathbb{D}^{\mathrm{can}} : (\mathrm{Rep}_q^{\mathrm{mxd}}(G))_c \to \mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G)_{\mathrm{loc.fin.dim}}$$
and
$$\mathbb{D}^{\mathrm{contr}} \simeq \Upsilon \circ \mathbb{D}^{\mathrm{can}} : (\mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G))_c \to (\mathrm{Rep}_q^{\widetilde{\mathrm{mxd}}}(G))_{\mathrm{loc.fin.dim}}.$$

8.5.7. *Proof of Theorem 8.3.5.* By Proposition 8.5.3, we have

$$\mathbb{D}^{\mathrm{can}}(\mathbb{M}^{\check{\lambda}}_{q^{-1},\mathrm{mxd}}) \simeq (\mathbb{D}^{\mathrm{contr}})^{-1} \circ \Upsilon(\mathbb{M}^{\check{\lambda}}_{q^{-1},\mathrm{mxd}}),$$

while the RHS identifies, according to Proposition 8.4.7 with

$$(\mathbb{D}^{\mathrm{contr}})^{-1}(\mathbb{M}^{\vee,\check{\lambda}+2\check{\rho}}_{q^{-1},\widetilde{\mathrm{mxd}}}[-d]),$$

and the latter identifies, according to (8.17), with

$$\mathbb{M}^{-\check{\lambda}-2\check{\rho}}_{q,\mathrm{mxd}}[d],$$

as required.

□

Part III: Kac-Moody vs quantum group representations

9. A conjectural extension of the Kazhdan-Lusztig equivalence

In this section we take our field of coefficients k to be $\mathbb{C}$, and we let b and κ be related by the formula (0.7).

In this section we will formulate the main conjecture of this paper (Conjecture 9.2.2), which compares the categories $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$ and $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$.

Technical remark: in order to unburden the notation, from this section on we will choose a uniformizer $t \in \mathcal{O}$, thereby trivializing the line ω_x (see Sect. 2.5.3).

9.1. **The Kazhdan-Lusztig equivalence.** In this subsection we take $-\kappa$ to be a negative integral level (although whatever we will say goes through for rational levels, see Remark 2.5.2).

We will recall the statement of the Kazhdan-Lusztig equivalence of [KL].

9.1.1. Consider the category

$$\mathrm{KL}(G,-\kappa) := \widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa}.$$

We recall from Sect. 1.1.2 that the category $\mathrm{KL}(G,-\kappa)$ is *by definition* compactly generated by objects of the form

$$\mathrm{Ind}^{\widehat{\mathfrak{g}}_{-\kappa}}_{\mathfrak{g}(\mathcal{O})}(V), \tag{9.1}$$

where V is a *finite-dimensional* representation of $G(\mathcal{O})$.

The category $\mathrm{KL}(G,-\kappa)$ carries a t-structure for which the forgetful functor $\mathrm{KL}(G,-\kappa) \to \mathrm{Vect}$ is t-exact. Note, however, that it is *not true* that $\mathrm{KL}(G,-\kappa)$ is left-separated complete in its t-structure.

Remark 9.1.2. Let us emphasize again the relationship between $\mathrm{KL}(G,-\kappa)$ and the abelian category $\mathrm{KL}(G,-\kappa)^\heartsuit$:

The former is the ind-completion of the full subcategory of $D^b(\mathrm{KL}(G,-\kappa)^\heartsuit)$ generated under finite colimits by objects (9.1).

Note that this relationship is the same as that between $\mathrm{Rep}_q(G)_{\mathrm{ren}}$ and the abelian category $(\mathrm{Rep}_q(G))^\heartsuit$.

9.1.3. For $\check{\lambda} \in \check{\Lambda}^+$ recall the notation for the Weyl modules

$$\mathbb{V}^{\check{\lambda}}_{-\kappa} := \mathrm{Ind}^{\widehat{\mathfrak{g}}_{-\kappa}}_{\mathfrak{g}(\mathcal{O})}(V^{\check{\lambda}}) \in \mathrm{KL}(G,-\kappa),$$

see (1.3). The objects $\mathbb{V}^{\check{\lambda}}_{-\kappa}$ are compact and they generate $\mathrm{KL}(G,-\kappa)$.

9.1.4. Let $\text{D-mod}_{-\kappa}(\text{Gr}_G)^{G(\mathcal{O})}$ denote the monoidal category of spherical D-modules on the affine Grassmannian Gr_G. We have a monoidal action of $\text{D-mod}_{-\kappa}(\text{Gr}_G)^{G(\mathcal{O})}$ on $\text{KL}(G, -\kappa)$,

$$\mathcal{F}, \mathcal{M} \mapsto \mathcal{F} \star \mathcal{M}.$$

A basic piece of structure that we need is the (classical) Geometric Satake is a monoidal functor

$$\text{Sat} : \text{Rep}(\check{G}) \to \text{D-mod}_{-\kappa}(\text{Gr}_G)^{G(\mathcal{O})}.$$

We normalize it so that for $\mu \in \Lambda^+$, we have a canonical map

$$\mathbb{V}^{\mu}_{-\kappa} \to \text{Sat}(V^{\mu}) \star \mathbb{V}^{0}_{-\kappa}.$$

Here V^{μ} is the irreducible object of $\text{Rep}(\check{G})^{\heartsuit}$ of highest weight μ; when we write $\mathbb{V}^{\mu}_{-\kappa}$, we consider μ as an element of $\check{\Lambda}$ via $\kappa(\lambda, -)$.

For a pair of elements $\mu \in \Lambda^+$ and $\check{\lambda}^+ \in \check{\Lambda}^+$ we have a canonical map

$$\mathbb{V}^{\mu+\check{\lambda}}_{-\kappa} \to \text{Sat}(V^{\mu}) \star \mathbb{V}^{\check{\lambda}}_{-\kappa}. \tag{9.2}$$

9.1.5. Let q be the $\mathbb{C}^*$-valued quadratic form on $\check{\Lambda}$ corresponding to κ. Note that the assumption that κ is integral implies that the lattice Λ_H (see Sect. 6.3.1) identifies with Λ, and the group H identifies with the Langands dual $\check{G}$ of G.

The theorem of Kazhdan and Lusztig of [KL] states the existence of an equivalence of abelian categories

$$\mathsf{F}_{-\kappa} : (\text{KL}(G, -\kappa)^{\heartsuit} \simeq \text{Rep}_q(G)^{\heartsuit}, \tag{9.3}$$

with the property that for $\check{\lambda} \in \check{\Lambda}^+$ there exists a canonical isomorphism

$$\mathsf{F}_{-\kappa}(\mathbb{V}^{\check{\lambda}}_{-\kappa}) \simeq \mathcal{V}^{\check{\lambda}}_q. \tag{9.4}$$

In what follows we will use two more properties of the equivalence (9.3), which is not stated in the paper, but which can be deduced from it:

- The action of $\text{Rep}(\check{G})^{\heartsuit}$ on $\text{KL}(G, -\kappa)^{\heartsuit}$ corresponds under $\mathsf{F}_{-\kappa}$ to the action of $\text{Rep}(\check{G})^{\heartsuit}$ on $\text{Rep}_q(G)^{\heartsuit}$ via pullback by quantum Frobenius.
- Under the above identification, the image under $\mathsf{F}_{-\kappa}$ of the map (9.2) is the canonical map

$$\mathcal{V}^{\mu+\check{\lambda}}_q \to \text{Frob}^*_q(V^{\mu}) \star \mathcal{V}^{\check{\lambda}}_q. \tag{9.5}$$

9.1.6. From (9.3), by passing to the bounded derived categories and ind-completing the corresponding full subcategories, we obtain an equivalence of DG categories:

$$\mathsf{F}_{-\kappa} : \text{KL}(G, -\kappa) \simeq \text{Rep}_q(G)_{\text{ren}}. \tag{9.6}$$

The compatibility of the equivalence (9.3) with the action of $\text{Rep}(\check{G})^{\heartsuit}$ at the abelian level implies the compatibility of the equivalence (9.6) with the action of $\text{Rep}(\check{G})$.

Remark 9.1.7. The equivalence (9.3) satisfying (9.4) for irrational levels as well (but in this case we have neither the Hecke action nor quantum Frobenius).

Note, however, that such an equivalence taken "as-is" (i.e., without taking into account the braided monoidal structures) is not very interesting as both categories are semi-simple.

9.2. **Conjectural extension to the Iwahori case.**

9.2.1. We now consider the category $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$. We propose the following extension of the Kazhdan-Lusztig equivalence:

Conjecture 9.2.2. *There exists an equivalence*

$$\mathsf{F}_{-\kappa} : \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} \simeq \operatorname{Rep}_q^{\mathrm{mxd}}(G) \tag{9.7}$$

with the following properties:

(i) *The square*

$$\begin{array}{ccc} \mathrm{KL}(G,-\kappa) := \widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q(G)_{\mathrm{ren}} \\ \Big\downarrow {\scriptstyle \mathbf{oblv}_{G(\mathcal{O})/I}} & & \Big\downarrow {\scriptstyle \mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q^{\mathrm{mxd}}(G) \end{array} \tag{9.8}$$

commutes.

(ii) *For $\mu \in \Lambda$, the action of J_μ on $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ corresponds to the action of $k^\mu \in \operatorname{Rep}(\check{T})$ on $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$.*

(iii) *For $\check{\lambda} \in \check{\Lambda}$, we have*

$$\mathsf{F}_{-\kappa}(\mathbb{W}^{\check{\lambda}}_{-\kappa}) \simeq \mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}.$$

There is one more expected property of the functor (9.7) that has to do with the structure on both sides of categories over the stack $\check{\mathfrak{n}}/\operatorname{Ad}(\check{B})$; we will discuss it in Sect. 11.2.1.

Remark 9.2.3. Conjecture 9.2.2, satisfying (i) and (iii) applies also for κ irrational. However, in this case it reduces to the known statement that both categories identify with the finite-dimensional $\mathfrak{g}\text{-mod}^B$ so that

$$\mathbb{M}^{\check{\lambda}}_{-\kappa} \leftrightarrow M^\lambda \leftrightarrow \mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}.$$

Here we are using Proposition 3.2.2, which says that for κ irrational $\mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq \mathbb{M}^{\check{\lambda}}_{-\kappa}$.

9.2.4. We will now run some consistency checks on Conjecture 9.2.2.

From the commutativity of the square (9.8), we deduce the commutativity of the following two squares:

$$\begin{array}{ccc} \mathrm{KL}(G,-\kappa) := \widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q(G)_{\mathrm{ren}} \\ \Big\uparrow {\scriptstyle \mathrm{Av}_!^{G(\mathcal{O})/I}} & & \Big\uparrow {\scriptstyle \mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q^{\mathrm{mxd}}(G) \end{array} \tag{9.9}$$

and

$$\begin{array}{ccc} \mathrm{KL}(G,-\kappa) := \widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q(G)_{\mathrm{ren}} \\ \Big\uparrow {\scriptstyle \mathrm{Av}_*^{G(\mathcal{O})/I}} & & \Big\uparrow {\scriptstyle \mathbf{coind}_{\mathrm{mxd}\to\mathrm{big}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q^{\mathrm{mxd}}(G), \end{array}$$

where the vertical arrows are obtained by passing to left and right adjoints in (9.8), respectively. Note, however, that since $G(\mathcal{O})/I \simeq G/B$ is proper, we have a canonical isomorphism

$$\mathrm{Av}_*^{G(\mathcal{O})/I} \simeq \mathrm{Av}_!^{G(\mathcal{O})/I}[-2d],$$

see Sect. 1.1.6.

We note that this is consistent with Conjecture 7.3.7, which says that

$$\mathbf{coind}_{\mathrm{mxd}\to\mathrm{big}} \simeq \mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}}[-2d].$$

9.2.5. Points (ii) and (iii) of Conjecture 9.2.2 are compatible due to the relations

$$J_\mu \star \mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq \mathbb{W}^{\mu+\check{\lambda}}_{-\kappa} \text{ and } k^\mu \otimes \mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}} \simeq \mathbb{M}^{\mu+\check{\lambda}}_{q,\mathrm{mxd}}.$$

(Note that for the first isomorphism we use the trivialization of ω_x.)

9.2.6. Let us now investigate the compatibility between points (i) and (iii) of Conjecture 9.2.2. Namely, evaluating both circuits of the diagram (9.9) on $\mathbb{W}^{\check{\lambda}}_{-\kappa}$ we obtain an isomorphism:

$$F_{-\kappa}(\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\check{\lambda}}_{-\kappa})) \simeq \mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}),$$

where $F_{-\kappa}$ is the original Kazhdan-Lusztig equivalence.

Recall that according to (5.6), we have

$$\mathbf{ind}_{\mathrm{mxd}\to\mathrm{big}}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}}) \simeq \mathcal{V}^{\check{\lambda}}_q.$$

Recall also that if $\check{\lambda} \in \check{\Lambda}^+$, then $\mathbb{W}^{\check{\lambda}}_{-\kappa} \simeq \mathbb{M}^{\check{\lambda}}_{-\kappa}$ (see Corollary 3.4.3), and since

$$\mathrm{Av}_!^{G/B}(M^{\check{\lambda}}) = V^{\check{\lambda}} \in \mathrm{Rep}(G),$$

we obtain $\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{M}^{\check{\lambda}}_{-\kappa}) \simeq \mathbb{V}^{\check{\lambda}}_{-\kappa}$, and hence

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\check{\lambda}}_{-\kappa}) \simeq \mathbb{V}^{\check{\lambda}}_{-\kappa}, \tag{9.10}$$

establishing the desired consistency.

9.2.7. We take (9.10) as the *definition* of $\mathbb{V}^{\check{\lambda}}_{-\kappa}$ for $\check{\lambda} \in \check{\Lambda}$ that is not necessarily dominant.

To summarize, we obtain the following isomorphism that references only the *original Kazhdan-Lusztig functor* $\mathsf{F}_{-\kappa}$ of (9.6)

$$F_{-\kappa}(\mathbb{V}^{\check{\lambda}}_{-\kappa}) \simeq \mathcal{V}^{\check{\lambda}}_q. \tag{9.11}$$

Ihe isomorphism (9.11) is a basic property of $\mathsf{F}_{-\kappa}$ for $\check{\lambda}$ dominant, and it follows from Conjecture 9.2.2 for all $\check{\lambda}$. However, in Sect. 10.6.5 we will prove:

Theorem 9.2.8. *The isomorphism* (9.11) *holds (unconditionally) for all* $\check{\lambda} \in \check{\Lambda}$.

9.3. Digression: dual Weyl modules.

9.3.1. For $\check{\lambda} \in \check{\Lambda}^+$ we introduce the dual Weyl module $\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa} \in \mathrm{KL}(G,-\kappa)$ by requiring

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa}}(\mathbb{V}^{\check{\lambda}'}_{-\kappa}, \mathbb{V}^{\vee,\check{\lambda}}_{-\kappa}) = \begin{cases} k \text{ for } \check{\lambda}' = \check{\lambda}, \\ 0 \text{ for } \check{\lambda}' \neq \check{\lambda},\ \check{\lambda}' \in \check{\Lambda}^+. \end{cases}$$

We have the following assertion:

Lemma 9.3.2. *For* $\check{\lambda} \in \check{\Lambda}^+$ *there exists a canonical isomorphism*

$$\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa} \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}).$$

Proof. We have:

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O})}_{-\kappa}}(\mathbb{V}^{\check{\lambda}'}_{-\kappa}, \mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{M}^{\vee,\check{\lambda}}_{-\kappa})) \simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{I}_{-\kappa}}(\mathbb{V}^{\check{\lambda}'}_{-\kappa}, \mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}).$$

The BGG resolution of $V^{\check{\lambda}}$ with terms $M^{w(\check{\lambda}'+\check{\rho})-\check{\rho}}$ implies that $\mathbb{V}^{\check{\lambda}'}_{-\kappa}$ admits a resolution with terms of the form $\mathbb{M}^{w(\check{\lambda}'+\check{\rho})-\check{\rho}}_{-\kappa}$. Note that for all $w \neq 1$, the weight $w(\check{\lambda}'+\check{\rho})-\check{\rho}$ is non-dominant, and hence not equal to $\check{\lambda}$. Hence, we obtain that

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{I}_{-\kappa}}(\mathbb{V}^{\check{\lambda}'}_{-\kappa}, \mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}) \simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{I}_{-\kappa}}(\mathbb{M}^{\check{\lambda}'}_{-\kappa}, \mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}),$$

and the assertion follows.

□

9.3.3. We now define

$$\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa} := \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{w_0(\check{\lambda})-2\check{\rho}}_{-\kappa})[-d] \tag{9.12}$$

for *any* $\check{\lambda} \in \check{\Lambda}$.

We claim:

Lemma 9.3.4. *For $\check{\lambda} \in \check{\Lambda}^+$, the definition of $\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa}$ agrees with the initial one.*

Proof. First, we claim that for $\check{\lambda}$ dominant, we have

$$j_{w_0,*} \star \mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa} \simeq \mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}.$$

Indeed, it suffices to show that

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{M}^{\check{\lambda}'}_{-\kappa}, j_{w_0,*} \star \mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa}) \simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(j_{w_0,!} \star \mathbb{M}^{\check{\lambda}'}_{-\kappa}, \mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa}) = \begin{cases} k \text{ for } \check{\lambda}' = \check{\lambda}, \\ 0 \text{ otherwise.} \end{cases}$$

For this it suffices to show that the cone of the canonical map

$$j_{w_0,!} \star \mathbb{M}^{\check{\lambda}'}_{-\kappa} \to \mathbb{M}^{w_0(\check{\lambda}')-2\check{\rho}}_{-\kappa}$$

is an extension of Verma modules of highest weights different from $w_0(\check{\lambda}) - 2\check{\rho}$. For the latter, it suffices to show the corresponding assertion for the map for the finite-dimensional $\mathfrak{g}$:

$$j_{w_0,!} \star M^{\check{\lambda}'} \to M^{w_0(\check{\lambda}')-2\check{\rho}},$$

which in turn follows from the (valid) isomorphism

$$j_{w_0,*} \star M^{\vee,w_0(\check{\lambda})-2\check{\rho}} \simeq M^{\vee,\check{\lambda}}, \quad \check{\lambda} \in \check{\Lambda}^+.$$

Now, using Corollary 3.6.6, we have

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa})[-d] \simeq \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa})[-d] \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa})[d] \simeq$$

$$\simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0,*} \star \mathbb{M}^{\vee,w_0(\check{\lambda})-2\check{\rho}}_{-\kappa}) \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{M}^{\vee,\check{\lambda}}_{-\kappa}) \simeq \mathbb{V}^{\vee,\check{\lambda}}_{-\kappa},$$

as required.

□

9.3.5. Note that taking into account Corollary 7.4.11, we obtain that Theorem 9.2.8 is equivalent to the following statement:

Corollary 9.3.6. *For any $\check{\lambda} \in \check{\Lambda}$, we have $F_{-\kappa}(\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa}) \simeq \mathcal{V}^{\vee,\check{\lambda}}_q$.*

Finally, note that for $\check{\lambda} \in \check{\Lambda}^+$, the assertion of Corollary 9.3.6 follows formally from Lemma 9.3.4: indeed, it is clear that the original Kazhdan-Lusztig functor satisfies:

$$\mathsf{F}_{-\kappa}(\mathbb{V}^{\vee,\check{\lambda}}_{-\kappa}) \simeq \mathcal{V}^{\vee,\check{\lambda}}_q.$$

9.4. **Digression: Kazhdan-Lusztig equivalence for positive level.** Let κ be the positive level, opposite of the negative level $-\kappa$. In this subsection we will discuss some consequences that the original Kazhdan-Lusztig equivalence has for the category $\mathrm{KL}(G, \kappa)$.

9.4.1. Recall that we have the duality identifications

$$\mathrm{KL}(G, \kappa) \simeq (\mathrm{KL}(G, -\kappa))^\vee \text{ and } \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}} \simeq (\mathrm{Rep}_q(G)_{\mathrm{ren}})^\vee,$$

the former given by Proposition 1.3.4 and the latter by (8.8).

Thus, starting from the equivalence

$$\mathsf{F}_{-\kappa} : \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}_q(G)_{\mathrm{ren}}$$

by duality, we obtain an equivalence

$$\mathsf{F}_{\kappa} : \mathrm{KL}(G, \kappa) \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}. \tag{9.13}$$

9.4.2. Note that the compatibility of the equivalence F_κ of (9.13) with the Hecke action reads as follows:

$$\mathsf{F}_\kappa(\mathrm{Sat}(V)\star\mathcal{M})\simeq \mathrm{Frob}_q^*(V^\tau)\otimes M,$$

where τ is the Cartan involution on $\check{G}$.

This is due to the fact that Verdier duality

$$\mathbb{D}:(\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})})_c\to(\text{D-mod}_{\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})})_c$$

satisfies

$$\mathbb{D}(\mathrm{Sat}(V))\simeq \mathrm{Sat}((V^\tau)^\vee). \tag{9.14}$$

9.4.3. Note that the equivalence $\mathsf{F}_\kappa:\mathrm{KL}(G,\kappa)\simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}$ satisfies:

$$\mathsf{F}_\kappa(\mathbb{V}^\lambda_\kappa)\simeq \mathcal{V}^{\vee,\check\lambda}_{q^{-1}},\quad \check\lambda\in\check\Lambda^+.$$

This follows from the identifications

$$\mathbb{D}(\mathbb{V}^{\check\lambda}_{-\kappa})\simeq\mathbb{V}^{-w_0(\check\lambda)}_\kappa \text{ and } \mathbb{D}(\mathcal{V}_q^{-w_0(\check\lambda)})\simeq\mathcal{V}^{\vee,\check\lambda}_{q^{-1}}.$$

9.4.4. For any $\check\lambda\in\check\Lambda$, set

$$\mathbb{V}^\lambda_\kappa:=\mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{D}(\mathbb{W}^{-w_0(\check\lambda)}_{-\kappa})).$$

Note that for $\check\lambda\in\check\Lambda^+$, this is consistent with the definition of $\mathbb{V}^{\check\lambda}_\kappa$ as $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}(\mathcal{O})}(V^{\check\lambda})$. Indeed,

$$\mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{D}(\mathbb{W}^{-w_0(\check\lambda)}_{-\kappa}))\simeq\mathbb{D}\left(\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{-w_0(\check\lambda)}_{-\kappa})\right)\simeq\mathbb{D}(\mathbb{V}^{-w_0(\check\lambda)}_{-\kappa})\simeq\mathbb{V}^{\check\lambda}_\kappa.$$

Note that Theorem 9.2.8 is equivalent to the following:

Theorem 9.4.5. *For any $\check\lambda\in\check\Lambda$, we have $\mathsf{F}_k(\mathbb{V}^{\check\lambda}_\kappa)\simeq\mathcal{V}^{\vee,\check\lambda}_{q^{-1}}$.*

Me emphasize again that the assertion of Theorem 9.4.5 follows from the usual Kazhdan-Lusztig equivalence for $\check\lambda\in\check\Lambda^+$.

9.5. **Extension to the Iwahori case for positive level.** In this subsection we will assume Conjecture 9.2.2 and deduce some consequences for the category $\widehat{\mathfrak{g}}\text{-mod}^I_\kappa$.

9.5.1. Recall now that we have the equivalences

$$\widehat{\mathfrak{g}}\text{-mod}^I_\kappa\simeq(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa})^\vee \text{ and } \mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G)\simeq(\mathrm{Rep}^{\mathrm{mxd}}_q(G))^\vee,$$

see Proposition 1.3.4 for the former and Theorem 8.3.5 for the latter.

Thus, starting from the conjectural equivalence

$$\mathsf{F}_{-\kappa}:\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}\simeq\mathrm{Rep}^{\mathrm{mxd}}_q(G),$$

by duality we obtain an equivalence

$$\mathsf{F}_\kappa:\widehat{\mathfrak{g}}\text{-mod}^I_\kappa\simeq\mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G).$$

9.5.2. Let us explore the properties of this equivalence that follow from the properties of (9.7).

First, the diagram

$$\begin{array}{ccc} \mathrm{KL}(G,\kappa) & \xrightarrow{\mathsf{F}_\kappa} & \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}} \\ {\scriptstyle \mathbf{oblv}_{G(\mathcal{O})/I}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{big}\to\mathrm{mxd}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_\kappa & \xrightarrow{\mathsf{F}_\kappa} & \mathrm{Rep}^{\mathrm{mxd}}_{q^{-1}}(G). \end{array} \tag{9.15}$$

is commutative. This follows from the commutativity of (9.8) by duality, see Sect. 1.3.8 and Proposition 8.3.3.

9.5.3. Let us define the object

$$J^{\mathbb{D}}_\mu \in \text{D-mod}_\kappa(\text{Fl}^{\text{aff}}_G)^I$$

as the Verdier dual of J_μ. For example, for μ dominant, $J^{\mathbb{D}}_\mu \simeq j_{\mu,*}$ and $J^{\mathbb{D}}_{-\mu} = j_{-\mu,!}$.

Then the functor F_κ intertwines the convolution action of $J^{\mathbb{D}}_\mu$ on $\widehat{\mathfrak{g}}\text{-mod}^I_\kappa$ with the functor $k^{-\mu} \otimes -$ on $\text{Rep}^{\text{mxd}}_{q^{-1}}(G)$.

9.5.4. Assuming the existence of $\mathsf{F}_{-\kappa}$ and hence that of F_κ, we obtain the following:

Proposition 9.5.5. *Under the equivalence*

$$\mathsf{F}_\kappa : \widehat{\mathfrak{g}}\text{-mod}^I_\kappa \simeq \text{Rep}^{\text{mxd}}_{q^{-1}}(G),$$

for $\check\lambda \in \check\Lambda$ *the functor*

$$\text{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check\lambda} : \widehat{\mathfrak{g}}\text{-mod}^I_\kappa \to \text{Vect}$$

corresponds to the functor

$$\text{C}^\cdot(U^{\text{Lus}}_{q^{-1}}(N), -)^{\check\lambda} : \text{Rep}^{\text{mxd}}_{q^{-1}}(G) \to \text{Vect}\,.$$

Proof. By Theorem 8.3.5, the functor

$$\text{C}^\cdot(U^{\text{Lus}}_{q^{-1}}(N), -)^{\check\lambda} : \text{Rep}^{\text{mxd}}_{q^{-1}}(G) \to \text{Vect}$$

is given by the pairing with $\mathbb{M}^{-\check\lambda-2\check\rho}_{q,\text{mxd}}[d]$, while by Corollary 2.4.4, the functor

$$\text{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check\lambda} : \widehat{\mathfrak{g}}\text{-mod}^I_\kappa \to \text{Vect}$$

is given by the pairing with $\mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[d]$. Now the assertion follows from the isomorphism

$$\mathsf{F}_{-\kappa}(\mathbb{W}^{\check\mu}_{-\kappa}) \simeq \mathbb{M}^{\check\mu}_{q,\text{mxd}}.$$

□

9.5.6. Bu juxtaposing Proposition 9.5.5 with the diagram (9.15), we obtain the following assertion, which, however, can be proved unconditionally:

Corollary 9.5.7. *Under the equivalence*

$$\mathsf{F}_\kappa : \text{KL}(G,\kappa) \simeq \text{Rep}_{q^{-1}}(G)_{\text{ren}},$$

for $\check\lambda \in \check\Lambda$ *the functor*

$$\text{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check\lambda} : \text{KL}(G,\kappa) \to \text{Vect}$$

corresponds to the functor

$$\text{C}^\cdot(U^{\text{Lus}}_{q^{-1}}(N), -)^{\check\lambda} : \text{Rep}_{q^{-1}}(G)_{\text{ren}} \to \text{Vect}\,.$$

Remark 9.5.8. Corollary 9.5.7 reproduces the result of [Liu, Theorem 5.3.1]. The proof that we will give is close in spirit to one in *loc.cit.*, but is *logically inequivalent* to it.

Proof. The functor

$$\text{C}^\cdot(U^{\text{Lus}}_{q^{-1}}(N), -)^{\check\lambda} : \text{Rep}_{q^{-1}}(G)_{\text{ren}} \to \text{Vect}$$

is given by $\mathcal{H}om_{\text{Rep}_{q^{-1}}(G)_{\text{ren}}}(\mathcal{V}^{\check\lambda}_{q^{-1}}, -)$, i.e., by the pairing with the object

$$(\mathcal{V}^{\check\lambda}_{q^{-1}})^\vee \simeq \mathcal{V}^{\vee,-w_0(\check\lambda)}_q.$$

The functor $\text{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check\lambda} : \widehat{\mathfrak{g}}\text{-mod}^I_\kappa \to \text{Vect}$ is given by the pairing with $\mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[d]$, and hence as a functor $\text{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)^{\check\lambda} : \text{KL}(G,\kappa) \to \text{Vect}$ by the pairing by the pairing with

$$\text{Av}^{G(\mathcal{O})/I}_*(\mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[d]) \simeq \text{Av}^{G(\mathcal{O})/I}_!(\mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[-d]) =: \mathbb{V}^{\vee,-w_0(\check\lambda)}_{-\kappa}.$$

Now the assertion follows from Corollary 9.3.6.

□

10. Cohomology of the small quantum group via Kac-Moody algebras

The goal of this section is to prove Theorem 9.2.8. The idea is to bootstrap the assertion for any $\check{\lambda}$ from the case when $\check{\lambda}$ is dominant. In the process of doing so we will need to describe the counterpart on the Kac-Moody side of *baby Verma modules* for quantum groups.

As a byproduct we will describe the functor on the Kazhdan-Lusztig category that corresponds to the functor of cohomology with respect to $u_q(N)$ for quantum groups.

10.1. **The Drinfeld-Plücker formalism.** The thrust of Sects. 10.1-10.3 is to give an expression of the baby Verma module $\mathbb{M}^{\check{\lambda}}_{q,\text{small}}$ in terms of restrictions of (dual) Weyl modules for the big quantum group (we need such a description in order to be able to transfer it to the Kazhdan-Lusztig category via the functors $\mathsf{F}_{-\kappa}$ or F_{κ}). The framework for doing so is provided by the Drinfeld-Plücker formalism[3].

In this subsection we recall some material from [Ga2, Sect. 6].

10.1.1. We start with the following observation.

Let $\mathcal{C}$ be a DG category equipped with an action of $\text{Rep}(\check{G})$. Let

$$\{c^\mu,\, \mu\in\Lambda\}\in\mathcal{C}$$

be a collection of objects equipped with a *Drinfeld-Plücker data*, i.e., a homotopy-coherent system of tensor-compatible maps

$$(V^{\mu_1})^\vee\star c^{\mu_2}\to c^{-\mu_1+\mu_2}. \tag{10.1}$$

To the family $\{c^\mu\}$ we can attach the object

$$c:=\underset{\nu\in\Lambda}{\oplus}\ \underset{\mu\in\Lambda^+}{\text{colim}}\, V^\mu\star c^{-\nu-\mu}\in\mathcal{C}.$$

In the formation of the colimit the transition maps are as follows: for $\mu_2=\mu_1+\mu$, the corresponding map is

$$V^{\mu_1}\star c^{-\mu_1-\nu}\to V^{\mu_1}\star V^{\mu_2}\star (V^{\mu_2})^\vee\star c^{-\mu_1-\nu}\to V^{\mu_1+\mu_2}\star c^{-\mu_1-\mu_2-\nu}.$$

Let us denote the tautological maps $V^\mu\star c^{-\mu-\nu}\to c$ by ϕ_μ. In particular, we have the maps

$$\phi_0: c^{-\nu}\to c.$$

According to [Ga2, Sect. 6], the above object c has the following pieces of structure.

10.1.2. First off, c is a Hecke eigen-object, i.e., it carries a tensor-compatible system of maps

$$V\star c\simeq c\otimes \underline{V},\quad V\in\text{Rep}(\check{G}), \tag{10.2}$$

where $\underline{V}\in\text{Vect}$ is the vector space underlying V.

Explicitly, the maps (10.2) are given as follows: for $\eta\in\Lambda^+$ large compared to V, we have

$$V\star V^\eta\star c^{-\nu-\eta}\simeq(\underset{\epsilon}{\oplus}V^{\eta+\epsilon}\otimes\underline{V}(\epsilon))\star c^{-\nu-\eta}\simeq\underset{\epsilon}{\oplus}V^{\eta+\epsilon}\star c^{(-\nu+\epsilon)-\eta-\epsilon}\otimes\underline{V}(\epsilon).$$

Note that in terms of the Hecke structure, the map $\phi_\mu: V^\mu\star c^{-\mu-\nu}\to c$ can be expressed via ϕ_0 as

$$V^\mu\star c^{-\mu-\nu}\overset{\text{Id}\otimes\phi_0}{\longrightarrow}V^\mu\otimes c\simeq c\otimes\underline{V}^\mu\to c, \tag{10.3}$$

where the last arrow is given by the *projection* onto the highest weight line.

10.1.3. Another piece of structure on c is an action of the algebraic group $\check{B}$. This action is compatible with the Hecke structure, i.e., the isomorphisms (10.2) are $\check{B}$-equivariant, where the $\check{B}$-action on the LHS is induced by that on c and on the RHS it is diagonal action.

[3]The terminology "Drinfeld-Plücker as well as the abstract framework for this formalism was suggested to us by S. Raskin.

10.1.4. The above compatibility means that c lifts to an object $c^{\rm enh}$ of the category

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}$$

along the forgetful functor

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{\mathbf{oblv}_{\check{B}}}{\longrightarrow} \operatorname{Vect} \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{\mathbf{coind}_{\check{G}}}{\longrightarrow} \operatorname{Rep}(\check{G}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \simeq \mathcal{C}.$$

Moreover:

- The direct summand $\underset{\mu\in\Lambda^+}{\operatorname{colim}} V^\mu \star c^{-\mu-\nu}$ of c corresponds to the weight ν-component of c with respect to the action of $\check{T} \subset \check{B}$, i.e., $\mathbf{inv}_{\check{T}}(k^{-\nu} \otimes c)$;
- The map $\phi_0 : c^{-\nu} \to c$ factors through $c^{-\nu} \to \mathbf{inv}_{\check{B}}(k^{-\nu} \otimes c) \to c$.

10.1.5. We note that the latter property combined with (10.3) determines the $\check{B}$-action on all of c.

Indeed, for a point $b \in \check{B}$, the composite map

$$V^\mu \star c^{-\nu-\mu} \overset{\phi_\mu}{\longrightarrow} c \overset{b}{\to} c$$

identifies with

$$V^\mu \star c^{-\mu-\nu} \overset{\operatorname{Id}\otimes\phi_0}{\longrightarrow} V^\mu \star c \simeq c \otimes \underline{V}^\mu \overset{(-\nu-\mu)(b)\cdot\otimes b\cdot}{\longrightarrow} c \otimes \underline{V}^\mu \to c,$$

where $(-\nu-\mu)(b)\cdot$ stands for the operation of multiplication by the scalar $(-\nu-\mu)(b)$.

10.2. **Digression: a conceptual explanation.** We will now present, following S. Raskin, a conceptual meaning of Drinfeld-Plücker structures, and of the construction

$$\{c^\mu\} \rightsquigarrow c^{\rm enh}$$

of the previous subsection.

10.2.1. Consider the category

$$\mathcal{C} \otimes \operatorname{Rep}(\check{T})$$

as acted on by $\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{T})$.

Consider the base affine space $\overline{\check{G}/\check{N}}$ of $\check{G}$, as acted on by $\check{G}\times\check{T}$. We can view $\mathcal{O}_{\overline{\check{G}/\check{N}}}$ as a commutative algebra object in $\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{T})$.

By definition, the category of families $\{c^\mu\}$ in Sect. 10.1.1, denoted $\operatorname{DrPl}(\mathcal{C})$, is

$$\mathcal{O}_{\overline{\check{G}/\check{N}}}\text{-mod}(\mathcal{C} \otimes \operatorname{Rep}(\check{T})).$$

10.2.2. In terms of this interpretation, the construction

$$\{c^\mu\} \rightsquigarrow c^{\rm enh}$$

is the functor

$$\jmath^* : \operatorname{DrPl}(\mathcal{C}) \to \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}$$

given by

$$\mathcal{O}_{\overline{\check{G}/\check{N}}}\text{-mod}(\mathcal{C} \otimes \operatorname{Rep}(\check{T})) \simeq \mathcal{O}_{\overline{\check{G}/\check{N}}}\text{-mod}(\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{T})) \underset{\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})}{\otimes} (\mathcal{C} \otimes \operatorname{Rep}(\check{T})) \simeq$$

$$\simeq \operatorname{QCoh}((\overline{\check{G}/\check{N}})/(\check{G}\times\check{T})) \underset{\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})}{\otimes} (\mathcal{C} \otimes \operatorname{Rep}(\check{T})) \to$$

$$\to \operatorname{QCoh}((\check{G}/\check{N})/(\check{G}\times\check{T})) \underset{\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})}{\otimes} (\mathcal{C} \otimes \operatorname{Rep}(\check{T})) \simeq \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})\otimes\operatorname{Rep}(\check{T})}{\otimes} (\mathcal{C} \otimes \operatorname{Rep}(\check{T})) \simeq$$

$$\simeq \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C},$$

where the arrow

$$\operatorname{QCoh}((\overline{\check{G}/\check{N}})/(\check{G}\times\check{T})) \to \operatorname{QCoh}((\check{G}/\check{N})/(\check{G}\times\check{T}))$$

is pullback with respect to the open embedding

$$\jmath : \check{G}/\check{N} \to \overline{\check{G}/\check{N}}.$$

10.2.3. The functor $\jmath^*$ is the left adjoint to a *fully faithful* functor denoted $\jmath_*$, corresponding to the direct image functor

$$\mathrm{QCoh}((\check{G}/\check{N})/(\check{G}\times\check{T})) \to \mathrm{QCoh}((\overline{\check{G}/\check{N}})/(\check{G}\times\check{T})).$$

Explicitly, the functor $\jmath_*$ sends to an object $c' \in \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathcal{C}$ to the family $\{c^\mu\}$ with

$$c^\mu := \mathbf{coind}_{\check{B}\to\check{G}}(k^\mu \otimes c').$$

The transition maps in this family are given by the canonical identifications

$$(V^\mu)^\vee \simeq \mathbf{coind}_{\check{B}\to\check{G}}(k^{-\mu}), \quad \mu \in \Lambda^+,$$

where we identify, by definition,

$$V^\mu := \mathbf{ind}_{\check{B}\to\check{G}}(k^\mu).$$

10.3. **The baby Verma object via the Drinfeld-Plücker formalism.** We will now show how the object

$$\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \in \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}}$$

arises following the pattern of Sect. 10.1.1.

10.3.1. We take

$$c^\mu := \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}+\mu}), \tag{10.4}$$

with the maps (10.1) given by the maps

$$\begin{multline}\mathrm{Frob}_q^*((V^{\mu_1})^\vee) \otimes \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}+\mu_2}) \simeq \\ \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}\left(\mathrm{Frob}_q^*((V^{\mu_1})^\vee)|_{\mathrm{Rep}_q(B)} \otimes k^{\check{\lambda}+\mu_2}\right) \to \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-\mu_1+\mu_2})\end{multline} \tag{10.5}$$

that come from the natural projections

$$\mathrm{Frob}_q^*((V^\mu)^\vee)|_{\mathrm{Rep}_q(B)} \to k^{-\mu}.$$

We claim that the resulting object

$$c^{\mathrm{enh}} \in \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} = \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}}$$

identifies canonically with $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})$.

10.3.2. First, note that the functor

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} \overset{\mathbf{oblv}_{\check{B}}}{\longrightarrow} \mathrm{Vect} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}} \overset{\mathbf{coind}_{\check{G}}}{\longrightarrow} \mathrm{Rep}_q(G)_{\mathrm{ren}} \tag{10.6}$$

identifies with the composite

$$\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \overset{\mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}}}{\longrightarrow} \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}} \overset{\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}}}{\longrightarrow} \mathrm{Rep}_q(G)_{\mathrm{ren}}.$$

We have:

$$\begin{multline}\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}} \circ \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}} \circ \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \simeq \mathbf{coind}_{\mathrm{sml}\to\mathrm{big}} \circ \mathbf{coind}_{\mathrm{sml}^+\to\mathrm{sml}}(k^{\check{\lambda}}) \simeq \\ \simeq \mathbf{coind}_{\mathrm{sml}^+\to\mathrm{big}}(k^{\check{\lambda}}) \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}} \circ \mathbf{coind}_{\mathrm{sml}^+\to\mathrm{Lus}^+}(k^{\check{\lambda}}) \simeq \\ \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}\left(\mathrm{Frob}_q^*(\mathcal{O}_{\check{B}}) \otimes k^{\check{\lambda}}\right).\end{multline} \tag{10.7}$$

We now note that $\mathcal{O}_{\check{B}} \in \operatorname{Rep}(\check{B})$ can be written as

$$\underset{\nu\in\Lambda}{\oplus} \underset{\mu\in\Lambda^+}{\operatorname{colim}} \mathbf{oblv}_{\check{G}\to\check{B}}(V^\mu)\otimes k^{-\nu-\mu}. \tag{10.8}$$

Hence, the RHS in (10.7) can be rewritten as

$$\underset{\nu\in\Lambda}{\oplus} \underset{\mu\in\Lambda^+}{\operatorname{colim}} \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}\left(\operatorname{Frob}_q^*(V^\mu)\otimes k^{\check{\lambda}-\nu-\mu}\right),$$

which we finally rewrite as

$$\underset{\nu\in\Lambda}{\oplus} \underset{\mu\in\Lambda^+}{\operatorname{colim}} \operatorname{Frob}_q^*(V^\mu)\otimes \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-\nu-\mu}). \tag{10.9}$$

Thus, we have identified the image of $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})$ under the functor (10.6) with the object $c\in\operatorname{Rep}_q(G)_{\mathrm{ren}}$ corresponding to the family (10.4).

10.3.3. We claim that the Hecke property and the $\check{B}$-action on

$$\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}}\circ \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}}\circ \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \tag{10.10}$$

coincide with those on the colimit (10.9), specified by the procedures in Sects. 10.1.2 and 10.1.3.

First off, in the formation of the colimit (10.9), for each ν, we can replace the index set $\{\mu\in\check{\Lambda}^+\}$ by its cofinal coset consisting of those μ, for which $\check{\lambda}-\nu-\mu$ belongs to $-\check{\Lambda}^+\subset\check{\Lambda}$. In this case, the terms appearing in the colimit belong to $(\operatorname{Rep}_q(G)_{\mathrm{ren}})^\heartsuit$.

Hence, it suffices to check the corresponding assertions at the level of homotopy categories (i.e., homotopy-coherence is automatic).

10.3.4. Now, the fact that Hecke structure on (10.10) is given, in terms of its presentation as (10.9), by the procedure of Sect. 10.1.2 follows from the corresponding property of $\mathcal{O}_{\check{B}}$: the isomorphisms

$$\mathbf{oblv}_{\check{G}\to\check{B}}(V)\otimes\mathcal{O}_{\check{B}}\simeq\mathcal{O}_{\check{B}}\otimes\underline{V}$$

are given by

$$\mathbf{oblv}_{\check{G}\to\check{B}}(V)\otimes\mathbf{oblv}_{\check{G}\to\check{B}}(V^\eta)\otimes k^{-\nu-\eta}\simeq\left(\underset{\epsilon}{\oplus}\mathbf{oblv}_{\check{G}\to\check{B}}(V^{\eta+\epsilon})\otimes\underline{V}(\epsilon)\right)\otimes k^{-\nu-\eta}\simeq$$
$$\simeq\underset{\epsilon}{\oplus}\mathbf{oblv}_{\check{G}\to\check{B}}(V^{\eta+\epsilon})\otimes k^{(-\nu+\epsilon)-\eta-\epsilon}\otimes\underline{V}(\epsilon).$$

10.3.5. The fact that the $\check{B}$-actions agree follows from the fact that (at the level of homotopy categories), the Hecke structure determines the $\check{B}$-action (see Sect. 10.1.5), combined with the fact that the maps ϕ_0

$$\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-\nu})\to\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}\circ\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{Lus}^+}(k^{\check{\lambda}})\simeq$$
$$\simeq\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}}\circ\mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}}\circ\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})$$

identify with

$$\mathbf{inv}_{\check{B}}\left(k^{-\nu}\otimes\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}}\circ\mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}}\circ\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})\right)\to$$
$$\to\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}}\circ\mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}}\circ\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}).$$

10.4. **The semi-infinite IC sheaf.** We will now start the process of transferring the baby Verma module to the Kac-Moody side, i.e., we would like to describe the object of

$$\mathrm{Rep}(\check{B}) \underset{\check{G}}{\otimes} \mathrm{KL}(G, -\kappa)$$

corresponding under $\mathsf{F}_{-\kappa}$ to

$$\mathbf{coind}_{\mathrm{Lus}^+ \to \frac{1}{2}}(k^{\check{\lambda}}) \in \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}}.$$

A key tool for this will be a certain geometric object, introduced in [Ga2] under the same "semi-infinite IC sheaf".

10.4.1. Let $\mathrm{IC}^{\frac{\infty}{2}}_{-\kappa} \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})}$ be the object introduced in [Ga2, Sect. 2.3]. Explicitly,

$$\underset{\mu \in \Lambda^+}{\mathrm{colim}}\, t^{-\mu} \cdot \mathrm{Sat}(V^\mu)[\langle \mu, 2\check{\rho} \rangle].$$

Let $\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ be its graded version, i.e.,

$$\underset{\nu \in \Lambda}{\oplus} t^{-\nu} \cdot \mathrm{IC}^{\frac{\infty}{2}}_{\kappa}.$$

It was shown in [Ga2, Sect. 6], the object $c = \overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ can be obtained by the Drifeld-Plücker formalism of Sect. 10.1.1 starting from the collection of objects

$$c^\mu := \delta_{t^\mu, \mathrm{Gr}}[\langle -\mu, 2\check{\rho} \rangle] \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})},$$

and the maps

$$\delta_{t^{\mu_2}, \mathrm{Gr}}[\langle -\mu_2, 2\check{\rho} \rangle] \star \mathrm{Sat}((V^{\mu_1})^\vee) \to \delta_{t^{-\mu_1+\mu_2}, \mathrm{Gr}}[\langle \mu_1 - \mu_2, 2\check{\rho} \rangle], \tag{10.11}$$

that come by adjunction from the canonical maps

$$\delta_{t^\mu, \mathrm{Gr}}[\langle -\mu, 2\check{\rho} \rangle] \to \mathrm{Sat}(V^\mu).$$

10.4.2. In particular, $\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ carries a Hecke structure and a $\check{B}$-action. Let $(\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa})^{\mathrm{enh}}$ denote the resulting object of

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{B}).$$

The following is established in *loc.cit.*:

Theorem 10.4.3. *The tautological map* $\delta_{1,\mathrm{Gr}} \to \overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ *identifies*

$$\delta_{1,\mathrm{Gr}} \simeq \mathbf{inv}_{\check{B}}(\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}).$$

10.4.4. A key observation is that $\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ is $N(\mathcal{K})$-equivariant, see [Ga2, Proposition 2.3.7(a)]. Hence, we obtain that $\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}$ is naturally an object of $\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{N(\mathcal{K})\cdot T(\mathcal{O})}$.

Since the forgetful functor

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{N(\mathcal{K})\cdot T(\mathcal{O})} \to \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})}$$

is fully faithful, we obtain that $(\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa})^{\mathrm{enh}}$ is naturally an object of

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{N(\mathcal{K})\cdot T(\mathcal{O})} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{B}).$$

10.5. **The semi-infinite IC sheaf, Iwahori version.**

10.5.1. Recall now that according to [Ga2, Sect. 5], the functor

$$\mathrm{Av}_*^{I^-} : \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})} \to \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{I^-\cdot T(\mathcal{O})},$$

when restricted to

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{N(\mathcal{K})\cdot T(\mathcal{O})} \subset \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})}$$

defines an equivalence onto

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{I} \subset \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{I^-\cdot T(\mathcal{O})}.$$

Set

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa} := \mathrm{Av}_*^{I^-}(\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}) \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{I}.$$

It is equipped with a Hecke structure and a compatible action of $\check{B}$. Let $(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa})^{\mathrm{enh}}$ denote the resulting object of

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{I} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{B}).$$

From Theorem 10.4.3 we obtain:

Corollary 10.5.2. *The tautological map* $\delta_{1,\mathrm{Gr}} \to \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa}$ *defines an isomorphism*

$$\delta_{1,\mathrm{Gr}} \simeq \mathbf{inv}_{\check{B}}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa}).$$

10.5.3. Note that by construction, $(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa})^{\mathrm{enh}}$, viewed as an object of

$$\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{B}),$$

can be obtained by the procedure of Sect. 10.1.1 applied to the family of objects

$$c^\mu := \mathrm{Av}_*^{I^-}(\delta_{t^\mu,\mathrm{Gr}})[-\langle \mu, 2\check{\rho}\rangle] \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})},$$

and the maps

$$\mathrm{Av}_*^{I^-}(\delta_{t^{\mu_2},\mathrm{Gr}})[\langle -\mu_2, 2\check{\rho}\rangle] \star \mathrm{Sat}((V^{\mu_1})^\vee) \to \mathrm{Av}_*^{I^-}(\delta_{t^{\mu_1+\mu_2},\mathrm{Gr}})[\langle \mu_1-\mu_2, 2\check{\rho}\rangle],$$

induced by (10.11).

10.5.4. Consider now another family, namely,

$$'c^\mu := J_\mu \star \delta_{1,\mathrm{Gr}},$$

and the maps

$$J_{\mu_2} \star \delta_{1,\mathrm{Gr}} \star \mathrm{Sat}((V^{\mu_1})^\vee) \to J_{-\mu_1+\mu_2} \star \delta_{1,\mathrm{Gr}}$$

that come by adjunction from the canonical maps

$$J_\mu \star \delta_{1,\mathrm{Gr}} \simeq j_{\mu,!} \star \delta_{1,\mathrm{Gr}} \simeq \pi_*(j_{\mu,!}) \to \mathrm{Sat}(V^\mu), \quad \mu \in \check{\Lambda}^+,$$

where $\pi : \mathrm{Fl}_G^{\mathrm{aff}} \to \mathrm{Gr}_G$.

We claim that the resulting object

$$'c^{\mathrm{enh}} \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{T(\mathcal{O})} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}(\check{B})$$

will be canonically isomorphic to

$$c^{\mathrm{enh}} = (\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa})^{\mathrm{enh}}.$$

This follows from the fact that if $\mu \in \Lambda^+$, we have an evident identification,

$$'c^{-\mu} \simeq c^{-\mu},$$

whereas $\Lambda^+ \subset \Lambda$ is cofinal.

Since the forgetful functor $\text{D-mod}_{-\kappa}(\text{Gr}_G)^I \to \text{D-mod}_{-\kappa}(\text{Gr}_G)^{T(\mathcal{O})}$ is fully faithful, we obtain that the isomorphism $c^{\text{enh}} \simeq {}'c^{\text{enh}}$ holds also in

$$\text{D-mod}_{-\kappa}(\text{Gr}_G)^I \underset{\text{Rep}(\check{G})}{\otimes} \text{Rep}(\check{B})$$

10.5.5. In particular, we obtain an isomorphism

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa} \simeq \underset{\nu}{\oplus} \underset{\mu}{\text{colim}}\, j_{-\nu-\mu,*} \star \text{Sat}(V^\mu),$$

where the colimit is taken over the set of those μ for which $\nu + \mu \in \Lambda^+$.

The latter presentation makes it clear that $\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa}$ lies in $(\text{D-mod}_{-\kappa}(\text{Gr}_G)^I)^\heartsuit[d]$: indeed, for $\nu \in \Lambda^{++}$, we have

$$J_{-\nu} \star \delta_{1,\text{Gr}} \simeq j_{-\mu,*} \star \delta_{1,\text{Gr}} \in (\text{D-mod}_{-\kappa}(\text{Gr}_G)^I)^\heartsuit[d],$$

while convolution with objects of the form $\text{Sat}(V)$ for $V \in \text{Rep}(\check{G})^\heartsuit$ is a t-exact endo-functor of $\text{D-mod}_{-\kappa}(\text{Gr}_G)^I$.

10.5.6. Let inv denote the equivalence

$$\text{D-mod}_{-\kappa}(\text{Gr}_G)^I \simeq \text{D-mod}_\kappa(\text{Fl}^{\text{aff}}_G)^{G(\mathcal{O})} \tag{10.12}$$

given by the inversion on $G(\mathcal{K})$. We normalize it so that

$$\text{inv}(\delta_{1,\text{Gr}}) \simeq \pi^*(\delta_{1,\text{Gr}}).$$

Note that

$$\text{inv}(\mathcal{F} \star \text{Sat}(V)) \simeq \text{Sat}(V^\tau) \star \text{inv}(\mathcal{F}), \quad V \in \text{Rep}(\check{G}),$$

where V^τ is obtained by the action of the Cartan involution τ on V (in particular, $(V^\mu)^\tau \simeq V^{-w_0(\tau)}$).

Denote

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa} := \text{inv}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa}) \in \text{D-mod}_\kappa(\text{Fl}^{\text{aff}}_G)^{G(\mathcal{O})}.$$

This object has a natural Hecke structure, and a compatible $\check{B}$-action. Explicitly,

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa} \simeq \underset{\nu}{\oplus} \underset{\mu}{\text{colim}}\, \text{Sat}(V^{-w_0(\mu)}) \star j_{\nu+\mu,*},$$

where the colimit is taken over the set of those μ for which $\nu + \mu \in \Lambda^+$.

Denote by $(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa})^{\text{enh}}$ the corresponding object of

$$\text{Rep}(\check{B}) \underset{\text{Rep}(\check{G})}{\otimes} \text{D-mod}_\kappa(\text{Fl}^{\text{aff}}_G)^{G(\mathcal{O})}.$$

From Corollary 10.5.2 we obtain:

Corollary 10.5.7. *The tautological map* $\pi^*(\delta_{1,\text{Gr}}) \to \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa}$ *defines an isomorphism*

$$\pi^*(\delta_{1,\text{Gr}}) \simeq \mathbf{inv}_{\check{B}}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa}).$$

10.5.8. By construction, $\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\text{inv}}_{\kappa}$, equipped with the above pieces of structure, arises by the procedure of Sect. 10.1.1 from the collection of objects

$$c^\mu := \text{Av}_*^{G(\mathcal{O})/I}(J^{\mathbb{D}}_{-\mu}) \in \text{D-mod}_\kappa(\text{Fl}^{\text{aff}}_G)^{G(\mathcal{O})}$$

where we regard $\text{D-mod}_\kappa(\text{Fl}_G)^{G(\mathcal{O})}$ as acted on by $\text{Rep}(\check{G})$ via $\text{Sat} \circ \tau$. The corresponding maps

$$\text{Sat}((V^{-w_0(\mu_1)})^\vee) \star \text{Av}_*^{G(\mathcal{O})/I}(J^{\mathbb{D}}_{-\mu_2}) \to \text{Av}_*^{G(\mathcal{O})/I}(J^{\mathbb{D}}_{\mu_1-\mu_2}), \quad \mu_1 \in \Lambda^+ \tag{10.13}$$

are obtained by adjunction from the canonical maps

$$\text{Av}_*^{G(\mathcal{O})/I}(J^{\mathbb{D}}_{-\mu}) = \text{Av}_*^{G(\mathcal{O})/I}(j_{-\mu,!}) \to \pi^*(\text{Sat}(V^{-w_0(\mu)})), \quad \mu \in \Lambda^+.$$

10.6. **The baby Verma object via Wakimoto modules: positive level case.** In this subsection we will be able to carry out the program indicated in the preamble to Sect. 10.4 but for the Kazhdan-Lusztig category at the positive level.

This will lead to the proof of Theorem 9.4.5, and by duality, to that of Theorem 9.2.8.

10.6.1. Recall that the (dual) Kazhdan-Lusztig equivalence

$$\mathsf{F}_\kappa : \mathrm{KL}(G,\kappa) \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}$$

respects the action of $\mathrm{Rep}(\check{G})$, where $\mathrm{Rep}(\check{G})$ acts on $\mathrm{KL}(G,\kappa)$ via $\mathrm{Sat}\circ\tau$.

Hence, it induces an equivalence

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G,\kappa) \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}, \tag{10.14}$$

where we identify the right-hand side with $\mathrm{Rep}^{\frac{1}{2}}_{q^{-1}}(G)_{\mathrm{ren}}$.

Recall that for $\check{\lambda} \in \check{\Lambda}$, we have the object

$$\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \in \mathrm{Rep}^{\frac{1}{2}}_{q^{-1}}(G)_{\mathrm{ren}}.$$

In this subsection we will identify the image of this object under the equivalence (10.14).

10.6.2. Consider the object

$$(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_\kappa)^{\mathrm{enh}} \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa}) \in \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G,\kappa).$$

We claim:

Theorem 10.6.3. *The object* $(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_\kappa)^{\mathrm{enh}} \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa})$ *corresponds to* $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})$ *under the equivalence* (10.14)*.*

Remark 10.6.4. The proof of Theorem 10.6.3 that we give below will be "artificial" in that it will result from some explicit calculation. However, later on, in Sect. 11, we will show "why" Theorem 10.6.3 should hold, based on an additional property of the conjectural equivalence (9.7).

10.6.5. Before we prove Theorem 10.6.3, let us show that it implies Theorem 9.4.5, and hence Theorem 9.2.8. Indeed, consider the objects

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_\kappa \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa}) \in \mathrm{KL}(G,\kappa)$$

and

$$\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}} \circ \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}} \circ \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}) \in \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}},$$

regarded as equipped with $\check{B}$-actions.

According to Theorem 10.6.3, these two objects correspond to one another under the equivalence F_κ. Hence, so do the corresponding objects obtained by applying $\mathbf{inv}_{\check{B}}$.

However,

$$\mathbf{inv}_{\check{B}}\left(\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}} \circ \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml}} \circ \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})\right) \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}}) =: \mathcal{V}^{\vee,w_0(\check{\lambda})}_{q^{-1}},$$

while according to Corollary 10.5.7, we have

$$\mathbf{inv}_{\check{B}}\left(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_\kappa \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa})\right) \simeq \mathbf{inv}_{\check{B}}\left(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_\kappa\right) \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa}) \simeq\simeq$$

$$\simeq \pi^*(\delta_{1,\mathrm{Gr}}) \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa}) \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(\mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa})) =: \mathbb{V}^{w_0(\check{\lambda})}_\kappa,$$

as desired.

□

10.6.6. The rest of this subsection is devoted to the proof of Theorem 10.6.3. We will show that the image of $(\overset{\bullet}{\mathcal{F}}{}_\kappa^{\frac{\infty}{2},\mathrm{inv}})^{\mathrm{enh}} \star \mathbb{D}(\mathbb{W}_{-\kappa}^{-\check\lambda})$ under the functor (10.14) and $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check\lambda})$ are obtained by the Drinfeld-Plücker formalism of Sect. 10.1.1 from equivalent families of objects c^μ and $'c^\mu$.

On the one hand, by Sect. 10.5.8, the object $(\overset{\bullet}{\mathcal{F}}{}_\kappa^{\frac{\infty}{2},\mathrm{inv}})^{\mathrm{enh}} \star \mathbb{D}(\mathbb{W}_{-\kappa}^{-\check\lambda})$ is obtained from the family of objects

$$c^\mu := \mathrm{Av}_*^{G(\mathcal{O})/I}(J_{-\mu}^{\mathbb{D}}) \star \mathbb{D}(\mathbb{W}_{-\kappa}^{-\check\lambda}) =: \mathbb{V}_\kappa^{w_0(\check\lambda+\mu)},$$

where the transition maps

$$\mathrm{Sat}((V^{-w_0(\mu_1)})^\vee) \star \mathbb{V}_\kappa^{w_0(\check\lambda+\mu_2)} \to \mathbb{V}_\kappa^{w_0(\check\lambda-\mu_1+\mu_2)} \tag{10.15}$$

are obtained from the maps (10.13) by convolution.

On the other hand, by Sects. 10.3.1-10.3.3, the object $\mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check\lambda})$ is obtained from the collection of objects

$$'c^\mu := \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda+\mu}) =: \mathcal{V}_{q^{-1}}^{\vee,w_0(\check\lambda+\mu)},$$

and the transtition maps

$$\begin{aligned} \mathrm{Frob}_{q^{-1}}^*((V^{\mu_1})^\vee) \otimes \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda+\mu_2}) \simeq \\ \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}\left(\mathrm{Frob}_q^*((V^{\mu_1})^\vee)|_{\mathrm{Rep}_q(B)} \otimes k^{\check\lambda+\mu_2}\right) \to \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda-\mu_1+\mu_2}). \end{aligned} \tag{10.16}$$

It suffices to show that the families $\mathsf{F}_\kappa(c^{-\mu})$ and $'c^{-\mu}$ can be identified for μ running over a subset cofinal in Λ. We take this subset to consist of those μ for which $\check\lambda-\mu$ is anti-dominant.

10.6.7. Indeed, if $\check\lambda-\mu$ is anti-dominant, we do know that

$$\mathsf{F}_\kappa(\mathbb{V}_\kappa^{w_0(\check\lambda-\mu)}) \simeq \mathcal{V}_{q^{-1}}^{\vee,w_0(\check\lambda-\mu)}.$$

Since these objects lie in $(\mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}})^\heartsuit$, the compatibility with the transition maps is sufficient to check at the level of 1-morphisms (i.e., the compatibility with homotopy-coherence is automatic).

10.6.8. On the one hand, we note that the maps (10.16) are obtained by duality from the maps (9.5)

$$\mathcal{V}_q^{\mu_1-(\check\lambda+\mu_2)} \to \mathrm{Frob}_q^*(V^{\mu_1}) \otimes \mathcal{V}_q^{-(\check\lambda+\mu_2)},$$

which are the images under $\mathsf{F}_{-\kappa}$ of the maps (9.2)

$$\mathbb{V}_{-\kappa}^{\mu_1-(\check\lambda+\mu_2)} \to \mathrm{Sat}(V^{\mu_1}) \star \mathbb{V}_{-\kappa}^{-(\check\lambda+\mu_2)}. \tag{10.17}$$

On the other hand, the transition maps (10.15) are obtained by duality from the maps

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}_{-\kappa}^{\mu_1-\check\lambda-\mu_2}) \simeq \mathrm{Av}_!^{G(\mathcal{O})/I}(J_{\mu_1} \star \mathbb{W}_{-\kappa}^{-\check\lambda-\mu_2}) \to \mathrm{Sat}(V^{\mu_1}) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}_{-\kappa}^{-\check\lambda-\mu_2}), \tag{10.18}$$

induced by the maps

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(J_\mu) = \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{\mu,!}) \to \pi^!(\mathrm{Sat}(V^\mu)), \quad \mu \in \Lambda^+. \tag{10.19}$$

10.6.9. Recall that for $\check\lambda' \in \check\Lambda^+$ we identify $\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}_{-\kappa}^{\check\lambda'})$ with $\mathbb{V}_{-\kappa}^{\check\lambda'}$ via

$$\mathbb{W}_{-\kappa}^{\check\lambda'} \simeq \mathbb{M}_{-\kappa}^{\check\lambda'}$$

and

$$\begin{aligned} \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{M}_{-\kappa}^{\check\lambda'}) = \mathrm{Av}_!^{G(\mathcal{O})/I}\left(\mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_{-\kappa}}(M^{\check\lambda'})\right) \simeq \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_{-\kappa}}\left(\mathrm{Av}_!^{G(\mathcal{O})/I}(M^{\check\lambda'})\right) \simeq \\ \simeq \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_{-\kappa}}\left(\mathrm{Av}_!^{G/B}(M^{\check\lambda'})\right) \simeq \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})}^{\widehat{\mathfrak{g}}_{-\kappa}}(V^{\check\lambda'}) \simeq \mathbb{V}_{-\kappa}^{\check\lambda'}. \end{aligned}$$

So we need to show that for $\check{\lambda}' \in \check{\Lambda}^+$ and $\mu \in \Lambda^+$, the map

$$\mathbb{V}^{\mu+\check{\lambda}'}_{-\kappa} \simeq \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{M}^{\mu+\check{\lambda}'}_{-\kappa}) \simeq \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{\mu,!} \star j_{-\mu,*} \star \mathbb{M}^{\mu+\check{\lambda}'}_{-\kappa}) \overset{(3.2)}{\longrightarrow}$$
$$\to \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{\mu,!} \star \mathbb{M}^{\check{\lambda}'}_{-\kappa}) \overset{(10.19)}{\longrightarrow} \mathrm{Sat}(V^\mu) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{M}^{\check{\lambda}'}_{-\kappa}) \simeq \mathrm{Sat}(V^\mu) \star \mathbb{V}^{\check{\lambda}'}_{-\kappa}$$

coincides with the canonical map

$$\mathbb{V}^{\mu+\check{\lambda}'}_{-\kappa} \to \mathrm{Sat}(V^\mu) \star \mathbb{V}^{\check{\lambda}'}_{-\kappa}.$$

However, this follows by unwinding the definitions by tracking the image of the highest weight vector.

10.7. **The baby Verma object via Wakimoto modules: the negative level case.** Even though we have already proved Theorem 9.2.8, we would now like it to prove it more directly, without appealing to positive vs negative level duality.

For this we will need to carry out the program indicated in the preamble to Sect. 10.4 directly in the negative level case.

10.7.1. Consider the original Kazhdan-Lusztig equivalence

$$\mathsf{F}_{-\kappa} : \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}_q(G)_{\mathrm{ren}},$$

and the induced equivalence

It induces an equivalence

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}}. \tag{10.20}$$

In this subsection we will identify the image of

$$\mathbf{ind}_{\mathrm{Lus}^+ \to \frac{1}{2}}(k^{\check{\lambda}}) \in \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}} \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\mathrm{ren}}$$

as an object of $\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G, -\kappa)$ under the equivalence (10.20).

10.7.2. Consider the following variant of the object $(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa})^{\mathrm{enh}}$, denoted

$$(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} \in \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})}.$$

Namely,

$$(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} := \left((\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} \star j_{w_0,*} \right)^{\tau} [d],$$

where τ is the Cartan involution on $\check{G}$ (normalized so that it preserves $\check{B}$ and acts as $\mu \mapsto -w_0(\mu)$ on the weights).

Explicitly, $(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}$ is obtained by the Drinfeld-Plücker formalism of Sect. 10.1.1 from the collection of objects

$$c^\mu := \mathrm{Av}_!^{G(\mathcal{O})/I}(J_\mu),$$

and the transition maps

$$\mathrm{Sat}((V^{\mu_1})^\vee) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(J_{\mu_2}) \to \mathrm{Av}_!^{G(\mathcal{O})/I}(J_{-\mu_1+\mu_2}) \tag{10.21}$$

coming by adjunction from the maps

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(J_\mu) = \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{\mu,!}) \to \pi^!(\mathrm{Sat}(V^\mu)), \quad \mu \in \Lambda^+.$$

The corresponding object $\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \in \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})}$ is given by

$$\underset{\nu \in \Lambda}{\oplus} \underset{\mu \in \Lambda^+}{\mathrm{colim}}\, \mathrm{Sat}(V^\mu) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(J_{-\mu-\nu}).$$

10.7.3. We claim:

Theorem 10.7.4. *Under the equivalence* (10.20), *the object* $\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})$ *corresponds to*

$$(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}\star J_{-2\rho}[-d]\star \mathbb{W}^{\check{\lambda}}_{-\kappa}.$$

10.7.5. Before giving the proof, let us note that Theorem 10.7.4 gives another proof for Theorem 9.2.8.

Indeed, let us apply $\mathbf{inv}_{\check{B}}$ to

$$\mathbf{coind}_{\check{G}}\circ \mathbf{oblv}_{\check{B}}(\mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}))$$

and

$$\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}\star J_{-2\rho}[-d]\star \mathbb{W}^{\check{\lambda}}_{-\kappa}.$$

On the quantum group side we get

$$\mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}\circ \mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}})\simeq \mathbf{coind}_{\frac{1}{2}\to\mathrm{big}}\circ \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}+2\check{\rho}-2\rho})\simeq$$
$$\simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}+2\check{\rho}-2\rho})\simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-2\rho})[-d].$$

On the Kac-Moody side, by Corollary 10.5.7, we get

$$\mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0,*}\star J_{-2\rho}[-d]\star \mathbb{W}^{\check{\lambda}}_{-\kappa})[d]\simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0,*}\star J_{-2\rho}\star \mathbb{W}^{\check{\lambda}}_{-\kappa})\simeq$$
$$\simeq \mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0,*}\star \mathbb{W}^{\check{\lambda}-2\rho}_{-\kappa})\simeq \pi^!(\delta_{1,\mathrm{Gr}})\star \mathbb{W}^{\check{\lambda}-2\rho}_{-\kappa}[-d]\simeq \mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\check{\lambda}-2\rho}_{-\kappa})[-d].$$

Thus, we obtain

$$\mathsf{F}_{-\kappa}(\mathrm{Av}_!^{G(\mathcal{O})/I}(\mathbb{W}^{\check{\lambda}-2\rho}_{-\kappa}))[-d]\simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check{\lambda}-2\rho})[-d],$$

as desired.

□

10.7.6. The rest of the subsection is devoted to the proof of Theorem 10.7.4. We will deduce it from Theorem 10.7.4 using an idea involving duality. On the first pass, this will look like an artificial procedure, but in Sect. 10.8 we will explain a conceptual framework that it fits in.

Let us start with a family of objects $\{c^\mu\}\in\mathcal{C}$ as in Sect. 10.1.1; let us assume $\mathcal{C}$ is compactly generated, and that the objects c^μ are compact.

We define the dual family $\{c^{\vee,\mu}\}\in\mathcal{C}^\vee$ by setting

$$c^{\vee,\mu}:=\mathbb{D}(c^{-\mu+2\rho})[-d],$$

where $\mathbb{D}$ denotes the canonical contravariant equivalence $\mathcal{C}^c\to(\mathcal{C}^\vee)^c$. Let $c^{\vee,\mathrm{enh}}$ denote the resulting object of $\mathrm{Rep}(\check{B})\underset{\mathrm{Rep}(\check{G})}{\otimes}\mathcal{C}^\vee$.

10.7.7. We apply the above procedure first to $\mathcal{C}=\mathrm{KL}(G,\kappa)$ and c^μ being

$$\mathrm{Av}_*^{G(\mathcal{O})/I}(J^{\mathbb{D}}_{-\mu})\star\mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa})$$

and the transition maps obtained from the maps (10.13) by convolution.

Note that the corresponding dual family $c^{\vee,\mu}$ is

$$\mathrm{Av}_!^{G(\mathcal{O})/I}(J_{\mu-2\rho})\star\mathbb{W}^{-\check{\lambda}}_{-\kappa}[-d]$$

and the transition maps obtained from the maps (10.21) by convolution. Hence the resulting object $c^{\vee,\mathrm{enh}}$ identifies with

$$(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}\star J_{-2\rho}[-d]\star \mathbb{W}^{-\check{\lambda}}_{-\kappa}.$$

10.7.8. We now perform the same procedure on the quantum group side. Take $\mathcal{C} := \operatorname{Rep}_{q^{-1}}(G)_{\mathrm{ren}}$. We start with the family c^μ

$$\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda+\mu})$$

and the transition maps (10.5). This family corresponds to the one on the Kac-Moody side under F_κ, by Sects. 10.6.7-10.6.9.

Consider the dual family $c^{\vee,\mu}$. We obtain that in order to prove Theorem 10.7.4 it suffices to show that the corresponding object

$$c^{\vee,\mathrm{enh}} \in \operatorname{Rep}(\check B) \underset{\operatorname{Rep}(\check G)}{\otimes} \operatorname{Rep}_q(G)_{\mathrm{ren}} \simeq \operatorname{Rep}_q^{\frac12}(G)_{\mathrm{ren}}$$

identifies with $\mathbf{ind}_{\mathrm{Lus}^+\to\frac12}(k^{-\check\lambda})$.

We will give two proofs of this fact. One, more direct but less conceptual, right below, and an essentially equivalent but more conceptual one in Sect. 10.8.4.

10.7.9. Recall that according to Corollary 7.4.8,

$$\mathbf{ind}_{\mathrm{Lus}^+\to\frac12}(k^{-\check\lambda}) \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\frac12}(k^{-\check\lambda+2\check\rho-2\rho}),$$

and according to 10.3.1-10.3.3 it corresponds to ${}'c^{\mathrm{enh}}$ given by the family

$${}'c^\mu := \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{-\check\lambda+\mu+2\check\rho-2\rho})$$

and transition maps (10.5).

10.7.10. Hence, it suffices to show that the families $c^{\vee,-\mu}$ and ${}'c^{-\mu}$ are equivalent for μ belonging to a cofinal subset in Λ. We take the subset in question to consist of those μ for which $-\check\lambda-\mu+2\check\rho-2\rho$ is anti-dominant.

First, for an individual μ, we have

$$c^{\vee,-\mu} = (\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda+\mu+2\rho}))^\vee[-d] \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{-\check\lambda-\mu-2\rho})[-d],$$

which by Corollary 7.4.11 identifies with

$$\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{-\check\lambda-\mu+2\check\rho-2\rho}) \simeq {}'c^{-\mu}.$$

Now, when $-\check\lambda-\mu+2\check\rho-2\rho \in -\check\Lambda^+$, these objects belong to $(\operatorname{Rep}_q(G)_{\mathrm{ren}})^\heartsuit$. Hence, in order to show that the two coincide as systems, it is enough to do so at the level of 1-morphisms (homotopy-coherence is automatic).

10.7.11. Thus, we have to show that for $\check\lambda' \in \check\Lambda$ and $\mu \in \Lambda^+$ the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Frob}_q^*((V^\mu)^\vee)\otimes \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\mu+\check\lambda'})[-d] & \longrightarrow & \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda'})[-d] \\
\sim\downarrow & & \downarrow\sim \\
\operatorname{Frob}_q^*((V^\mu)^\vee)\otimes \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\mu+\check\lambda'+2\check\rho}) & \longrightarrow & \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda'+2\check\rho}).
\end{array}$$

This follows by juxtaposing the following two commutative diagrams

$$\begin{array}{ccc}
\operatorname{Frob}_q^*((V^\mu)^\vee)\otimes \mathbf{ind}_{\mathrm{Lus}^+\to\frac12}(k^{\mu+\check\lambda'}) & \longrightarrow & \mathbf{ind}_{\mathrm{Lus}^+\to\frac12}(k^{\check\lambda'}) \\
\sim\downarrow & & \downarrow\sim \\
\operatorname{Frob}_q^*((V^\mu)^\vee)\otimes \mathbf{coind}_{\mathrm{Lus}^+\to\frac12}(k^{\mu+\check\lambda'+2\check\rho-2\rho}) & \longrightarrow & \mathbf{coind}_{\mathrm{Lus}^+\to\frac12}(k^{\check\lambda'+2\check\rho-2\rho}),
\end{array}$$

which follows from the construction, and

$$\begin{array}{ccc}
(V^\mu)^\vee\otimes \mathbf{ind}_{\check B\to\check G}(k^\mu\otimes V)[-d] & \longrightarrow & \mathbf{ind}_{\check B\to\check G}(V)[-d] \\
\downarrow & & \downarrow \\
(V^\mu)^\vee\otimes \mathbf{coind}_{\check B\to\check G}(k^{\mu+2\rho}\otimes V) & \longrightarrow & \mathbf{coind}_{\check B\to\check G}(k^{2\rho}\otimes V),
\end{array}$$

which is a property of Serre duality on $\check{G}/\check{B}$.

□

10.8. **Drinfeld-Plücker formalism and duality.** In this subsection we will give a conceptual explanation of the duality procedure of Sect. 10.7.6.

10.8.1. Since $\mathcal{C}$ was assumed compactly generated, we obtain that $\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}$ is also compactly generated, and hence, dualizable. Moreover, we have a canonical identification

$$\left(\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}\right)^{\vee} \simeq \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{\vee}, \tag{10.22}$$

so that that the functor

$$\mathbf{oblv}_{\check{G}\to\check{B}} : \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{\vee} \to \mathcal{C}^{\vee}$$

is the dual of the functor

$$\mathbf{coind}_{\check{B}\to\check{G}} : \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \to \mathcal{C}.$$

Explicitly, for $c \in \mathcal{C}^c$ and $V \in \operatorname{Rep}(\check{B})^c$, we have

$$\mathbb{D}(V \otimes c) \simeq V^{\vee} \otimes \mathbb{D}(c)$$

as objects in the two sides of (10.22), respectively.

10.8.2. Let c' be a compact object of $\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}$. Tautologically, we have

$$\mathbf{coind}_{\check{B}\to\check{G}}(\mathbb{D}(c')) \simeq \mathbb{D}(\mathbf{ind}_{\check{B}\to\check{G}}(c'))$$

as objects in $\mathcal{C}^{\vee}$.

Recall also that

$$\mathbf{ind}_{\check{B}\to\check{G}}(c') \simeq \mathbf{coind}_{\check{B}\to\check{G}}(k^{2\rho} \otimes c')[d],$$

and recall the functor

$$\jmath^* : \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \to \operatorname{DrPl}(\mathcal{C}).$$

Hence, we obtain the following expression for $\jmath_*(\mathbb{D}(c'))$:

$$\jmath_*(\mathbb{D}(c'))^{\mu} \simeq \mathbb{D}(\jmath_*(c)^{-\mu+2\rho})[-d]. \tag{10.23}$$

This is the origin for the duality procedure in Sect. 10.7.6.

10.8.3. We now consider the situation when

$$\mathcal{C} := \operatorname{Rep}_{q^{-1}}(G)_{\mathrm{ren}},$$

so that $\mathcal{C}^{\vee} = \operatorname{Rep}_q(G)_{\mathrm{ren}}$ and

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} = \operatorname{Rep}^{\frac{1}{2}}_{q^{-1}}(G)_{\mathrm{ren}},$$

$$(\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C})^{\vee} \simeq \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{\vee} \simeq \operatorname{Rep}^{\frac{1}{2}}_{q}(G)_{\mathrm{ren}}. \tag{10.24}$$

By unwinding the constructions we obtain that the resulting pairing

$$\operatorname{Rep}^{\frac{1}{2}}_{q}(G)_{\mathrm{ren}} \otimes \operatorname{Rep}^{\frac{1}{2}}_{q^{-1}}(G)_{\mathrm{ren}} \to \operatorname{Vect}$$

is given by

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{H}om_{\operatorname{Rep}^{\frac{1}{2}}_{q}(G)_{\mathrm{ren}}}(k, \mathcal{M}_1 \otimes \mathcal{M}_2^{\sigma}),$$

where σ is a canonical equivalence

$$(\operatorname{Rep}^{\frac{1}{2}}_{q}(G)_{\mathrm{ren}})^{\mathrm{rev}} \to \operatorname{Rep}^{\frac{1}{2}}_{q^{-1}}(G)_{\mathrm{ren}}.$$

In particular, we obtain that with respect to the duality (10.24) we have:

$$\mathbb{D}(\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda})) \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{-\check\lambda}). \tag{10.25}$$

10.8.4. We will now give a conceptual proof of the identification stated Sect. 10.7.8, i.e., that for the family

$$c^{\vee,\mu} := (\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda-\mu+2\rho}))^\vee[-d]$$

the resulting object $c^{\vee,\mathrm{enh}}$ identifies with $\mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{-\check\lambda})$.

Indeed, this follows from (10.23) and (10.25) using the fact that the functor $\jmath_*$ is fully faithful and that the initial family

$$c^\mu := \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda+\mu})$$

identifies with $\jmath_*(\mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\check\lambda}))$.

10.9. **Cohomology of the small quantum group.** In this subsection we will show how Theorems 10.7.4 and 10.6.3 allow to express the functor of cohomology with respect to $u_q(N)$ on the Kazhdan-Lusztig side.

10.9.1. Consider the functor

$$\mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Vect},$$

given by

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(u_q(N), \mathcal{M})^{\check\lambda}.$$

In this subsection we will explain what functor it corresponds to under the equivalences

$$\mathsf{F}_{-\kappa} : \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}_q(G)_{\mathrm{ren}} \text{ and } \mathsf{F}_\kappa : \mathrm{KL}(G, \kappa) \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}.$$

10.9.2. Recall the object

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa} := \mathrm{Av}_*^{I^-}(\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{-\kappa}) \in \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^I.$$

It is acted on by $\check B$, and in particular by $\check T$. Set

$$\mathcal{F}^{\frac{\infty}{2}}_{-\kappa} := \mathbf{inv}_{\check T}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{-\kappa}).$$

Explicitly,

$$\mathcal{F}^{\frac{\infty}{2}}_{-\kappa} \simeq \underset{\mu}{\mathrm{colim}}\, j_{-\mu,*} \star \mathrm{Sat}(V^\mu).$$

10.9.3. Recall also the object

$$\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \in \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})},$$

see Sect. 10.7.2. It is also acted on by $\check B$ and hence by $\check T$.

Let

$$\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2}}_{\kappa} \in \text{D-mod}_\kappa(\mathrm{Gr}_G)^I$$

be the image of $\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}$ under the inversion equivalence (10.12). Set

$$\mathcal{F}^{-,\frac{\infty}{2}}_{\kappa} := \mathbf{inv}_{\check T}(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2}}_{\kappa}).$$

Explicitly,

$$\mathcal{F}^{-,\frac{\infty}{2}}_{\kappa} \simeq \underset{\mu}{\mathrm{colim}}\, J^{\mathbb{D}}_\mu \star \mathrm{Sat}(V^{-w_0(\mu)})[2d].$$

10.9.4. We will prove:

Theorem 10.9.5.
(a) *Under the equivalence* $\mathsf{F}_{-\kappa} : \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}_q(G)_{\mathrm{ren}}$, *the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(u_q(N), \mathcal{M})^{\check{\lambda}}, \quad \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Vect}$$

corresponds to

$$\mathcal{M} \mapsto \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check{\lambda}}_{-\kappa}, \mathcal{F}^{\frac{\infty}{2}}_{-\kappa} \star \mathcal{M}).$$

(b) *Under the equivalence* $\mathsf{F}_{\kappa} : \mathrm{KL}(G, \kappa) \simeq \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}$, *the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(u_{q^{-1}}(N), \mathcal{M})^{\check{\lambda}}, \quad \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}} \to \mathrm{Vect}$$

corresponds to

$$\mathcal{M} \mapsto \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{F}^{-,\frac{\infty}{2}}_{\kappa} \star \mathcal{M})^{\check{\lambda}}[-2d].$$

Remark 10.9.6. Note that we can rewrite the RHS in Theorem 10.9.5(a) also as

$$\mathcal{M} \mapsto \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^{T(\mathcal{O})}_{-\kappa}}(\mathbb{W}^{\check{\lambda}}_{-\kappa}, \mathrm{IC}^{\frac{\infty}{2}}_{-\kappa} \star \mathcal{M}).$$

Similarly, set

$$\mathrm{IC}^{-,\frac{\infty}{2}}_{\kappa} = w_0 \cdot \mathrm{IC}^{\frac{\infty}{2}}_{\kappa} \in \text{D-mod}_\kappa(\mathrm{Gr}_G)^{N^-(\mathcal{K})\cdot T(\mathcal{O})},$$

and note that

$$\mathcal{F}^{-,\frac{\infty}{2}}_{\kappa} \simeq \mathrm{Av}^{I/T(\mathcal{O})}_*(\mathrm{IC}^{-,\frac{\infty}{2}}_{\kappa})[2d] \simeq \mathrm{Av}^{N(\mathcal{O})}_*(\mathrm{IC}^{-,\frac{\infty}{2}}_{\kappa})[2d].$$

Then the RHS in Theorem 10.9.5(b) can we rewritten as

$$\mathcal{M} \mapsto \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathrm{IC}^{-,\frac{\infty}{2}}_{\kappa} \star \mathcal{M})^{\check{\lambda}}.$$

10.9.7. The rest of this section is devoted to the proof of Theorem 10.9.5.

First, we note that the functor

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(u_q(N), \mathcal{M})^{\check{\lambda}}, \quad \mathrm{Rep}_q(G)_{\mathrm{ren}} \to \mathrm{Vect}$$

can be interpreted as the functor of pairing with the object

$$\mathbf{coind}_{\mathrm{sml,grd}\to\mathrm{big}} \circ \mathbf{coind}_{\mathrm{sml,grd}^+\to\mathrm{sml,grd}}(k^{-\check{\lambda}}) \in \mathrm{Rep}_{q^{-1}}(G)_{\mathrm{ren}}.$$

Note also that the above object

$$\mathbf{coind}_{\mathrm{sml,grd}\to\mathrm{big}} \circ \mathbf{coind}_{\mathrm{sml,grd}^+\to\mathrm{sml,grd}}(k^{-\check{\lambda}}) \simeq \mathbf{coind}_{\mathrm{sml,grd}^+\to\mathrm{big}}(k^{-\check{\lambda}})$$

can be canonically identified with

$$\mathbf{inv}_{\check{T}}\left(\mathbf{coind}_{\mathrm{sml}^+\to\mathrm{big}}(k^{-\check{\lambda}})\right) \simeq \mathbf{inv}_{\check{T}}\left(\mathbf{coind}_{\mathrm{sml}\to\mathrm{big}} \circ \mathbf{coind}_{\mathrm{sm}^+\to\mathrm{sm}}(k^{-\check{\lambda}})\right).$$

10.9.8. Hence, applying Theorems 10.6.3 and Theorem 10.7.4, respectively, we rewrite the corresponding functors on the Kazhdan-Lusztig side as

$$\mathcal{M} \mapsto \langle \mathcal{F}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa} \star \mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{\kappa}), \mathcal{M}\rangle$$

(for point (a)), and

$$\mathcal{M} \mapsto \langle \mathcal{F}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star J_{-2\rho}[-d] \star \mathbb{W}^{-\check{\lambda}+2\rho-2\check{\rho}}_{-\kappa}, \mathcal{M}\rangle$$

(for point (b)), respectively (here for point (b) we have also used Corollary 7.4.8 to pass from $\mathbf{coind}_{\mathrm{sm}^+\to\mathrm{sm}}(k^{-\check{\lambda}})$ to $\mathbf{ind}_{\mathrm{sm}^+\to\mathrm{sm}}(k^{-\check{\lambda}+2\rho-2\check{\rho}})$).

We have:

$$\langle \mathcal{F}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa} \star \mathbb{D}(\mathbb{W}^{\check{\lambda}}_{-\kappa}), \mathcal{M}\rangle \simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check{\lambda}}_{-\kappa}, \mathcal{F}^{\frac{\infty}{2}}_{-\kappa} \star \mathcal{M}),$$

thereby proving point (a).

For point (b) we have:

$$\langle \mathcal{F}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star J_{-2\rho}[-d] \star \mathbb{W}^{-\check\lambda+2\rho-2\check\rho}_{-\kappa}, \mathcal{M}\rangle \simeq \langle \mathcal{F}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star \mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[-d], \mathcal{M}\rangle \simeq \langle \mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[d], \mathcal{F}^{-,\frac{\infty}{2}}_{\kappa} \star \mathcal{M}\rangle[-2d].$$

Finally, we recall that for $\mathcal{M}' \in \widehat{\mathfrak{g}}\text{-mod}^I_\kappa$ we have

$$\langle \mathbb{W}^{-\check\lambda-2\check\rho}_{-\kappa}[d], \mathcal{M}'\rangle \simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathcal{M}')^{\check\lambda},$$

by Corollary 2.4.4.

□

11. Compatibility with the Arkhipov-Bezrukavnikov action action

In this section we introduce one more requirement on the conjectural equivalence (9.7). This will provide a conceptual explanation of the isomorphisms of Theorems 10.7.4 and 10.6.3.

In the next section we will use it to give an interpretation of the functor of cohomology with respect to $U_q^{\mathrm{DK}}(N)$ on the Kac-Moody side.

11.1. **The Arkhipov-Bezrukavnikov action: a reminder.**

11.1.1. Recall that in [AB, Sect. 3] a monoidal functor

$$\mathrm{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check B)) \to \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \tag{11.1}$$

was constructed.

We will need the following pieces of information regarding this functor:

- For $\mu \in \Lambda$,
$$\mathsf{q}^*(k^\mu) \mapsto J_\mu,$$
where q denotes the projection
$$\check{\mathfrak{n}}/\operatorname{Ad}(\check B) \to \mathrm{pt}/\check B.$$
- The resulting action of $\mathrm{Rep}(\check G)$ on $\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^I$ obtained via
$$\mathrm{Rep}(\check G) \overset{\mathbf{oblv}_{\check G\to\check B}}{\longrightarrow} \mathrm{Rep}(\check B) \overset{\mathsf{q}^*}{\to} \mathrm{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check B)) \tag{11.2}$$
and the action of $\text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I$ on $\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^I$, is given by
$$\mathcal{F} \mapsto \mathcal{F} \star \mathrm{Sat}(-).$$

11.1.2. Thus, if $\mathcal{C}$ is a category equipped with an action of $G(\mathcal{K})$ at level $-\kappa$ (see Sect. 1.2.2), then the category $\mathcal{C}^I$ acquires an action of $\mathrm{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check B))$ with the following properties:

- For $c \in \mathcal{C}$ and $\mu \in \Lambda$, we have
$$\mathsf{q}^*(k^\mu) \otimes c := J_\mu \star c.$$
- The forgetful functor $\mathbf{oblv}_{G(\mathcal{O})/I} : \mathcal{C}^{G(\mathcal{O})} \to \mathcal{C}^I$ intertwines the $\mathrm{Rep}(\check G)$-action on $\mathcal{C}^{G(\mathcal{O})}$ coming from Sat and the $\mathrm{Rep}(\check G)$-action on $\mathcal{C}^I$ coming from (11.2).

11.1.3. It follows that the functor $\mathbf{oblv}_{G(\mathcal{O})/I} : \mathcal{C}^{G(\mathcal{O})} \to \mathcal{C}^I$ canonically factors as

$$\mathcal{C}^{G(\mathcal{O})} \overset{\mathbf{oblv}_{\check G\to\check B}}{\longrightarrow} \mathrm{Rep}(\check B) \underset{\mathrm{Rep}(\check G)}{\otimes} \mathcal{C}^{G(\mathcal{O})} \overset{(\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}}}{\longrightarrow} \mathcal{C}^I,$$

and its left adjoint $\mathrm{Av}_!^{G(\mathcal{O})/I}$ factors as

$$\mathcal{C}^{G(\mathcal{O})} \overset{\mathbf{ind}_{\check G\to\check B}}{\longleftarrow} \mathrm{Rep}(\check B) \underset{\mathrm{Rep}(\check G)}{\otimes} \mathcal{C}^{G(\mathcal{O})} \overset{(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}}{\longleftarrow} \mathcal{C}^I,$$

where $(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}$ is the left adjoint of $(\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}}$, and is a functor of $\mathrm{Rep}(\check B)$-module categories.

11.1.4. Similarly, $\mathrm{Av}_*^{G(\mathcal{O})/I}$ factors as

$$\mathcal{C}^{G(\mathcal{O})} \overset{\mathbf{coind}_{\check{G}\to\check{B}}}{\longleftarrow} \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})} \overset{(\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}}}{\longleftarrow} \mathcal{C}^I,$$

where $(\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}}$ is the right adjoint of $(\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}}$, and is a functor of of $\mathrm{Rep}(\check{B})$-modules categories.

Recall now that

$$\mathrm{Av}_!^{G(\mathcal{O})/I} \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}[2d] \text{ and } \mathbf{ind}_{\check{G}\to\check{B}}(-) \simeq \mathbf{coind}_{\check{G}\to\check{B}}(k^{2\rho}\otimes -)[d].$$

It follows formally that we have a canonical isomorphism of functors of $\mathrm{Rep}(\check{B})$-modules categories

$$(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}} \simeq (\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}} \circ (J_{-2\rho}\star -)[d].$$

11.1.5. The following result was established in [FG3, Theorem 7.3.1]:

Theorem 11.1.6. *The functor*

$$(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}} : \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \to \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})}$$

is given by

$$\mathcal{F}' \mapsto (\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} \star J_{-2\rho}\star \mathcal{F}'[-d].$$

Corollary 11.1.7. *For any $\mathcal{C}$ with an action of $G(\mathcal{K})$ at level $-\kappa$, the functors*

$$(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}} \textit{ and } (\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}},$$

are given by convolution with the objects

$$(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} \star J_{-2\rho}[-d] \textit{ and } (\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}[-2d],$$

respectively.

11.1.8. Let us now consider the situation at the positive level. We define the monoidal functor

$$\mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B})) \to \text{D-mod}_{\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \tag{11.3}$$

by applying monoidal duality

$$\mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B}))_c \to \mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B}))_c$$

(i.e., the naive duality on perfect complexes) and Verdier duality

$$\mathbb{D} : (\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I)_c \to (\text{D-mod}_{\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I)_c.$$

Thus, for a category $\mathcal{C}$ with an action of $G(\mathcal{K})$ at level κ, we obtain an action of $\mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B}))$ on $\mathcal{C}^I$ with the following properties:

- For $c\in\mathcal{C}$ and $\mu\in\Lambda$, we have
$$\mathsf{q}^*(k^\mu)\otimes c := J^{\mathbb{D}}_{-\mu}\star c.$$
- The forgetful functor $\mathbf{oblv}_{G(\mathcal{O})/I} : \mathcal{C}^{G(\mathcal{O})} \to \mathcal{C}^I$ intertwines the $\mathrm{Rep}(\check{G})$-action on $\mathcal{C}^{G(\mathcal{O})}$ coming from $\mathrm{Sat}\circ\tau$ and the $\mathrm{Rep}(\check{G})$-action on $\mathcal{C}^I$ coming from (11.2).

11.1.9. We still have the pair of functors

$$(\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}} : \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})} \rightleftarrows \mathcal{C}^I : (\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}},$$

and it follows formally from Theorem 11.1.6 that the functor $(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}$ is given by convolution with the object

$$(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa})^{\mathrm{enh}} \star J^{\mathbb{D}}_{2\rho}[d] \in \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \text{D-mod}_{\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})}.$$

11.2. **Compatibility with the equivalences $\mathsf{F}_{-\kappa}$ and F_κ.**

11.2.1. We are now ready state one more expected property of the conjectural equivalence $\mathsf{F}_{-\kappa}$:

The equivalence

$$\mathsf{F}_{-\kappa}:\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}\to \operatorname{Rep}_q^{\mathrm{mxd}}(G)$$

intertwines the $\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))$-action on $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ and on $\operatorname{Rep}_q^{\mathrm{mxd}}(G)$ (the latter is from Conjecture 7.2.3). Moreover, (9.8) is a commutative diagram of categories acted on by $\operatorname{Rep}_q(G)$.

As a formal consequence we obtain the following commutative diagram

$$\begin{array}{ccc} \operatorname{Rep}(\check{B})\underset{\operatorname{Rep}(\check{G})}{\otimes}\operatorname{KL}(G,-\kappa) & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}(\check{B})\underset{\operatorname{Rep}(\check{G})}{\otimes}\operatorname{Rep}_q(G)_{\mathrm{ren}} \\ {\scriptstyle (\operatorname{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{r}_{\text{baby-ren}\to\mathrm{ren}}\circ\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} & \xrightarrow{\mathsf{F}_{-\kappa}} & \operatorname{Rep}_q^{\mathrm{mxd}}(G) \end{array} \tag{11.4}$$

of categories acted on by $\operatorname{Rep}(\check{B})$.

11.2.2. Let us show how commutative diagram (11.4) explains the result of Theorem 10.7.4: indeed, we apply both circuits to $\mathbb{W}^{\check{\lambda}}_{-\kappa}$, noting that

$$\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathbb{M}^{\check{\lambda}}_{q,\mathrm{mxd}})\simeq \mathbb{M}^{\check{\lambda}}_{q,\frac{1}{2}}\simeq \mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}),$$

and use Corollary 11.1.7.

11.2.3. By duality, we obtain that the equivalence

$$\mathsf{F}_{\kappa}:\widehat{\mathfrak{g}}\text{-mod}^I_{\kappa}\to \operatorname{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)$$

intertwines the $\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))$-action on $\widehat{\mathfrak{g}}\text{-mod}^I_{\kappa}$ from Sect. 11.1.8, and on $\operatorname{Rep}_{q^{-1}}^{\mathrm{mxd}}(G)$, and we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Rep}(\check{B})\underset{\operatorname{Rep}(\check{G})}{\otimes}\operatorname{KL}(G,\kappa) & \xrightarrow{\mathsf{F}_{\kappa}} & \operatorname{Rep}(\check{B})\underset{\operatorname{Rep}(\check{G})}{\otimes}\operatorname{Rep}_{q^{-1}}(G)_{\mathrm{ren}} \\ {\scriptstyle (\operatorname{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{r}_{\text{baby-ren}\to\mathrm{ren}}\circ\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}} \\ \widehat{\mathfrak{g}}\text{-mod}^I_{\kappa} & \xrightarrow{\mathsf{F}_{\kappa}} & \operatorname{Rep}_{q^{-1}}^{\mathrm{mxd}}(G) \end{array} \tag{11.5}$$

of categories acted on by $\operatorname{Rep}(\check{B})$.

11.2.4. Note that (11.5) gives a conceptual explanation of Theorem 10.6.3. Indeed, let us apply both circuits of the diagram to $\mathbb{D}(\mathbb{W}^{-\check{\lambda}-2\rho}_{-\kappa})[d]$.

On the one hand,

$$\mathsf{F}_{\kappa}(\mathbb{D}(\mathbb{W}^{-\check{\lambda}-2\rho}_{-\kappa})[d])\simeq \mathbb{D}^{\mathrm{can}}(\mathbb{M}^{-\check{\lambda}-2\rho}_{q,\mathrm{mxd}})[-d]\simeq \mathbb{M}^{\check{\lambda}+2\rho-2\check{\rho}}_{q^{-1},\mathrm{mxd}},$$

and

$$\mathbf{ind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathbb{M}^{\check{\lambda}+2\rho-2\check{\rho}}_{q^{-1},\mathrm{mxd}})\simeq \mathbf{ind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}+2\rho-2\check{\rho}}_{q^{-1},\mathrm{mxd}})\simeq \mathbf{coind}_{\mathrm{Lus}^+\to\frac{1}{2}}(k^{\check{\lambda}}_{q^{-1},\mathrm{mxd}}).$$

On the other hand,

$$(\operatorname{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}}(\mathbb{D}(\mathbb{W}^{-\check{\lambda}-2\rho}_{-\kappa})[d])\simeq (\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa})^{\mathrm{enh}}\star J^{\mathbb{D}}_{2\rho}[d]\star\mathbb{D}(\mathbb{W}^{-\check{\lambda}-2\rho}_{-\kappa}[d])\simeq$$
$$\simeq (\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa})^{\mathrm{enh}}\star\mathbb{D}(J_{2\rho}\star\mathbb{W}^{-\check{\lambda}-2\rho}_{-\kappa})\simeq (\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},\mathrm{inv}}_{\kappa})^{\mathrm{enh}}\star\mathbb{D}(\mathbb{W}^{-\check{\lambda}}_{-\kappa}),$$

as required.

11.3. **More on spherical vs Iwahori.** The material in this subsection is included for the sake of completeness. We will not need in the sequel.

11.3.1. We return to the general setting of Sect. 11.1.2. The following is established in [FG3, Main Theorem 4, Sect. 5][4]:

Theorem 11.3.2. *The functor of* $\operatorname{Rep}(\check{B})$*-module categories*

$$(\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}} : \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \to \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})}$$

factors canonically as

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \xrightarrow{\iota^*} \operatorname{QCoh}(\mathrm{pt}/\check{B}) \underset{\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \overset{\mathfrak{r}_{\mathrm{geom}}}{\to}$$
$$\to \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})},$$

where the functor $\mathfrak{r}_{\mathrm{geom}}$ *is fully faithful.*

Remark 11.3.3. The assertion of Theorem 11.3.2 is in fact a formal consequence of Bezrukavnikov's theorem in [Bez], which we will review in Sect. 13.1. The corresponding assertion on the coherent side is that the functor

$$\operatorname{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}})/\operatorname{Ad}(\check{B}) \xrightarrow{\iota^*} \operatorname{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}})/\operatorname{Ad}(\check{B})$$

factors as

$$\operatorname{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}})/\operatorname{Ad}(\check{B}) \xrightarrow{\iota^*} \operatorname{QCoh}(\mathrm{pt}/\check{B}) \underset{\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \operatorname{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}})/\operatorname{Ad}(\check{B}) \to$$
$$\to \operatorname{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}})/\operatorname{Ad}(\check{B}),$$

where the second arrow is fully faithful (the latter can be seen from the theory of singular of [AriG]).

11.3.4. It follows formally from Theorem 11.3.2 that for $\mathcal{C}$ as in Sect. 11.1.2, the functor

$$(\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}} : \mathcal{C}^I \to \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})}$$

factors canonically as

$$\mathcal{C}^I \xrightarrow{\iota^*} \operatorname{QCoh}(\mathrm{pt}/\check{B}) \underset{\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \mathcal{C}^I \overset{\mathfrak{r}_{\mathcal{C}}}{\to} \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})},$$

where the functor $\mathfrak{r}_{\mathcal{C}}$ is fully faithful.

Remark 11.3.5. The essential image of the functor

$$\mathfrak{r}_{\mathcal{C}} : \operatorname{QCoh}(\mathrm{pt}/\check{B}) \underset{\operatorname{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \mathcal{C}^I \to \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})} \tag{11.6}$$

can be described explicitly via the *derived Satake equivalence.*

Namely, the category $\mathcal{C}^{G(\mathcal{O})}$ is acted on by the monoidal category $\operatorname{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt})/\check{G})$, and hence the category $\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})}$ is acted on by the monoidal category $\operatorname{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt})/\check{B})$. Hence, objects in $\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})}$ admit a singular support, which is a conical $\operatorname{Ad}(\check{B})$-invariant Zariski-closed subset in $\check{\mathfrak{g}}^\vee$ (which is automatically contained in the nilpotent cone). It follows from [AriG, Proposition7.4.3] and Bezrukavnikov's theory [Bez] that the image of the functor $\mathfrak{r}_{\mathcal{C}}$ of (11.6) is the full subcategory consisting of objects whose singular support is contained in $(\check{\mathfrak{g}}/\check{\mathfrak{n}})^\vee \subset \check{\mathfrak{g}}^\vee$.

[4]The statement of [FG3, Main Theorem 4, Sect. 5] contains a typo: the functor Υ goes in the opposite direction.

11.3.6. Let us apply this to $\mathcal{C} = \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}$. We obtain that the functor

$$(\mathrm{Av}_!^{G(\mathcal{O})/I})^{\mathrm{enh}} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I \to \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G, -\kappa)$$

factors via a fully faithful functor

$$\mathfrak{r}_{\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}} : \mathrm{QCoh}(\mathrm{pt}/\check{B}) \underset{\mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B}))}{\otimes} \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I \to \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{KL}(G, -\kappa).$$

Recall that on the quantum group side we have

$$\mathrm{QCoh}(\mathrm{pt}/\check{B}) \underset{\mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B}))}{\otimes} \mathrm{Rep}_q^{\mathrm{mxd}}(G) \simeq \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}.$$

Thus, the functor $\mathfrak{r}_{\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}}$ corresponds to the fully faithful embedding

$$\mathfrak{r}_{\text{baby-ren}\to\text{ren}} : \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}} \to \mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G)_{\text{ren}}.$$

12. Cohomology of the De Concini-Kac quantum group via Kac-Moody algebras

In this section we will show how Conjecture 9.2.2 allows to interpret the functor of cohomology with respect to $U_q^{\mathrm{DK}}(N)$ on the Kac-Moody side of the equivalences $\mathsf{F}_{-\kappa}$ and F_κ.

12.1. The statement.

12.1.1. Let us assume the existence of the equivalence

$$\mathsf{F}_{-\kappa} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I \simeq \mathrm{Rep}_q^{\mathrm{mxd}}(G) \tag{12.1}$$

that satisfies the additional compatibility of Sect. 11.2.1, and hence by duality also of the equivalence

$$\mathsf{F}_{\kappa} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa}^I \simeq \mathrm{Rep}_{q^{-1}}^{\mathrm{mxd}}(G) \tag{12.2}$$

with the corresponding additional compatibility.

For $\check{\lambda} \in \check{\Lambda}$, on the quantum group side we consider the functor

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-), -)^{\check{\lambda}} : \mathrm{Rep}_q^{\mathrm{mxd}}(G) \to \mathrm{Vect}\,.$$

We now wish to describe what this functor corresponds to on the Kac-Moody side.

12.1.2. We will prove:

Theorem 12.1.3. *Assume Conjecture 9.2.2. Then:*

(a) *Under the equivalence* (12.1)*, the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-), -)^{\check{\lambda}}$$

corresponds to the functor

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), -)^{\check{\lambda}} : \widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I \to \mathrm{Vect}\,.$$

(b) *Under the equivalence* (12.2)*, the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_{q^{-1}}^{\mathrm{DK}}(N^-), -)^{\check{\lambda}}$$

corresponds to the functor

$$\mathcal{M} \mapsto \mathrm{C}^{\frac{\infty}{2}}\left(\mathfrak{n}^-(\mathcal{K}), \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, j_{-\mu,*} \star j_{\mu,*} \star \mathcal{M}\right)^{\check{\lambda}}.$$

12.1.4. As a plausibility check for Theorem 12.1.3(a), let us check that both sides of the theorem evaluate in the same way on

$$\mathsf{F}_{-\kappa}(\mathbb{W}^{\check\lambda}_{-\kappa}) \simeq \mathbb{M}^{\check\lambda}_{q,\mathrm{mxd}}.$$

Indeed,

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathbb{W}^{\check\lambda}_{-\kappa})^{\check\lambda'} = \begin{cases} k[-d] \text{ if } \check\lambda' = \check\lambda + 2\check\rho \\ 0 \text{ otherwise,} \end{cases}$$

by Proposition 2.4.7, and

$$\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-), \mathbb{M}^{\check\lambda}_{q,\mathrm{mxd}})^{\check\lambda'} = \begin{cases} k[-d] \text{ if } \check\lambda' = \check\lambda + 2\check\rho \\ 0 \text{ otherwise,} \end{cases}$$

by Corollary 7.4.4.

12.1.5. As a formal consequence of Theorem 12.1.3 we obtain the following statement (which is conjectural, since Theorem 12.1.3 assumes Conjecture 9.2.2) pertaining to the original Kazhdan-Lusztig equivalence:

Conjecture 12.1.6.
(a) *Under the equivalence* $\mathsf{F}_{-\kappa} : \mathrm{KL}(G, -\kappa) \simeq \mathrm{Rep}_q(G)$*, the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-), -)^{\check\lambda}$$

corresponds to the functor

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), -)^{\check\lambda}.$$

(b) *Under the equivalence* $\mathsf{F}_{\kappa} : \mathrm{KL}(G, \kappa) \simeq \mathrm{Rep}_{q^{-1}}(G)$*, the functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_{q^{-1}}^{\mathrm{DK}}(N^-), -)^{\check\lambda}$$

corresponds to the functor

$$\mathcal{M} \mapsto \mathrm{C}^{\frac{\infty}{2}}\left(\mathfrak{n}^-(\mathcal{K}), \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, j_{-\mu,*} \star j_{\mu,*} \star \mathcal{M}\right)^{\check\lambda}.$$

The rest of this section is devoted to the proof of Theorem 12.1.3.

12.2. **Cohomology of** $U_q^{\mathrm{DK}}(N^-)$ **via the cohomology of** $U_q^{\mathrm{Lus}}(N)$**.** We begin the proof of Theorem 12.1.3 by showing that there is a certain categorical procedure that allows to express the cohomology of $U_q^{\mathrm{DK}}(N^-)$ in terms of the cohomology of $U_q^{\mathrm{Lus}}(N)$.

The rest of the proof will essentially consist of applying the same procedure on the Kac-Moody side.

12.2.1. Let $\mathcal{C}$ be a category equipped with an action of $\mathrm{Rep}(\check G)$. For $\mu \in \Lambda^+$ consider the following endo-functor of $\mathrm{Rep}(\check B) \underset{\mathrm{Rep}(\check G)}{\otimes} \mathcal{C}$,

$$c \mapsto P_\mu(c) := k^{-\mu} \otimes \left(\mathbf{oblv}_{\check G\to\check B} \circ \mathbf{coind}_{\check B\to\check G}(k^{w_0(\mu)} \otimes c)\right).$$

Pick a representative $w_0' \in \mathrm{Norm}_{\check G}(\check T)$ of $w_0 \in W$. The action of w_0' trivializes the lowest weight line in every V^μ; in particular we obtain a canonical map of $\check B$-modules

$$V^{-w_0(\mu)} \to k^{-\mu} \tag{12.3}$$

or, equivalently, an identification

$$V^{-w_0(\mu)} \simeq (V^\mu)^\vee.$$

The choice of w_0' defines on the family of functors $\mu \rightsquigarrow P_\mu$ a structure of directed family with the transition maps being

$$k^{-\mu_2} \otimes \Big(\mathbf{oblv}_{\check{G}\to\check{B}} \circ \mathbf{coind}_{\check{B}\to\check{G}}(k^{w_0(\mu_2)} \otimes c)\Big) \to$$
$$\to k^{-\mu_2} \otimes V^{-w_0(\mu_1)} \otimes (V^{-w_0(\mu_1)})^\vee \otimes \Big(\mathbf{oblv}_{\check{G}\to\check{B}} \circ \mathbf{coind}_{\check{B}\to\check{G}}(k^{w_0(\mu_2)} \otimes c)\Big) \overset{(12.3)}{\longrightarrow}$$
$$\to k^{-\mu_1-\mu_2} \otimes (V^{-w_0(\mu_1)})^\vee \otimes \Big(\mathbf{oblv}_{\check{G}\to\check{B}} \circ \mathbf{coind}_{\check{B}\to\check{G}}(k^{w_0(\mu_2)} \otimes c)\Big) \to$$
$$\to k^{-\mu_1-\mu_2} \otimes \Big(\mathbf{oblv}_{\check{G}\to\check{B}} \circ \mathbf{coind}_{\check{B}\to\check{G}}(k^{w_0(\mu_1+\mu_2)} \otimes c)\Big),$$

where the last arrow comes from the canonical map

$$(V^\mu)^\vee \to \mathbf{coind}_{\check{B}\to\check{G}}(k^{-\mu}), \quad \mu \in \Lambda^+$$

dual to $\mathbf{ind}_{\check{B}\to\check{G}}(k^\mu) \simeq V^\mu$.

Denote

$$P := \underset{\mu\in\Lambda^+}{\operatorname{colim}}\, P_\mu.$$

12.2.2. Take $\mathcal{C} := \operatorname{Rep}_q(G)$, and consider the above family of endo-functors of

$$\operatorname{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}} \simeq \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \operatorname{Rep}_q(G).$$

Consider now the following family of endo-functors of $\operatorname{Rep}_q^{\text{mxd}}(G)$:

$$\mathcal{M} \mapsto \widetilde{P}_\mu(\mathcal{M}) := \mathbf{oblv}_{\frac{1}{2}\to\text{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ P_\mu \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\text{mxd}\to\frac{1}{2}}(\mathcal{M}),$$

which we can also rewrite as

$$\mathcal{M} \mapsto k^{-\mu} \otimes \Big(\mathbf{oblv}_{\text{big}\to\text{mxd}} \circ \mathbf{coind}_{\text{mxd}\to\text{big}}(k^{w_0(\mu)} \otimes \mathcal{M})\Big).$$

We claim:

Proposition 12.2.3. *The functor*

$$\mathcal{M} \mapsto \mathrm{C}^\cdot(U_q^{\text{DK}}(N^-), \mathcal{M})^{\check{\lambda}}, \quad \operatorname{Rep}_q^{\text{mxd}}(G) \to \operatorname{Vect}$$

identifies canonically with

$$\underset{\mu\in\Lambda^+}{\operatorname{colim}}\, \mathrm{C}^\cdot(U_q^{\text{Lus}}(N), \widetilde{P}_\mu(\mathcal{M}))^{w_0(\check{\lambda})}.$$

The rest of this subsection is devoted to the proof of Proposition 12.2.3.

12.2.4. We will deduce Proposition 12.2.3 from the following statement in the abstract context of Sect. 12.2.1.

Note that the action of w_0 defines an endo-functor of $\operatorname{Rep}(\check{T})$. A choice of a lift of w_0 to an element $w_0' \in \operatorname{Norm}_{\check{G}}(\check{T})$ endows this endo-functor with a structure of endo-functor of $\operatorname{Rep}(\check{T})$ as a module over $\operatorname{Rep}(\check{G})$. In particular, we have a well-defined endo-functor

$$\operatorname{Rep}(\check{T}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{w_0'}{\to} \operatorname{Rep}(\check{T}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}.$$

We claim:

Proposition 12.2.5. *The functor P identifies canonically with the composition*

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{\mathbf{oblv}_{\check{B}\to\check{T}}}{\longrightarrow} \operatorname{Rep}(\check{T}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{w_0'}{\to} \operatorname{Rep}(\check{T}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C} \overset{\mathbf{coind}_{\check{T}\to\check{B}}}{\longrightarrow} \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}.$$

12.2.6. Let us first show how Proposition 12.2.5 implies Proposition 12.2.3.

First off, we note that both sides are continuous functors out of $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$ (for the LHS this follows from the definition of $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$, and for the RHS from Corollary 5.2.7). Hence, we can assume that $\mathcal{M}$ is compact. Denote $\mathcal{M}' := \mathbf{coind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathcal{M})$. This is a compact object in $\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}$.

We rewrite

$$\mathrm{C}^{\cdot}(U_q^{\mathrm{Lus}}(N), \widetilde{P}_\mu(\mathcal{M}))^{w_0(\check{\lambda})} \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{mxd}}(G)}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\mathrm{mxd}}, \mathbf{oblv}_{\frac{1}{2}\to\mathrm{mxd}} \circ \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ P_\mu \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}} \circ \mathbf{coind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathcal{M})\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\frac{1}{2}}, \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ P_\mu \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right),$$

and hence

$$\underset{\mu\in\Lambda^+}{\mathrm{colim}}\ \mathrm{C}^{\cdot}(U_q^{\mathrm{Lus}}(N), \widetilde{P}_\mu(\mathcal{M}))^{w_0(\check{\lambda})} \simeq$$
$$\simeq \underset{\mu\in\Lambda^+}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\frac{1}{2}}, \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ P_\mu \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{baby-ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\frac{1}{2}}, \mathfrak{s}_{\text{ren}\to\text{baby-ren}} \circ P \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\frac{1}{2}}, P \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right)$$

Now using Proposition 12.2.5 we rewrite the latter expression as

$$\mathcal{H}om_{\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\text{ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\frac{1}{2}}, \mathbf{coind}_{\check{T}\to\check{B}} \circ w_0' \circ \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right)$$

and further by adjunction as

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\mathrm{sml}}, w_0' \circ \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right).$$

Since both sides are compact as objects in $\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\text{ren}}$, and since the functor $\mathfrak{s}$ is fully faithful on compact objects, the latter expression maps isomorphically to

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\mathrm{sml}}, \mathfrak{s} \circ w_0' \circ \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\mathrm{sml}}, w_0' \circ \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{s} \circ \mathfrak{r}_{\text{baby-ren}\to\text{ren}}(\mathcal{M}')\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)}\left(\mathbb{M}^{w_0(\check{\lambda})}_{q,\mathrm{sml}}, w_0' \circ \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{s}_{\text{baby}}(\mathcal{M}')\right).$$

Next, we rewrite

$$\mathrm{C}^{\cdot}(U_q^{\mathrm{DK}}(N^-), \mathcal{M})^{\check{\lambda}} \simeq \mathrm{C}^{\cdot}\left(u_q(N^-), \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml,grd}} \circ \mathfrak{s}_{\text{baby}} \circ \mathbf{coind}_{\mathrm{mxd}\to\frac{1}{2}}(\mathcal{M})\right)^{\check{\lambda}} \simeq$$
$$\simeq \mathrm{C}^{\cdot}\left(u_q(N^-), \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{s}_{\text{baby}}(\mathcal{M}')\right)^{\check{\lambda}},$$

which we further rewrite as

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)}\left(\mathbf{ind}_{\mathrm{sml}^-\to\mathrm{sml,grd}}(k^{\check{\lambda}}), \mathbf{oblv}_{\check{B}\to\check{T}} \circ \mathfrak{s}_{\text{baby}}(\mathcal{M}')\right). \tag{12.4}$$

We have

$$\mathrm{Rep}_q^{\mathrm{sml,grd}}(G) \simeq \mathrm{Rep}(\check{T}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Rep}_q(G).$$

Let us apply the functor w_0' and use the identification

$$w_0'(\mathbf{ind}_{\mathrm{sml}^-\to\mathrm{sml,grd}}(k^{\check{\lambda}})) \simeq \mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml,grd}}(k^{w_0(\check{\lambda})}).$$

Thus, we rewrite the expression in (12.4) as

$$\mathcal{H}om_{\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)}\left(\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml,grd}}(k^{w_0(\check{\lambda})}), w_0'\cdot\mathbf{oblv}_{\check{B}\to\check{T}}\circ\mathfrak{s}_{\mathrm{baby}}(\mathcal{M}')\right),$$

as desired.

12.2.7. *Proof of Proposition 12.2.5.* In order to prove Proposition 12.2.5 it suffices to consider the universal case, i.e., $\mathcal{C}=\mathrm{Rep}(\check{G})$. We need to establish an isomorphism between the two endo-functors of $\mathrm{Rep}(\check{B})$ as functors between $\mathrm{Rep}(\check{G})$-module categories.

We first construct a natural transformation

$$P\to\mathbf{coind}_{\check{T}\to\check{B}}\circ w_0'\circ\mathbf{oblv}_{\check{B}\to\check{T}}. \tag{12.5}$$

To do so, we need to construct a compatible family of natural transformations

$$P_\mu\to\mathbf{coind}_{\check{T}\to\check{B}}\circ w_0'\circ\mathbf{oblv}_{\check{B}\to\check{T}},\quad \mu\in\Lambda^+.$$

By adjunction, the latter amounts to a compatible system of natural transformations

$$\mathbf{oblv}_{\check{G}\to\check{T}}\circ\mathbf{coind}_{\check{B}\to\check{G}}(k^{w_0(\mu)}\otimes -)\to w_0'\circ(k^{w_0(\mu)}\otimes\mathbf{oblv}_{\check{B}\to\check{T}}(-)).$$

These natural transformations are obtained from the natural transformation

$$\mathbf{oblv}_{\check{G}\to\check{T}}\circ\mathbf{coind}_{\check{B}\to\check{G}}\simeq w_0'\circ\mathbf{oblv}_{\check{G}\to\check{T}}\circ\mathbf{coind}_{\check{B}\to\check{G}}\simeq$$
$$\simeq w_0'\circ\mathbf{oblv}_{\check{B}\to\check{T}}\circ\mathbf{oblv}_{\check{G}\to\check{B}}\circ\mathbf{coind}_{\check{B}\to\check{G}}\to w_0'\circ\mathbf{oblv}_{\check{B}\to\check{T}}.$$

In order to check that (12.5) is an isomorphism, it is enough to do so on the objects $k^\nu\in\mathrm{Rep}(\check{B})$. We have

$$P(k^\nu)\simeq\underset{\mu\in\Lambda^+}{\mathrm{colim}}\,k^{w_0(\mu)}\otimes\mathbf{oblv}_{\check{G}\to\check{B}}((V^{\mu-\nu})^\vee)\overset{w_0'}{\simeq}\underset{\mu\in\Lambda^+}{\mathrm{colim}}\,k^{w_0(\mu)}\otimes\mathbf{oblv}_{\check{G}\to\check{B}}(V^{w_0(-\mu+\nu)}),$$

which maps isomorphically to $\mathbf{coind}_{\check{T}\to\check{B}}(k^{w_0(\nu)})$ (see (10.8)), as required.

12.3. The P_μ functors on the geometric side.

12.3.1. Let $\mathcal{C}$ be a category with an action of $G(\mathcal{K})$ at level $-\kappa$. For $\mu\in\Lambda^+$, consider the endo-functor of $\mathcal{C}^I$, denoted $\widetilde{P}''_\mu$ of $\mathcal{C}^I$, given by convolution with the object $j_{-\mu,*}\star j_{\mu,*}\star j_{w_0,*}[-d]$. These functors form a directed family under the transitions maps specified in Corollary 3.4.7.

Denote

$$\widetilde{P}'':=\underset{\mu\in\Lambda^+}{\mathrm{colim}}\,\widetilde{P}''_\mu.$$

12.3.2. Consider now the object of $\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I$ equal to

$$j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*},\quad \mu\in\Lambda^+. \tag{12.6}$$

We note that for $\mu\in\Lambda^{++}$, we have

$$j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\in(\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I)^\heartsuit[d].$$

This follows from the fact that

$$p_*(j_{-\mu,*})\in(\text{D-mod}_{-\kappa}(\mathrm{Gr}_G^{\mathrm{aff}})^I)^\heartsuit[d]\text{ and }\mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0(\mu),*})\in\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})},$$

combined with [FG3, Lemma 9.1.4].

We consider the submonoid $\{0\}\cup\Lambda^{++}\subset\Lambda^+$; note that the resulting subcategory $\Lambda^{++}\subset\Lambda^+$ is cofinal.

We claim that the objects (12.6) form a directed family indexed by Λ^{++}. Namely, the transition maps are induced by the maps

$$p^*(\delta_{1,\mathrm{Gr}})\to j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*},\quad \mu\in\Lambda^{++} \tag{12.7}$$

constructed as follows:

By adjunction, the datum of a map (12.7) is equivalent to that of a map

$$\delta_{1,\mathrm{Gr}} \to j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star p_*(j_{w_0(\mu),*})$$

and further to that of a map

$$p_*(j_{\mu,!}) \to p^*(\delta_{1,\mathrm{Gr}}) \star p_*(j_{w_0(\mu),*}). \tag{12.8}$$

We note:

$$p_*(j_{w_0(\mu),*}) \simeq p_*(j_{w_0(\mu)\cdot w_0,*})[d] = p_*(j_{w_0\cdot\mu,*})[d].$$

The sought-for map (12.8) is

$$p_*(j_{\mu,!}) \simeq j_{w_0,!} \star p_*(j_{w_0\cdot\mu,!}) \to p^!(\delta_{1,\mathrm{Gr}})[-d] \star p_*(j_{w_0\cdot\mu,!}) \simeq$$
$$\simeq p^*(\delta_{1,\mathrm{Gr}})[d] \star p_*(j_{w_0\cdot\mu,!}) \to p^*(\delta_{1,\mathrm{Gr}})[d] \star p_*(j_{w_0\cdot\mu,*}).$$

Remark 12.3.3. Consider the object

$$\underset{\mu\in\Lambda^{++}}{\mathrm{colim}}\, j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} \in \text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}^{\mathrm{aff}}_G)^I.$$

In Sect. 13.5 we will see that under Bezrukavnikov's equivalence

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G})$$

the above object corresponds (up to a twist) to the dualizing sheaf on the big Schubert cell in

$$(\check{G}/\check{B} \times \check{G}/\check{B})^o/\check{G} \subset (\check{G}/\check{B} \times \check{G}/\check{B})/\check{G} \subset (\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G}.$$

12.3.4. Let $\widetilde{P}'_\mu$ denote the endo-functor of $\mathcal{C}^I$ given by convolution with $j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*}$. Denote

$$\widetilde{P}' := \underset{\mu\in\Lambda^{++}}{\mathrm{colim}}\, \widetilde{P}'_\mu.$$

Note that we have a natural transformation between the families

$$\widetilde{P}'_\mu \to \widetilde{P}''_\mu,$$

namely

$$j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} \to j_{-\mu,*} \star j_{w_0,*}[-d] \star j_{w_0(\mu),*} \simeq j_{-\mu,*} \star j_{w_0\cdot w_0(\mu),*}[-d] \simeq$$
$$\simeq j_{-\mu,*} \star j_{\mu\cdot w_0,*}[-d] \simeq j_{-\mu,*} \star j_{\mu,*} \star j_{w_0,*}[-d].$$

We will prove:

Theorem 12.3.5. *The resulting natural transformation $\widetilde{P}' \to \widetilde{P}''$ is an isomorphism.*

12.4. **Relation to semi-infinite cohomology on the Kac-Moody side: negative level case.** In this subsection we will use the formalism of the functors P_μ to prove point (a) of Theorem 12.1.3.

12.4.1. Let P_μ be the endo-functor of

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})}$$

defined in Sect. 12.2.1.

Define the endo-functor $\widetilde{P}_\mu$ of $\mathcal{C}^I$ by

$$\widetilde{P}_\mu := (\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}} \circ P_\mu \circ (\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}}.$$

Denote

$$\widetilde{P} := \underset{\mu\in\Lambda^{+}}{\mathrm{colim}}\, \widetilde{P}_\mu.$$

Note that we have a term-wise isomorphism

$$\widetilde{P}_\mu \simeq \widetilde{P}'_\mu. \tag{12.9}$$

We will prove:

Proposition 12.4.2. *The term-wise isomorphism* (12.9) *lifts to an isomorphism of directed families for* $\mu \in \Lambda^{++}$.

Remark 12.4.3. Recall that the transition maps for the family $\{\widetilde{P}_\mu\}$ came from the transition maps for the family $\{P_\mu\}$, and those depended on a choice of a lift of $w_0 \in W$ to an element $w_0' \in \mathrm{Norm}_{\check{G}}(\check{T})$. However, in the course of the proof of Proposition 12.4.2 we will see that Geometric Satake provides a particular choice of such a lift.

Corollary 12.4.4. *There exists an isomorphism of endo-functors* $\widetilde{P} \simeq \widetilde{P}'$.

12.4.5. Let us assume Proposition 12.4.2 (and hence Corollary 12.4.4) and prove Theorem 12.1.3(a):

According to Corollary 3.4.7, we have:

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}^-(\mathcal{K}), \mathcal{M})^{\check{\lambda}} \simeq \underset{\mu \in \Lambda^+}{\mathrm{colim}}\, \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{w_0(\check{\lambda})}_{-\kappa}, j_{-\mu,*} \star j_{\mu,*} \star j_{w_0,*}[-d] \star \mathcal{M}) \simeq$$
$$\simeq \mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{w_0(\check{\lambda})}_{-\kappa}, \widetilde{P}''(\mathcal{M})),$$

which according to Theorem 12.3.5 identifies with

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{w_0(\check{\lambda})}_{-\kappa}, \widetilde{P}'(\mathcal{M})),$$

and further, by Corollary 12.4.4 with

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{w_0(\check{\lambda})}_{-\kappa}, \widetilde{P}(\mathcal{M})).$$

Now, by Sect. 11.2.1, the equivalence $\mathsf{F}_{-\kappa}$ intertwines the endo-functor $\widetilde{P}$ on $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ with the functor $\widetilde{P}$ on $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$. Hence, the assertion of Theorem 12.1.3(a) follows from Proposition 12.2.3.

□

12.5. **Relation to semi-infinite cohomology on the Kac-Moody side: positive level case.** In this subsection we will use the formalism of the functors P_μ to prove point (b) of Theorem 12.1.3.

12.5.1. Let $\mathcal{C}$ be a category acted on (strongly) by $G(\mathcal{K})$ at level κ. We define the endo-functors $\widetilde{P}''_\mu$ of $\mathcal{C}^I$ to be given by convolution with the objects

$$j_{w_0,*}[-d] \star j_{w_0(\mu),*} \star j_{-w_0(\mu),*}, \quad \mu \in \Lambda^+.$$

Set

$$\widetilde{P}'' := \underset{\mu \in \Lambda^+}{\mathrm{colim}}\, \widetilde{P}''_\mu.$$

Define the endo-functors $\widetilde{P}'_\mu$ to be given by convolutions with the objects

$$j_{\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{-w_0(\mu),*}.$$

These functors form a directed family by a procedure similar to that in Sect. 12.3.2. Set

$$\widetilde{P}' := \underset{\mu \in \Lambda^{++}}{\mathrm{colim}}\, \widetilde{P}'_\mu.$$

As in Sect. 12.3.4 we have a natural transformation between the families

$$\widetilde{P}'_\mu \to \widetilde{P}''_\mu.$$

From Theorem 12.3.5 we obtain:

Corollary 12.5.2. *The resulting natural transformation* $\widetilde{P}' \to \widetilde{P}''$ *is an isomorphism.*

Proof. The assertion of Corollary 12.5.2 follows from the fact that the situation at the positive level is obtained from that at the negative level by applying the inversion equivalence

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \to \text{D-mod}_{\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I.$$

□

12.5.3. Let P_μ be the directed family of endo-functors of

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathcal{C}^{G(\mathcal{O})}$$

defined in Sect. 12.2.1. We have a term-wise identification

$$\widetilde{P}_\mu \simeq \widetilde{P}'_\mu. \tag{12.10}$$

As in Proposition 12.4.2 we prove:

Proposition 12.5.4. *The term-wise isomorphism* (12.10) *lifts to an isomorphism of directed families for* $\mu \in \Lambda^+$.

Hence:

Corollary 12.5.5. *There exists an isomorphism of endo-functors* $\widetilde{P} \simeq \widetilde{P}'$.

12.5.6. We are now ready to deduce Theorem 12.1.3(b):

We have:

$$\mathrm{C}^{\frac{\infty}{2}}\left(\mathfrak{n}^-(\mathcal{K}), \underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*} \star j_{\mu,*} \star \mathcal{M}\right)^{\check{\lambda}} \simeq$$

$$\simeq \mathrm{C}^{\frac{\infty}{2}}\left(\mathfrak{n}(\mathcal{K}), j_{w_0,*}[-d] \star \left(\underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*} \star j_{\mu,*}\right) \star \mathcal{M}\right)^{w_0(\check{\lambda})} \simeq$$

$$\simeq \mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \widetilde{P}''(\mathcal{M}))^{w_0(\check{\lambda})},$$

which according to Corollary 12.5.2 identifies with

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \widetilde{P}'(\mathcal{M}))^{w_0(\check{\lambda})},$$

and further, according to Corollary 12.5.5, with

$$\mathrm{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \widetilde{P}(\mathcal{M}))^{w_0(\check{\lambda})}.$$

Now, by Sect. 11.2.1, the equivalence F_κ intertwines the endo-functor $\widetilde{P}$ on $\widehat{\mathfrak{g}}\text{-mod}^I_\kappa$ with the functor $\widetilde{P}$ on $\operatorname{Rep}^{\mathrm{mxd}}_{q^{-1}}(G)$. Hence, the assertion of Theorem 12.1.3(a) follows from Proposition 12.2.3.

□

12.6. **Proof of Theorem 12.3.5.**

12.6.1. Although the assertion of Theorem 12.3.5 is geometric (talks about D-modules on the affine flag scheme), we will use representation theory to prove it. Namely, choose an integral weight $\check{\lambda} \in \check{\Lambda}$ which is κ-admissible (see Sect. 3.6.1) and *regular*, which means that the inequalities in (3.4) are strict.

Then the Kashiwara-Tanisaki localization theorem says that the functor

$$\mathcal{F} \mapsto \mathcal{F} \star \mathbb{M}^{\check{\lambda}}_{-\kappa}, \quad \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \to \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa} \tag{12.11}$$

is conservative (in addition to being t-exact).

Indeed, under the equivalence

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \simeq \text{D-mod}_{(-\kappa,\lambda)}(\mathrm{Fl}^{\mathrm{aff}}_G)^I,$$

the functor (12.11) corresponds to the functor

$$\Gamma(\mathrm{Fl}^{\mathrm{aff}}_G, -) : \text{D-mod}_{(-\kappa,\lambda)}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \to \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}.$$

12.6.2. Hence, it suffices to show that the resulting map

$$\underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\star \mathbb{M}^{\check\lambda}_{-\kappa}[d]\to \underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*}\star j_{\mu,*}\star j_{w_0,*}\star \mathbb{M}^{\check\lambda}_{-\kappa} \tag{12.12}$$

is an isomorphism.

We will show that both sides of (12.12) yield an object $\mathcal{M}\in\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ that satisfies:

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check\lambda'}_{-\kappa},\mathcal{M})=\begin{cases} k \text{ if } \check\lambda'=w_0(\check\lambda)-2\check\rho;\\ 0 \text{ otherwise.}\end{cases}$$

This would imply that the map (12.12) is an isomorphism: indeed it is easy to see that it is non-zero, while the objects $\mathbb{W}^{\check\lambda'}_{-\kappa}$ generate $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$.

Remark 12.6.3. Note that the above object $\mathcal{M}$ is *not* one of the affine dual Verma modules: the latter are right-orthogonal to affine Verma modules, whereas our $\mathcal{M}$ is right-orthogonal to the Wakimoto modules.

12.6.4. The fact that the functor (12.11) is t-exact and conservative implies that $\mathbb{M}^{\check\lambda}_{-\kappa}$ is irreducible.

This, in turn implies that the canonical map

$$\mathbb{M}^{\check\lambda}_{-\kappa}\to\mathbb{W}^{\check\lambda}_{-\kappa}$$

is an isomorphism (indeed, the two sides have equal formal characters).

12.6.5. Now the fact that the RHS in (12.12) yields an object with the desired orthogonality property follows from Corollary 3.4.10.

12.6.6. For the LHS we have:

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}\left(\mathbb{W}^{\check\lambda'}_{-\kappa},\underset{\mu\in\Lambda^+}{\operatorname{colim}}\, j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\star\mathbb{W}^{\check\lambda}_{-\kappa}[d]\right)\simeq$$
$$\simeq\underset{\mu\in\Lambda^+}{\operatorname{colim}}\,\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check\lambda'}_{-\kappa}, j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\star\mathbb{W}^{\check\lambda}_{-\kappa}[d]).$$

For an individual μ, we have:

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\check\lambda'}_{-\kappa}, j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\star\mathbb{W}^{\check\lambda}_{-\kappa}[d])\simeq$$
$$\simeq\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(j_{\mu,!}\star\mathbb{W}^{\check\lambda'}_{-\kappa}, p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*}\star\mathbb{W}^{\check\lambda}_{-\kappa}[d])\simeq$$
$$\simeq\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{W}^{\mu+\check\lambda'}_{-\kappa}, p^*(\delta_{1,\mathrm{Gr}})\star\mathbb{W}^{w_0(\mu)+\check\lambda}_{-\kappa}[d])\simeq$$
$$\simeq\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}\left(\mathbb{W}^{\mu+\check\lambda'}_{-\kappa},\mathbf{oblv}_{G(\mathcal{O})/I}\circ\mathrm{Av}^{G(\mathcal{O})/I}_*(\mathbb{W}^{w_0(\mu)+\check\lambda}_{-\kappa})[d]\right)\simeq$$
$$\simeq\mathcal{H}om_{\mathrm{KL}(G,-\kappa)}(\mathrm{Av}^{G(\mathcal{O})/I}_!(\mathbb{W}^{\mu+\check\lambda'}_{-\kappa}),\mathrm{Av}^{G(\mathcal{O})/I}_*(\mathbb{W}^{w_0(\mu)+\check\lambda}_{-\kappa})[d]).$$

Assume now that μ is large enough so that $\mu+\lambda'$ is dominant and $w_0(\mu)+\check\lambda+2\check\rho$ is anti-dominant. In this case

$$\mathrm{Av}^{G(\mathcal{O})/I}_!(\mathbb{W}^{\mu+\check\lambda'}_{-\kappa})\simeq\mathbb{V}^{\mu+\check\lambda'}_{-\kappa}\text{ and }\mathrm{Av}^{G(\mathcal{O})/I}_*(\mathbb{W}^{w_0(\mu)+\check\lambda}_{-\kappa})[d]\simeq\mathbb{V}^{\vee,\mu+w_0(\check\lambda)-2\check\rho}_{-\kappa},$$

and the desired orthogonality is manifest.

□

12.7. **Proof of Proposition 12.4.2.**

12.7.1. Since the objects

$$j_{-\mu,*}\star p^*(\delta_{1,\mathrm{Gr}})\star j_{w_0(\mu),*},\quad \mu\in\Lambda^{++}$$

lie in the heart of the t-structure, it suffices to show that the transition maps coincide for individual 1-morphisms (i.e., higher homotopy coherence is automatic).

12.7.2. Recall that the term-wise isomorphism

$$\widetilde{P}_\mu \simeq \widetilde{P}'_\mu$$

comes from the (tautological) identifications

$$\mathrm{Av}_*^{G(\mathcal{O})/I} \simeq \mathbf{coind}_{\check{B}\to\check{G}} \circ (\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}} \text{ and } \mathbf{oblv}_{G(\mathcal{O})/I} \simeq (\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}} \circ \mathbf{oblv}_{\check{G}\to\check{B}}.$$

Recall also that according to Corollary 11.1.7, the functor $(\mathrm{Av}_*^{G(\mathcal{O})/I})^{\mathrm{enh}}$ is given by convolution with the object

$$(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}[-2d] \in \text{D-mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})}.$$

12.7.3. Let us now recall the construction of the transition maps for the family $\{\widetilde{P}_\mu\}$.

First, let us note that Geometric Satake defines an identification

$$\mathrm{Sat}(V^{-w_0(\mu)}) \simeq \mathrm{Sat}((V^\mu)^\vee), \tag{12.13}$$

thereby fixing a choice of a lift $w_0' \in \mathrm{Norm}_{\check{G}}(\check{T})$.

Indeed,

$$\mathrm{Sat}(V^\mu) \simeq \mathrm{IC}_\mu, \tag{12.14}$$

so (12.13) is the assertion that the objects

$$\mathrm{IC}_\mu \text{ and } \mathrm{IC}_{-w_0(\mu)}$$

of $\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ are monoidal duals of each other. However, this follows from the fact that the operation of passage to the monoidal dual in $\text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ is given by

$$\mathcal{F} \mapsto \mathbb{D}(\mathcal{F}^{\mathrm{inv}}),$$

where

$$\mathrm{inv} : \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})} \to \text{D-mod}_{\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})}$$

is induced by inversion, and $\mathbb{D} : \text{D-mod}_{\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})} \to \text{D-mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ is Verdier duality.

12.7.4. Thus, the assertion of Proposition 12.4.2 amounts to the verification of the commutativity of the following diagram

$$\begin{array}{ccc}
p^*(\delta_{1,\mathrm{Gr}}) & \longrightarrow & \mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}[-2d]\right) \\
& & \downarrow \\
\downarrow & & \mathrm{IC}_{-w_0(\mu)} \star \mathrm{IC}_\mu \star \mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}[-2d]\right) \\
& & \downarrow \\
j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} & \longrightarrow & j_{-\mu,*} \star \mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}[-2d]\right) \star j_{w_0(\mu),*}
\end{array}$$

where the lower right vertical arrow comes from the canonical maps

$$\mathrm{IC}_{-w_0(\mu)} \to p_*(j_{-\mu,*})$$

and

$$\mathrm{IC}_\mu \star \mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}\right) \to \mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}} \star j_{w_0(\mu),*}\right) \tag{12.15}$$

Let us rewrite the map (12.15) more explicitly. First off, we have

$$\mathbf{coind}_{\check{B}\to\check{G}}\left((\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa})^{\mathrm{enh}}\right) \simeq \mathbf{inv}_{\check{B}}\left(\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}\right).$$

By unwinding the definitions, we obtain that the map

$$\mathrm{IC}_\mu \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \to \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star j_{w_0(\mu),*},$$

obtained from (12.15), and viewed as a map of objects of $\mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ equipped with an action of $\check{B}$, equals the composition

$$\mathrm{IC}_\mu \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} = \mathrm{Sat}(V^\mu) \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \simeq \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \otimes \underline{V^\mu} \to$$
$$\to \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \otimes k^{w_0(\mu)} \simeq \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star j_{w_0(\mu),*},$$

where the projection $V^\mu \to k^{w_0(\mu)}$ uses the specified choice of the lift $w_0' \in \mathrm{Norm}_{\check{G}}(\check{T})$, and the isomorphism

$$\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \otimes k^{w_0(\mu)} \simeq \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star j_{w_0(\mu),*}$$

follows from the construction of $\overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}$ as

$$\underset{\nu}{\oplus} \underset{\mu' \in \Lambda^+}{\mathrm{colim}}\, \mathrm{Sat}(V^{\mu'}) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{-\nu-\mu',*}).$$

To summarize, we need to establish the commutativity of the following diagram

$$\begin{array}{ccc}
p^*(\delta_{1,\mathrm{Gr}}) & \longrightarrow & \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}[-2d] \\
 & & \downarrow \\
\downarrow & & \mathrm{IC}_{-w_0(\mu)} \star \mathrm{IC}_\mu \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}[-2d] \\
 & & \downarrow \\
j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} & \longrightarrow & j_{-\mu,*} \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa}[-2d] \star j_{w_0(\mu),*}.
\end{array}$$

12.7.5. Let us now describe explicitly the map

$$\mathrm{IC}_\mu \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \to \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star j_{w_0(\mu),*}.$$

By unwinding the constructions, we obtain that this map fits into the commutative diagrams

$$\begin{array}{ccc}
\mathrm{IC}_\mu \star \mathrm{IC}_{-w_0(\mu)} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{-\nu+w_0(\mu),*}) & \longrightarrow & \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{-\nu+w_0(\mu),*}) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{IC}_\mu \star \mathrm{Sat}(V^{-w_0(\mu)}) \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{-\nu+w_0(\mu),*}) & & \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{-\nu,*}) \star j_{w_0(\mu),*} \\
\downarrow & & \downarrow \\
\mathrm{IC}_\mu \star \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} & \longrightarrow & \to \overset{\bullet}{\mathcal{F}}{}^{-,\frac{\infty}{2},\mathrm{inv}}_{-\kappa} \star j_{w_0(\mu),*}
\end{array}$$

for $\nu \in \Lambda$.

Hence, we are reduced to showing the commutativity of the next diagram

$$\begin{array}{ccc}
p^*(\delta_{1,\mathrm{Gr}}) & \longrightarrow & \mathrm{IC}_{-w_0(\mu)} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{w_0(\mu),*})[-2d] \\
 & & \downarrow \\
\downarrow & & \mathrm{IC}_{-w_0(\mu)} \star \mathrm{IC}_\mu \star \mathrm{IC}_{-w_0(\mu)} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{w_0(\mu),*})[-2d] \\
 & & \downarrow \\
j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} & \longrightarrow & j_{-\mu,*} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{w_0(\mu),*})[-2d].
\end{array}$$

12.7.6. We note, however, that the composite right vertical arrow coincides with the map

$$\mathrm{IC}_{-w_0(\mu)} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{w_0(\mu),*}) \to j_{-\mu,*} \star \mathrm{Av}_!^{G(\mathcal{O})/I}(j_{w_0(\mu),*})$$

induced by the map $\mathrm{IC}_{-w_0(\mu)} \to p_*(j_{-\mu,*})$.

Identifying $\mathrm{Av}_!^{G(\mathcal{O})/I}(-)[-2d] \simeq \mathrm{Av}_*^{G(\mathcal{O})/I}$, we obtain that it suffices to show that the map

$$\delta_{1,\mathrm{Gr}} \to j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star p_*(j_{w_0(\mu),*})$$

of (12.7) equals the composition

$$p^*(\delta_{1,\mathrm{Gr}}) \to \mathrm{IC}_{-w_0(\mu)} \star \mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0(\mu),*}) \to j_{-\mu,*} \star \mathrm{Av}_*^{G(\mathcal{O})/I}(j_{w_0(\mu),*}),$$

which is an elementary verification.

□

13. Interpretation in terms of coherent sheaves

In this final section we will show that the (conjectural) equivalence (9.7) allows to compare a *regular block* of $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$ with the category of ind-coherent sheaves on the Steinberg variety for $\check{G}$,

13.1. **Bezrukavnikov's theory: recollections.**

13.1.1. Bezrukavnikov's theory of [Bez] states the existence of an equivalence of monoidal categories

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G}) \tag{13.1}$$

and of their module categories

$$\text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^I \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}), \tag{13.2}$$

where $\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}} := G(\mathcal{K})/\overset{\circ}{I}$.

Under the above equivalences, the direct image functor along

$$\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G} \to \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G})$$

corresponds to the pullback functor along $\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}} \to \mathrm{Fl}_G^{\mathrm{aff}}$ shifted by $[\dim(T)]$.

13.1.2. We normalize the equivalence (13.1) so that the object $J_\mu \in \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I$ goes over to the image if $k^\mu \in \mathrm{Rep}(\check{B})$ under the composite functor

$$\mathrm{Rep}(\check{B}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{B}) \overset{\mathfrak{q}^*}{\to} \mathrm{QCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B})) \overset{-\otimes \omega_{\check{\mathfrak{n}}/\mathrm{Ad}(\check{B})}}{\longrightarrow} \mathrm{IndCoh}(\check{\mathfrak{n}}/\mathrm{Ad}(\check{B})) \simeq$$
$$\simeq \mathrm{IndCoh}(\widetilde{\check{\mathcal{N}}}/\check{G}) \overset{*\text{-pshfwd}}{\longrightarrow} \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G}).$$

In particular the monoidal unit $\delta_{1,\mathrm{Fl}} \in \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I$ corresponds to the image of $k \in \mathrm{Rep}(\check{B})$ under the above functor (which is a monoidal unit in $\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G})$, as it should be).

13.1.3. Along with the equivalences (13.1) and (13.2) one proves their spherical counterparts:

$$\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})} \simeq \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathcal{N}}})/\check{G}) \tag{13.3}$$

and

$$\text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^{G(\mathcal{O})} \simeq \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}). \tag{13.4}$$

The equivalence (13.3) has the following features:

- For $\mu \in \Lambda$, it sends $\mathrm{Av}_!^{G(\mathcal{O})/I}(J_\mu)[-d] \in \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})}$ to the image of $k^\mu \in \mathrm{Rep}(\check{B})$ under the functor

$$\mathrm{Rep}(\check{B}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{B}) \overset{-\otimes\omega_{\mathrm{pt}/\check{B}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{B}) \to \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}), \tag{13.5}$$

where the last arrow is $*$-pushforward along

$$\mathrm{pt}/\check{B} \to (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \check{\mathfrak{n}})/\mathrm{Ad}(\check{B}) \simeq (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}.$$

- For $\mu \in \Lambda^+$ it sends $p^!(\mathrm{Sat}(V^\mu))[-d] \in \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})}$ to the image of $V^\mu \in \mathrm{Rep}(\check{G})$ under the composition of $\mathbf{oblv}_{\check{G}\to\check{B}} : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{B})$ and the functor (13.5).

13.1.4. The equivalences (13.1) and (13.3) are related via the commutative diagram

$$\begin{array}{ccc}
\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I & \longrightarrow & \mathrm{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}) \\
{\scriptstyle \mathrm{Av}_!^{G(\mathcal{O})/I}[-d]}\downarrow & & \downarrow \\
\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})} & \longrightarrow & \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}),
\end{array}$$

where the right vertical arrow is

$$\mathrm{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}) \simeq \mathrm{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{B}) \overset{\text{!-plbck}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{B}) \overset{*\text{-pshfwd}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G})$$

or, by passing to right adjoints, via the diagram

$$\begin{array}{ccc}
\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I & \longrightarrow & \mathrm{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}) \\
{\scriptstyle \mathbf{oblv}_{G(\mathcal{O})/I}}\uparrow & & \uparrow \\
\text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^{G(\mathcal{O})} & \longrightarrow & \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}),
\end{array} \tag{13.6}$$

where the right vertical arrow is the functor

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}) \overset{\text{!-plbck}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{B}) \overset{k^{-2\rho}\otimes-}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{B}) \overset{*\text{-pshfwd}}{\longrightarrow}$$
$$\to \mathrm{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{B}) \simeq \mathrm{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}).$$

The functors (13.2) and (13.4) are related in a similar way.

13.1.5. Note that it follows that the image of

$$J_{2\rho} \star p^*(\delta_{1,\mathrm{Gr}})[d] \in \text{D-mod}_{-\kappa}(\mathrm{Fl}_G^{\mathrm{aff}})^I$$

under the equivalence (13.1) is the image of the dualizing sheaf on $\mathrm{QCoh}((\check{G}/\check{B} \times \check{G}/\check{B})/\check{G}$ under

$$\mathrm{IndCoh}((\check{G}/\check{B} \times \check{G}/\check{B})/\check{G}) \overset{*\text{-pshfwd}}{\longrightarrow} \mathrm{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\mathcal{N}}})/\check{G}).$$

13.2. **Regular block categories.**

13.2.1. The linkage principle for $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$ says that

$$\mathcal{H}om_{\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}}(\mathbb{M}^{\check{\lambda}_1}_{-\kappa}, \mathbb{M}^{\check{\lambda}_2}_{-\kappa}) = 0$$

unless $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}$ lie in the same orbit of the "dotted" action of $W^{\text{aff,ext}} := W \ltimes \Lambda$ on $\check{\Lambda}$, i.e., when $\check{\lambda}_2$ is not of the form

$$(\mu \cdot w) \cdot \check{\lambda}_1 := \mu + w(\check{\lambda}_1 + \check{\rho}) - \rho, \quad \mu \in \Lambda,\, w \in W.$$

Fix a *regular* κ-admissible weight $\check{\lambda}_0$. Let

$$\mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}) \subset \widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$$

be the full subcategory generated by affine Verma modules $\mathbb{M}^{\check{\lambda}}_{-\kappa}$ for $\check{\lambda} \in W^{\text{aff,ext}} \cdot \check{\lambda}_0$. By the above, this subcategory is actually a direct summand of $\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}$.

Since the objects $\mathbb{M}^{\check{\lambda}}_{-\kappa}$ and $\mathbb{W}^{\check{\lambda}}_{-\kappa}$ have the same decomposition series, we have

$$\mathbb{W}^{\check{\lambda}}_{-\kappa} \in \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}) \Leftrightarrow \check{\lambda} \in W^{\text{aff,ext}} \cdot \check{\lambda}_0.$$

13.2.2. The Kashiwara-Tanisaki localization theorem of [KT] (see also [FG1, Theorem 5.5]) says that there exists a canonical t-exact equivalence

$$\text{D-mod}_{(-\kappa,\check{\lambda}_0)}(\widetilde{\text{Fl}}^{\text{aff}}_G)^I \simeq \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}), \tag{13.7}$$

given by taking sections on $\widetilde{\text{Fl}}^{\text{aff}}_G$ and then T-invariants.

The composite functor

$$\text{D-mod}_{-\kappa}(\text{Fl}^{\text{aff}}_G)^I \to \text{D-mod}_{-\kappa}(\widetilde{\text{Fl}}^{\text{aff}}_G)^I \simeq \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa})$$

is given by

$$\mathcal{F} \mapsto \mathcal{F} \star \mathbb{M}^{\check{\lambda}_0}_{-\kappa}.$$

Composing with (13.2) we thus obtain an equivalence

$$\text{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}) \tag{13.8}$$

13.2.3. The linkage principle for the small quantum group implies that

$$\mathcal{H}om_{\text{Rep}_q^{\text{sml,grd}}(G)}(\mathbb{M}^{\check{\lambda}_1}_{q,\text{sml}}, \mathbb{M}^{\check{\lambda}_2}_{q,\text{sml}}) = 0$$

unless unless $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}$ lie in the same orbit of the dotted action of $W^{\text{aff,ext}}$ on $\check{\Lambda}$.

Using the grading, it follows that we also have

$$\mathcal{H}om_{\text{Rep}_q^{\text{mxd}}(G)}(\mathbb{M}^{\check{\lambda}_1}_{q,\text{mxd}}, \mathbb{M}^{\check{\lambda}_2}_{q,\text{mxd}}) = 0$$

unless $\check{\lambda}_2 \in W^{\text{aff,ext}} \cdot \check{\lambda}_1$.

Let

$$\mathsf{Bl}(\text{Rep}_q^{\text{mxd}}(G)) \subset \text{Rep}_q^{\text{mxd}}(G)$$

be the full subcategory generated by the standard objects $\mathbb{M}^{\check{\lambda}}_{q,\text{mxd}}$ for $\check{\lambda} \in W^{\text{aff,ext}} \cdot \check{\lambda}_0$. By the above, this subcategory is actually a direct summand of $\text{Rep}_q^{\text{mxd}}(G)$.

13.2.4. Thus, from Conjecture 9.2.2 we deduce:

Conjecture 13.2.5. *There exists an equivalence*

$$\mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}) \simeq \mathsf{Bl}(\text{Rep}_q^{\text{mxd}}(G)).$$

Combining with (13.8), we obtain:

Conjecture 13.2.6. *There exists an equivalence*

$$\text{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \mathsf{Bl}(\text{Rep}_q^{\text{mxd}}(G)).$$

13.2.7. Thus, we obtain a string if equivalences

$$\text{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \text{D-mod}_{-\kappa}(\widetilde{\text{Fl}}^{\text{aff}}_G)^I \simeq \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}^I_{-\kappa}) \simeq \mathsf{Bl}(\text{Rep}^{\text{mxd}}_q(G)). \tag{13.9}$$

If we assume also the compatibility of $\mathsf{F}_{-\kappa}$ specified in Sect. 11.2.1, then these equivalences respect the action of $\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B})) \simeq \text{QCoh}(\widetilde{\widetilde{\mathcal{N}}}/\check{G})$.

Under the equivalences (13.9) we have the following correspondence of objects

$$\Delta^{\text{IndCoh}}_* \circ (- \otimes \omega_{\check{\mathfrak{n}}/\check{B}}) \circ \mathsf{q}^*(k^\mu) \;\leftrightarrow\; J_\mu \;\leftrightarrow\; \mathbb{W}^{\mu+\check{\lambda}_0}_{-\kappa} \;\leftrightarrow\; \mathbb{M}^{\mu+\check{\lambda}_0}_q, \tag{13.10}$$

13.2.8. We obtain that for $\mu \in \Lambda \subset \check{\Lambda}$, the functor

$$\text{C}^\cdot(U^{\text{Lus}}_q(N), -)^{\mu+\check{\lambda}_0} : \mathsf{Bl}(\text{Rep}^{\text{mxd}}_q(G)) \to \text{Vect}$$

corresponds to the functor on $\text{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G})$ given by

$$\text{IndCoh}((\widetilde{\widetilde{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \xrightarrow{\Delta^!} \text{IndCoh}(\widetilde{\widetilde{\mathcal{N}}}/\check{G}) \simeq \text{IndCoh}(\check{\mathfrak{n}}/\check{B}) \xrightarrow{-\otimes\omega^{-1}_{\check{\mathfrak{n}}/\check{B}}} \text{QCoh}(\check{\mathfrak{n}}/\check{B}) \xrightarrow{\mathsf{q}_*}$$

$$\to \text{QCoh}(\text{pt}/\check{B}) \xrightarrow{\mathcal{H}om(k^\mu,-)} k.$$

Remark 13.2.9. The composite conjectural equivalence

$$\text{IndCoh}(\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}) \simeq \mathsf{Bl}(\text{Rep}^{\text{mxd}}_q(G)), \tag{13.11}$$

viewed as categories acted on by $\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))$ implies the following:

Take an element $\chi \in \check{\mathfrak{n}}$, and let us tensor both sides of (13.11) with Vect over $\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))$, where

$$\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B})) \to \text{Vect}$$

corresponds to the evaluation at χ. We obtain that the resulting category

$$\mathsf{Bl}(\text{Rep}^\chi_q(G)) := \text{Vect} \underset{\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \mathsf{Bl}(\text{Rep}^{\text{mxd}}_q(G))$$

identifies with

$$\text{Vect} \underset{\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \text{IndCoh}(\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}). \tag{13.12}$$

Note that (13.12) is a full subcategory of

$$\text{IndCoh}(\text{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}),$$

where $\text{pt} \to \check{\mathfrak{g}}$ is the image of χ under $\check{\mathfrak{n}} \to \check{\mathfrak{g}}$. Here the (derived) scheme

$$\text{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}$$

is the derived Springer fiber over χ.

Remark 13.2.10. One actually expects something a little stronger than the above equivalence

$$\text{Vect} \underset{\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \text{IndCoh}(\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}) \simeq \mathsf{Bl}(\text{Rep}^\chi_q(G))_\chi. \tag{13.13}$$

Namely, let $\text{Rep}^\chi_q(G)_{\text{ren}}$ be the following renormalized version of

$$\text{Rep}^\chi_q(G) := \text{Vect} \underset{\text{QCoh}(\check{\mathfrak{n}}/\operatorname{Ad}(\check{B}))}{\otimes} \text{Rep}^{\text{mxd}}_q(G).$$

Namely, $\text{Rep}^\chi_q(G)_{\text{ren}}$ is the ind-completion of the full (but not cocomplete) subcategory of comprised of objects whose image under the forgetful functor $\text{Rep}^\chi_q(G) \to \text{Rep}^{\text{mxd}}_q(G)$ is compact.

Let $\mathsf{Bl}(\mathrm{Rep}_q^{\chi}(G)_{\mathrm{ren}})$ be the corresponding direct summand of $\mathrm{Rep}_q^{\chi}(G)_{\mathrm{ren}}$. Then we expect an equivalence

$$\mathrm{IndCoh}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}) \simeq \mathsf{Bl}(\mathrm{Rep}_q^{\chi}(G)_{\mathrm{ren}}). \tag{13.14}$$

Note that for $\chi = 0$, we have

$$\mathrm{Rep}_q^0(G) \simeq \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\text{baby-ren}} \text{ and } \mathrm{Rep}_q^0(G)_{\mathrm{ren}} \simeq \mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}},$$

so in this case (13.14) recovers the equivalence

$$\mathrm{IndCoh}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}) \simeq \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}),$$

see (13.17) below.

13.3. **Spherical case and relation to the** [ABG] **equivalence.**

13.3.1. Let $\mathsf{Bl}(\mathrm{KL}(G,-\kappa))$ denote the premiage of $\mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I)$ under the forgetful functor

$$\mathbf{oblv}_{G(\mathcal{O})/I} : \mathsf{Bl}(\mathrm{KL}(G,-\kappa)) \to \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I).$$

This is a direct summand of $\mathsf{Bl}(\mathrm{KL}(G,-\kappa))$ generated by the Weyl modules

$$\mathbb{V}_{-\kappa}^{\check{\lambda}}, \quad \check{\lambda} \in \check{\Lambda}^+ \cap (W^{\mathrm{aff,ext}} \cdot \check{\lambda}_0).$$

The Kashiwara-Tanisaki equivalence induces an equivalence

$$\text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^{G(\mathcal{O})} \to \mathsf{Bl}(\mathrm{KL}(G,-\kappa))$$

that makes the following diagram commute:

$$\begin{array}{ccc} \text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^I & \longrightarrow & \mathsf{Bl}(\widehat{\mathfrak{g}}\text{-mod}_{-\kappa}^I) \\ \uparrow & & \uparrow \\ \text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^{G(\mathcal{O})} & \longrightarrow & \mathsf{Bl}(\mathrm{KL}(G,-\kappa)). \end{array}$$

13.3.2. Let

$$\mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}) \subset \mathrm{Rep}_q(G)_{\mathrm{ren}}$$

be the full subcategory generated by the Weyl modules

$$\mathcal{V}_q^{\check{\lambda}}, \quad \check{\lambda} \in \check{\Lambda}^+ \cap (W^{\mathrm{aff,ext}} \cdot 0).$$

The linkage principle for $\mathrm{Rep}_q(G)_{\mathrm{ren}}$ says that $\mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}})$ is actually a direct summand of $\mathrm{Rep}_q(G)_{\mathrm{ren}}$.

The Kazhdan-Lusztig equivalence $\mathsf{F}_{-\kappa}$ induces an equivalence

$$\mathsf{Bl}(\mathrm{KL}(G,-\kappa)) \to \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}).$$

13.3.3. Composing, we obtain a string of equivalences

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \text{D-mod}_{-\kappa}(\widetilde{\mathrm{Fl}}_G^{\mathrm{aff}})^{G(\mathcal{O})} \simeq \mathsf{Bl}(\mathrm{KL}(G,-\kappa)) \simeq \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}), \tag{13.15}$$

compatible with those in (13.9) under the functors specified earlier.

The composite equivalence

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}) \tag{13.16}$$

is the equivalence established in [ABG].

13.3.4. Note that under the equivalence (13.16), the image of

$$k^\mu \in \mathrm{Rep}(\check{B}), \quad \mu \in \Lambda$$

under the functor

$$\mathrm{Rep}(\check{B}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{B}) \overset{-\otimes\omega_{\mathrm{pt}/\check{B}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{B}) \overset{*\text{-pshfwd}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \check{\mathfrak{b}})/\check{B}) \simeq \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G})$$

corresponds to

$$\mathcal{V}_q^{\mu+\check{\lambda}_0}[-d] \simeq \mathbf{ind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\mu+\check{\lambda}_0})[-d] \simeq \mathbf{coind}_{\mathrm{Lus}^+\to\mathrm{big}}(k^{\mu+\check{\lambda}_0+2\check{\rho}}) \simeq \mathcal{V}_q^{\vee,\mu+\check{\lambda}_0+2\check{\rho}} \in \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}).$$

The latter identification seems well-known for $\mu \in -\Lambda^+$, but may be new for more general μ.

13.3.5. The equivalence (13.16) induces a diagram of equivalences:

$$(13.17)\qquad \begin{array}{ccc}
\mathrm{IndCoh}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}) & \longrightarrow & \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml}}(G)_{\mathrm{ren}}) \\
{\scriptstyle !\text{-plbck}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{sml.grd}\to\mathrm{sml}}} \\
\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T}) & \longrightarrow & \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}) \\
{\scriptstyle !\text{-plbck}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\frac{1}{2}\to\mathrm{sml.grd}}} \\
\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}) & \longrightarrow & \mathsf{Bl}(\mathrm{Rep}_q^{\frac{1}{2}}(G)_{\mathrm{ren}}) \\
{\scriptstyle !\text{-plbck}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}\mathrm{big}\to\frac{1}{2}} \\
\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) & \longrightarrow & \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}).
\end{array}$$

13.3.6. Let i denote the embedding of the unit point

$$\mathrm{pt} \to \check{G}/\check{B}.$$

We obtain that with respect to the equivalence

$$(13.18)\qquad \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T}) \simeq \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}),$$

the image of $k^\mu \in \mathrm{Rep}(\check{T})$ under the functor

$$\mathrm{Rep}(\check{T}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{\mathsf{i}_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T})$$

corresponds to the object

$$\mathbf{ind}_{\mathrm{sml}^+\to\mathrm{sml.grd}}(k^{\mu+\check{\lambda}_0})[-d] \in \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}).$$

13.3.7. Let i^- denote the embedding of the point w_0

$$\mathrm{pt} \to \check{G}/\check{B}.$$

We obtain that under the equivalence (13.18) the image of $k^\mu \in \mathrm{Rep}(\check{T})$ under the functor

$$\mathrm{Rep}(\check{T}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{(\mathsf{i}^-)_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T})$$

corresponds to the object

$$\mathbf{ind}_{\mathrm{sml}^-\to\mathrm{sml.grd}}(k^{\mu+\check{\lambda}_0})[-d] \in \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}).$$

13.3.8. From the above identifications, we obtain that for $\mu \in \Lambda \subset \check{\Lambda}$, the functor

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T}) \to \mathrm{Vect}$$

given by

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T}) \overset{\mathsf{i}^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega^{-1}_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^\mu,-)}{\longrightarrow} \mathrm{Vect}$$

corresponds to the functor

$$\mathrm{C}^\cdot(u_q(N),-)^{\mu+\check{\lambda}_0}[d] : \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}) \to \mathrm{Vect}\,.$$

Similarly, the functor

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{T}) \overset{(\mathsf{i}^-)^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega^{-1}_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^\mu,-)}{\longrightarrow} \mathrm{Vect}$$

$$\mathrm{C}^\cdot(u_q(N^-),-)^{\mu+\check{\lambda}_0}[d] : \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{sml,grd}}(G)_{\mathrm{ren}}) \to \mathrm{Vect}\,.$$

13.4. **Cohomology of the DK quantum group via coherent sheaves.** We will now give an interpretation of the functors $\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-),-)^{\check{\lambda}}$ on $\mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{mxd}}(G))$ and $\mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}})$ (for some specific values of $\check{\lambda}$) in terms of coherent sheaves.

For the duration of this subsection we will assume Conjecture 13.2.5.

13.4.1. We obtain that for $\check{\lambda}$ of the form $\mu + \check{\lambda}_0$, the functor

$$\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-),-)^{\mu+\check{\lambda}_0}[d] : \mathsf{Bl}(\mathrm{Rep}_q^{\mathrm{mxd}}(G)) \to \mathrm{Vect} \tag{13.19}$$

corresponds to the functor

$$\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \mathrm{IndCoh}((\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}) \overset{!\text{-plbck}}{\longrightarrow}$$
$$\to \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B}) \overset{!\text{-plbck}}{\longrightarrow} \mathrm{IndCoh}((\check{G}/\check{B})/\check{B}) \overset{!\text{-plbck}}{\longrightarrow}$$
$$\to \mathrm{IndCoh}((\check{G}/\check{B})/\check{T}) \overset{(\mathsf{i}^-)^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega^{-1}_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^\mu,-)}{\longrightarrow} \mathrm{Vect}\,.$$

Note, however, that the composite map

$$\mathrm{pt}/\check{T} \overset{\mathsf{i}^-}{\to} (\check{G}/\check{B})/\check{T} \to (\check{G}/\check{B})/\check{B} \to (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B} \to (\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B} \simeq (\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}$$

appearing in the above formula identifies with

$$\mathrm{pt}/\check{T} \simeq ((\check{G}/\check{B}) \times (\check{G}/\check{B}))^o/\check{G} \subset (\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G},$$

where $((\check{G}/\check{B}) \times (\check{G}/\check{B}))^o \subset (\check{G}/\check{B}) \times (\check{G}/\check{B})$ is the open Bruhat cell, which is naturally an open subset in $(\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}$.

Hence, we obtain that the functor (13.19) corresponds to

$$\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \overset{!\text{-plbck}}{\longrightarrow} \mathrm{IndCoh}(((\check{G}/\check{B}) \times (\check{G}/\check{B}))^o/\check{G}) \simeq \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^\mu,-)}{\longrightarrow} \mathrm{Vect}\,.$$

13.4.2. Note now that we have a Cartesian diagram

$$\begin{array}{ccc} (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \check{\mathfrak{b}}^-)/\check{T} & \longrightarrow & \mathrm{pt}/\check{T} \\ \downarrow & & \downarrow \\ (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B} & \longrightarrow & (\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{B} \end{array}$$

and also a Cartesian diagram

$$\begin{array}{ccc} (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \check{\mathfrak{b}}^-)/\check{T} & \longrightarrow & \mathrm{pt}/\check{T} \\ \downarrow & & \downarrow \\ (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G} & \xrightarrow{r} & (\check{G}/\check{B})/\check{G}, \end{array}$$

where the right vertical arrow is the composite

$$\mathrm{pt}/\check{T} \xrightarrow{\mathfrak{i}^-} (\check{G}/\check{B})/\check{T} \to (\check{G}/\check{B})/\check{G},$$

and the bottom horizontal arrow, denoted r, is

$$(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G} \to \widetilde{\check{\mathfrak{g}}}/\check{G} \to (\check{G}/\check{B})/\check{G}.$$

From here, combined with Conjecture 13.2.6 (and taking into account (13.6)), we obtain:

Conjecture 13.4.3. *Under the* [ABG] *equivalence*

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \simeq \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}),$$

the functor

$$\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N^-), -)^{\mu+\check{\lambda}_0}[d] : \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}) \to \mathrm{Vect}$$

corresponds to the composition

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \overset{r_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}((\check{G}/\check{B})/\check{G}) \overset{!\text{-plbck}}{\longrightarrow} \mathrm{IndCoh}((\check{G}/\check{B})/\check{T}) \overset{(\mathfrak{i}^-)^!}{\longrightarrow}$$

$$\to \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega^{-1}_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^{\mu+2\rho},-)}{\longrightarrow} \mathrm{Vect}.$$

13.4.4. Equivalently, we obtain that the functor

$$\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N), -)^{\mu+\check{\lambda}_0}[d] : \mathsf{Bl}(\mathrm{Rep}_q(G)_{\mathrm{ren}}) \to \mathrm{Vect}$$

corresponds to the composition

$$\mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G}) \overset{r_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}((\check{G}/\check{B})/\check{G}) \overset{!\text{-plbck}}{\longrightarrow} \mathrm{IndCoh}((\check{G}/\check{B})/\check{T}) \overset{\mathfrak{i}^!}{\longrightarrow}$$

$$\to \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega^{-1}_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}/\check{T}) \simeq \mathrm{Rep}(\check{T}) \overset{\mathcal{H}om(k^{\mu+2\rho},-)}{\longrightarrow} \mathrm{Vect}.$$

13.4.5. As a reality check, let us compare the latter expression for $\mathrm{C}^\cdot(U_q^{\mathrm{DK}}(N), -)^{\mu+\check{\lambda}_0}[d]$ with one for $\mathrm{C}^\cdot(u_q(N), -)^{\mu+\check{\lambda}_0}[d]$ given in Sect. 13.3.8.

Namely, we note that for $\mathcal{M} \in u_q(N)\text{-mod}(\mathrm{Rep}_q(T))$ the object

$$\mathbf{coind}_{\mathrm{DK}^+\to\mathrm{sm}^+} \circ \mathbf{oblv}_{\mathrm{sm}^+\to\mathrm{DK}^+}(\mathcal{M})$$

admits a filtration with associated graded isomorphic to

$$\mathcal{M} \otimes \mathrm{Sym}(\check{\mathfrak{n}}[-1]).$$

Similarly, the object

$$r^* \circ \mathfrak{i}_*^{\mathrm{IndCoh}}(k^\mu)$$

has a canonical filtration with the associated being the image under

$$\mathrm{Rep}(\check{T}) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{T}) \overset{-\otimes\omega_{\mathrm{pt}/\check{T}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{pt}/\check{T}) \overset{\mathfrak{i}_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}})/\check{G})$$

of the objects

$$k^{\mu} \otimes \mathrm{Sym}((\check{\mathfrak{g}}/\check{\mathfrak{b}})^{\vee}[1]) \simeq k^{\mu+2\rho} \otimes \mathrm{Sym}(\check{\mathfrak{n}}^{\vee}[1]).$$

13.5. **The big Schubert cell on the D-modules side.**

13.5.1. In this subsection, for completeness, we will prove the following result:

Theorem 13.5.2. *Under the equivalence* (13.1)*, the object*

$$J_{2\rho} \star \left(\underset{\mu\in\Lambda^{++}}{\mathrm{colim}}\, j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*} \in \mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \right)[d]$$

corresponds to the direct image of the dualizing sheaf along the map

$$(\check{G}/\check{B} \times \check{G}/\check{B})^o/\check{G} \to (\widetilde{\widetilde{\check{\mathcal{N}}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{G}.$$

The rest of this subsection is devoted to the proof of this theorem.

13.5.3. Consider the setting of Sect. 12.4.1 with $\mathcal{C} = \mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I$. We have the diagram

$$\begin{array}{ccc} \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})} & \xrightarrow{(\mathbf{oblv}_{G(\mathcal{O})/I})^{\mathrm{enh}}} & \mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I \\ \mathbf{oblv}_{\check{G}\to\check{B}} \uparrow & & \\ \mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^{G(\mathcal{O})} & & \end{array} \tag{13.20}$$

By Proposition 12.4.2, the $\widetilde{P}$ endofunctor of $\mathrm{D\text{-}mod}_{-\kappa}(\mathrm{Fl}^{\mathrm{aff}}_G)^I$ associated with this diagram is given by convolution with the object

$$\underset{\mu\in\Lambda^{++}}{\mathrm{colim}}\, j_{-\mu,*} \star p^*(\delta_{1,\mathrm{Gr}}) \star j_{w_0(\mu),*}.$$

By (13.6), under Bezrukavnikov's equivalence, the diagram (13.20) corresponds to the following diagram:

$$\begin{array}{ccccc} \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{B}) & \xrightarrow{\mathsf{q}^*(k^{-2\check{\rho}})\otimes\iota_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{B}) & \xrightarrow{\sim} & (\widetilde{\widetilde{\check{\mathcal{N}}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{G} \\ {}^{!\text{-plbck}}\uparrow & & & & \\ \mathrm{IndCoh}((\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{G}). & & & & \end{array}$$

Thus, we need to show that the value of the corresponding $\widetilde{P}$ endofunctor of $\mathrm{IndCoh}(\widetilde{\widetilde{\check{\mathcal{N}}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{B})$ on $\Delta_*^{\mathrm{IndCoh}}(\omega_{\widetilde{\check{\mathcal{N}}}/\check{G}})$ produces the object isomorphic to the direct image of the dualizing sheaf along the map

$$(\check{G}/\check{B} \times \check{G}/\check{B})^o/\check{G} \to (\widetilde{\widetilde{\check{\mathcal{N}}}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{G} \simeq (\check{\mathfrak{n}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\widetilde{\check{\mathcal{N}}}})/\check{B}$$

tensored by $\mathsf{q}^*(k^{-2\check{\rho}})[-d]$.

13.5.4. The right adjoint of $\mathsf{q}^*(k^{-2\check{\rho}})\otimes\iota_*^{\mathrm{IndCoh}}$ identifies with $k^{2\check{\rho}}\otimes\iota^!$. And we have a Cartesian diagram

$$\begin{array}{ccccc} \mathrm{pt}/\check{B} & \longrightarrow & \check{\mathfrak{n}}/\check{B} & \xrightarrow{\sim} & \widetilde{\widetilde{\mathcal{N}}}/\check{G} \\ {\scriptstyle\iota}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Delta} \\ (\mathrm{pt}\underset{\check{\mathfrak{g}}}{\times}\widetilde{\widetilde{\mathcal{N}}})/\check{B} & \xrightarrow{\iota} & (\check{\mathfrak{n}}\underset{\check{\mathfrak{g}}}{\times}\widetilde{\widetilde{\mathcal{N}}})/\check{B} & \xrightarrow{\sim} & (\widetilde{\widetilde{\mathcal{N}}}\underset{\check{\mathfrak{g}}}{\times}\widetilde{\widetilde{\mathcal{N}}})/\check{G}, \end{array}$$

hence,

$$\iota^!(\Delta_*^{\mathrm{IndCoh}}(\omega_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}))\simeq\iota_*^{\mathrm{IndCoh}}(\omega_{\mathrm{pt}/\check{B}}).$$

Since the map $\iota:\mathrm{pt}/\check{B}\to(\mathrm{pt}\underset{\check{\mathfrak{g}}}{\times}\widetilde{\widetilde{\mathcal{N}}})/\check{B}$ factors as

$$\mathrm{pt}/\check{B}\xrightarrow{\mathsf{i}}(\check{G}/\check{B})/\check{B}\to(\mathrm{pt}\underset{\check{\mathfrak{g}}}{\times}\widetilde{\widetilde{\mathcal{N}}})/\check{B},$$

it suffices to show that the value of the endofunctor of $\mathrm{IndCoh}((\check{G}/\check{B})/\check{B})$ given by P on the object

$$k^{2\check{\rho}}\otimes\mathsf{i}_*^{\mathrm{IndCoh}}(\omega_{\mathrm{pt}/\check{B}})\in\mathrm{IndCoh}((\check{G}/\check{B})/\check{B}) \tag{13.21}$$

produces the dualizing sheaf on the big Schubert cell in $(\check{G}/\check{B})/\check{B}$, shifted by $[-d]$.

13.5.5. Consider the diagram

$$\begin{array}{ccc} \mathrm{QCoh}((\check{G}/\check{B})/\check{B}) & \xrightarrow{-\otimes\omega_{(\check{G}/\check{B})/\check{B}}} & \mathrm{IndCoh}((\check{G}/\check{B})/\check{B}) \\ {\scriptstyle *\text{-plbck}}\uparrow & & \uparrow{\scriptstyle !\text{-plbck}} \\ \mathrm{QCoh}(\mathrm{pt}/\check{G}) & \xrightarrow{-\otimes\omega_{\mathrm{pt}/\check{G}}} & \mathrm{IndCoh}(\mathrm{pt}/\check{G}). \end{array}$$

The object (13.21) identifies with

$$\mathsf{i}_*(\mathcal{O}_{\mathrm{pt}/\check{B}})[-d]\otimes\omega_{(\check{G}/\check{B})/\check{B}}.$$

Hence, it suffices to show that the endofunctor of $\mathrm{QCoh}((\check{G}/\check{B})/\check{B})$ given by P on $\mathsf{i}_*(\mathcal{O}_{\mathrm{pt}/\check{B}})$ produces the structure sheaf of the big Schubert cell. However, the latter is manifest.

References

[AB] S. Arkhipov, R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands duality*, Israel J. of Math. **170** (2009), 135–185.

[ABG] S. Arkhipov, R. Bezrukavnikov and V. Ginzburg, *Quantum groups, the loop Grassmannian and the Springer resolution*, JAMS **17** (2004), 595–678.

[ABBGM] S. Arkhipov, R. Bezrukavnikov, A. Braverman, D. Gaitsgory and I. Mirković, *Modules over the quantum group and semi-infinite flag manifold*, Transformation Groups **10** (2005), 279–362.

[AG1] S. Arkhipov and D. Gaitsgory, *Another realization of the category of modules over the small quantum group*, Adv. in Math. **173** (2003), 114–143.

[AG2] S. Arkhipov and D. Gaitsgory, *D-modules on the loop group via chiral algebras*, IMRN (2002) no. 4, 165–210.

[APW] H. H. Andersen, P. Polo and K. Wen, *Representations of quantum algebras*, Invent. Math. **104**, 1–59 (1991).

[BD] A. Beilinson and V. Drinfeld, *Chiral Algebras*, AMS Colloquium Publications **51** (2004).

[AriG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves, and the geometric Langlands conjecture*, Selecta Math. N.S. **21** (2015), 1–199.

[Bez] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ.IHES **123**, (2016), 1–67.

[Et] P. Etingof et al, *Tensor categories*, Mathematical Surveys and Monographs **205**, AMS, Providence RI (2015).

[FF] B. Feigin and E.!Frenkel, *A family of representations of affine Lie algebras*, Russ. Math. Surv. **43** (1988) 221–222.

[Fr1] E. Frenkel *Determinant formulas for the free field representations of the Virasoro and Kac-Moody algebras*, Physics letters B **286** (1992), 71–77.

[Fr2] E. Frenkel *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005) 297–404.

[FG1] E. Frenkel and D. Gaitsgory, *Localization of $\widehat{\mathfrak{g}}$-modules on the affine Grassmannian*, joint with E. Frenkel, Ann. of Math. **170** (2009), 1339–1381.

[FG2] E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, in: Algebraic Geometry and Number Theory, Progr. Math. **253** (2006), 69–260, Birkhuser Boston.

[FG3] E. Frenkel and D. Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, Represent. Theory **13** (2009), 470–608.

[Ga1] D. Gaitsgory, *Ind-coherent sheaves*, Moscow Mathematical J. **13** (2013), 399–528.

[Ga2] D. Gaitsgory, *The semi-infinite intersection cohomology sheaf*, Adv. in Math. **327** (2018), 789–868.

[Ga3] D. Gaitsgory, *Sheaves of categories and the notion of 1-affineness*, Contemporary Mathematics **643** (2015), 1–99.

[Geo] University of Georgia VIGRE algebra group, *An analog of Kostant's theorem for the cohomology of quantum groups*, Proc. AMS **138** (2010), 85-99.

[GR] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, Vol. 1: Correspondences and Duality*, Mathematical surveys and monographs **221** (2017), AMS, Providence, RI.

[GY] D. Gaitsgory and A. Yom Din, *An analog of the Deligne-Lusztig duality for (g,K)-modules*, Adv. Math. **333** (2018), 212–265.

[KL] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, J. Amer. Math. Soc. **6** (1993), 905–1011 and **7** (1994), 335–453.

[KT] M. Kashiwara and T. Tanisaki, *The Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J. **77** (1995) 21–62.

[Liu] Chia-Cheng Liu, *Semi-infinite cohomology and Kazhdan-Lusztig equivalence at the positive level*, arXiv:1807.01773.

[Lu] J. Lurie, *Higher Algebra*, available at http://www.math.harvard.edu/∼lurie